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Tb theorem for the generalized singular integral operator on product Lipschitz spaces with para-accretive functions

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Abstract. By developing the Littlewood-Paley characterization for product homogeneous Lipschitz spaces $\mathrm{Lip}(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ and $\mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, and establishing a density argument for $\mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ in the weak sense, we give a Tb theorem for the generalized singular integral operator on $\mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, where $b(x, y) = b_1(x)b_2(y)$, b_1, b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Contents

1. Introduction and main results

Early in 1952, Calderón and Zygmund [1] introduced the singular integrals with convolution kernels and proved that these operators are bounded on the $L^p(\mathbb{R}^n)$ spaces with $1 < p < \infty$, which extended the related results for Hilbert transform on $L^p(\mathbb{R})$ and Riesz transforms on $L^p(\mathbb{R}^n)$. Later, people paid more attention to the Calderón-Zygmund operators with non-convolution kernels. To be more precise, assume $k(x, y)$ is a continuous function with $x \neq y$, satisfying the following estimates for some $\epsilon > 0$:

$$|k(x, y)| \leq C|x - y|^{-n}, \tag{1.1}$$

$$|k(x, y) - k(x', y)| \leq C|x - x'|^{\epsilon}|x - y|^{-n-\epsilon}, \tag{1.2}$$

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for all x, x' and $y \in \mathbb{R}^n$ with $|x-x'| \leq |x-y|/2$, and

$$|k(x,y) - k(x,y')| \leq C|y-y'|^{\epsilon}|x-y|^{-n-\epsilon}, \tag{1.3}$$

for all x, y and $y' \in \mathbb{R}^n$ with $|y-y'| \leq |x-y|/2$, the smallest such constant C in (1.1), (1.2) and (1.3) is denoted by $|k|_{CZ}$. A Calderón-Zygmund singular integral operator T is a continuous linear operator from $C_0^\infty(\mathbb{R}^n)$ into its dual associated to the kernel $k(x,y)$ above, which can be represented by

$$\langle Tf, g\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k(x,y)f(y)g(x)dydx$$

for test functions f and g in C_0^∞, whose supports are disjoint. It was well-known that the L^2-boundedness of convolution singular operators follows from the Plancherel theorem. However, we cannot apply the Plancherel theorem to obtain the L^2-boundedness of non-convolution singular integral operators. So, it is necessary to develop new methods to obtain the L^2-boundedness. The remarkable $T1$ theorem provides a general criterion for the L^2-boundedness of Calderón-Zygmund singular integral operators, see [2] and [6] among others. If the Calderón-Zygmund operator T is bounded on L^2, the norm of T is defined by $\|T\|_{CZ} = \|T\|_{L^2\mapsto L^2} + |k|_{CZ}$. The $T1$ theorem has also been extended for Besov and Triebel-Lizorkin spaces. For the endpoint boundedness, there are also analogous $T1$ criterions for Calderón-Zygmund singular integral operators, see for example [13].

However, the $T1$ theorem cannot be applied to the following Cauchy integral

$$\mathcal{C}(f)(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \frac{f(y)}{(x-y)+i(a(x)-a(y))}dy,$$

where the function $a(x)$ satisfies the Lipschitz condition. Meyer first observed that $\mathcal{C}(b) = 0$ provided $b(x) = 1 + ia'(x)$. If one replaces the function 1 in the $T1$ theorem by an accretive function b which is a bounded complex-valued function satisfies $0 < \delta \leq \mathrm{Re}(b(x))$ for almost everywhere, then this result would imply the L^2-boundedness of $\mathcal{C}(f)$ on all Lipschitz curves. Mclntosh and Meyer [19] (see also[20]) proved such a Tb theorem, where b is an accretive function. Finally, David, Journé and Semmes [3] proved a new Tb theorem with replacing the function 1 by the so-called para-accretive functions b (see Definition 1.1 below). Moreover, they verified that the para-accretivity is also necessary in the sense that the Tb theorem holds for a bounded function b, then b is para-accretive.

In order to extend the Tb theorem to Hardy spaces, Han, Lee and Lin [10] developed a new class of Hardy spaces $H_b^p(\mathbb{R}^n)$ associated to a para-accretive functions b, which can be expressed equivalently as the space of distributions such that their Littlewood-Paley g-functions associated to b belong to $L^p(\mathbb{R}^n)$. It is shown that if $T^*(b) = 0$, the Calderón-Zygmund operator is bounded from the classical Hardy spaces to the new Hardy spaces $H_b^p(\mathbb{R}^n)$ for $n/(n+1) < p \leq 1$.

It's well known that the Lipschitz spaces on $\mathbb{R}^n$ play an important role in harmonic analysis and partial differential equations (see [7, 16, 18, 21]). Fefferman and Stein [4] proved that the dual spaces of $H^1(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ are $BMO(\mathbb{R}^n)$ and the Lipschitz space $\mathrm{Lip}(n(1/p-1))(\mathbb{R}^n)$ for $n/(n+1) < p < 1$, respectively. Similarly, Han, Lee and Lin [11] prove that the dual space of $H^p_b(\mathbb{R}^n)$ is $\mathrm{Lip}_b(n(1/p-1))(\mathbb{R}^n)$ for $n/(n+1) < p < 1$. More precisely, denote $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n) = \{f : f = bg, g \in \mathrm{Lip}(\alpha)(\mathbb{R}^n)\}$, where $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$ is the classical Lipschitz space of order α. Recently, the authors [22] gave the Littlewood-Paley characterization for the Lipschitz spaces with para-accretive function, and founded a Tb criteria for the boundedness of Calderón-Zygmund operators. More precisely, the authors proved that the Calderón-Zygmund operator is bounded from $\mathrm{Lip}_b(\alpha)(\mathbb{R}^n)$ to the classical Lipschitz spaces $\mathrm{Lip}(\alpha)(\mathbb{R}^n)$, $0 < \alpha < \epsilon$, if and only if $T(b) = 0$.

In this paper, we consider the product space $\mathbb{R}^n \times \mathbb{R}^m$ along with two parameter family of dilations $(x, y) \mapsto (\delta_1 x, \delta_2 y)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\delta_i > 0$, $i = 1, 2$, instead of the classical one-parameter dilation. Fefferman and Stein [5] generalized the Calderón-Zygmund operators of convolution type to the two-parameter setting and obtained the boundedness of $L^p(\mathbb{R}^n \times \mathbb{R}^m)$. Journé [15] introduced the product non-convolution singular integral operators (see Definition 1.3) and provided the $T1$ theorem on this product setting. Han, Lee, Lin and Lin [13] extended the $T1$ theorem to product Hardy spaces $H^p(\mathbb{R}^n \times \mathbb{R}^m)$, and also proved in [12] a Tb theorem on product spaces $\mathbb{R}^n \times \mathbb{R}^m$, where $b(x, y) = b_1(x)b_2(y)$, and b_1, b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Recently, Hart [14] gave a bilinear Tb theorem for singular integral operators of Calderón-Zygmund type, and proved product Lebesgue space bounds for bilinear Riesz transforms defined on Lipschitz curves. In 2019, Lee, Li and Lin [17] introduced the Hardy spaces associated with para-accretive functions in product domains and established the endpoint version of product Tb theorem with respect to Hardy spaces $H^p_{b_1,b_2}(\mathbb{R}^n \times \mathbb{R}^m)$ and the dual spaces $CMO^p_{b_1,b_2}(\mathbb{R}^n \times \mathbb{R}^m)$. However, it is an open problem whether the dual spaces of product Hardy spaces are the product Lipschitz spaces. The authors in [23] established a necessary and sufficient condition for the boundedness of product Calderón-Zygmund operators on the product Lipschitz spaces.

A natural question is how to give a Tb theorem for the product non-convolution singular integral operators on product Lipschitz spaces. The main purpose of this paper is to address this question.

We will recall some definitions about para-accretive function, Lipschitz spaces, test function spaces and an approximation to the identity. We begin with recalling para-accretive function and product Lipschitz spaces on $\mathbb{R}^n \times \mathbb{R}^m$.

Definition 1.1 (Para-accretive function [3])**.** A bounded complex-valued function b defined on $\mathbb{R}^n$ is said to be para-accretive if there exist constants

$C, \delta > 0$ such that, for all cubes $Q \subset \mathbb{R}^n$, there is a $Q' \subset Q$ with $\delta|Q| \leq |Q'|$ satisfying

$$\frac{1}{|Q|}\left|\int_{Q'} b(x)dx\right| \geq C > 0.$$

Note that, by the Lebesgue differentiation theorem, $b^{-1}(x)$ is also bounded.

Definition 1.2. Let $0 < \alpha_1, \alpha_2 < 1$. A function f on $\mathbb{R}^{n+m}$ is said to belong to product homogeneous Lipschitz spaces, $\mathrm{Lip}(\alpha_1, \alpha_2)$, if there exists some constants C, such that for $x, u \in \mathbb{R}^n, y, v \in \mathbb{R}^m$,

$$|f(x-u, y-v) - f(x, y-v) - f(x-u, y) + f(x, y)| \leq C|u|^{\alpha_1}|v|^{\alpha_2}. \quad (1.4)$$

If $f \in \mathrm{Lip}(\alpha_1, \alpha_2)$, then $\|f\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}$, the norm of f, is defined by the smallest constant C in (1.4). Denote

$$\mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m) = \{f : f = bg, g \in \mathrm{Lip}(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)\},$$

where $b(x, y) = b_1(x)b_2(y)$, b_1, b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If $f \in \mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, then $f = bg$ where $g \in \mathrm{Lip}(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, and the norm of f is defined by $\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} = \|g\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}$.

