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Higher dimensional simplicial complexity

Ayse Borat

Abstract. In this paper we generalize the simplicial complexity which is defined by Gonzalez in [5], to higher dimensions. We introduce some of its properties such as its relation with the topological complexity and the relation between the dimensions of the simplicial complexity. At the last section an example of a motion planner for a complex of S^1 is given.

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1. Introduction

Topological complexity, defined by Farber in [2], is a tool that measures how far a space is from admitting a motion planner. More precisely, for a topological space X, topological complexity of X (denoted by $\mathrm{TC}(X)$) is defined as the Schwarz genus of the fibration $\pi : PX \to X \times X$ given by $\pi(\gamma) = (\gamma(0), \gamma(1))$ where PX stands for the path space of X. For more details, see [3], [2].

Rudyak generalized this definiton to the higher dimensions in his paper [7] and this new notion is further studied by Basabe, Gonzalez, Rudyak and Tamaki in [1]. The following is the definition of higher dimensional topological complexity.

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Definition 1.1. [7] Consider the wedge sum of n closed intervals $[0,1]_i$ for $i = 1, 2, \ldots, n$ where the zeros 0_i's are identified, and denote it by J_n ($n \in \mathbb{N}$). For a path-connected space X, denote by X^{J_n} the function space of paths with n-legs. Then there is a fibration $e_n : X^{J_n} \to X^n$ given by $e_n(f) = (f(1_1), f(1_2), \ldots, f(1_n))$. The n-dimensional topological complexity is defined to be the Schwarz genus of e_n and is denoted by $TC_n(X)$.

Schwarz defined this genus in "nonreduced" terms. Throughout this paper we will use the reduced version.

The concept of topological complexity is considered in the combinatorial realm by Tanaka in [9] and in the simplicial realm by Fernandez-Ternero, Macias-Virgos, Minuz and Vilches in [4] and by Gonzalez in [5]. Even though the authors of [4] and [5] consider the simplicial version of "topological complexity", the viewpoints in the papers [4] and [5] are different. The basic difference is that [5] makes use of the subdivision functor to recast (and not only estimate) TC in combinatorial terms.

In this paper we will focus on the simplicial complexity as it is given by Gonzalez in [5]. In the second section, we will construct the higher dimensional simplicial complexity and will give some of its properties. In the last section, an example of a (3-dimensional) motion planner for a complex of S^1 will be introduced.

2. Higher dimensional simplicial complexity

The following lemma is a generalization of Lemma 4.21 in [3].

Lemma 2.1. *Let X be a path-connected space and consider the fibration e_n as described in Definition 1.1. Then $U \subseteq X^n$ admits a section $s : U \to X^{J_n}$ if and only if there is some map $g\colon U \to X$ such that each composition $f_i : U \overset{\iota}{\hookrightarrow} X^n \xrightarrow{\text{proj}_i} X$ is homotopic to g, where $\text{proj}_i : X^n \to X$ is the projection to the i-th factor.*

Proof. Suppose for each $i \in \{1, \cdots, n\}$, we have homotopies $H^i : U \times I \to X$ such that $H^i(x, 0) = g(x)$ and $H^i(x, 1) = f_i(x_1, \cdots, x_n) = x_i$ where $x = (x_1, \cdots, x_n) \in U$.

Define a map $s : U \to X^{J_n}$ such that the n-legged path $s(x)\colon J^n \to X$ is defined as follows. The i-th leg in $s(x)$ is given by the path $H^i_x : I \to X$ and these legs are identified at $H^i_x(0_i) = g(x)$. Notice that $s(x)(1_i) = f_i(x) = x_i$ for $1 \leq i \leq n$. Hence $e_n \circ s = id_U$ holds.

On the other hand, suppose that $s : U \to X^{J_n}$ is a section such that $e_n \circ s = id_U$. In other words, s satisfies $s(x)(1_i) = x_i$ for each $x \in U$. Moreover, let the wedge point be given by $g(x)$ for each $s(x)$. Then for each i, we can define homotopies $H^i : U \times I \to X$ given by

$$H^i(x, t_i) = s(x)(t_i)$$

which satisfies $H^i(x, 0_i) = g(x)$ and $H^i(x, 1_i) = x_i = f_i(x)$. □

Definition 2.2. Let $\varphi, \psi \colon K \to L$ be two simplicial maps between simplicial complexes. φ and ψ are said to be contiguous, denoted by $\varphi \sim_c \psi$, if $\sigma = \{v_0, \dots, v_n\}$ is a simplex in K, then

$$\varphi(\sigma) \cup \psi(\sigma) = \{\varphi(v_0), \dots, \varphi(v_n), \psi(v_0), \dots, \psi(v_n)\}$$

constitute a simplex in L .

Definition 2.3. [5] Let K, L be complexes. $\varphi, \psi : K \to L$ are called c-contiguous if there exists a sequence of maps $h_0, h_1, \cdots, h_c : K \to L$ satisfying $h_0 = \varphi$, $h_c = \psi$ and that for each $i = 1, 2, \cdots, c$, the pair (h_{i-1}, h_i) is contiguous.

Throughout this paper, all complexes and their products will be taken in the category of ordered complexes. This is required as the topological realization of the product is homeomorphic to the product of the topological realizations of the factors.

