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Strongly surjective maps from certain two-complexes with trivial top-cohomology onto the projective plane

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Abstract. For the model two-complex K of the group presentation $\langle x, y \,|\, x^{k+1}yxy\rangle$, with $k \geq 1$ odd, we describe representatives for all free and based homotopy classes of maps from K into the projective plane. As a result we classify the homotopy classes containing only surjective maps. With this approach we get an answer, for maps into the real projective plane, to a classical question in topological root theory, which is known so far, in dimension two, only for maps into the sphere, the torus and the Klein bottle. The answer follows by proving that for all $k \geq 1$ odd, the two-complex K has trivial second integer cohomology group and, for $k \geq 3$ odd, there exist strongly surjective maps from K onto the real projective plane. For $k = 1$, there does not exist such a strongly surjective map.

Contents

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1. Introduction and main theorem

The Hopf-Whitney Classification Theorem [10, Corollary 6.19, p. 244] implies that, for a finite and connected n-dimensional complex X (an n-complex, for short), the set $[X;S^n]$ of the free homotopy classes of maps from X into the n-sphere S^n is in a one-to-one correspondence with the integer cohomology group $H^n(X;\mathbb{Z})$. Thus, there exists a map from X onto S^n whose free homotopy class contains only surjective maps if and only if $H^n(X;\mathbb{Z}) \neq 0$. Such a map is called a *strong surjection* or a *strongly surjective map*.

The composition of a strong surjection from X onto S^n with the double covering map $\mathfrak{p} : S^n \to \mathbb{R}\mathrm{P}^n$ provides a strong surjection from X onto the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. Hence, the assumption $H^n(X;\mathbb{Z}) \neq 0$ implies the existence of a strong surjection from X onto $\mathbb{R}\mathrm{P}^n$. In this article, we prove, for $n = 2$, that the converse implication does not hold true. Namely, we show the existence of strongly surjective maps from K onto $\mathbb{R}\mathrm{P}^2$ for a certain two-complex K for which $H^2(K;\mathbb{Z}) = 0$.

This work concerns a central problem in topological root theory, namely, to know for what closed n-manifold Y, the nullity of the top integer cohomology group of an n-complex X forces the non-existence of strong surjections from X into Y.

Besides the relationship with the Hopf-Whitney Classification Theorem, the relevance of the problem lies in the fact that, by the Universal Coefficient Theorem for Cohomology [8, Theorem 3.2, p. 195], an n-complex X with $H^n(X;\mathbb{Z}) = 0$ is (co)homologically like a $(n-1)$-complex, since such a nullity is equivalent to $H_n(X;\mathbb{Z}) = 0$ and $H_{n-1}(X;\mathbb{Z})$ torsion free; that is, the invariant $H^n(\,\cdot\,;\mathbb{Z})$ is not able to detect the existence of n-cells even when the inclusion $X^{n-1} \hookrightarrow X$ of the $(n-1)$-skeleton of X into X is not a homotopy equivalence.

As a consequence of the classification theorem for surfaces, in dimension two it is more feasible to completely solve the problem. However, the first contributions [1, 2] were presented in dimension three. In the 2000's, C. Aniz answers the problem (proposed by D. L. Gonçalves) for the following 3-manifolds: the cartesian product $S^1 \times S^2$, the non-orientable S^1-bundle over S^2 and the orbit space of S^3 with respect to the action of the Quaternion group. He showed that only in the second case there exists a 3-complex X with $H^3(X;\mathbb{Z}) = 0$ and a strong surjection from X onto the corresponding 3-manifold.

The first conclusive answer in dimension two, other than that provide by the Hopf-Whitney Classification Theorem, was present in 2016 in [4], in which the first author built a countable collection of two-complexes with trivial second integer cohomology group and, from each of them, there exists a strong surjection onto the torus $S^1 \times S^1$. By composing each such strong surjection with the double covering map from the torus onto the Klein bottle, we get a strong surjection onto the Klein bottle.

There is no other known conclusive answer to the problem. Therefore, in dimension two, there are answers only for maps into the sphere, the torus and the Klein bottle.

However, there exist partial answers for maps into the projective plane $\mathbb{R}\mathrm{P}^2$. We refer to the results presented in [5, 3]. In fact, in [5] the germ of the conclusive answer is present: it is shown that a cohomological condition implies the non existence of a strongly surjective map. Here, we give a more complete answer as a consequence of our main theorem:

Theorem 1.1. *Let K be the model two-complex of the group presentation $\mathcal{P} = \langle x, y \,|\, x^{k+1}yxy \rangle$, with $k \geq 1$ odd. Then $H^2(K;\mathbb{Z}) = 0$ and we have:*

1. *If $k = 1$, then $[K;\mathbb{R}\mathrm{P}^2] \equiv [K;\mathbb{R}\mathrm{P}^2]^* \equiv \{1\} \sqcup \{\bar{0}\}$ and both the homotopy classes contain non-surjective maps.*
2. *If $k = 2p - 1 \geq 3$, then $[K;\mathbb{R}\mathrm{P}^2]^* \equiv \{1\} \sqcup \mathbb{Z}_k$ and $[K;\mathbb{R}\mathrm{P}^2] \equiv \{1\} \sqcup \mathbb{Z}_p$. The free homotopy classes corresponding to 1 and $\bar{0}$ contain non-surjective maps and the remaining $p - 1$ classes contain only surjective maps.*

The notation used in Theorem 1.1 is detailed in the text. We anticipate that the symbol $\equiv$ indicates bijection between sets (without preserving any algebraic structure).

We describe the structure of this article, highlighting the steps of the proof of Theorem 1.1. In Section 2 we introduce notations and recall some results regarding the action of the fundamental group over based homotopy classes. In Section 3 we describe in detail the free and the based homotopy classes of self-maps of the projective plane and we prove that the action of $\pi_1(\mathbb{R}\mathrm{P}^2)$ on the set $[\mathbb{R}\mathrm{P}^2;\mathbb{R}\mathrm{P}^2]^*_{id}$ exchanges based homotopy classes of maps of opposite *twisted degree*. In Section 4 we finally consider the model two-complex K of the group presentation $\mathcal{P} = \langle x, y \,|\, x^{k+1}yxy \rangle$, with $k \geq 1$ odd, and we prove that, for the unique twisted integer coefficient system β over K, other than the trivial one, the corresponding twisted cohomology group $H^2(K;_\beta\mathbb{Z})$ is cyclic of order k. In Section 5 we build a special map $\omega : K \to \mathbb{R}\mathrm{P}^2$ for which the induced homomorphism on twisted cohomology groups, namely $\omega^* : H^2(\mathbb{R}\mathrm{P}^2;_\varrho\mathbb{Z}) \to H^2(K;_\beta\mathbb{Z})$, corresponds to the natural epimorphism $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$, and so ω is strongly surjective, for $k \neq 1$. Section 6 consists of the proof of Theorem 1.1. The proof follows from a complete description of representatives for all the free and based homotopy classes of maps from K into $\mathbb{R}\mathrm{P}^2$. The main step is the proof that each based map $f : K \to \mathbb{R}\mathrm{P}^2$ inducing the homomorphism β on fundamental groups is based homotopic to a map $f_n = h_n \circ \omega$, in which ω is the special map built in Section 5 and $h_n : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ is a map of twisted degree n, where n is an odd integer in the set $\{-k, -k+1 \ldots, k-1, k\}$. Consequently, the action of $\pi_1(K)$ on $[K;\mathbb{R}\mathrm{P}^2]^*_\beta$ can be obtained from the action of $\pi_1(\mathbb{R}\mathrm{P}^2)$ on $[\mathbb{R}\mathrm{P}^2;\mathbb{R}\mathrm{P}^2]^*_{id}$ and the induced homomorphism on twisted cohomology groups by f_n is not trivial for $n \neq \pm k$. This then forces f_n to be strongly surjective.