We also need some basic definitions and notations to introduce the product singular integral operators. Let $C_0^\eta(\mathbb{R}^n)$ denote the space of continuous functions f with compact support such that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\eta} < \infty.$$

Let $C_0^\eta(\mathbb{R}^n \times \mathbb{R}^m)$ denote the space of continuous functions f with compact support such that

$$\|f\|_\eta := \sup_{\substack{x_1 \neq y_1 \\ x_2 \neq y_2}} \frac{|f(x_1, x_2) - f(y_1, x_2) - f(x_1, y_2) + f(y_1, y_2)|}{|x_1 - y_1|^\eta |x_2 - y_2|^\eta} < \infty.$$

Definition 1.3 (Product Calderón-Zygmund operator [15])**.** Let $K(x_1, x_2, y_1, y_2)$ be a locally integrable function defined on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \setminus \{(x_1, x_2, y_1, y_2) : x_1 = y_1 \text{ or } x_2 = y_2\}$, which satisfies the size estimate

$$|K(x_1, x_2, y_1, y_2)| \leq \frac{C}{|x_1 - y_1|^n |x_2 - y_2|^m} \quad (1.5)$$

for some $C > 0$. Furthermore, one has the following smoothness estimates, for some $\varepsilon > 0$,

$$\begin{aligned}
\left|K(x_1, x_2, y_1, y_2) - K(x_1', x_2, y_1, y_2)\right| &\leq C\frac{|x_1 - x_1'|^\varepsilon}{|x_1 - y_1|^{n+\varepsilon}|x_2 - y_2|^m}, \\
\left|K(x_1, x_2, y_1, y_2) - K(x_1, x_2', y_1, y_2)\right| &\leq C\frac{|x_2 - x_2'|^\varepsilon}{|x_1 - y_1|^n|x_2 - y_2|^{m+\varepsilon}}, \quad (1.6) \\
\left|K(x_1, x_2, y_1, y_2) - K(x_1, x_2, y_1', y_2)\right| &\leq C\frac{|y_1 - y_1'|^\varepsilon}{|x_1 - y_1|^{n+\varepsilon}|x_2 - y_2|^m},
\end{aligned}$$

$$\left|K(x_1,x_2,y_1,y_2)-K(x_1,x_2,y_1,y_2')\right|\le C\frac{|y_2-y_2'|^\varepsilon}{|x_1-y_1|^n|x_2-y_2|^{m+\varepsilon}}$$

for $2|x_1-x_1'|\le|x_1-y_1|$, $2|x_2-x_2'|\le|x_2-y_2|$, $2|y_1-y_1'|\le|x_1-y_1|$, $2|y_2-y_2'|\le|x_2-y_2|$, respectively. One also has that

$$\begin{aligned}&\big|[K(x_1,x_2,y_1,y_2)-K(x_1',x_2,y_1,y_2)]\\&\qquad-[K(x_1,x_2',y_1,y_2)-K(x_1',x_2',y_1,y_2)]\big|\\&\le C\frac{|x_1-x_1'|^\varepsilon}{|x_1-y_1|^{n+\varepsilon}}\frac{|x_2-x_2'|^\varepsilon}{|x_2-y_2|^{m+\varepsilon}}\end{aligned}\tag{1.7}$$

for $2|x_1-x_1'|\le|x_1-y_1|$ and $2|x_2-x_2'|\le|x_2-y_2|$;

$$\begin{aligned}&\big|[K(x_1,x_2,y_1,y_2)-K(x_1,x_2,y_1',y_2)]\\&\qquad-[K(x_1,x_2,y_1,y_2')-K(x_1,x_2,y_1',y_2')]\big|\\&\le C\frac{|y_1-y_1'|^\varepsilon}{|x_1-y_1|^{n+\varepsilon}}\frac{|y_2-y_2'|^\varepsilon}{|x_2-y_2|^{m+\varepsilon}}\end{aligned}\tag{1.8}$$

for $2|y_1-y_1'|\le|x_1-y_1|$ and $2|y_2-y_2'|\le|x_2-y_2|$.

We say that an operator T is a product Calderón-Zygmund singular integral operator if T is a continuous linear operator from $C_0^\eta(\mathbb{R}^n\times\mathbb{R}^m)$, $\eta>0$, into $(C_0^\eta(\mathbb{R}^n\times\mathbb{R}^m))'$ associated with the kernel K satisfying (1.5), (1.6), (1.7) and (1.8), such that the operator T can be represented by

$$\langle Tf_1\otimes f_2,g_1\otimes g_2\rangle=$$
$$\iint_{\mathbb{R}^n\times\mathbb{R}^m}\iint_{\mathbb{R}^n\times\mathbb{R}^m}g_1(x_1)g_2(x_2)K(x_1,x_2,y_1,y_2)f_1(y_1)f_2(y_2)dx_1dx_2dy_1dy_2,$$

for the test functions $f_1,g_1\in C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp}f_1\cap\operatorname{supp}g_1=\varnothing$, and $f_2,g_2\in C_0^\eta(\mathbb{R}^m)$ with $\operatorname{supp}f_2\cap\operatorname{supp}g_2=\varnothing$.

Suppose that T is such a product Calderón-Zygmund singular integral operator on $C_0^\eta(\mathbb{R}^n\times\mathbb{R}^m)$, T is said to be a product Calderón-Zygmund operator if T extends to be a bounded operator on $L^2(\mathbb{R}^{n+m})$.

Definition 1.4 (Generalized singular integral operator [12])**.** Suppose $b(x,y)$ $=b_1(x)b_2(y)$, where b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $\eta>0$, we say that an operator T is a generalized singular integral operator if T is a continuous linear operator from $bC_0^\eta(\mathbb{R}^n\times\mathbb{R}^m)$ into $(bC_0^\eta(\mathbb{R}^n\times\mathbb{R}^m))'$ associated with the kernel K satisfying (1.5), (1.6), (1.7) and (1.8) such that

$$\begin{aligned}&\langle M_bTM_bf_1\otimes f_2,g_1\otimes g_2\rangle\\&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_2(x_2)\,b_1(x_1)\,g_1(x_1)\,g_2(x_2)\\&\quad\times K(x_1,x_2,y_1,y_2)\,b_2(y_2)\,b_1(y_1)\,f_1(y_1)\,f_2(y_2)\,dx_1dx_2dy_1dy_2,\end{aligned}\tag{1.9}$$

for the test functions $f_1, g_1 \in C_0^\eta(\mathbb{R}^n)$ with $\operatorname{supp} f_1 \cap \operatorname{supp} g_1 = \varnothing$, and $f_2, g_2 \in C_0^\eta(\mathbb{R}^m)$ with $\operatorname{supp} f_2 \cap \operatorname{supp} g_2 = \varnothing$, where M_b denotes the multiplication operator by b, that is, $M_b f = bf$.

In order to introduce the main results, we need the following notations.

$$C_{b_1,0}^\eta(\mathbb{R}^n) = \left\{ \psi \in C_0^\eta(\mathbb{R}^n) : \int_{\mathbb{R}^n} \psi(y) b_1(y) dy = 0 \right\},$$

and similarly,

$$C_{b_2,0}^\eta(\mathbb{R}^m) = \left\{ \psi \in C_0^\eta(\mathbb{R}^m) : \int_{\mathbb{R}^m} \psi(y) b_2(y) dy = 0 \right\}.$$

Let T be a generalized singular integral operator and test functions $f_1, g_1 \in C_0^\eta(\mathbb{R}^n)$, $f_2, g_2 \in C_0^\eta(\mathbb{R}^m)$. We define the operator T_1 by the following

$$\langle b_2 g_2, \langle b_1 g_1, T_1(b_1 f_1) \rangle \, b_2 f_2 \rangle = \langle M_b T M_b f_1 \otimes f_2, g_1 \otimes g_2 \rangle .$$

Here we remark that the operator $\langle b_1 g_1, T_1(b_1 f_1) \rangle : b_2 C_0^\eta(\mathbb{R}^m) \mapsto (b_2 C_0^\eta(\mathbb{R}^m))'$ is a singular integral operator on $\mathbb{R}^m$ with kernel $\big\langle b_1 g_1, T_1(b_1 f_1) \big\rangle (x_2, y_2) = \big\langle b_1 g_1, \widetilde{K}_2(x_2, y_2) b_1 f_1 \big\rangle$, where, for each $x_2, y_2 \in \mathbb{R}^m$, $\widetilde{K}_2(x_2, y_2)$ is a Calderón-Zygmund operator acting on function on $\mathbb{R}^n$ with the kernel $\widetilde{K}_2(x_2, y_2)(x_1, y_1) = K(x_1, x_2, y_1, y_2)$.