Consider an approximation

$$\iota : \mathrm{Sd}^b(K^n) \to \mathrm{Sd}^{b-1}(K^n)$$

to the identity on $||K^n||$ where $\mathrm{Sd}^b(K^n)$ denotes the b-fold barycentric subdivision. As in [5], for simplicity, denote the iterated composition of these maps by ι as well, such as

$$\iota : \mathrm{Sd}^b(K^n) \to \mathrm{Sd}^{b'}(K^n).$$

For $i \in \{1, 2, \cdots, n\}$, define the map π_i as the composition

$$\pi_i : \mathrm{Sd}^b(K^n) \xrightarrow{\iota} K^n \xrightarrow{\mathrm{proj}_i} K$$

where proj_i is the projection map to the i-th factor.

Definition 2.4. Let K be an ordered complex. For $b, c_1, c_2, \cdots, c_n \in \mathbb{Z}_{\geq 0}$, if there are $k + 1$ subcomplexes $J_0, J_1, \cdots, J_k$ covering $\mathrm{Sd}^b(K^n)$ satisfying that on each J_i (where $i = 0, 1, \cdots, k$) there exists a map $g_i : J_i \to K$ such that each map

$$f_j^i : J_i \hookrightarrow \mathrm{Sd}^b(K^n) \xrightarrow{\pi_j} K \qquad \text{(where } j = 1, 2, \cdots, n)$$

is c_j-contiguous to $g_i : J_i \to K$ on J_i, then the least possible number k is called $(b, c_1, c_2, \cdots, c_n)$-simplicial complexity of K or simply n-dimensional (b, C)-simplicial complexity where $C = (c_1, c_2, \cdots, c_n)$.

We will denote it by $\mathrm{SC}^b_{C,n}(K)$ or if C is given explicitly we can omit the subindex n and denote it by $\mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K)$.

If no such finite coverings exist, we will set $\mathrm{SC}^b_{C,n}(K) = \infty$.

Proposition 2.5. *Given $C = (c_1, c_2, \cdots, c_n)$ and $D = (d_1, d_2, \cdots, d_n)$. Then*

$$\mathrm{SC}^b_{C,n}(K) \geq \mathrm{SC}^b_{D,n}(K)\,, \quad \textit{if} \quad c_i \leq d_i \quad \textit{for each } i.$$

Proposition 2.5 allows us to make the following definition.

$$\mathrm{SC}^b_n(K) := \lim_{c_1\to\infty} \lim_{c_2\to\infty} \cdots \lim_{c_n\to\infty} \mathrm{SC}^b_{C,n}(K).$$

Note that, here the "limit" is just a notation and what we really mean is the stabilized value. Observe that

$$\mathrm{SC}^b_n(K) \leq \mathrm{SC}^b_{C,n}(K)$$

holds for any $C = (c_1, c_2, \cdots, c_n)$.

Lemma 2.6. $\mathrm{SC}^b_n(K)$ *does not depend on the chosen approximations $\iota : \mathrm{Sd}^b(K^n) \to \mathrm{Sd}^{b-1}(K^n)$ to the identity.*

Proof. Let

$$\iota, \bar{\iota} : \mathrm{Sd}^b(K^n) \to \mathrm{Sd}^{b-1}(K^n)$$

be different approximations to the identity. If $C = (c_1, c_2, \cdots, c_n)$, let us denote the corresponding n-dimensional (b, C)-simplicial complexities by $\mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K)$ and $\overline{\mathrm{SC}}^b_{(c_1,c_2,\cdots,c_n)}(K)$, respectively.

Note that $\iota, \bar{\iota}$ are 1-contiguous by Lemma 3.5.4 in [8]. Hence

$$f_i : J \hookrightarrow \mathrm{Sd}^b(K^n) \xrightarrow{\pi_i} K$$

is 1-contiguous to

$$\bar{f}_i : J \hookrightarrow \mathrm{Sd}^b(K^n) \xrightarrow{\overline{\pi}_i} K$$

(for each $i = 1, 2, \cdots, n$) by Lemma 3.5.1 in [8]. Therefore

$$\mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K) \geq \overline{\mathrm{SC}}^b_{(c_1+1,c_2+1,\cdots,c_n+1)}(K)$$

$$\geq \mathrm{SC}^b_{(c_1+2,c_2+2,\cdots,c_n+2)}(K).$$

Thus the limits $\mathrm{SC}^b_n(K)$ and $\overline{\mathrm{SC}}^b_n(K)$ must be equal. □

As given in [5], if $\lambda : \mathrm{Sd}^1(J) \to J$ is a simplicial approximation to the identity, then $\alpha \circ \lambda$ and $\iota \circ \beta$ which fit in the following diagram are 1-contiguous

$$\begin{array}{ccc} \mathrm{Sd}^1(J) & \xrightarrow{\beta} & \mathrm{Sd}^{b+1}(K^n) \\ \big\downarrow \lambda & & \big\downarrow \iota \\ J & \xrightarrow{\alpha} & \mathrm{Sd}^b(K^n) \end{array}$$

where α is the obvious embedding and β is the corresponding (subdivided) embedding.

Therefore it follows that $\mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K) \geq \mathrm{SC}^{b+1}_{(c_1+1,c_2+1,\cdots,c_n+1)}(K)$.

Lemma 2.7. $\mathrm{SC}^0_n(K) \geq \mathrm{SC}^1_n(K) \geq \mathrm{SC}^2_n(K) \geq \cdots$.