Throughout the text, for the sake of simplicity, we call a finite and connected two-dimensional CW-complex by a two-complex. We also simplify f is a continuous map by f is a map. Furthermore, we consider the cyclic group $\mathbb{Z}_2 = \{1, -1\}$ with its multiplicative structure and, where appropriated, we identify an automorphism $\tau \in \mathrm{Aut}(\mathbb{Z})$ with its value $\tau(1)$.

We believe that the approach developed in this article can be useful to extend Theorem 1.1 for any two-complex with trivial second integer cohomology group. We conjecture that given a two-complex K with $H^2(K;\mathbb{Z}) = 0$, the set $[K;\mathbb{R}\mathrm{P}^2]$ is finite and, for each $\alpha \in \hom(\Pi;\mathbb{Z}_2)$ we have: (i) there exists a bijection between $[K;\mathbb{R}\mathrm{P}^2]^*_\alpha$ and $H^2(K;_\alpha\mathbb{Z})$; (ii) there exists one and only one homotopy classe in $[K;\mathbb{R}\mathrm{P}^2]^*_\alpha$ which is not strongly surjective (this class corresponds to the trivial element of $H^2(K;_\alpha\mathbb{Z})$ under the bijection claimed in (i)).

To finish this introduction, we would like to point out the following two problems, which consist to study the question analyzed on this work in the following cases: (i) maps $K \to \mathbb{R}\mathrm{P}^n$ from an n-dimensional CW-complex K into the n-dimensional projective space, for $n > 2$; (ii) maps $K \to \mathbb{R}\mathrm{P}^2$ where K is a CW-complex of dimension > 2.

2. Actions of π_1 on based homotopy classes

Let K be a two-complex with fundamental group $\Pi = \pi_1(K)$ and take a 0-cell e^0 in K to be its base-point. Consider the real projective plane $\mathbb{R}\mathrm{P}^2$ with its minimal cellular structure, namely $\mathbb{R}\mathrm{P}^2 = c^0 \cup c^1 \cup c^2$, and take c^0 to be the base-point.

In what follows, we distinguish free homotopies and based homotopies starting at a given based map $f : K \to \mathbb{R}\mathrm{P}^2$. We observe that, by the Cellular Approximation Theorem, each map from K into $\mathbb{R}\mathrm{P}^2$ is freely homotopic to a based map. Hence, in order to study free or based homotopy classes, we can assume that a homotopy class always admite a representative given *a priori* by a map which is based. We define:

- $[K;\mathbb{R}\mathrm{P}^2]$ is the set of free homotopy classes $[f]$ of maps $f : K \to \mathbb{R}\mathrm{P}^2$.
- $[K;\mathbb{R}\mathrm{P}^2]^*$ is the set of based homotopy classes $[f]^*$ of based maps $f : K \to \mathbb{R}\mathrm{P}^2$.
- $[K;\mathbb{R}\mathrm{P}^2]^*_\alpha$ is the set of based homotopy classes $[f]^*$ of based maps $f : K \to \mathbb{R}\mathrm{P}^2$ such that $\alpha = f_\# : \pi_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^2)$.

It follows that

$$[K;\mathbb{R}\mathrm{P}^2]^* = \bigsqcup_{\alpha \in \hom(\Pi;\mathbb{Z}_2)} [K;\mathbb{R}\mathrm{P}^2]^*_\alpha.$$

We are identifying, in this description, the group $\hom(\Pi;\mathbb{Z}_2)$ with the group $\hom(\pi_1(K);\pi_1(\mathbb{R}\mathrm{P}^2))$.

The fundamental group $\pi_1(\mathbb{R}\mathrm{P}^2)$ acts on the set $[K;\mathbb{R}\mathrm{P}^2]^*$ and, following [10, Chapter V, Corollary 4.4], $[K;\mathbb{R}\mathrm{P}^2]$ corresponds to the quotient set of

$[K;\mathbb{R}\mathrm{P}^2]^*$ by this action, what we indicate by

$$[K;\mathbb{R}\mathrm{P}^2] \equiv \frac{[K;\mathbb{R}\mathrm{P}^2]^*}{\pi_1(\mathbb{R}\mathrm{P}^2)}.$$

We recall, in a general context, how the action of $\pi_1(Y)$ on $[X;Y]^*$ is defined. Consider based spaces (X,x_0) and (Y,y_0). Let $f_0, f_1 : X \to Y$ be based maps and let $u : I \to Y$ be a loop in Y based at y_0. Suppose there exists a homotopy $F : X \times I \to Y$, starting at f_0 and ending at f_1, such that $F(x_0,t) = u(t)$. Then we say that f_0 is freely homotopic to f_1 along to u and we write $f_0 \simeq_u f_1$. If u is the constant path at the base-point y_0, we say that f_0 is based homotopic to f_1 and we write $f_0 \simeq_* f_1$. We have:

(i) Given a based map $f_0 : X \to Y$ and a loop u in Y based at y_0, then $f_0 \simeq_u f_1$ for some based map $f_1 : X \to Y$.

(ii) If $f_0 \simeq_u f_1$ and $f_0 \simeq_v f_2$ and $u \simeq v\,(\mathrm{rel}.\partial I)$, then $f_1 \simeq_* f_2$.

(iii) If $f_0 \simeq_u f_1$ and $f_1 \simeq_v f_2$, then $f_0 \simeq_{uv} f_2$.

This defines the action of $\pi_1(Y)$ on $[X;Y]^*$. Thus: given a based map $f_0 : X \to Y$ and an element $[u] \in \pi_1(Y)$ represented by a loop u in Y based at y_0, there exists a based map $f_1 : X \to Y$ such that $f_0 \simeq_u f_1$, and we define the action of $[u]$ on $[f_0]^*$ to be $[f_1]^*$, that is,

$$[u][f_0]^* = [f_1]^*.$$

Returning to our approach, let $\sigma : I \to \mathbb{R}\mathrm{P}^2$ be the loop in $\mathbb{R}\mathrm{P}^2$ based at c^0 whose trajectory encircles once the 1-cell c^1. Then $[\sigma] \in \pi_1(\mathbb{R}\mathrm{P}^2)$ is the generator of $\pi_1(\mathbb{R}\mathrm{P}^2)$. Furthermore, σ induces the identity automorphism

$$\hat{\sigma} : \pi_1(\mathbb{R}\mathrm{P}^2) \to \pi_1(\mathbb{R}\mathrm{P}^2) \quad \text{given by} \quad \hat{\sigma}([u]) = [\sigma^{-1}][u][\sigma] = [u].$$

Lemma 2.1. *Each subset* $[K;\mathbb{R}\mathrm{P}^2]^*_\alpha$ *of* $[K;\mathbb{R}\mathrm{P}^2]^*$ *is invariant by the action of* $\pi_1(\mathbb{R}\mathrm{P}^2)$.