Moreover, if f_1 is a bounded C^η function, then for all $g_1 \in C_{b_1,0}^\eta(\mathbb{R}^n)$ and all $f_2, g_2 \in C_0^\eta(\mathbb{R}^m)$, $\langle M_b T M_b f_1 \otimes f_2, g_1 \otimes g_2 \rangle$ is well defined. Particularly, we define $T_1(b_1) = 0$ if and only if

$$\langle M_b T M_b 1 \otimes f_2, g_1 \otimes g_2 \rangle = 0 \tag{1.10}$$

for all $g_1 \in C_{b_1,0}^\eta(\mathbb{R}^n)$ and $f_2, g_2 \in C_0^\eta(\mathbb{R}^m)$. Similarly, we can also define $T_1^*(b_1) = 0$ if and only if

$$\langle M_b T M_b f_1 \otimes f_2, 1 \otimes g_2 \rangle = 0$$

for all $f_1 \in C_{b_1,0}^\eta(\mathbb{R}^n)$ and $f_2, g_2 \in C_0^\eta(\mathbb{R}^m)$.

Exchanging the role of indices we get the meaning of $T_2(b_2) = 0$ and $T_2^*(b_2) = 0$.

Definition 1.5 (Product test function [9, 12])**.** Suppose $b(x, y) = b_1(x) b_2(y)$, where b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $i = 1, 2$, fix $\gamma_i > 0, \beta_i \in (0, 1]$. A function f defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a test function of type $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ with width $d_1, d_2 > 0$ if f satisfies the following conditions:

(i) $|f(x, y)| \le C \frac{d_1^{\gamma_1}}{(d_1 + |x - x_0|)^{n+\gamma_1}} \frac{d_2^{\gamma_2}}{(d_2 + |y - y_0|)^{m+\gamma_2}}$;

(ii) $|f(x, y) - f(x', y)| \le C \left(\frac{|x - x'|}{d_1 + |x - x_0|} \right)^{\beta_1} \frac{d_1^{\gamma_1}}{(d_1 + |x - x_0|)^{n+\gamma_1}} \frac{d_2^{\gamma_2}}{(d_2 + |y - y_0|)^{m+\gamma_2}}$ for $|x - x'| \le (d_1 + |x - x_0|)/2$;

(iii) $|f(x,y)-f(x,y')| \le C\frac{d_1^{\gamma_1}}{(d_1+|x-x_0|)^{n+\gamma_1}}\left(\frac{|y-y'|}{d_2+|y-y_0|}\right)^{\beta_2}\frac{d_2^{\gamma_2}}{(d_2+|y-y_0|)^{m+\gamma_2}}$ for $|y-y'| \le (d_2+|y-y_0|)/2$;

(iv) $|[f(x,y)-f(x,y')]-[f(x',y)-f(x',y')]|$
$\le C\left(\frac{|x-x'|}{d_1+|x-x_0|}\right)^{\beta_1}\frac{d_1^{\gamma_1}}{(d_1+|x-x_0|)^{n+\gamma_1}}\left(\frac{|y-y'|}{d_2+|y-y_0|}\right)^{\beta_2}\frac{d_2^{\gamma_2}}{(d_2+|y-y_0|)^{m+\gamma_2}}$
for $|x-x'| \le (d_1+|x-x_0|)/2$ and $|y-y'| \le (d_2+|y-y_0|)/2$;

(v) $\int_{\mathbb{R}^n} f(x,y)b_1(x)dx = 0$ for all $y \in \mathbb{R}^m$;

(vi) $\int_{\mathbb{R}^m} f(x,y)b_2(y)dy = 0$ for all $x \in \mathbb{R}^n$.

If f is a test function of type $(\beta_1,\beta_2,\gamma_1,\gamma_2)$ centered at $(x_0,y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ with width $d_1, d_2 > 0$, we write $f \in \mathcal{M}(x_0,y_0;d_1,d_2;\beta_1,\beta_2;\gamma_1,\gamma_2)$ and we define the norm by

$$\|f\|_{\mathcal{M}(x_0,y_0;d_1,d_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \text{(i), (ii), (iii) and (iv) hold}\}.$$

We denote by $\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ the class of $\mathcal{M}(x_0,y_0;d_1,d_2;\beta_1,\beta_2;\gamma_1,\gamma_2)$ with $d_1 = d_2 = 1$ for fixed $(x_0,y_0) \in \mathbb{R}^n \times \mathbb{R}^m$. It is easy to see that

$$\mathcal{M}(x_1,x_2;d_1,d_2;\beta_1,\beta_2;\gamma_1,\gamma_2) = \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$$

with an equivalent norm for all $(x_1,x_2) \in \mathbb{R}^n \times \mathbb{R}^m$. We can check that the space $\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ is a Banach space. The dual space $(\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ consists of all linear functionals $\mathcal{L}$ from $\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)}, \quad \text{for all } f \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2).$$

We denote $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ and $f \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$. As in the case of non product spaces, we denote by $\mathring{\mathcal{M}}$ $(\beta_1,\beta_2;\gamma_1,\gamma_2)$ the completion of the space $\mathcal{M}(\varepsilon,\varepsilon;\varepsilon,\varepsilon)$ in $\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ when $0 < \beta_1,\beta_2,\gamma_1,\gamma_2 < \varepsilon$. As usual, we write

$$b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2) = \{f|f = bg \ \text{ for some } g \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)\}.$$

If $f \in b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$ and $f = bg$ for $g \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$, then the norm is defined by $\|f\|_{b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)} = \|g\|_{\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)}$.

To state the Calderón type reproducing formula, we also need the definition of an approximation to the identity.

Definition 1.6 (Approximation to the identity [8, 17])**.** Suppose b_1 is a para-accretive functions on $\mathbb{R}^n$. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called an approximation to the identity associated to b_1 if $S_k(x,y)$, the kernels of S_k, are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that there exist $C > 0$ and some $0 < \varepsilon \le 1$ for all $k \in \mathbb{Z}$ and all x, x', y and $y' \in \mathbb{R}^n$,

(i) $|S_k(x,y)| \le C\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$;

(ii) $|S_k(x,y) - S_k(x',y)| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$ for $|x-x'| \le (2^{-k}+|x-y|)/2$;

(iii) $|S_k(x,y) - S_k(x,y')| \le C\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$ for $|y-y'| \le (2^{-k}+|x-y|)/2$;

(iv) $|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]|$
$\le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon}\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k}+|x-y|)^{n+\varepsilon}}$ for $|x-x'| \le (2^{-k}+|x-y|)/2$ and $|y-y'| \le (2^{-k}+|x-y|)/2$;

(v) $\int_{\mathbb{R}^n} S_k(x,y)b_1(x)dy = 1$ for all $k \in \mathbb{Z}, x \in \mathbb{R}^n$;

(vi) $\int_{\mathbb{R}^n} S_k(x,y)b_1(y)dx = 1$ for all $k \in \mathbb{Z}, y \in \mathbb{R}^n$.

Similarly, let b_2 be a para-accretive function on $\mathbb{R}^m$, we have the approximation operators $\{\dot{S}_j\}_{j\in\mathbb{Z}}$ to the identity associated to b_2, and their kernels $\dot{S}_j(x,y)$ satisfying the similar conditions as that above. Set $D_k = S_k - S_{k-1}$, $\dot{D}_j = \dot{S}_j - \dot{S}_{j-1}$.

Remark 1.7. The existence of such an approximation to the identity follows from [3]. By Coifman's construction, if b_1 is para-accretive one can construct an approximation to the identity of order θ such that $S_k(x,y)$ has compact support when one variable is fixed, that is, there is a constant $C > 0$ such that for all $k \in \mathbb{Z}$, $S_k(x,y) = 0$ if $|x-y| > C2^{-k}$. Similarly, we have $\dot{S}_j(x,y) = 0$ if $|x-y| > C2^{-j}$ for b_2.

The first result of this paper is the following Littlewood-Paley characterization of the product Lipschitz spaces $\mathrm{Lip}(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ and $\mathrm{Lip}_b(\alpha_1,\alpha_2)$ $(\mathbb{R}^n \times \mathbb{R}^m)$.

Theorem 1.8. *Suppose $b(x,y) = b_1(x)b_2(y)$, where b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For $\alpha_1,\alpha_2 \in (0,\varepsilon)$, $\beta_1,\beta_2 \in (0,\varepsilon)$, $\gamma_1 \in (\alpha_1,\varepsilon)$, $\gamma_2 \in (\alpha_2,\varepsilon)$, we have*

(i) $f \in \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $f \in (\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ and

$$\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j f(x,y)| \le C < \infty.$$

Moreover,

$$\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} \approx \sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j f(x,y)|. \tag{1.11}$$

(ii) $f \in \mathrm{Lip}(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if $f \in (b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ and

$$\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j (bf)(x,y)| \le C < \infty.$$

Moreover,

$$\|f\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \approx \sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j (bf)(x,y)|. \tag{1.12}$$

The main result of this paper is the Tb criterion for the boundedness of product non-convolution singular integral operators on product Lipschitz spaces.

Theorem 1.9. *Suppose $b(x,y) = b_1(x)b_2(y)$, where b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let T be a generalized singular integral operator and bounded on $L^2(\mathbb{R}^{n+m})$, then T is bounded from $\mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ to $\mathrm{Lip}(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ for $\alpha_1,\alpha_2 \in (0,\varepsilon)$ if and only if $T_1(b_1) = T_2(b_2) = 0$.*

The organization of this paper is as follows. In Section 2, we will give the proof of Theorem 1.8. We will devote to the proof of Theorem 1.9 in Section 3.

Throughout this paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. We use the notation $A \approx B$ to denote that there exists a positive constant C such that $C^{-1}B \le A \le CB$. Let $j \wedge j'$ be the minimum of j and j'.