Proof.

$$\begin{aligned} \mathrm{SC}^b_n(K) &= \mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K)\,, \qquad \text{for large } c_1, c_2, \cdots, c_n \\ &\geq \mathrm{SC}^{b+1}_{(c_1+1,c_2+1,\cdots,c_n+1)}(K) \\ &\geq \lim_{c_1+1\to\infty} \lim_{c_2+1\to\infty} \cdots \lim_{c_n+1\to\infty} \mathrm{SC}^{b+1}_{(c_1+1,c_2+1,\cdots,c_n+1)}(K) \\ &= \mathrm{SC}^{b+1}_n(K). \end{aligned}$$

The first and the last equalities follow from the definitions whereas the first inequality follows from the discussion above. □

From the decreasing sequence in Lemma 2.7, we will define *n-dimensional simplicial complexity* as the stabilized value in this sequence. For a complex K, its n-th simplicial complexity will be denoted by $\mathrm{SC}_n(K)$.

Theorem 2.8. *If K is a complex,* $\mathrm{TC}_n(||K||) \leq \mathrm{SC}_n(K)$.

Proof. $$\begin{aligned} \mathrm{SC}_n(K) &= \mathrm{SC}^b_{(c_1,c_2,\cdots,c_n)}(K)\,, \qquad \text{for large } b, c_1, c_2, \cdots, c_n \\ &\geq \mathrm{TC}_n(||K||). \end{aligned}$$

The equality follows from the definition of $\mathrm{SC}_n(K)$ while the inequality follows from Lemma 2.7 and the fact that topological realization of c_i-contiguous maps are homotopic. □

Theorem 2.9. *If K is a finite complex, then* $\mathrm{TC}_n(||K||) = \mathrm{SC}_n(K)$.

Proof. It suffices to show that $\mathrm{TC}_n(||K||) \geq \mathrm{SC}_n(K)$.

Let $\mathrm{TC}_n(||K||) = k$. Then there exists $U_0, U_1, \cdots, U_k \subset ||K||^n$ covering $||K||^n$ such that for each $i \in \{0, 1, \cdots, k\}$, each $f^i_j : U_i \hookrightarrow ||K||^n \xrightarrow{\pi_i} ||K||$ (for $j = 1, 2, \cdots, n$) is homotopic to some $g_i : U_i \to ||K||$ by Lemma 2.1.

Let us choose $b \in \mathbb{Z}^+$ which is large enough so that the realization of each simplex of $\mathrm{Sd}^b(K^n)$ is contained in some U_i. For each i, construct a subcomplex L_i of $\mathrm{Sd}^{b_i}(K^n)$ so that it contains these simplices whose realization is contained in U_i.

Since each f^i_j is homotopic to g_i over U_i, each f^i_j is homotopic to g_i over $||L_i||$ as well. By Lemma 3.5.6 in [8], since K is finite, for each $j \in \{1, \cdots, n\}$, there are $d_j, c_j \in \mathbb{Z}^+$ such that $f^i_j : U_i \to ||K||^n \to ||K||$ and $g_i : U_i \to ||K||$ have simplicial approximations $\psi_j, \psi_g : \mathrm{Sd}^{b+d_j}(L^n_i) \to \mathrm{Sd}^{b+d_j}(K^n) \to K$ which are c_j-contiguous. Hence $\mathrm{SC}^{b+d_1+\cdots+d_n}_{(c_1,\cdots,c_n)}(K) \leq k$. Thus $\mathrm{SC}_n(K) \leq k$. □

Corollary 2.10. *If K and L are two finite complexes whose topological realizations are homotopy equivalent, then they have* $\mathrm{SC}_n(K) = \mathrm{SC}_n(L)$.

Corollary 2.11. *Let K be a finite complex. Then $||K||$ is contractible if and only if* $\mathrm{SC}_n(K) = 0$.

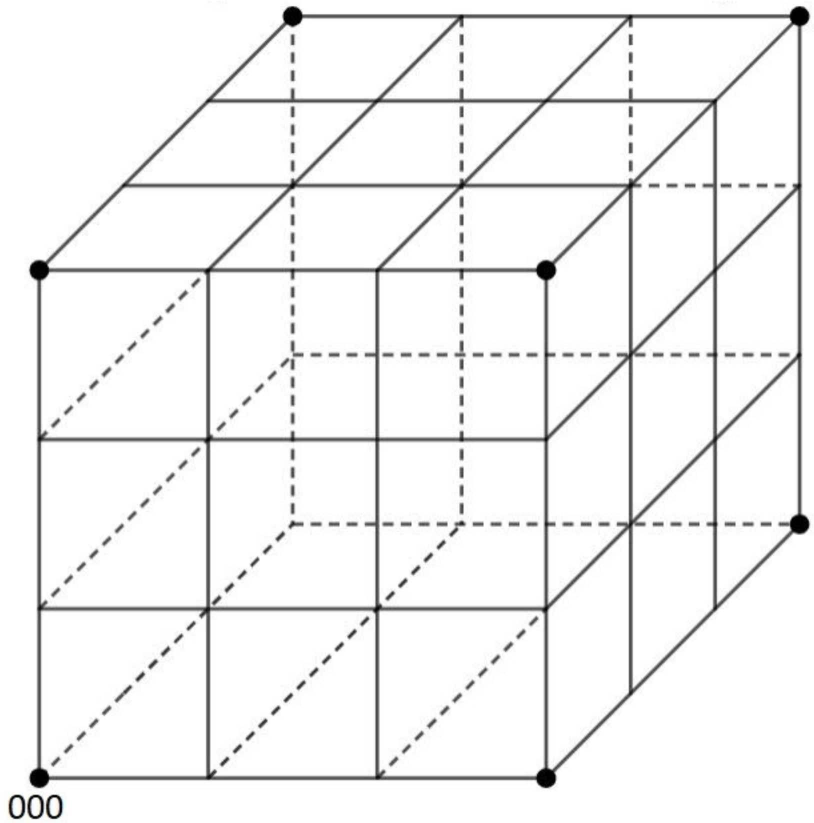
Corollary 2.12. *If K is a finite complex,* $\mathrm{SC}_n(K) \leq \mathrm{SC}_{n+1}(K)$.