Proof. Consider the generator $[\sigma]$ of $\pi_1(\mathbb{R}\mathrm{P}^2)$. Given a based homotopy class $[f_0]^* \in [K;\mathbb{R}\mathrm{P}^2]^*_\alpha$, we take a based map $f_1 : K \to \mathbb{R}\mathrm{P}^2$ such that $f_0 \simeq_\sigma f_1$, that is, there exists a homotopy $H : f_0 \simeq f_1$ such that $H(e^0,t) = \sigma(t)$. Then $(f_1)_\# = \hat{\sigma} \circ (f_0)_\# = id \circ \alpha = \alpha$ and, by definition, $[\sigma][f_0]^* = [f_1]^*$. □

It follows that

$$[K;\mathbb{R}\mathrm{P}^2] \equiv \bigsqcup_{\alpha \in \hom(\Pi;\mathbb{Z}_2)} \frac{[K;\mathbb{R}\mathrm{P}^2]^*_\alpha}{\pi_1(\mathbb{R}\mathrm{P}^2)}.$$

In the case in which K is *aspherical* (has contractible universal covering), Theorem 4.12 of [9] provides, for each $\alpha \in \hom(\Pi;\mathbb{Z}_2)$, a bijection

$$[K;\mathbb{R}\mathrm{P}^2]^*_\alpha \equiv H^2(K;{}_\alpha\mathbb{Z}),$$

in which $H^2(K;{}_\alpha\mathbb{Z})$ is the second cohomology group of K with the local integer coefficient system $\alpha : \Pi \to \mathbb{Z}_2 \approx \mathrm{Aut}(\mathbb{Z})$. We explore this fact next.

3. Self-maps of the projective plane

In this section we present an analysis of the free and based homotopy classes of self-maps of the real projective plane. In a sense, what we present explains certain facts that can be inferred from [6, Proposition 2.1].

Throughout the section, $\mathfrak{p} : S^2 \to \mathbb{R}\mathrm{P}^2$ is the double covering map, $\mathfrak{a} : S^2 \to S^2$ is the antipodal map, $\mathbb{Z}^{odd}$ is the set of the odd integers and $\mathbb{Z}_+^{odd}$ is the set of the non-negative ones.

We consider the sphere S^2 as the suspension of S^1, that is, the quotient space obtained from the cylinder $S^1 \times [-1,1]$ by collapsing $S^1 \times \{-1\}$ to a single point (the south pole) and $S^1 \times \{1\}$ to another single point (the north pole). Thus, we can write a point of S^2 as a class $\lceil e^{i\theta}, \tau \rceil$ in which $(e^{i\theta}, \tau) \in S^1 \times [-1,1]$. We take:

$$s_1^0 = \lceil 1, 0 \rceil \text{ to be the base-point in } S^2;$$

$$s_2^0 = -s_1^0 = \lceil -1, 0 \rceil \text{ to be the antipodal point of } s_1^0;$$

$$s_1^1 = \text{ the half-equator arc } \lceil e^{i\theta}, 0 \rceil, \text{ for } 0 \leq \theta \leq \pi, \text{ from } s_1^0 \text{ to } s_2^0.$$

$$\sigma = \mathfrak{p}(s_1^1) \text{ to be the loop representing the generator of } \pi_1(\mathbb{R}\mathrm{P}^2).$$

The orientation of a loop provides over $\mathbb{R}\mathrm{P}^2$ the local integer coefficient system

$$\varrho : \pi_1(\mathbb{R}\mathrm{P}^2) \to \mathrm{Aut}(\mathbb{Z}) \text{ given by } \varrho(1) = 1 \text{ and } \varrho(-1) = -1.$$

Next, we consider the cohomology group $H^2(\mathbb{R}\mathrm{P}^2; {}_{\varrho}\mathbb{Z})$ with the local integer coefficient system ϱ.

We stablish the following one-to-one correspondences:

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^* \equiv \mathbb{Z}_2 \sqcup \mathbb{Z}^{odd} \quad \text{and} \quad [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2] \equiv \mathbb{Z}_2 \sqcup \mathbb{Z}_+^{odd}.$$

All maps given *a priori* will be considered to be based.

Firstly, we write

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^* = [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_0^* \sqcup [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_{id}^*,$$

in which the subscripts 0 and *id* indicate that the corresponding maps induce the trivial and the identity homomorphism on fundamental groups, respectively.

It follows by Lemma 2.1 that

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2] \equiv \frac{[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_0^*}{\pi_1(\mathbb{R}\mathrm{P}^2)} \sqcup \frac{[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_{id}^*}{\pi_1(\mathbb{R}\mathrm{P}^2)}.$$

Let $h : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ be a based map and take $\widetilde{h} : S^2 \to S^2$ to be the based lifting of $h \circ \mathfrak{p}$ through $\mathfrak{p} : S^2 \to \mathbb{R}\mathrm{P}^2$, so that we have the commutative diagram:

$$\begin{array}{ccc} S^2 & \xrightarrow{\widetilde{h}} & S^2 \\ {\scriptstyle \mathfrak{p}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}} \\ \mathbb{R}\mathrm{P}^2 & \xrightarrow{h} & \mathbb{R}\mathrm{P}^2 \end{array}$$

We claim that $\widetilde{h}$ is necessarily either even or odd; in fact, for each $x \in S^2$, we have either $\widetilde{h}(-x) = \widetilde{h}(x)$ or $\widetilde{h}(-x) = -\widetilde{h}(x)$, and so $\langle \widetilde{h}(x), \widetilde{h}(-x) \rangle = \pm 1$. By continuity, the map $x \mapsto \langle \widetilde{h}(x), \widetilde{h}(-x) \rangle$ is constant equal to either 1 or -1, and so $\widetilde{h}$ is either even or odd.

If $h : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ induces the trivial homomorphism on fundamental groups, that is, $[h]^* \in [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_0$, then h lifts through $\mathfrak{p}$ to a based map $\overline{h} : \mathbb{R}\mathrm{P}^2 \to S^2$. Now,

$$[\mathbb{R}\mathrm{P}^2; S^2]^* \equiv H^2(\mathbb{R}\mathrm{P}^2; \mathbb{Z}) \approx \mathbb{Z}_2,$$

and we can describe the two classes $[\overline{h}_{00}]^*$ and $[\overline{h}_{01}]^*$ by means of its representing maps, namely, $\overline{h}_{00} : \mathbb{R}\mathrm{P}^2 \to S^2$ is the constant map and $\overline{h}_{01} : \mathbb{R}\mathrm{P}^2 \to S^2$ is the quotient map that collapses the one-skeleton $S^1 \subset \mathbb{R}\mathrm{P}^2$ to the base-point of S^2. Defining the composed maps $h_{0i} = \mathfrak{p} \circ \overline{h}_{0i} : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ for $i = 0, 1$, we have

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_0 = \{[h_{00}]^*, [h_{01}]^*\} \equiv \mathbb{Z}_2.$$

Since h_{00} and h_{01} lift through $\mathfrak{p}$ and obviously $[\mathbb{R}\mathrm{P}^2; S^2]^* \equiv [\mathbb{R}\mathrm{P}^2; S^2]$, it follows that

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_0 \equiv [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_0 \equiv \mathbb{Z}_2.$$

We remark that both the liftings $\widetilde{h}_{00} = \overline{h}_{00} \circ \mathfrak{p}$ (= *constant*) and $\widetilde{h}_{01} = \overline{h}_{01} \circ \mathfrak{p}$ are even self-maps of S^2. Now, if $\widetilde{h} : S^2 \to S^2$ is even, then $\widetilde{h} = \widetilde{h} \circ \mathfrak{a}$, and so $\deg(\widetilde{h}) = -\deg(\widetilde{h})$, which forces $\deg(\widetilde{h}) = 0$ and, therefore, $\widetilde{h}$ is homotopically trivial, which does not imply that h is itself homotopically trivial. This is what happens with the map $\widetilde{h}_1$, that is, $\widetilde{h}_1$ is homotopic to the constant map, but the maps $\overline{h}_1$ and h_1 are not.