2. Littlewood-Paley characterization for product Lipschitz spaces

Before the proof of Theorem 1.8, we recall two continuous versions of the Calderón type reproducing formula.

Proposition 2.1 (Continuous Calderón type reproducing formula [12, 17])**.** *Suppose that b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. $\{S_k\}_{k\in\mathbb{Z}}$ and $\{\dot{S}_j\}_{j\in\mathbb{Z}}$ are approximations to the identity defined as in Definition 1.6. Then there exist four families of operators $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{\dot{D}}_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{\widetilde{\dot{D}}}_j\}_{j\in\mathbb{Z}}$ such that, for all $f \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$,*

$$
\begin{aligned}
f(x,y) &= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \widetilde{D}_k M_{b_1} \widetilde{\dot{D}}_j M_{b_2} D_k M_{b_1} \dot{D}_j M_{b_2} f(x,y) \\
&= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} D_k M_{b_1} \dot{D}_j M_{b_2} \widetilde{\widetilde{D}}_k M_{b_1} \widetilde{\widetilde{\dot{D}}}_j M_{b_2} f(x,y),
\end{aligned}
\tag{2.1}
$$

the series converge in the L^p-norm, $1 < p < \infty$, and in the $\mathcal{M}(\beta_1',\beta_2';\gamma_1',\gamma_2')$-norm for $\beta_1' < \beta_1$, $\beta_2' < \beta_2$ and $\gamma_1' < \gamma_1, \gamma_2' < \gamma_2$. The series also converge in sense of the space $(b\mathcal{M}(\beta_1',\beta_2';\gamma_1',\gamma_2'))'$ for $\beta_1' < \beta_1$, $\beta_2' < \beta_2$ and $\gamma_1' < \gamma_1, \gamma_2' < \gamma_2$. Moreover, all $\widetilde{D}_k(x,y)$, the kernels of $\widetilde{D}_k$, satisfy the following estimates: for $0 < \varepsilon' < \varepsilon$, where ε is the regularity exponent of S_k, there exists a constant $C > 0$ such that

(i) $|\widetilde{D}_k(x,y)| \le C\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}$;

(ii) $|\widetilde{D}_k(x,y) - \widetilde{D}_k(x',y)| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}$ *for* $2|x-x'| \le 2^{-k}+|x-y|$;

(iii) $\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)b_1(x)dx = 0$ *for all* $k \in \mathbb{Z}$ *and* $y \in \mathbb{R}^n$;
(iv) $\int_{\mathbb{R}^n} \widetilde{D}_k(x,y)b_1(y)dy = 0$ *for all* $k \in \mathbb{Z}$ *and* $x \in \mathbb{R}^n$.

And $\widetilde{\widetilde{D}}_k(x,y)$*, the kernels of* $\widetilde{\widetilde{D}}_k$*, satisfy the same conditions above but with interchanging the positions of* x *and* y*.*

Similarly, $\widetilde{\dot{D}}_j(x,y)$*, the kernel of* $\widetilde{\dot{D}}_j$*, satisfy the following estimates:*

(i) $|\widetilde{\dot{D}}_j(x,y)| \le C\frac{2^{-j\varepsilon'}}{(2^{-j}+|x-y|)^{m+\varepsilon'}}$;
(ii) $|\widetilde{\dot{D}}_j(x,y) - \widetilde{\dot{D}}_j(x',y)| \le C\left(\frac{|x-x'|}{2^{-j}+|x-y|}\right)^{\varepsilon'} \frac{2^{-j\varepsilon'}}{(2^{-j}+|x-y|)^{m+\varepsilon'}}$ *for* $2|x-x'| \le 2^{-j} + |x-y|$;
(iii) $\int_{\mathbb{R}^m} \widetilde{\dot{D}}_j(x,y)b_2(x)dx = 0$ *for all* $j \in \mathbb{Z}$ *and* $y \in \mathbb{R}^m$;
(iv) $\int_{\mathbb{R}^m} \widetilde{\dot{D}}_j(x,y)b_2(y)dy = 0$ *for all* $j \in \mathbb{Z}$ *and* $x \in \mathbb{R}^m$.

And $\widetilde{\widetilde{\dot{D}}}_j(x,y)$*, the kernels of* $\widetilde{\widetilde{\dot{D}}}_j$*, also satisfy the same conditions above but with interchanging the positions of* x *and* y*.*

We also have

$$\begin{aligned} f(x,y) &= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} M_{b_1}\widetilde{D}_k^{\sharp} M_{b_2}\widetilde{\dot{D}}_j^{\sharp} M_{b_1} D_k M_{b_2}\dot{D}_j f(x,y) \\ &= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} M_{b_1} D_k M_{b_2}\dot{D}_j M_{b_1}\widetilde{\widetilde{D}}_k^{\sharp} M_{b_2}\widetilde{\widetilde{\dot{D}}}_j^{\sharp} f(x,y), \end{aligned} \tag{2.2}$$

where the series converges in the L^p*-norm,* $1 < p < \infty$*, in the* $b\mathcal{M}(\beta_1',\beta_2';\gamma_1',\gamma_2')$*-norm for* $\beta_1' < \beta_1, \beta_2' < \beta_2$*,* $\gamma_1' < \gamma_1, \gamma_2' < \gamma_2$*, and in* $(\mathcal{M}(\beta_1',\beta_2';\gamma_1',\gamma_2'))'$ *for* $\beta_1 < \beta_1', \beta_2 < \beta_2'$*,* $\gamma_1 < \gamma_1', \gamma_2 < \gamma_2'$*. Moreover,* $\widetilde{D}_k^{\sharp}(x,y)$*, the kernel of* $\widetilde{D}_k^{\sharp}$*, and* $\widetilde{\widetilde{D}}_k^{\sharp}(x,y)$*, the kernel of* $\widetilde{\widetilde{D}}_k^{\sharp}$*, satisfy the same conditions as* $\widetilde{D}_k(x,y)$ *and* $\widetilde{\widetilde{D}}_k(x,y)$*, respectively.* $\widetilde{\dot{D}}_j^{\sharp}(x,y)$*, the kernel of* $\widetilde{\dot{D}}_j^{\sharp}$*, and* $\widetilde{\widetilde{\dot{D}}}_j^{\sharp}(x,y)$*, the kernel of* $\widetilde{\widetilde{\dot{D}}}_j^{\sharp}$*, satisfy the same conditions as* $\widetilde{\dot{D}}_j(x,y)$ *and* $\widetilde{\widetilde{\dot{D}}}_j(x,y)$*, respectively.*

Now we give the proof of Theorem 1.8. For (i), firstly, we prove that if $f \in \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, then $f \in (\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$. To see this, let $g \in \mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2)$, we only need to check the inner product $\langle f,g\rangle$ is well defined. Since $f \in \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, there exists a function $h \in \mathrm{Lip}(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$ such that $f = bh$ and $\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} = \|h\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}$. By $\int_{\mathbb{R}^n} g(x,y)b_1(x)dx = 0$ and $\int_{\mathbb{R}^m} g(x,y)b_2(y)dy = 0$, we have

$$\begin{aligned} &|\langle f,g\rangle| \\ &= \left|\int_{\mathbb{R}^m}\int_{\mathbb{R}^n} b_1(x)b_2(y)h(x,y)g(x,y)dxdy\right| \end{aligned}$$

$$= \left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} b_1(x) b_2(y) \left[h(x,y) - h(x_0,y) - h(x,y_0) + h(x_0,y_0) \right] g(x,y) dx dy \right|$$
$$\le \|h\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |b_1(x)||b_2(y)||x-x_0|^{\alpha_1} |y-y_0|^{\alpha_2} |g(x,y)| dx dy$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \frac{|x-x_0|^{\alpha_1}}{(1+|x-x_0|)^{n+\gamma_1}} \frac{|y-y_0|^{\alpha_2}}{(1+|y-y_0|)^{m+\gamma_2}} dx dy$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} < \infty.$$

Furthermore, by $\int_{\mathbb{R}^n} D_k(x_1,y_1) b_1(y_1) dy_1 = 0$ and $\int_{\mathbb{R}^m} \dot{D}_j(x_2,y_2) b_2(y_2) dy_2 = 0$, we have

$$\left| D_k \dot{D}_j f\,(x_1,x_2) \right|$$
$$= \left| \iint_{\mathbb{R}^n \times \mathbb{R}^m} D_k\,(x_1,y_1)\, \dot{D}_j\,(x_2,y_2)\, f\,(y_1,y_2)\, dy_1 dy_2 \right|$$
$$= \Bigg| \iint_{\mathbb{R}^n \times \mathbb{R}^m} D_k\,(x_1,y_1)\, \dot{D}_j\,(x_2,y_2) \left(\frac{f(y_1,y_2)}{b_1(y_1)b_2(y_2)} - \frac{f(x_1,y_2)}{b_1(x_1)b_2(y_2)} \right.$$
$$\left. - \frac{f(y_1,x_2)}{b_1(y_1)b_2(x_2)} + \frac{f(x_1,x_2)}{b_1(x_1)b_2(x_2)} \right) b_1\,(y_1)\, b_2\,(y_2)\, dy_1 dy_2 \Bigg|$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} \iint_{\mathbb{R}^n \times \mathbb{R}^m} |D_k\,(x_1,y_1)\,||\dot{D}_j\,(x_2,y_2)\,|\, |x_1-y_1|^{\alpha_1}\, |x_2-y_2|^{\alpha_2}$$
$$\times |b_1(y_1)||b_2(y_2)| dy_1 dy_2$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} 2^{-k\alpha_1} 2^{-j\alpha_2}$$
$$\times \iint_{\mathbb{R}^n \times \mathbb{R}^m} \frac{2^{-k(\varepsilon-\alpha_1)}}{(2^{-k}+|x_1-y_1|)^{n+\varepsilon-\alpha_1}} \frac{2^{-j(\varepsilon-\alpha_2)}}{(2^{-j}+|x_2-y_2|)^{m+\varepsilon-\alpha_2}} dy_1 dy_2$$
$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} 2^{-k\alpha_1} 2^{-j\alpha_2}.$$