Proof. This follows directly from Proposition 3.3 in [7], i.e., $\mathrm{TC}_n(X) \leq \mathrm{TC}_{n+1}(X)$, and Theorem 2.9. □

3. An example for SC_3

The following example is provided by Carlos Ortiz and Jesus Gonzalez. It took 6 days of computer calculation and the computer implementation used in this example will be described in [6] by Ortiz, Lara, and Gonzalez.

Example 3.1. This example gives a SC_3 motion planning algorithm with 5 rules for a simplicial complex of S^1. Let K be a complex for S^1 given by $\{01, 12, 20\}$ where $0, 1, 2$ are vertices. $K \times K \times K$ is represented as a cube with opposite walls identified in the usual way.

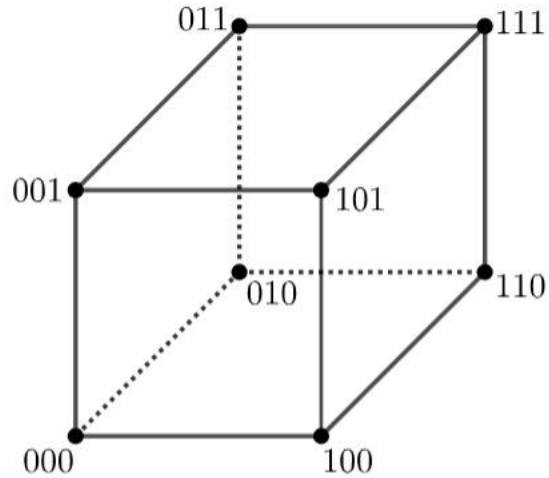


$K \times K \times K$ has 27-vertices which are named as follows.

$v_1 := 000$	$v_{10} := 100$	$v_{19} := 200$
$v_2 := 001$	$v_{11} := 101$	$v_{20} := 201$
$v_3 := 002$	$v_{12} := 102$	$v_{21} := 202$
$v_4 := 010$	$v_{13} := 110$	$v_{22} := 210$
$v_5 := 011$	$v_{14} := 111$	$v_{23} := 211$
$v_6 := 012$	$v_{15} := 112$	$v_{24} := 212$
$v_7 := 020$	$v_{16} := 120$	$v_{25} := 220$
$v_8 := 021$	$v_{17} := 121$	$v_{26} := 221$
$v_9 := 022$	$v_{18} := 122$	$v_{27} := 222$

Let $\pi_i : K \times K \times K \to K$, for $i = 1, 2, 3$, denote the i-th projection. For instance, $\pi_1(v_7) = 0$ and $\pi_2(v_{18}) = 2$.

Notice that $K \times K \times K$ is made out of 27 little cubes each of which is similar to the one in the figure below.



The above cube is made out of 6 3-simplices. These are

{ 000, 001, 011, 111 }

{ 000, 001, 101, 111 }

{ 000, 010, 011, 111 }

{ 000, 010, 110, 111 }

{ 000, 100, 101, 111 }

{ 000, 100, 110, 111 }

We will distribute the 162 resulting 3-simplices into 5 domains S1, S2, S3, S4 and S5, which will assemble a 3-dimensional simplicial motion planner.

The domains

Facets of domain 1 (52 in total):

$$
\begin{aligned}
S_1 = \big\{&\{v_2, v_3, v_6, v_{24}\}, \{v_4, v_{13}, v_{14}, v_{17}\}, \{v_2, v_3, v_6, v_{15}\}, \\
&\{v_1, v_{19}, v_{22}, v_{24}\}, \{v_1, v_3, v_{21}, v_{27}\}, \{v_1, v_4, v_6, v_{15}\}, \\
&\{v_{11}, v_{20}, v_{21}, v_{27}\}, \{v_1, v_4, v_{22}, v_{24}\}, \{v_{11}, v_{12}, v_{21}, v_{27}\}, \\
&\{v_{13}, v_{14}, v_{17}, v_{26}\}, \{v_1, v_4, v_5, v_{14}\}, \{v_{10}, v_{19}, v_{21}, v_{27}\}, \\
&\{v_{11}, v_{12}, v_{21}, v_{24}\}, \{v_4, v_{13}, v_{16}, v_{17}\}, \{v_1, v_7, v_{25}, v_{27}\}, \\
&\{v_{11}, v_{20}, v_{26}, v_{27}\}, \{v_2, v_{20}, v_{26}, v_{27}\}, \{v_1, v_4, v_{22}, v_{23}\},
\end{aligned}
$$