On the other hand, it follows from Borsuk-Ulam Theorem (in its version presented in [7, Chapter 2, § 6, p. 91]) that if $\widetilde{h} : S^2 \to S^2$ is odd, then $\deg(\widetilde{h})$ is odd. By the way, maps $\widetilde{h} : S^2 \to S^2$ of arbitrary odd degree they do exist: for, given an odd integer k, the suspension $\widetilde{h}_k : S^2 \to S^2$ of the map $S^1 \ni z \mapsto z^k \in S^1$ is odd and has degree k.

Each such an odd map $\widetilde{h}_k : S^2 \to S^2$ induces on the quotient a based map $h_k : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$, and it is easy to see that h_k induces the identity homomorphism of fundamental groups, because the map $\mathfrak{p} \circ \widetilde{h}_k$ maps the 1-cell s^1_1 in S^2 onto k times the 1-cell c^1 in $\mathbb{R}\mathrm{P}^2$. Thus, for each odd k, we have $[h_k]^* \in [\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_{id}$.

Since the degree classifies the homotopy classes of self-maps of S^2, it follows from the Lifting Homotopy Property that for two odd integers $k \neq l$, the corresponding maps $h_k, h_l : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ are not based homotopic.

Therefore, the function $[h]^* \mapsto \deg(\widetilde{h})$ provides a one-to-one correspondence

$$[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_{id} \equiv \mathbb{Z}^{odd}.$$

Equivalently, this bijection can be written as $[h]^* \mapsto d_\varrho(h)$, in which the last number is the *twisted degree* of h, that is, the integer $d_\varrho(h)$ such that the homomorphism

$$h^* : H^2(\mathbb{R}\mathrm{P}^2; {}_\varrho\mathbb{Z}) \to H^2(\mathbb{R}\mathrm{P}^2; {}_\varrho\mathbb{Z}),$$

induced by h on cohomology groups with the non-trivial local integer coefficient system ϱ, corresponds to the multiplication by $d_\varrho(h)$. This will be clearer after Section 5.

Now, for each odd $k > 0$, both the maps $\widetilde{h}_{-k}$ and $\mathfrak{a} \circ \widetilde{h}_k$ have the same degree $-k$, and so they are freely homotopic, but not based homotopic, since $\mathfrak{a} \circ \widetilde{h}_k$ is not even based.

We present a special free homotopy $\widetilde{H} : \tilde{h}_{-k} \simeq \mathfrak{a} \circ \widetilde{h}_k$. We define $\widetilde{H} : S^2 \times I \to S^2$ by

$$\widetilde{H}\big(\lceil e^{i\theta}, \tau\rceil, t\big) = \big\lceil \mathbf{r}(t\pi)\cdot e^{i(2t-1)k\theta}, (1-2t)\tau\big\rceil,$$

in which $z \mapsto \mathbf{r}(t\pi)\cdot z$ is the positive rotation under angle $t\pi$ in the complex plane. We have:

$$\widetilde{H}\big(\lceil e^{i\theta}, \tau\rceil, 0\big) = \lceil e^{-ik\theta}, \tau\rceil = \widetilde{h}_{-k}\big(\lceil e^{i\theta}, \tau\rceil\big),$$
$$\widetilde{H}\big(\lceil e^{i\theta}, \tau\rceil, 1\big) = \lceil -e^{ik\theta}, -\tau\rceil = -\widetilde{h}_k\big(\lceil e^{i\theta}, \tau\rceil\big) = \mathfrak{a} \circ \widetilde{h}_k\big(\lceil e^{i\theta}, \tau\rceil\big).$$

Hence, $\widetilde{H}$ is really a free homotopy starting at $\widetilde{h}_{-k}$ and ending at $\mathfrak{a} \circ \widetilde{h}_k$. Such a homotopy is not based, since the trajectory of the path $t \mapsto \widetilde{H}(\lceil 1, 0\rceil, t)$ is the half-equator arc s^1_1.

Now, we observe that, since k is odd, $\widetilde{H}$ is odd in the first coordinate, that is,

$$\widetilde{H}\big(-\lceil e^{i\theta}, \tau\rceil, t\big) = -\widetilde{H}\big(\lceil e^{i\theta}, \tau\rceil, t\big).$$

Thus, $\widetilde{H}$ induces to quotient a free homotopy $H : \mathbb{R}\mathrm{P}^2 \times I \to \mathbb{R}\mathrm{P}^2$ starting at h_{-k} and ending at h_k (since $h_k = \mathfrak{p}(\mathfrak{a} \circ \widetilde{h}_k)$). Moreover, the trajectory of the path $t \mapsto H(c^0, t)$ is the loop σ whose path homotopy class is the generator of $\pi_1(\mathbb{R}\mathrm{P}^2)$.

We have proved the following proposition:

Proposition 3.1. *The action of $\pi_1(\mathbb{R}\mathrm{P}^2)$ on $[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]^*_{id}$ exchanges the based homotopy classes $[h_k]^*$ and $[h_{-k}]^*$ and so the function $[h] \mapsto |d_\varrho(h)|$ provides a bijection $[\mathbb{R}\mathrm{P}^2; \mathbb{R}\mathrm{P}^2]_{id} \equiv \mathbb{Z}^{odd}_+$.*

4. The model two-complex of $\mathcal{P} = \langle x, y \mid x^{k+1}yxy\rangle$

Let K be the model two-complex of the presentation $\mathcal{P} = \langle x, y \mid x^{k+1}yxy\rangle$, with $k \geq 1$ odd, that is, the two-complex with a single 0-cell e^0, two 1-cells $e^1_x \cup e^1_y$ and a single two-cell e^2 which is attached on the one-skeleton

$K^1 = e^0 \cup e^1_x \cup e^1_y$ by spelling the word $r = x^{k+1}yxy$. We take the 0-cell e^0 to be the base-point of K.

The fundamental group of K is the group $\Pi = F(x,y)/N(r)$ presented by $\mathcal{P}$. Let $\bar{x}$ and $\bar{y}$ in Π be the images of x and y, respectively, by the natural homomorphism $F(x,y) \to \Pi$ from the free group $F(x,y)$ onto Π.

In what follows, we consider the cohomology groups of K with local integer coefficient systems, which we call *twisted cohomology groups*, for short.