It means that

$$\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1} 2^{j\alpha_2} |D_k \dot{D}_j f(x,y)| \le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}.$$

We now prove the converse implication of Theorem 1.8 (i). Suppose that $f \in (\mathcal{M}(\beta_1,\beta_2,\ \gamma_1,\gamma_2))'$ and $\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1} 2^{j\alpha_2} |D_k \dot{D}_j f(x,y)| \le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}$. We first show that f is a continuous function. Recalling the Calderón type reproducing formula (2.2) for $f \in (\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$,

$$f(x,y) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} M_{b_1} \widetilde{D}_k^{\sharp} M_{b_2} \widetilde{\dot{D}}_j^{\sharp} M_{b_1} D_k M_{b_2} \dot{D}_j f(x,y).$$

We split $\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}}$ by the sums over (i)$j \ge 0, k \ge 0$; (ii)$j \ge 0, k < 0$; (iii)$j < 0, k \ge 0$; (iv)$j < 0, k < 0$, and write $f = f_1 + f_2 + f_3 + f_4$ in $(\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ for corresponding j and k. We will show that f_i $(i = 1,2,3,4)$ are continuous functions.

For the first case, using the size conditions of $\widetilde{D}_k$ and $\widetilde{D}_j$, we get

$$\begin{aligned}
&|f_1(x,y)| \\
&= \Big|\sum_{k\geq 0}\sum_{j\geq 0}\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x)\widetilde{D}_k^{\sharp}(x,x_1)b_2(y)\widetilde{\dot{D}}_j^{\sharp}(y,x_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x_1) \\
&\qquad \times D_k(x_1,y_1)b_2(x_2)\dot{D}_j(x_2,y_2)f(y_1,y_2)dy_1dy_2dx_1dx_2\Big| \\
&\leq C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{k\geq 0}\sum_{j\geq 0}2^{-k\alpha_1}2^{-j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}|b(x,y)||b(x_1,x_2)| \\
&\qquad \times \left|\widetilde{D}_k^{\sharp}(x,x_1)\right|\left|\widetilde{\dot{D}}_j^{\sharp}(y,x_2)\right|dx_1dx_2 \\
&\leq C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{k\geq 0}\sum_{j\geq 0}2^{-k\alpha_1}2^{-j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-x_1|)^{n+\varepsilon'}} \\
&\qquad \times \frac{2^{-j\varepsilon'}}{(2^{-j}+|y-x_2|)^{m+\varepsilon'}}dx_1dx_2 \\
&\leq C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{k\geq 0}\sum_{j\geq 0}2^{-k\alpha_1}2^{-j\alpha_2} \\
&\leq C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}.
\end{aligned}$$

So the series for f_1 is converges uniformly in x, it implies that f_1 is a continuous function.

For $g\in\mathcal{M}(\beta_1,\beta_2,\gamma_1,\gamma_2)$, by the cancellation condition $\int_{\mathbb{R}^m} g(x,y)b_2(y)dy = 0$, we can write

$$\begin{aligned}
&\langle f_2,g\rangle \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{k\geq 0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x)\widetilde{D}_k^{\sharp}(x,x_1)b_2(y)\widetilde{\dot{D}}_j^{\sharp}(y,x_2)g(x,y)dxdy \\
&\qquad \times \iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x_1)D_k(x_1,y_1)b_2(x_2)\dot{D}_j(x_2,y_2)f(y_1,y_2)dy_1dy_2dx_1dx_2 \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{k\geq 0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x)\widetilde{D}_k^{\sharp}(x,x_1)b_2(y)\Big(\widetilde{\dot{D}}_j^{\sharp}(y,x_2) \\
&\qquad - \widetilde{\dot{D}}_j^{\sharp}(y_0,x_2)\Big)g(x,y)dxdy\ b_1(x_1)b_2(x_2)D_k\dot{D}_jf(x_1,x_2)dx_1dx_2 \\
&= \Big\langle \sum_{k\geq 0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x)\widetilde{D}_k^{\sharp}(x,x_1)b_2(y)\left(\widetilde{\dot{D}}_j^{\sharp}(y,x_2)-\widetilde{\dot{D}}_j^{\sharp}(y_0,x_2)\right) \\
&\quad \times b_1(x_1)b_2(x_2)D_k\dot{D}_jf(x_1,x_2)dx_1dx_2,\ g(x,y)\Big\rangle.
\end{aligned}$$

We focus on the series in the inner product. Noticing that b_1, b_2 are bounded functions, we have

$$\begin{aligned}&\Bigg|\sum_{k\geq 0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m} b_1(x)\widetilde{D}_k^\sharp(x,x_1)b_2(y)\left(\widetilde{\dot{D}}_j^\sharp(y,x_2)-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\right)b_1(x_1)\\
&\qquad\qquad \times b_2(x_2)D_k\dot{D}_j f(x_1,x_2)dx_1dx_2\Bigg|\\
&\leq \|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}\\
&\qquad\qquad \times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left|\widetilde{D}_k^\sharp(x,x_1)\left(\widetilde{\dot{D}}_j^\sharp(y,x_2)-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\right)\right|dx_1dx_2.\end{aligned}$$

If $|y-y_0|\leq\frac{2^{-j}}{2}$, using the smoothness condition on $\widetilde{\dot{D}}_j^\sharp$, it will be that

$$\begin{aligned}&\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Bigg|\widetilde{D}_k^\sharp(x,x_1)\bigg(\widetilde{\dot{D}}_j^\sharp(y,x_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\bigg)\Bigg|dx_1dx_2\\
&\leq C\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}\int_{\mathbb{R}^n}\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-x_1|)^{n+\varepsilon'}}dx_1\\
&\qquad \times\int_{\mathbb{R}^m}\left(\frac{|y-y_0|}{2^{-j}+|y-x_2|}\right)^{\varepsilon'}\frac{2^{-j\varepsilon'}}{(2^{-j}+|y-x_2|)^{m+\varepsilon'}}dx_2 \qquad (2.3)\\
&\leq C\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}2^{j\varepsilon'}|y-y_0|^{\varepsilon'}\int_{\mathbb{R}^m}\frac{2^{-j\varepsilon'}}{(2^{-j}+|y-x_2|)^{m+\varepsilon'}}dx_2\\
&\leq C\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j(\alpha_2-\varepsilon')}|y-y_0|^{\varepsilon'}\\
&\leq C|y-y_0|^{\varepsilon'}.\end{aligned}$$

If $|y-y_0|>\frac{2^{-j}}{2}$, using the size conditions of $\widetilde{D}_k^\sharp$ and $\widetilde{\dot{D}}_j^\sharp$, we have

$$\begin{aligned}&\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Bigg|\widetilde{D}_k^\sharp(x,x_1)\bigg(\widetilde{\dot{D}}_j^\sharp(y,x_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\bigg)\Bigg|dx_1dx_2\\
&\leq C\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j\alpha_2}\int_{\mathbb{R}^n}\frac{2^{-k\varepsilon'}}{(2^{-k}+|x-x_1|)^{n+\varepsilon'}}dx_1\int_{\mathbb{R}^m}\left(\frac{2|y-y_0|}{2^{-j}}\right)^{\varepsilon'}\\
&\qquad \times\left(\frac{2^{-j\varepsilon'}}{(2^{-j}+|y-x_2|)^{m+\varepsilon'}}+\frac{2^{-j\varepsilon'}}{(2^{-j}+|y_0-x_2|)^{m+\varepsilon'}}\right)dx_2 \qquad (2.4)\end{aligned}$$

$$\leq C\sum_{k\geq 0}\sum_{j<0}2^{-k\alpha_1}2^{-j(\alpha_2-\varepsilon')}|y-y_0|^{\varepsilon'}$$
$$\leq C|y-y_0|^{\varepsilon'}.$$

Thus, we obtain that for any given large $L>0$, the series for f_2 is converges uniformly for $|y-y_0|\leq L$ in the distribution sense. This means that f_2 is a continuous function on any compact subset in $\mathbb{R}^{n+m}$ and hence, it is continuous on $\mathbb{R}^{n+m}$. Similarly, f_3 is also a continuous function.