$$\{v_1, v_4, v_6, v_{24}\}, \{v_4, v_6, v_{15}, v_{18}\}, \{v_{10}, v_{19}, v_{22}, v_{23}\},$$
$$\{v_{10}, v_{19}, v_{22}, v_{24}\}, \{v_1, v_4, v_5, v_{23}\}, \{v_4, v_{13}, v_{15}, v_{18}\},$$
$$\{v_2, v_3, v_{21}, v_{27}\}, \{v_2, v_3, v_{21}, v_{24}\}, \{v_4, v_{13}, v_{16}, v_{18}\},$$
$$\{v_{14}, v_{17}, v_{26}, v_{27}\}, \{v_1, v_{19}, v_{21}, v_{27}\}, \{v_1, v_3, v_6, v_{24}\},$$
$$\{v_1, v_3, v_{21}, v_{24}\}, \{v_1, v_{19}, v_{21}, v_{24}\}, \{v_{10}, v_{19}, v_{21}, v_{24}\},$$
$$\{v_1, v_7, v_9, v_{27}\}, \{v_{10}, v_{12}, v_{21}, v_{27}\}, \{v_{10}, v_{19}, v_{25}, v_{27}\},$$
$$\{v_2, v_{20}, v_{21}, v_{24}\}, \{v_{11}, v_{17}, v_{26}, v_{27}\}, \{v_4, v_5, v_{14}, v_{17}\},$$
$$\{v_{10}, v_{12}, v_{21}, v_{24}\}, \{v_1, v_3, v_9, v_{27}\}, \{v_2, v_8, v_{26}, v_{27}\},$$
$$\{v_2, v_{20}, v_{21}, v_{27}\}, \{v_2, v_3, v_9, v_{27}\}, \{v_{13}, v_{16}, v_{17}, v_{26}\},$$
$$\{v_1, v_3, v_6, v_{15}\}, \{v_{11}, v_{20}, v_{21}, v_{24}\}, \{v_1, v_4, v_{13}, v_{15}\},$$
$$\{v_1, v_4, v_{13}, v_{14}\}, \{v_1, v_{19}, v_{22}, v_{23}\}, \{v_2, v_8, v_9, v_{27}\},$$
$$\{v_1, v_{19}, v_{25}, v_{27}\}\}$$

Facets of domain 2 (43 in total):

$$S_2 = \{\{v_2, v_{11}, v_{14}, v_{15}\}, \{v_4, v_5, v_8, v_{17}\}, \{v_4, v_5, v_{23}, v_{26}\},$$
$$\{v_2, v_3, v_9, v_{18}\}, \{v_{13}, v_{16}, v_{25}, v_{26}\}, \{v_1, v_2, v_8, v_{17}\},$$
$$\{v_1, v_{19}, v_{25}, v_{26}\}, \{v_1, v_2, v_8, v_{26}\}, \{v_5, v_8, v_9, v_{27}\},$$
$$\{v_1, v_2, v_{20}, v_{26}\}, \{v_{10}, v_{13}, v_{22}, v_{24}\}, \{v_{13}, v_{22}, v_{25}, v_{26}\},$$
$$\{v_2, v_3, v_{12}, v_{15}\}, \{v_2, v_8, v_{17}, v_{18}\}, \{v_{10}, v_{13}, v_{15}, v_{24}\},$$
$$\{v_2, v_8, v_9, v_{18}\}, \{v_4, v_7, v_8, v_{26}\}, \{v_5, v_8, v_{26}, v_{27}\},$$
$$\{v_1, v_7, v_8, v_{17}\}, \{v_{10}, v_{13}, v_{22}, v_{23}\}, \{v_{10}, v_{12}, v_{15}, v_{24}\},$$
$$\{v_4, v_5, v_8, v_{26}\}, \{v_{11}, v_{12}, v_{15}, v_{24}\}, \{v_2, v_3, v_{12}, v_{18}\},$$
$$\{v_1, v_2, v_{11}, v_{17}\}, \{v_5, v_6, v_9, v_{18}\}, \{v_5, v_{23}, v_{26}, v_{27}\},$$
$$\{v_1, v_2, v_{11}, v_{14}\}, \{v_5, v_6, v_9, v_{27}\}, \{v_4, v_7, v_8, v_{17}\},$$
$$\{v_{13}, v_{22}, v_{23}, v_{26}\}, \{v_1, v_7, v_8, v_{26}\}, \{v_1, v_7, v_{25}, v_{26}\},$$
$$\{v_2, v_{11}, v_{12}, v_{15}\}, \{v_5, v_8, v_9, v_{18}\}, \{v_2, v_{11}, v_{17}, v_{18}\},$$
$$\{v_5, v_8, v_{17}, v_{18}\}, \{v_4, v_7, v_{25}, v_{26}\}, \{v_4, v_{22}, v_{25}, v_{26}\},$$
$$\{v_4, v_{22}, v_{23}, v_{26}\}, \{v_{11}, v_{14}, v_{15}, v_{24}\}, \{v_1, v_{19}, v_{20}, v_{26}\},$$
$$\{v_2, v_{11}, v_{12}, v_{18}\}\}$$

Facets of domain 3 (35 in total):