Since the group Π has two generators, $\bar{x}$ and $\bar{y}$, and in the word $r = x^{k+1}yxy$ the sums of the powers of the letters x and y are respectively $k+2$ (which is odd) and 2, we have just one local integer coefficient systems over K, other than the trivial one, namely, the system

$$\beta : \Pi \to \mathrm{Aut}(\mathbb{Z}) \quad \text{given by} \quad \beta(\bar{x}) = 1 \text{ and } \beta(\bar{y}) = -1.$$

Proposition 4.1. *Let K be the model two-complex of the group presentation $\mathcal{P} = \langle x, y \,|\, x^{k+1}yxy \rangle$, with $k \geq 1$ odd. We have:*

(1) $H^2(K;\mathbb{Z}) = 0$ *and* $H^2(K;_\beta\mathbb{Z}) \approx \mathbb{Z}/k\mathbb{Z}$.
(2) K *is aspherical.*

Proof. The first statement of (1) follows from a straightforward analysis of the cellular co-chain complex of K and the second one is announciated in [5, Example 7.3], but without details. Since next we need to identify explicitly a generator of $H^2(K;_\beta\mathbb{Z})$, we provide a detailed calculation of the group $H^2(K;_\beta\mathbb{Z})$ after this proof. Assertion (2) follows from (1) and [5, Proposition 4.1]. □

Remark 4.2. Before proceeding to the calculations of the twisted cohomology group $H^2(K;_\beta\mathbb{Z})$, we observe that, for $k \neq 0$ even, the two-complex K is also aspherical. This fact follows from [5, Section 4], in which it is remarked that a one-relator model two-complex is aspherical if and only if the single relator of its presentation is not freely trivial and has period one. Thus, in order to have K non-aspherical we should take $k = 0$. On the other hand, if $k \geq 1$ is even, then $H^2(K;\mathbb{Z}) \approx \mathbb{Z}_2$ and so K would not be an interesting two-complex from the viewpoint of the inspiring problem of this article.

Returning to the case $k \geq 1$ odd, we compute the twisted cohomology group of $H^2(K;_\beta\mathbb{Z})$. We use the procedure and the notations presented in [5, Section 3]. Briefly:

- $\xi_\beta : \mathbb{Z}[\Pi] \to \mathbb{Z}$ is the β-augmentation function, that is, the function defined by $\xi_\beta(\sum_k n_i\pi_i) = \sum_i n_i\beta(\pi_i)$.
- $\|\cdot\| : \mathbb{Z}[F(x,y)] \to \mathbb{Z}[\Pi]$ is the natural extension on group rings of the natural homomorphism $F(x,y) \to \Pi = F(x,y)/N(r)$.
- $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y} : F(x,y) \to \mathbb{Z}[F(x,y)]$ are the Reidmeister-Fox derivatives.

For the relator word $r = x^{k+1}yxy$, we have:

$$\frac{\partial r}{\partial x} = (1 + x + \cdots + x^k) + x^{k+1}y \text{ and so } \xi_\beta\big(\|\frac{\partial r}{\partial x}\|\big) = (k+1) - 1 = k,$$

$$\frac{\partial r}{\partial y} = x^{k+1}(1 + yx) \text{ and so } \xi_\beta\big(\|\frac{\partial r}{\partial y}\|\big) = 1(1-1) = 0.$$

Consider the cellular chains of K with its natural identifications and generators:

$$C_0(K) = H_0(K^0) \approx \mathbb{Z}\langle e^0\rangle,$$
$$C_1(K) = H_1(K^1, K^0_{\mathcal{P}}) \approx \mathbb{Z}^2\langle e^1_x, e^1_y\rangle,$$
$$C_2(K) = H_2(K, K^1) \approx \mathbb{Z}\langle e^2\rangle.$$

Let $\widetilde{K}$ be the universal covering space of K, endowed with its natural cellular structure. Select a 0-cell $\tilde{e}^0$ over e^0, a 1-cell $\tilde{e}^1_x$ over e^1_x, a 1-cell $\tilde{e}^1_y$ over e^1_y and a 2-cell $\tilde{e}^2$ over e^2. The group Π acts on the left (via covering transformation) on the cellular chain complex $C_q(\widetilde{K}) = H_q(\widetilde{K}^q, \widetilde{K}^{q-1})$ making it into a left $\mathbb{Z}[\Pi]$-module, so that we have identifications

$$C_0(\widetilde{K}) = \mathbb{Z}[\Pi]\langle \tilde{e}^0\rangle,$$
$$C_1(\widetilde{K}) = \mathbb{Z}[\Pi]^2\langle \tilde{e}^1_x, \tilde{e}^1_y\rangle,$$
$$C_2(\widetilde{K}) = \mathbb{Z}[\Pi]\langle \tilde{e}^2\rangle.$$

Via this identifications and considering the action $\beta : \Pi \to \mathrm{Aut}(\mathbb{Z})$, we have the corresponding twisted cellular chain complex of left $\mathbb{Z}[\Pi]$-modules

$$C^\beta_*(\widetilde{K}) : 0 \to C_2(\widetilde{K}) \xrightarrow{\tilde{\partial}^\beta_2} C_1(\widetilde{K}) \xrightarrow{\tilde{\partial}^\beta_1} C_0(\widetilde{K}) \to 0,$$

in which the boundaries operators are given by

$$\tilde{\partial}^\beta_1(\tilde{e}^1_x) = \xi_\beta(1 - \bar{x})\tilde{e}^0 = 0,$$
$$\tilde{\partial}^\beta_1(\tilde{e}^1_y) = \xi_\beta(1 - \bar{y})\tilde{e}^0 = 2\tilde{e}^0,$$
$$\tilde{\partial}^\beta_2(\tilde{e}^2) = \xi_\beta\big(\|\frac{\partial r}{\partial x}\|\big)\tilde{e}^1_x + \xi_\beta\big(\|\frac{\partial r}{\partial y}\|\big)\tilde{e}^1_y = k\tilde{e}^1_x.$$

Consider the corresponding twisted cellular co-chain complex

$$C^*_\beta(\widetilde{K}) : 0 \leftarrow \hom^\Pi(C_2(\widetilde{K}); \mathbb{Z}) \xleftarrow{\tilde{\delta}^\beta_2} \hom^\Pi(C_1(\widetilde{K}); \mathbb{Z}) \xleftarrow{\tilde{\delta}^\beta_1} \hom^\Pi(C_0(\widetilde{K}); \mathbb{Z}) \leftarrow 0.$$

In each $\hom^\Pi(C_i(\widetilde{K}); \mathbb{Z})$, the integers $\mathbb{Z}$ is seen as a left $\mathbb{Z}[\Pi]$-module via the action $\beta : \Pi \to \mathrm{Aut}(\mathbb{Z})$. The co-boundaries operators $\tilde{\delta}^\beta_*$ are defined by the usual dual form

$$\tilde{\delta}^\beta_*(\phi) = \phi \circ \tilde{\partial}^\beta_2.$$

Explicitly, a given co-chain $\phi \in \hom^{\Pi}(C_1(\widetilde{K});\mathbb{Z})$ is defined by its values $\phi(\tilde{e}^1_x)$ and $\phi(\tilde{e}^1_x)$, and the co-chain $\tilde{\delta}^{\beta}_2(\phi) : C_2(K) \to \mathbb{Z}$ is given by

$$\tilde{\delta}^{\beta}_2(\phi)(\tilde{e}^2) = \xi_\beta\Big(\|\frac{\partial r}{\partial x}\|\Big)\phi(\tilde{e}^1_x) + \xi_\beta\Big(\|\frac{\partial r}{\partial y}\|\Big)\phi(\tilde{e}^1_y) = k\phi(\tilde{e}^1_x).$$

Now, $\hom^{\Pi}(C_2(\widetilde{K});\mathbb{Z}) \approx \mathbb{Z}$ is generated by the co-chain $\phi^*_2 : C_2(\widetilde{K}) \to \mathbb{Z}$ given by

$$\phi^*_2(\tilde{e}^2) = 1, \text{ that is, the dual of the chain } \tilde{e}^2.$$