For the term f_4, we also use the cancellation conditions $\int_{\mathbb{R}^n} g(x,y)b_1(x)dx = 0$ and $\int_{\mathbb{R}^m} g(x,y)b_2(y)dy=0$,

$$\begin{aligned}&\big\langle f_4,g\big\rangle\\&=\Big\langle\sum_{k<0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x)\left(\widetilde{D}_k^\sharp(x,x_1)-\widetilde{D}_k^\sharp(x_0,x_1)\right)b_2(y)\Big(\widetilde{\dot{D}}_j^\sharp(y,x_2)\\&\qquad-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\Big)b_1(x_1)b_2(x_2)D_k\dot{D}_jf(x_1,x_2)dx_1dx_2,\ g(x,y)\Big\rangle.\end{aligned}$$

Similar with the methods of (2.3) and (2.4), we obtain

$$\begin{aligned}&\sum_{k<0}\sum_{j<0}\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x)\left(\widetilde{D}_k^\sharp(x,x_1)-\widetilde{D}_k^\sharp(x_0,x_1)\right)b_2(y)\\&\qquad\times\left(\widetilde{\dot{D}}_j^\sharp(y,x_2)-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\right)b_1(x_1)b_2(x_2)D_k\dot{D}_jf(x_1,x_2)dx_1dx_2\\&\leq C|x-x_0|^{\varepsilon'}|y-y_0|^{\varepsilon'}.\end{aligned}$$

Thus, f_4 is a continuous function on any compact subset in $\mathbb{R}^{n+m}$.

Now, we estimate $\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}$ as follows. For any $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^m$, we can choose $N_0,M_0\in\mathbb{Z}$ such that:

$$2^{-N_0-1}\leq|x-x_0|<2^{-N_0},\ 2^{-M_0-1}\leq|y-y_0|<2^{-M_0}.$$

The Calderón type reproducing formula (2.2) and the assumption $|D_k\dot{D}_jf(x,y)| \leq C2^{-k\alpha_1}2^{-j\alpha_2}$ tell us that

$$\begin{aligned}&\left|\frac{f(x,y)}{b(x,y)}-\frac{f(x_0,y)}{b(x_0,y)}-\frac{f(x,y_0)}{b(x,y_0)}+\frac{f(x_0,y_0)}{b(x_0,y_0)}\right|\\&=\Bigg|\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left(\widetilde{D}_k^\sharp(x,x_1)-\widetilde{D}_k^\sharp(x_0,x_1)\right)\left(\widetilde{\dot{D}}_j^\sharp(y,x_2)-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\right)\\&\qquad\times b_1(x_1)b_2(x_2)D_k\dot{D}_jf(x_1,x_2)dx_1dx_2\Bigg|\\&\leq C\Bigg(\sum_{k\leq N_0}\sum_{j\leq M_0}+\sum_{k>N_0}\sum_{j\leq M_0}+\sum_{k\leq N_0}\sum_{j>M_0}+\sum_{k>N_0}\sum_{j>M_0}\Bigg)2^{-k\alpha_1}2^{-j\alpha_2}\\&\qquad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left|\left(\widetilde{D}_k^\sharp(x,x_1)-\widetilde{D}_k^\sharp(x_0,x_1)\right)\left(\widetilde{\dot{D}}_j^\sharp(y,x_2)-\widetilde{\dot{D}}_j^\sharp(y_0,x_2)\right)\right|dx_1dx_2\end{aligned}$$

$:= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4.$

Using similar methods as in (2.3) and (2.4), it will be that

$$\mathcal{F}_1 \le C \sum_{k\le N_0} \sum_{j\le M_0} 2^{-k\alpha_1} 2^{-j\alpha_2} \frac{|x-x_0|}{2^{-k}} \frac{|y-y_0|}{2^{-j}}$$
$$\le C 2^{N_0(1-\alpha_1)} 2^{M_0(1-\alpha_2)} |x-x_0||y-y_0| \le C|x-x_0|^{\alpha_1}|y-y_0|^{\alpha_2}.$$
$$\mathcal{F}_2 \le C \sum_{k\le N_0} \sum_{j\le M_0} 2^{-k\alpha_1} 2^{-j\alpha_2} \frac{|x-x_0|}{2^{-k}}$$
$$\le C 2^{N_0(1-\alpha_1)} 2^{-M_0\alpha_2} |x-x_0| \le C|x-x_0|^{\alpha_1}|y-y_0|^{\alpha_2}.$$
$$\mathcal{F}_3 \le C \sum_{k\le N_0} \sum_{j\le M_0} 2^{-k\alpha_1} 2^{-j\alpha_2} \frac{|y-y_0|}{2^{-j}}$$
$$\le C 2^{-N_0\alpha_1} 2^{M_0(1-\alpha_2)} |y-y_0| \le C|x-x_0|^{\alpha_1}|y-y_0|^{\alpha_2}.$$
$$\mathcal{F}_4 \le C \sum_{k> N_0} \sum_{j> M_0} 2^{-k\alpha_1} 2^{-j\alpha_2} \le C 2^{-N_0\alpha_1} 2^{-M_0\alpha_2} \le C|x-x_0|^{\alpha_1}|y-y_0|^{\alpha_2}.$$

Therefore, we have

$$\left| \frac{f(x,y)}{b(x,y)} - \frac{f(x_0,y)}{b(x_0,y)} - \frac{f(x,y_0)}{b(x,y_0)} + \frac{f(x_0,y_0)}{b(x_0,y_0)} \right| \le C|x-x_0|^{\alpha_1}|y-y_0|^{\alpha_2},$$

and we can get the proof of (1.11).

Finally, using the same methods with the Calderón type reproducing formula (2.1), we can get the conclusion (1.12) in Theorem 1.8(ii).

Remark 2.2. We can also use $\{\widetilde{D}_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{\dot{D}}_j\}_{j\in\mathbb{Z}}$ and $\{\widetilde{\widetilde{D}}_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{\widetilde{\dot{D}}}_j\}_{j\in\mathbb{Z}}$ to characterize the Lipschitz spaces $\mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n\times\mathbb{R}^m)$ and $\mathrm{Lip}(\alpha_1,\alpha_2)(\mathbb{R}^n\times\mathbb{R}^m)$.

3. The boundedness of generalized singular integral operator

Before giving the details of the proof of Theorem 1.9, we introduce some new product Besov spaces, and prove some useful lemmas.

Definition 3.1. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$, $\{\dot{S}_j\}_{j\in\mathbb{Z}}$ are approximations to the identity associated to b_1, b_2, respectively. Assume $b(x,y) = b_1(x)b_2(y)$, $0 < \beta_1, \beta_2, \gamma_1, \gamma_2 < \varepsilon$, $-\varepsilon < \alpha_1, \alpha_2 < \varepsilon$ and $1 \le p, q \le \infty$. The homogeneous product Besov space $\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}(\mathbb{R}^n\times\mathbb{R}^m)$ is defined by

$$\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}(\mathbb{R}^n\times\mathbb{R}^m) = \left\{ f \in (\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))' : \|f\|_{\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}} < \infty \right\}$$

with the norm

$$\|f\|_{\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}} \equiv \left\{ \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} 2^{j\alpha_1 q} 2^{k\alpha_2 q} \left\| D_k \dot{D}_j(f) \right\|^q_{L^p(\mathbb{R}^{n+m})} \right\}^{\frac{1}{q}}.$$

The homogeneous product Besov spaces $b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}(\mathbb{R}^n\times\mathbb{R}^m)$ is defined by

$$b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}(\mathbb{R}^n\times\mathbb{R}^m)=\left\{f\in(b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))':\|f\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}}<\infty\right\}$$

with the norm

$$\|f\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}}\equiv\left\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}2^{j\alpha_1 q}2^{k\alpha_2 q}\left\|D_k\dot{D}_j(bf)\right\|^q_{L^p(\mathbb{R}^{n+m})}\right\}^{\frac{1}{q}}.$$

It's easy to see that $f\in b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}$ if and only if $bf\in\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}$. Moreover, we can show that the definition of $\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}$ and $b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}$ are independent of the choice of approximations to the identity. In fact, if $f\in b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}$, we can prove that

$$\|f\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{p,q,b}}\approx\left\{\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}2^{j\alpha_1 q}2^{k\alpha_2 q}\left\|E_k\dot{E}_j(bf)\right\|^q_{L^p(\mathbb{R}^{n+m})}\right\}^{\frac{1}{q}},$$

where E_k and $\dot{E}_j$ possess similar properties as D_k and $\dot{D}_j$, respectively. Similar to the one paremeter case, the key step is that using the Calderón type reproducing formula (2.1) to estimate $E_k\dot{E}_j(bf)$, where the following almost orthogonality estimate plays an important role.