$$S_3 = \big\{\{v_1, v_{10}, v_{13}, v_{14}\}, \{v_{10}, v_{11}, v_{14}, v_{23}\}, \{v_1, v_{10}, v_{11}, v_{14}\},$$
$$\{v_5, v_6, v_{15}, v_{18}\}, \{v_2, v_5, v_6, v_{15}\}, \{v_1, v_7, v_{16}, v_{18}\},$$
$$\{v_{10}, v_{12}, v_{18}, v_{27}\}, \{v_2, v_5, v_6, v_{24}\}, \{v_{10}, v_{16}, v_{18}, v_{27}\},$$
$$\{v_1, v_{10}, v_{16}, v_{18}\}, \{v_1, v_3, v_{12}, v_{18}\}, \{v_1, v_{10}, v_{12}, v_{18}\},$$
$$\{v_{10}, v_{19}, v_{20}, v_{26}\}, \{v_1, v_{10}, v_{16}, v_{17}\}, \{v_4, v_7, v_9, v_{18}\},$$
$$\{v_2, v_5, v_{23}, v_{24}\}, \{v_{10}, v_{16}, v_{25}, v_{27}\}, \{v_{11}, v_{20}, v_{23}, v_{24}\},$$
$$\{v_{10}, v_{19}, v_{20}, v_{23}\}, \{v_{10}, v_{13}, v_{14}, v_{23}\}, \{v_{10}, v_{11}, v_{20}, v_{26}\},$$
$$\{v_{10}, v_{16}, v_{17}, v_{26}\}, \{v_{10}, v_{11}, v_{17}, v_{26}\}, \{v_{10}, v_{11}, v_{20}, v_{23}\},$$
$$\{v_4, v_7, v_{16}, v_{18}\}, \{v_4, v_7, v_{16}, v_{17}\}, \{v_2, v_{20}, v_{23}, v_{24}\},$$
$$\{v_{11}, v_{14}, v_{23}, v_{24}\}, \{v_1, v_3, v_9, v_{18}\}, \{v_1, v_7, v_{16}, v_{17}\},$$
$$\{v_4, v_6, v_9, v_{18}\}, \{v_{10}, v_{16}, v_{25}, v_{26}\}, \{v_{10}, v_{19}, v_{25}, v_{26}\},$$
$$\{v_1, v_7, v_9, v_{18}\}, \{v_1, v_{10}, v_{11}, v_{17}\}\big\}$$

Facets of domain 4 (24 in total):

$$S_4 = \big\{\{v_{13}, v_{16}, v_{25}, v_{27}\}, \{v_{11}, v_{12}, v_{18}, v_{27}\}, \{v_{14}, v_{17}, v_{18}, v_{27}\},$$
$$\{v_{14}, v_{15}, v_{24}, v_{27}\}, \{v_4, v_{22}, v_{25}, v_{27}\}, \{v_1, v_2, v_5, v_{23}\},$$
$$\{v_4, v_7, v_9, v_{27}\}, \{v_1, v_2, v_5, v_{14}\}, \{v_4, v_6, v_9, v_{27}\},$$
$$\{v_2, v_5, v_{14}, v_{15}\}, \{v_{13}, v_{15}, v_{24}, v_{27}\}, \{v_{13}, v_{15}, v_{18}, v_{27}\},$$
$$\{v_1, v_{19}, v_{20}, v_{23}\}, \{v_{13}, v_{22}, v_{25}, v_{27}\}, \{v_5, v_{14}, v_{15}, v_{18}\},$$
$$\{v_5, v_{14}, v_{17}, v_{18}\}, \{v_{14}, v_{15}, v_{18}, v_{27}\}, \{v_1, v_2, v_{20}, v_{23}\},$$
$$\{v_4, v_7, v_{25}, v_{27}\}, \{v_4, v_6, v_{24}, v_{27}\}, \{v_{13}, v_{22}, v_{24}, v_{27}\},$$
$$\{v_{11}, v_{17}, v_{18}, v_{27}\}, \{v_{13}, v_{16}, v_{18}, v_{27}\}, \{v_4, v_{22}, v_{24}, v_{27}\}\big\}$$

Facets of domain 5 (8 in total):

$$S_5 = \big\{\{v_{14}, v_{23}, v_{24}, v_{27}\}, \{v_5, v_{23}, v_{24}, v_{27}\}, \{v_1, v_{10}, v_{12}, v_{15}\},$$
$$\{v_1, v_{10}, v_{13}, v_{15}\}, \{v_5, v_6, v_{24}, v_{27}\}, \{v_{13}, v_{14}, v_{23}, v_{26}\},$$
$$\{v_1, v_3, v_{12}, v_{15}\}, \{v_{14}, v_{23}, v_{26}, v_{27}\}\big\}$$

Now we will introduce the rules over these domains. In each case there are two contiguity chains, one is from π_1 to π_2 and the second is from π_1 to π_3. The rows below stand for the values at vertices (in the order indicated) of the simplicial maps in the chain.

Contiguity chains for domain 1

Vertices: $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9,$

$v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20},$

$v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}.$

Chain from π_1 to π_2:
[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 2, 2, 2, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 2, 2, 0, 2, 2, 2, 1, 1, 0, 1, 1, 1, 2, 0, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 2, 2, 2, 2, 2, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 2, 0, 2, 2, 2, 2]
[0, 0, 0, 0, 1, 0, 2, 2, 2, 0, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 2, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 2, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 0, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 0, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 1, 2, 2, 2]

Chain from π_1 to π_3:
[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 1, 2, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 0, 2, 2, 0, 1, 2, 1, 1, 0, 1, 1, 0, 0, 2, 2, 2, 2, 2, 0, 2, 2]
[0, 2, 2, 0, 0, 2, 0, 2, 2, 0, 1, 2, 1, 1, 0, 1, 1, 0, 0, 2, 2, 0, 0, 2, 0, 2, 2]
[0, 2, 2, 0, 0, 2, 0, 2, 2, 0, 1, 2, 1, 1, 0, 1, 1, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2]
[0, 2, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 0, 0, 1, 0, 0, 1, 2, 0, 1, 2, 0, 1, 2]
[0, 2, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2]
[0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2]

Contiguity chains for domain 2

Vertices: $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9,$

$v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20},$

$v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}.$

Chain from π_1 to π_2:
[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 1, 0, 2, 2, 2, 2, 1, 2, 2, 2]
[0, 0, 0, 0, 0, 0, 0, 0, 2, 1, 0, 0, 1, 1, 1, 2, 0, 0, 2, 2, 2, 2, 1, 2, 2, 0]
[0, 0, 0, 0, 2, 2, 2, 2, 2, 1, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 2, 2, 1, 2, 2, 2]
[0, 0, 0, 2, 2, 1, 2, 2, 2, 1, 0, 0, 2, 1, 1, 2, 2, 2, 0, 0, 2, 2, 1, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 1, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2]
[0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2]