Analogously, $\hom^{\Pi}(C_1(\widetilde{K});\mathbb{Z}) \approx \mathbb{Z}^2$ is generated by the co-chains $\phi^*_x, \phi^*_y : C_1(\widetilde{K}) \to \mathbb{Z}$ which are the dual of the chains $\tilde{e}^1_x$ and $\tilde{e}^1_y$, that is,

$$\phi^*_x(\tilde{e}^1_x) = 1,\ \phi^*_x(\tilde{e}^1_y) = 0, \quad \text{and} \quad \phi^*_y(\tilde{e}^1_x) = 0,\ \phi^*_y(\tilde{e}^1_y) = 1.$$

Thus, the co-boundary operator $\tilde{\delta}^{\beta}_2 : \hom^{\Pi}(C_1(\widetilde{K});\mathbb{Z}) \to \hom^{\Pi}(C_2(\widetilde{K});\mathbb{Z})$ is given by

$$\tilde{\delta}^{\beta}_2(\phi^*_x) = k\phi^*_2 \quad \text{and} \quad \tilde{\delta}^{\beta}_2(\phi^*_y) = 0.$$

Therefore,

$$H^2(K;{}_\beta\mathbb{Z}) \approx \frac{\hom^{\Pi}(C_2(\widetilde{K});\mathbb{Z})}{\operatorname{Im}(\tilde{\delta}^{\beta}_2)} \approx \frac{\mathbb{Z}\langle\phi^*_2\rangle}{\langle k\phi^*_2\rangle} \approx \frac{\mathbb{Z}}{k\mathbb{Z}}\big\langle \phi^*_2 + \langle k\phi^*_2\rangle\big\rangle.$$

Remark 4.3. Let us point out that the group $H^2(K;{}_\beta\mathbb{Z})$ depends on the word r, and not only on the sums of the powers of the letters x and y. For example, if we take $r = x^{k+2+n}y^2x^{-n}$, for $k \geq 1$ and $n \geq 0$, it can be shown that $H^2(K;{}_\beta\mathbb{Z}) \approx \mathbb{Z}_{k+2}$.

5. Maps from K into $\mathbb{R}\mathrm{P}^2$

In this section, we continue to consider the model two-complex K of the presentation $\mathcal{P} = \langle x, y \mid x^{k+1}yxy\rangle$, with $k \geq 1$ odd. Also, we keep considering the non-trivial local integer coefficient system $\varrho : \pi_1(\mathbb{R}\mathrm{P}^2) \to \mathrm{Aut}(\mathbb{Z})$.

Let us consider the cellular map $\omega : K \to \mathbb{R}\mathrm{P}^2$ defined naturally by collapsing the 1-cell e^1_x to the 0-cell c^0 of $\mathbb{R}\mathrm{P}^2$. It is possible to understand the map ω by considering K as the identification space obtained from the disc D^2 with identifications in its boundary $S^1 = \partial D^2$ respect to the word $r = x^{k+1}yxy$. We explain: first we divide S^1 into $k+4$ oriented arc segments, all of the them with the counter-clockwise orientation, enumerated from a selected point e^0 by $x, \ldots, x, y, x, y$. Then we collapse the first $k+1$ arcs indexed with the letter x to the point e^0 and we collapse the other arc indexed with the letter x to another point, we say, $-e^0$. That way, we obtain a new disc whose boundary are composed by two oriented arcs, both with the counter-clockwise orientation, indexed by the letter y. Then, by the collage of these arcs one with other, we obtain the projective plane.

We have described the map ω in such a way that it is easy to see that the 1-cell e^1_y is identified with the 1-cell c^1 and the interior of the 2-cell e^2 is mapped homeomorphically onto the interior of the 2-cell c^2.

Consider the induced homomorphism $\omega_\# : \Pi \to \pi_1(\mathbb{R}\mathrm{P}^2)$ on fundamental groups. Of course, $\omega_\#(\bar{x}) = 1$ and $\omega_\#(\bar{y}) = -1$. Hence, the map ω co-induces on K the local integer coefficient system $\varrho \circ \omega_\# : \Pi \to \mathrm{Aut}(\mathbb{Z})$ given by $\varrho \circ \omega_\#(\bar{x}) = 1$ and $\varrho \circ \omega_\#(\bar{y}) = -1$, that is, $\varrho \circ \omega_\# = \beta$. It follows that ω induces a homomorphism

$$\omega^* : H^2(\mathbb{R}\mathrm{P}^2;_\varrho\mathbb{Z}) \to H^2(K;_\beta\mathbb{Z}).$$

Proposition 5.1. *For each $k \geq 3$ odd, $\omega^* : H^2(\mathbb{R}\mathrm{P}^2;_\varrho\mathbb{Z}) \to H^2(K;_\beta\mathbb{Z})$ corresponds to the natural epimorphism $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$.*

The twisted cohomology group $H^2(\mathbb{R}\mathrm{P}^2;_\varrho\mathbb{Z})$ is well known to be infinite cyclic. However, we present the computation in order to identify an explicit generator. After that, we proceed to the proof of Proposition 5.1 itself.

Let us consider the projective plane $\mathbb{R}\mathrm{P}^2$ as the model two complex of the group presentation $\langle z \,|\, z^2 \rangle$, so that, $\mathbb{R}\mathrm{P}^2$ is endowed with its natural cellular structure $\mathbb{R}\mathrm{P}^2 = c^0 \cup c^1 \cup c^2$.

Let $\mathfrak{p} : S^2 \to \mathbb{R}\mathrm{P}^2$ be the universal covering map and consider the sphere S^2 with its cellular structure co-induced by $\mathfrak{p}$, so that, $S^2 = s_1^0 \cup s_2^0 \cup s_1^1 \cup s_2^1 \cup s_1^2 \cup s_2^2$, with $\mathfrak{p}(s_j^i) = c^i$, for $1 \leq i, j \leq 2$. As before, we choose the numeration of the cells so that the 1-cell s_1^1 starts at s_1^0 and ends at $s_2^0 = -s_1^0$, and s_1^2 is the 2-cell whose orientation makes $\tilde{\partial}_2(s_1^2) = s_1^1 + s_2^1$.

So, we take s_1^0, s_1^1 and s_1^2 to be the favourite cells of S^2, so that we have the following identifications of $\mathbb{Z}[\pi_1(\mathbb{R}\mathrm{P}^2)] = \mathbb{Z}[\mathbb{Z}_2]$-module:

$$C_0(\widetilde{\mathbb{R}\mathrm{P}^2}) = C_0(S^2) = \mathbb{Z}[\mathbb{Z}_2]\langle s_1^0 \rangle,$$
$$C_1(\widetilde{\mathbb{R}\mathrm{P}^2}) = C_1(S^2) = \mathbb{Z}[\mathbb{Z}_2]\langle s_1^1 \rangle,$$
$$C_2(\widetilde{\mathbb{R}\mathrm{P}^2}) = C_2(S^2) = \mathbb{Z}[\mathbb{Z}_2]\langle s_1^2 \rangle.$$