Lemma 3.2. *Suppose that b_1 and b_2 are para-accretive functions on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. E_k and $\dot{E}_j$ possess similar properties as D_k and $\dot{D}_j$, respectively. For $x_1,y_1\in\mathbb{R}^n$, $x_2,y_2\in\mathbb{R}^m$, we have*

$$\begin{aligned}&\left|E_kM_{b_1}\dot{E}_jM_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}(x_1,x_2,y_1,y_2)\right|\\&\le C2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+|x_1-y_1|)^{n+\varepsilon}}\frac{2^{-(j\wedge j')\varepsilon}}{(2^{-(j\wedge j')}+|x_2-y_2|)^{m+\varepsilon}}.\end{aligned}\tag{3.1}$$

Proof. For the sake of simplicity, we only consider the case $k>k'$ and $j>j'$, the other three cases are similar. By the cancellation condition $\int_{\mathbb{R}^n}E_k(x_1,z_1)\ b(z_1)dz_1=0$ and $\int_{\mathbb{R}^m}\dot{E}_j(x_2,z_2)b(z_2)dz_2=0$, we have

$$\begin{aligned}&\left|E_kM_{b_1}\dot{E}_jM_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}(x_1,x_2,y_1,y_2)\right|\\&=\Bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}E_k(x_1,z_1)b_1(z_1)\dot{E}_j(x_2,z_2)b_2(z_2)D_{k'}(z_1,y_1)b_1(y_1)\\&\qquad\times\dot{D}_{j'}(z_2,y_2)b_2(y_2)dz_1dz_2\Bigg|\\&=\Bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}E_k(x_1,z_1)b_1(z_1)\dot{E}_j(x_2,z_2)b_2(z_2)\left(D_{k'}(z_1,y_1)-D_{k'}(x_1,y_1)\right)\\&\qquad\times b_1(y_1)\left(\dot{D}_{j'}(z_2,y_2)-\dot{D}_{j'}(x_2,y_2)\right)b_2(y_2)dz_1dz_2\Bigg|\end{aligned}$$

$$\leq C \iint_{\mathbb{R}^n\times\mathbb{R}^m} |E_k(x_1,z_1)||\dot{E}_j(x_2,z_2)|\,|D_{k'}(z_1,y_1)-D_{k'}(x_1,y_1)| \times \left|\dot{D}_{j'}(z_2,y_2)-\dot{D}_{j'}(x_2,y_2)\right| dz_1dz_2.$$

We only give the estimate for $\int_{\mathbb{R}^n} |E_k(x_1,z_1)|\,|D_{k'}(z_1,y_1)-D_{k'}(x_1,y_1)|\,dz_1$. Note that $|x_1-z_1|<C2^{-k}$, $|y_1-z_1|<C2^{-k'}$, so we get $|x_1-y_1| = |x_1-z_1+z_1-y_1| \leq C2^{-k'}$. If $2|x_1-z_1|>2^{-k'}+|x_1-y_1|$, we have $2|x_1-z_1|/2^{-k'}>1$. Using the size condition on E_k and $D_{k'}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |E_k(x_1,z_1)|\,|D_{k'}(z_1,y_1)-D_{k'}(x_1,y_1)|\,dz_1\\
&\leq C\int_{\mathbb{R}^n} \frac{2^{-k\varepsilon}}{(2^{-k}+|x_1-z_1|)^{n+\varepsilon}}\left(\frac{2|x_1-z_1|}{2^{-k'}}\right)^{\varepsilon}\\
&\qquad\times\left(\frac{2^{-k'\varepsilon}}{(2^{-k'}+|z_1-y_1|)^{n+\varepsilon}}+\frac{2^{-k'\varepsilon}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon}}\right)dz_1\\
&\leq C\int_{\chi_{\{|z_1-x_1|\leq 2^{-k}\}}} 2^{kn}2^{k'\varepsilon}|x_1-z_1|^{\varepsilon}2^{k'n}dz_1\\
&\leq C2^{-(k-k')\varepsilon}2^{k'n}\chi_{\{|x_1-y_1|\leq C2^{-k'}\}}\\
&\approx C2^{-(k-k')\varepsilon}\frac{2^{-k'\varepsilon}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon}}.
\end{aligned} \tag{3.2}$$

If $2|x_1-z_1|\leq 2^{-k'}+|x_1-y_1|$, take virtue of the size condition on E_k and smooth condition on $D_{k'}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |E_k(x_1,z_1)|\,|D_{k'}(z_1,y_1)-D_{k'}(x_1,y_1)|\,dz_1\\
&\leq C\int_{\mathbb{R}^n} \frac{2^{-k\varepsilon}}{(2^{-k}+|x_1-z_1|)^{n+\varepsilon}}\left(\frac{|x_1-z_1|}{2^{-k'}+|x_1-y_1|}\right)^{\varepsilon}\\
&\qquad\times\frac{2^{-k'\varepsilon}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon}}dz_1\\
&\leq C\int_{\chi_{\{|z_1-x_1|\leq 2^{-k}\}}} 2^{kn}|z_1-x_1|^{\varepsilon}2^{k'(\varepsilon+n)}dz_1\\
&\approx C2^{-(k-k')\varepsilon}\frac{2^{-k'\varepsilon}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon}}.
\end{aligned} \tag{3.3}$$

Similarly, we can deal with $\int_{\mathbb{R}^m} |\dot{E}_j(x_2,z_2)|\left|\dot{D}_{j'}(z_2,y_2)-\dot{D}_{j'}(x_2,y_2)\right|dz_2$. So we get the estimate in (3.1).

□

We will use the following density argument in the homogeneous product Besov spaces.

Lemma 3.3. *Suppose that* $g \in (b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$ *with* $\beta_1,\beta_2,\gamma_1,\gamma_2 \in (0,\varepsilon)$ *and* $\|g\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} < \infty$ *for* $\alpha_1,\alpha_2 \in (-\varepsilon,\varepsilon)$*. Then there exists a sequence* $\{g_N\}_{N=1}^{\infty}$*,* $g_N \in \mathcal{M}(\varepsilon',\varepsilon';\varepsilon',\varepsilon')$ *with* $\max\{\alpha_1,\alpha_2\} < \varepsilon' < \varepsilon$*,* $N = 1,2,\cdots$ *, such that*

$$\lim_{N\to\infty} \|g-g_N\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} \to 0.$$

Proof. For $g \in (b\mathcal{M}(\beta_1,\beta_2;\gamma_1,\gamma_2))'$, we consider the Calderón type reproducing formula (2.1) in the distribution sense. For $N \in \mathbb{N}^+$, we can choose a sequence as following

$$g_N(x,y) = \sum_{|k'|\le N}\sum_{|j'|\le N} \widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g(x,y),$$

it is easy to see that $g_N \in \mathcal{M}(\varepsilon',\varepsilon';\varepsilon',\varepsilon')$. In the next, we will show

$$\|g-g_N\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} \to 0 \ as\ N\to\infty. \tag{3.4}$$

By the Minkowski inequality, we have

$$\begin{aligned}
&\|g-g_N\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\\
&\le \Bigg\|\sum_{|j'|>N}\sum_{|k'|>N} \widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g\Bigg\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\\
&\quad + \Bigg\|\sum_{|j'|\le N}\sum_{|k'|>N} \widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g\Bigg\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\\
&\quad + \Bigg\|\sum_{|j'|>N}\sum_{|k'|\le N} \widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g\Bigg\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\\
&:= \Phi_1+\Phi_2+\Phi_3.
\end{aligned}$$

For the first term Φ_1,

$$\begin{aligned}
\Phi_1 &= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Bigg|D_kM_{b_1}\dot{D}_jM_{b_2}\Bigg(\sum_{|k'|>N}\sum_{|j'|>N}\widetilde{D}_{k'}M_{b_1}\\
&\qquad\qquad \widetilde{\dot{D}}_{j'}M_{b_2}D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g\Bigg)(x,y)\Bigg|dxdy\\
&= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x,x_1)b_1(x_1)\dot{D}_j(y,x_2)b_2(x_2)\\
&\quad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{|k'|>N}\sum_{|j'|>N}\widetilde{D}_{k'}(x_1,x_3)b_1(x_3)\widetilde{\dot{D}}_{j'}(x_2,x_4)b_2(x_4)\\
&\quad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_{k'}(x_3,y_1)b_1(y_1)\dot{D}_{j'}(x_4,y_2)b_2(y_2)g(y_1,y_2)dy_1dy_2
\end{aligned}$$

$$\times dx_3dx_4dx_1dx_2\Big|dxdy$$

$$= \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} 2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big|\sum_{|k'|>N}\sum_{|k'|>N}\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_kM_{b_1}\dot{D}_jM_{b_2}$$

$$\widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}(x,y,x_3,x_4)\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_{k'}(x_3,y_1)b_1(y_1)$$

$$\times\dot{D}_{j'}(x_4,y_2)b_2(y_2)g(y_1,y_2)dy_1dy_2dx_3dx_4\Big|dxdy.$$

By the almost orthogonal estimates similar as in (3.1), we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big|D_kM_{b_1}\dot{D}_jM_{b_2}\widetilde{D}_{k'}M_{b_1}\widetilde{\dot{D}}_{j'}M_{b_2}(x,y,x_3,x_4)\Big|dxdy$$

$$\le C2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-(k'\wedge k)\varepsilon}}{(2^{-(k'\wedge k)}+|x-x_3|)^{n+\varepsilon}}$$

$$\times\frac{2^{-(j'\wedge j)\varepsilon}}{(2^{-(j'\wedge j)}+|y-x_4|)^{m+\varepsilon}}dxdy$$

$$\le C2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}.$$

It follows that

$$\Phi_1\le C\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\sum_{|k'|>N}\sum_{|j'|>N}2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}2^{-k'\alpha_1}2^{-j'\alpha_2}\ 2^{k'\alpha_1}2^{j'\alpha_2}$$

$$\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_{k'}(x_3,y_1)b_1(y_1)\dot{D}_{j'}(x_4,y_2)b_2(y_2)$$

$$\times g(y_1,y_2)dy_1dy_2\Big|dx_3dx_4$$

$$= C\sum_{|k'|>N}\sum_{|j'|>N}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{(k-k')\alpha_1}2^{(j-j')\alpha_2}2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}$$

$$\times 2^{k'\alpha_1}2^{j'\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big|D_{k'}M_{b_1}\dot{D}_{j'}M_{b_2}g(x_3,x_4)\Big|dx_3dx_4.$$

Since $\max\{\alpha_1,\alpha_2\}<\varepsilon$, we have

$$\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{(k-k')\alpha_1}2^{(j-j')\alpha_2}2^{-|k-k'|\varepsilon}2^{-|j-j'|\varepsilon}$$

$$\le\left(\sum_{k>k'}2^{(k-k')\alpha_1}2^{-(k-k')\varepsilon}+\sum_{k<k'}2^{(k-k')\alpha_1}2^{-(k'-k)\varepsilon}\right)$$

$$\times\left(\sum_{j>j'}2^{(j-j')\alpha_2}2^{-(j-j')\varepsilon}+\sum_{j<j'}2^{(j-j')\alpha_2}2^{-(j'-j)\varepsilon}\right)$$

$$\le C,$$

and

$$\Phi_1 \le C \sum_{|k'|>N} \sum_{|j'|>N} 2^{k'\alpha_1} 2^{j'\alpha_2} \left\| D_{k'} \dot{D}_{j'}(bg) \right\|_{L^1(\mathbb{R}^{n+m})}.$$

Considering $\|g\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} < \infty$, we get $\Phi_1 \to 0$ as $N \to \infty$. We can also use the same methods to obtain Φ_2, $\Phi_3 \to 0$ as $N \to \infty$. According the above estimates, we get (3.4).