Chain from π_1 to π_3:
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]$
$[0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 2, 1, 1, 1, 1, 0, 0, 2, 2, 2, 2, 2, 2, 2]$
$[0, 0, 0, 0, 0, 2, 0, 0, 2, 1, 1, 1, 2, 1, 1, 2, 1, 0, 0, 2, 2, 2, 2, 2, 2, 2]$
$[0, 0, 0, 0, 0, 2, 0, 0, 2, 2, 1, 1, 2, 1, 1, 0, 1, 0, 0, 0, 2, 2, 2, 0, 2, 0]$
$[0, 0, 0, 0, 0, 2, 0, 0, 0, 2, 1, 1, 2, 1, 1, 0, 1, 0, 0, 0, 0, 2, 2, 0, 0, 0]$
$[0, 1, 1, 0, 0, 2, 0, 0, 0, 2, 1, 1, 2, 1, 1, 2, 1, 0, 0, 1, 2, 2, 2, 0, 0, 2]$
$[0, 1, 1, 0, 0, 2, 0, 0, 0, 2, 1, 1, 2, 1, 2, 0, 0, 0, 1, 1, 0, 2, 2, 0, 0, 2]$
$[0, 1, 1, 0, 0, 2, 0, 0, 0, 2, 1, 1, 0, 1, 2, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 0]$
$[0, 1, 1, 0, 0, 0, 0, 1, 0, 2, 1, 1, 0, 1, 2, 0, 1, 0, 0, 1, 0, 0, 2, 0, 1, 0]$
$[0, 1, 1, 0, 0, 1, 0, 1, 1, 2, 1, 1, 0, 1, 2, 0, 1, 1, 0, 0, 0, 0, 2, 1, 1, 0]$
$[1, 1, 2, 0, 1, 1, 0, 1, 1, 2, 1, 2, 0, 1, 2, 0, 1, 1, 1, 1, 0, 0, 2, 0, 1, 0]$
$[0, 1, 2, 0, 1, 1, 0, 1, 1, 0, 1, 2, 0, 1, 2, 0, 1, 1, 0, 1, 0, 0, 2, 0, 1, 1]$
$[0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 1, 1, 1, 0, 1, 0, 1, 2, 0, 1, 1]$
$[0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 1, 2]$

Contiguity chains for domain 3

Vertices: $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9,$

$v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20},$

$v_{23}, v_{24}, v_{25}, v_{26}, v_{27}.$

Chain from π_1 to π_2:
$[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]$
$[1, 0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2]$
$[1, 0, 0, 1, 0, 0, 1, 1, 2, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 2, 2]$
$[1, 0, 0, 1, 0, 0, 2, 1, 2, 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 2, 2, 2, 2]$
$[1, 0, 1, 1, 0, 0, 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 0, 2, 2, 2, 2, 2, 2]$
$[2, 0, 2, 1, 0, 0, 2, 1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 0, 0, 2, 2, 2, 2, 2]$
$[2, 0, 2, 1, 0, 0, 2, 1, 2, 0, 2, 1, 2, 1, 2, 2, 1, 0, 0, 2, 0, 2, 2, 2]$
$[2, 0, 1, 1, 0, 1, 2, 1, 2, 0, 2, 1, 2, 1, 2, 2, 1, 0, 0, 2, 0, 2, 2, 2]$
$[2, 0, 1, 1, 0, 1, 2, 2, 2, 0, 2, 2, 2, 1, 2, 2, 1, 0, 0, 2, 0, 2, 2, 2]$
$[2, 0, 1, 1, 0, 1, 1, 2, 2, 0, 2, 2, 2, 1, 2, 2, 1, 0, 0, 0, 0, 0, 2, 2]$
$[2, 0, 1, 1, 1, 1, 1, 2, 2, 0, 2, 2, 2, 1, 2, 2, 1, 0, 0, 0, 0, 0, 2, 2]$
$[2, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 1, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2]$
$[2, 0, 1, 1, 1, 1, 2, 2, 0, 0, 2, 2, 0, 1, 2, 2, 2, 0, 0, 0, 0, 2, 2, 2]$
$[2, 1, 1, 1, 1, 1, 1, 2, 0, 0, 2, 0, 0, 1, 2, 2, 2, 0, 0, 0, 1, 2, 2, 2]$
$[2, 1, 2, 1, 1, 1, 2, 1, 0, 0, 2, 0, 0, 1, 2, 2, 2, 0, 0, 1, 1, 2, 2, 2]$
$[2, 1, 2, 1, 1, 1, 2, 2, 0, 0, 0, 0, 0, 1, 2, 2, 2, 0, 0, 1, 1, 2, 2, 2]$
$[0, 0, 0, 2, 1, 1, 2, 2, 0, 0, 0, 0, 0, 1, 2, 2, 2, 0, 0, 1, 1, 2, 2, 2]$
$[0, 0, 0, 1, 1, 1, 2, 2, 0, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 2, 2, 2]$