Via this identifications and considering the action $\varrho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$, we have the corresponding twisted cellular chain complex of left $\mathbb{Z}[\mathbb{Z}_2]$-modules

$$C_*^\varrho(S^2) : 0 \to C_2(S^2) \xrightarrow{\tilde{\partial}_2^\varrho} C_1(S^2) \xrightarrow{\tilde{\partial}_1^\varrho} C_0(S^2) \to 0,$$

in which the boundaries operators are given by

$$\tilde{\partial}_1^\varrho(s_1^1) = \xi_\varrho(1 - \bar{z})s_1^0 = 2s_1^0,$$

$$\tilde{\partial}_2^\varrho(s_1^2) = \xi_\varrho\big(\|\frac{\partial z^2}{\partial z}\|\big)s_1^1 = \xi_\varrho\big(\|1 + z\|\big)s_1^1 = 0.$$

Consider the corresponding twisted cellular co-chain complex

$$C_\varrho^*(S^2) : 0 \leftarrow \hom^{\mathbb{Z}_2}(C_2(S^2);\mathbb{Z}) \xleftarrow{\tilde{\delta}_2^\varrho} \hom^{\mathbb{Z}_2}(C_1(S^2);\mathbb{Z}) \xleftarrow{\tilde{\delta}_1^\varrho} \hom^{\mathbb{Z}_2}(C_0(S^2);\mathbb{Z}) \leftarrow 0.$$

In each $\hom^{\mathbb{Z}_2}(C_i(S^2);\mathbb{Z})$, the integers $\mathbb{Z}$ is seen as a left $\mathbb{Z}[\mathbb{Z}_2]$-module via the action $\varrho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$. The co-boundaries operators $\tilde{\delta}_*^\varrho$ are defined by the dual form $\tilde{\delta}_*^\varrho(\phi) = \phi \circ \tilde{\partial}_2^\varrho$.

Particularly, since $\tilde{\partial}_2^\varrho = 0$, also $\tilde{\delta}_*^\varrho = 0$. Therefore,

$$H^2(\mathbb{R}\mathrm{P}^2;{}_\varrho\mathbb{Z}) = \hom^{\mathbb{Z}_2}(C_2(S^2);\mathbb{Z}) \approx \mathbb{Z}\langle\varphi_2^*\rangle,$$

in which the generator is the co-chain $\varphi_2^* : C_2(S^2) \to \mathbb{Z}$ defined by $\varphi_2^*(s_1^2) = 1$, that is, φ_2^* is the dual of the chain s_1^2.

Proof of Proposition 5.1. Let $\mathfrak{q} : \widetilde{K} \to K$ be the universal covering map and let $\tilde{\omega} : \tilde{K} \to S^2$ be the unique cellular map satisfying $\tilde{\omega}(\tilde{e}^0) = s_1^0$ and $\mathfrak{p} \circ \tilde{\omega} = \omega \circ \mathfrak{q}$, that is, $\tilde{\omega}$ is the lifting of ω to universal coverings, starting at e_1^0. Then $\tilde{\omega}$ collapses $\tilde{e}_x^1$ to s_1^0 and maps the interior of the cells $\tilde{e}_y^1$ and $\tilde{e}^2$ homeomorphically onto the interior of s_1^1 and s_1^2, respectively.

Let consider the following commutative diagram, in which the vertical arrows are the homomorphisms induced by $\tilde{\omega}$ in level of chains:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_2(\tilde{K}) & \xrightarrow{\tilde{\partial}_2^\beta} & C_1(\tilde{K}) & \xrightarrow{\tilde{\partial}_1^\beta} & C_0(\tilde{K}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{\omega}_\#^2} & & \downarrow{\scriptstyle \tilde{\omega}_\#^1} & & \downarrow{\scriptstyle \tilde{\omega}_\#^0} & & \\
0 & \longrightarrow & C_2(S^2) & \xrightarrow{\tilde{\partial}_2^\varrho = 0} & C_1(S^2) & \xrightarrow{\tilde{\partial}_1^\varrho} & C_0(S^2) & \longrightarrow & 0
\end{array}$$

By the definition of the map $\tilde{\omega}$, the homomorphism $\tilde{\omega}_\#^0$, $\tilde{\omega}_\#^1$ and $\tilde{\omega}_\#^2$ are given, in terms of its values on the generators of its domains, by:

$$\tilde{\omega}_\#^0(\tilde{e}^0) = s_1^0 \quad \text{and} \quad \tilde{\omega}_\#^1(\tilde{e}_x^1) = 0,\ \tilde{\omega}_\#^1(\tilde{e}_y^1) = s_1^1 \quad \text{and} \quad \tilde{\omega}_\#^2(\tilde{e}^2) = s_1^2.$$

Now, we consider the corresponding commutative diagram in level of co-chains. To shorten, we denote $C^i(X) = \mathrm{Hom}^G(C_i(X);\mathbb{Z})$, for $(X,G) = (\tilde{K},\Pi)$ and $(X,G) = (S^2,\mathbb{Z}_2)$.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & C^2(\tilde{K}) & \xleftarrow{\tilde{\delta}_2^\beta} & C^1(\tilde{K}) & \xleftarrow{\tilde{\delta}_1^\beta} & C^0(\tilde{K}) & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle \tilde{\omega}_2^\#} & & \uparrow{\scriptstyle \tilde{\omega}_1^\#} & & \uparrow{\scriptstyle \tilde{\omega}_0^\#} & & \\
0 & \longleftarrow & C^2(S^2) & \xleftarrow{\tilde{\delta}_2^\varrho = 0} & C^1(S^2) & \xleftarrow{\tilde{\delta}_1^\varrho} & C^0(S^2) & \longleftarrow & 0
\end{array}$$

By duality, the homomorphisms $\tilde{\omega}_0^\#$, $\tilde{\omega}_1^\#$ and $\tilde{\omega}_2^\#$ are given, in terms of its values in the generators of its domains, by:

$$\tilde{\omega}_\#^0(\varphi_0^*) = \phi_0^* \quad \text{and} \quad \tilde{\omega}_1^\#(\varphi_1^*) = \phi_y^* \quad \text{and} \quad \tilde{\omega}_2^\#(\varphi_2^*) = \phi_2^*.$$

It follows that $\omega^* : H^2(\mathbb{R}\mathrm{P}^2;{}_\varrho\mathbb{Z}) \to H^2(K;{}_\beta\mathbb{Z})$ is given by:

$$\frac{C^2(S^2)}{\mathrm{Im}(\tilde{\delta}_2^\varrho)} \ni \varphi_2^* + 0 \mapsto \tilde{\omega}_2^\#(\varphi_2^*) + \mathrm{Im}(\tilde{\delta}_2^\beta) = \varphi_2^* + \langle k\varphi_2^*\rangle \in \frac{C^2(\tilde{K})}{\mathrm{Im}(\tilde{\delta}_2^\beta)}.$$

Therefore, ω^* corresponds to the natural epimorphism $\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$. □

Obviously, we present more calculations than necessary to prove Proposition 5.1. We do this in order to make more clear the induced homomorphisms by the maps involved.

6. Proof of the main theorem

This entire section is the proof of the main theorem of the article.