Chain from π_1 to π_3:
[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 1, 2]
[0, 2, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 2, 2, 2, 2, 1, 2]
[1, 2, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 2, 1, 2, 2, 2, 1, 2]
[1, 2, 2, 0, 0, 0, 1, 1, 2, 1, 2, 1, 1, 0, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2]
[2, 2, 2, 0, 0, 0, 1, 1, 2, 1, 1, 1, 1, 0, 1, 1, 1, 2, 2, 2, 2, 1, 1, 2]
[1, 0, 2, 0, 0, 0, 1, 1, 1, 1, 2, 1, 1, 0, 1, 1, 1, 2, 2, 2, 2, 1, 1, 1]
[1, 2, 2, 0, 0, 0, 0, 1, 1, 1, 2, 2, 1, 0, 0, 1, 1, 1, 2, 2, 2, 1, 1, 1]
[1, 2, 1, 0, 0, 0, 1, 1, 1, 1, 1, 2, 1, 0, 0, 1, 1, 1, 1, 2, 2, 0, 1, 1]
[1, 2, 1, 1, 0, 0, 0, 1, 1, 1, 1, 2, 1, 0, 0, 1, 0, 1, 1, 2, 2, 0, 1, 0]
[1, 2, 1, 0, 2, 0, 0, 1, 1, 1, 1, 2, 1, 0, 1, 1, 0, 1, 1, 2, 2, 0, 1, 0]
[1, 2, 1, 0, 2, 0, 0, 1, 1, 1, 1, 1, 1, 2, 0, 1, 0, 1, 2, 1, 2, 1, 1, 1]
[0, 2, 1, 0, 2, 0, 0, 1, 1, 1, 1, 0, 1, 2, 0, 1, 0, 1, 1, 1, 2, 0, 1, 1]
[0, 2, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 1, 2, 0, 1, 0, 0, 1, 1, 2, 0, 1, 1]
[0, 2, 0, 0, 2, 2, 0, 0, 0, 1, 0, 0, 1, 2, 0, 1, 0, 0, 1, 1, 2, 0, 1, 1]
[0, 1, 0, 0, 2, 2, 0, 0, 0, 1, 0, 0, 1, 2, 0, 1, 0, 0, 1, 1, 2, 0, 1, 0]
[0, 1, 0, 0, 2, 2, 0, 2, 0, 1, 2, 0, 1, 2, 0, 0, 2, 0, 1, 1, 2, 0, 1, 2]
[0, 1, 2, 0, 1, 2, 0, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 1, 2, 0, 1, 2]

Contiguity chains for domain 4

Vertices: $v_1, v_2, v_4, v_5, v_6, v_7, v_9,$
$v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20},$
$v_{22}, v_{23}, v_{24}, v_{25}, v_{27}.$

Chain from π_1 to π_2:
[0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 2, 0, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 2, 2, 2, 2, 2]
[0, 0, 1, 0, 1, 2, 2, 1, 2, 1, 1, 1, 2, 1, 1, 0, 0, 2, 0, 1, 2, 2]
[1, 0, 1, 1, 1, 2, 2, 1, 2, 1, 1, 1, 2, 1, 1, 0, 0, 1, 0, 1, 2, 2]
[0, 0, 1, 1, 1, 2, 2, 2, 2, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 1, 2, 2]
[0, 0, 1, 1, 1, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2, 0, 0, 1, 1, 1, 2, 2]

Chain from π_1 to π_3:
[0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2]
[0, 0, 2, 0, 2, 0, 2, 1, 2, 2, 1, 1, 2, 1, 1, 0, 2, 2, 2, 2, 2, 2]
[0, 0, 0, 0, 2, 0, 2, 1, 2, 2, 1, 1, 2, 1, 1, 0, 2, 0, 0, 2, 0, 2]
[0, 0, 0, 1, 2, 0, 2, 1, 2, 2, 1, 1, 2, 1, 1, 0, 2, 0, 0, 2, 0, 2]
[0, 0, 0, 1, 2, 0, 2, 1, 2, 2, 1, 1, 2, 1, 2, 0, 2, 0, 0, 2, 0, 2]
[0, 0, 0, 1, 2, 0, 0, 1, 2, 2, 1, 1, 0, 2, 2, 0, 0, 0, 0, 2, 0, 2]
[0, 1, 0, 1, 2, 0, 2, 1, 2, 2, 1, 1, 0, 2, 2, 0, 1, 0, 0, 2, 0, 2]
[0, 1, 0, 1, 2, 0, 2, 1, 2, 2, 1, 2, 0, 2, 2, 0, 1, 0, 1, 2, 0, 2]

[0, 1, 0, 1, 2, 0, 2, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 0, 1, 2, 0, 2]

Contiguity chains for domain 5

Vertices: $v_1, v_3, v_5, v_6,$

$v_{10}, v_{12}, v_{13}, v_{14}, v_{15}$

$v_{23}, v_{24}, v_{26}, v_{27}.$

Chain from π_1 to π_2:
[0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2]
[0, 0, 2, 2, 0, 0, 1, 1, 1, 2, 2, 2, 2]
[0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 2, 2]

Chain from π_1 to π_3:
[0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2]
[0, 0, 0, 2, 0, 0, 1, 2, 0, 2, 2, 1, 2]
[0, 2, 2, 2, 0, 0, 1, 1, 0, 2, 2, 1, 2]
[0, 2, 1, 2, 0, 2, 1, 1, 0, 1, 2, 1, 2]
[0, 2, 1, 2, 0, 2, 0, 1, 0, 1, 2, 1, 1]
[0, 2, 1, 2, 0, 2, 0, 1, 2, 1, 2, 1, 2]

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(Ayse Borat) Bursa Technical University, Faculty of Engineering and Natural Sciences, Department of Mathematics, Bursa, Turkey

ayse.borat@btu.edu.tr

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