Proof of Theorem 1.1. Via the natural identification $\mathrm{Aut}(\mathbb{Z}) \approx \mathbb{Z}_2$, we have $\hom(\Pi; \mathbb{Z}_2) = \{1, \beta\}$, in which 1 denotes the trivial homomorphism and β is the homomorphism given at the beginning of Section 4. Thus,

$$[K; \mathbb{R}\mathrm{P}^2]^* = [K; \mathbb{R}\mathrm{P}^2]^*_1 \sqcup [K; \mathbb{R}\mathrm{P}^2]^*_\beta.$$

Since K is aspherical, we have the following bijections given in [9, Theorem 4.2]:

$$[K; \mathbb{R}\mathrm{P}^2]^*_1 \equiv H^2(K; \mathbb{Z}) = 0 \ \text{ and } \ [K; \mathbb{R}\mathrm{P}^2]^*_\beta \equiv H^2(K; {}_\beta\mathbb{Z}) \approx \mathbb{Z}/k\mathbb{Z},$$

Hence, the sets $[K; \mathbb{R}\mathrm{P}^2]_1 \equiv [K; \mathbb{R}\mathrm{P}^2]^*_1$ have only one element, namely, the (free or based) homotopy class of the constant map at the 0-cell c^0. In the assertions (1) and (2) of Theorem 1.1, this element corresponds to 1 and the component $[K; \mathbb{R}\mathrm{P}^2]^*_1$ of $[K; \mathbb{R}\mathrm{P}^2]^*$, as well as the component $[K; \mathbb{R}\mathrm{P}^2]_1$ of $[K; \mathbb{R}\mathrm{P}^2]$, corresponds to $\{1\}$. This proves a part of the assertions (1) and (2) of Theorem 1.1.

On the other hand, the set $[K; \mathbb{R}\mathrm{P}^2]^*_\beta$ has k elements, and we describe them and also the elements of the set $[K; \mathbb{R}\mathrm{P}^2]_\beta \equiv [K; \mathbb{R}\mathrm{P}^2]^*_\beta / \pi_1(\mathbb{R}\mathrm{P}^2)$.

If $k = 1$, then $[K; \mathbb{R}\mathrm{P}^2]^*_\beta$ has only one element and so $[K; \mathbb{R}\mathrm{P}^2]_\beta \equiv [K; \mathbb{R}\mathrm{P}^2]^*_\beta$. In the statement of Theorem 1.1, assertion (1), this element corresponds to $\bar{0}$ and the component $[K; \mathbb{R}\mathrm{P}^2]^*_\beta$ of $[K; \mathbb{R}\mathrm{P}^2]^*$, as well as the component $[K; \mathbb{R}\mathrm{P}^2]_\beta$ of $[K; \mathbb{R}\mathrm{P}^2]$, corresponds to $\{\bar{0}\}$. Now, if $[f]^* \in [K; \mathbb{R}\mathrm{P}^2]^*_\beta$, then $f_\# = \beta$ and, since $H^2(K; {}_\beta\mathbb{Z}) = 0$, it follows from [5, Theorem 1.1] that f is homotopic to a non-surjective map.

We have completed the proof of the assertion (1) of Theorem 1.1.

To prove what is missing from assertion (2), we take $k = 2p - 1 \geq 3$.

Let $\mathrm{Odd}(k)$ be the set of the odd integers in the set $\{2 - k, \ldots, k - 2\}$. Then $\mathrm{Odd}(k)$ has $k - 1 = 2p - 2$ elements, being $p - 1$ of them positive and $p - 1$ of them negative.

For each $n \in \mathrm{Odd}(k)$ we define the map

$$f_n = h_n \circ \omega : K \to \mathbb{R}\mathrm{P}^2,$$

in which $\omega : K \to \mathbb{R}\mathrm{P}^2$ is the map built in Section 5 and $h_n : \mathbb{R}\mathrm{P}^2 \to \mathbb{R}\mathrm{P}^2$ is the based map whose twisted degree is $d_\varrho(h_n) = n$, as presented in Section 3. Additionally, define

$$f_0 = h_k \circ \omega : K \to \mathbb{R}\mathrm{P}^2.$$

For each $n \in \mathrm{Odd}(k) \cup \{0\}$, the homomorphism $(f_n)_\# : \Pi \to \pi_1(\mathbb{R}\mathrm{P}^2)$ is equal to β and the homomorphism $f_n^* : H^2(\mathbb{R}\mathrm{P}^2; {}_\varrho\mathbb{Z}) \to H^2(K; {}_\beta\mathbb{Z})$ corresponds to $\mu_n : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ given by $\mu_n(1) = n + k\mathbb{Z}$. It follows that, for each $n \in \mathrm{Odd}(k)$, the map f_n is strongly surjective and, for $m \neq n$ in $\mathrm{Odd}(k) \cup \{0\}$, the maps f_m and f_n are not based homotopic. Therefore,

$$[K; \mathbb{R}\mathrm{P}^2]^*_\beta = \big\{[f_n]^* : n \in \mathrm{Odd}(k) \cup \{0\}\big\} \equiv \mathbb{Z}_k.$$

However, since the maps h_n and h_{-n} are freely homotopic, also the maps f_n and f_{-n} are freely homotopic, for each $n \in \mathrm{Odd}(k)$, that is, the based homotopy classes $[f_n]^*$ and $[f_{-n}]^*$ are exchanged by the action of $\pi_1(\mathbb{R}\mathrm{P}^2)$ over $[K; \mathbb{R}\mathrm{P}^2]^*_\beta$. On the other hand, since k is odd, it is obvious that the remaining class $[f_0]^*$ is fixed by the action of $\pi_1(\mathbb{R}\mathrm{P}^2)$.

Therefore, taking $\mathrm{Odd}^+(k)$ to be the set of the positive integers in $\mathrm{Odd}(k)$, we have

$$\frac{[K; \mathbb{R}\mathrm{P}^2]^*_\beta}{\pi_1(\mathbb{R}\mathrm{P}^2)} \equiv \big\{[f_n] : n \in \mathrm{Odd}^+(k) \cup \{0\}\big\} \equiv \mathbb{Z}_p.$$

Since we have proved that each f_n, for $n \in \mathrm{Odd}^+(k)$, is strongly surjective, in order to finish the proof of Theorem 1.1 it remains to prove that f_0 is not strongly surjective. For this, consider the cellular map $g^1 : K^1 = S^1_x \vee S^1_y \to S^1$ which maps S^1_x encircling 2 times into S^1, in the positive orientation, and maps S^1_y encircling $k+2$ times in S^1, in the opposite orientation. The homomorphism $g^1_\# : F(x,y) \approx \pi_1(K^1) \to \pi_1(S^1) \approx \mathbb{Z}$ is given by $g^1_\#(x) = 2$ and $g^1_\#(y) = -(k+2)$. It follows that $g^1_\#(r) = 0$, and so g^1 extends to a map $\overline{g} : K \to S^1$. Composing $\overline{g}$ with the skeleton inclusion $l : S^1 \hookrightarrow \mathbb{R}\mathrm{P}^2$ we obtain a map $g : K \to \mathbb{R}\mathrm{P}^2$ such that $g_\# : \Pi \to \pi_1(\mathbb{R}\mathrm{P}^2)$ is equal to β and $g^* : H^2(\mathbb{R}\mathrm{P}^2; {}_\varrho\mathbb{Z}) \to H^2(K; {}_\beta\mathbb{Z})$ is trivial. It follows from the one-to-one correspondence $[K; \mathbb{R}\mathrm{P}^2]^*_\beta \equiv H^2(K; {}_\beta\mathbb{Z}) \approx \mathbb{Z}/k\mathbb{Z}$ that the based homotopy classes $[g]^*$ and $[f_0]^*$ are equal, since both correspond to the zero class in $H^2(K; {}_\beta\mathbb{Z})$. Therefore, f_0 is homotopic to the non-surjective map g. □

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