□

Using the same method, we can get a similar result in $\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}$. We omit the details.

Lemma 3.4. *Suppose that $f \in (\mathcal{M}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ with $\beta_1, \beta_2, \gamma_1, \gamma_2 \in (0, \varepsilon)$ and $\|f\|_{\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} < \infty$ for $\alpha_1, \alpha_2 \in (-\varepsilon, \varepsilon)$. Then there exists a sequence $\{f_N\}_{N=1}^{\infty}$, $f_N \in b\mathcal{M}(\varepsilon', \varepsilon'; \varepsilon', \varepsilon')$ with $\max\{\alpha_1, \alpha_2\} < \varepsilon' < \varepsilon$, $N = 1, 2, \cdots$, such that*

$$\lim_{N\to\infty} \|f - f_N\|_{\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}} \to 0.$$

Lemma 3.5. *If $f \in \mathrm{Lip}_b(\alpha_1, \alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, $g \in b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}(\mathbb{R}^n \times \mathbb{R}^m)$, $\alpha_1, \alpha_2 \in (0, \varepsilon)$, then*

$$\left|\langle f, g\rangle\right| \le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} \, \|g\|_{b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}}. \tag{3.5}$$

Proof. By dense argument in Lemma 3.3, it suffices to show (3.5) holds for $g \in \mathcal{M}(\varepsilon', \varepsilon'; \varepsilon', \varepsilon')$, $\max\{\alpha_1, \alpha_2\} < \varepsilon' < \varepsilon$. According to the construction of $\widetilde{D}_k$ and $\widetilde{\widetilde{D}}_k$, $\widetilde{\dot{D}}_j$ and $\widetilde{\widetilde{\dot{D}}}_j$, by the Calderón type reproducing formula (2.1), we have

$$\begin{aligned}
&\langle f, g\rangle \\
&= \left\langle f(x,y), \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widetilde{D}_k M_{b_1} \widetilde{\dot{D}}_j M_{b_2} D_k M_{b_1} \dot{D}_j M_{b_2} g(x,y) \right\rangle \\
&= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \widetilde{D}_k(x, x_1) b_1(x_1) \widetilde{\dot{D}}_j(y, x_2) b_2(x_2) f(x,y) dx dy \\
&\qquad \times \iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x_1, y_1) b_1(y_1) \dot{D}_j(x_2, y_2) b_2(y_2) g(y_1, y_2) dy_1 dy_2 dx_1 dx_2 \\
&= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \widetilde{\widetilde{D}}_k(x_1, x) \widetilde{\widetilde{\dot{D}}}_j(x_2, y) f(x,y) dx dy \; b_1(x_1) b_2(x_2) \\
&\qquad \times \iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x_1, y_1) b_1(y_1) \dot{D}_j(x_2, y_2) b_2(y_2) g(y_1, y_2) dy_1 dy_2 dx_1 dx_2 \\
&= \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \iint_{\mathbb{R}^n\times\mathbb{R}^m} \widetilde{\widetilde{D}}_k \widetilde{\widetilde{\dot{D}}}_j f(x_1, x_2) \; b_1(x_1) b_2(x_2) \; D_k \dot{D}_j (bg)(x_1, x_2) dx_1 dx_2.
\end{aligned}$$

Since b_1 and b_2 are bounded functions, by Remark 2.2, we have

$$\begin{aligned}\left|\langle f,g\rangle\right| &\le C \sup_{\substack{k,j\in\mathbb{Z},\\ x_1\in\mathbb{R}^n,x_2\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}\left|\widetilde{\widetilde{D}}_k\widetilde{\widetilde{\dot{D}}}_j f(x_1,x_2)\right| \\ &\quad\times\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{-k\alpha_1}2^{-j\alpha_2}\left\|D_k\dot{D}_j(bg)\right\|_{L^1(\mathbb{R}^{n+m})} \\ &\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\|g\|_{b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}}.\end{aligned}$$

□

The following density argument for the product Lipschitz spaces in the weak sense plays an important role in proving Theorem 1.9.

Lemma 3.6. *If $f\in \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n\times\mathbb{R}^m)$, $\alpha_1,\alpha_2\in(0,\varepsilon)$, then there exists a sequence $\{f_N\}\in L^2(\mathbb{R}^{n+m})\cap \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n\times\mathbb{R}^m)$ such that*

(i)

$$\|f_N\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)},$$

where the constant C is independent of f_N and f.

(ii) For any $g\in b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\lim_{N\to\infty}\langle f_N,g\rangle=\langle f,g\rangle.$$

Proof. Using the Calderón type reproducing formula (2.2), we construct the sequence $\{f_N\}$ as following,

$$f_N(x,y)=\sum_{|k'|<N}\sum_{|j'|<N}M_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}M_{b_1}D_{k'}M_{b_2}\dot{D}_{j'}f(x,y).$$

Obviously, $f_N\in L^2(\mathbb{R}^{n+m})$ and converges to f in the sense of distribution. To prove $f_N\in \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n\times\mathbb{R}^m)$, by Littlewood-Paley characterization (1.11) of the Lipschitz spaces, we need to show

$$\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n,y\in\mathbb{R}^m}}2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_jf_N(x,y)|\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}.$$

To control the term on the left side of the inequality above, using the almost orthogonal estimates (3.1), we have

$$\begin{aligned}&\left|D_k\dot{D}_jf_N(x,y)\right| \\ &=\left|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x,x_1)\dot{D}_j(y,x_2)f_N(x_1,x_2)dx_1dx_2\right| \\ &=\left|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x,x_1)\dot{D}_j(y,x_2)\sum_{|k'|<N}\sum_{|j'|<N}\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x_1)\widetilde{D}^{\sharp}_{k'}(x_1,x_3)\right. \\ &\qquad\times b_2(x_2)\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,x_4)\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x_3)D_{k'}(x_3,y_1)b_2(x_4)\dot{D}_{j'}(x_4,y_2)\end{aligned}$$

$$\times f(y_1,y_2)dy_1dy_2dx_3dx_4dx_1dx_2\Big|$$

$$= \Big|\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{|k'|<N}\sum_{|j'|<N} D_kM_{b_1}\dot{D}_jM_{b_2}\widetilde{D}^{\sharp}_{k'}M_{b_1}\widetilde{D}^{\sharp}_{j'}M_{b_2}(x,y,x_3,x_4)$$

$$\times\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_{k'}(x_3,y_1)\dot{D}_{j'}(x_4,y_2)f(y_1,y_2)dy_1dy_2dx_3dx_4\Big|$$

$$\le C\sum_{|k'|<N}\sum_{|j'|<N}2^{-|k'-k|\varepsilon}2^{-|j'-j|\varepsilon}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-(k'\wedge k)\varepsilon}}{(2^{-(k'\wedge k)}+|x-x_3|)^{n+\varepsilon}}$$

$$\times\frac{2^{-(j'\wedge j)\varepsilon}}{(2^{-(j'\wedge j)}+|y-x_4|)^{m+\varepsilon}}|D_{k'}\dot{D}_{j'}f(x_3,x_4)|dx_3dx_4$$

$$\le C\sup_{\substack{k',j'\in\mathbb{Z},\\ x_3\in\mathbb{R}^n,x_4\in\mathbb{R}^m}} 2^{k'\alpha_1}2^{j'\alpha_2}|D_{k'}\dot{D}_{j'}f(x_3,x_4)|$$

$$\times\sum_{|k'|<N}\sum_{|j'|<N}2^{-k'\alpha_1}2^{-j'\alpha_2}2^{-|k'-k|\varepsilon}2^{-|j'-j|\varepsilon}$$

$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{|k'|<N}\sum_{|j'|<N}2^{-k'\alpha_1}2^{-j'\alpha_2}2^{-|k'-k|\varepsilon}2^{-|j'-j|\varepsilon}.$$

We have,

$$2^{k\alpha_1}2^{j\alpha_2}\left|D_k\dot{D}_jf_N(x,y)\right|$$

$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\sum_{|k'|<N}\sum_{|j'|<N}2^{(k-k')\alpha_1}2^{(j-j')\alpha_2}2^{-|k'-k|\varepsilon}2^{-|j'-j|\varepsilon}$$

$$\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)},$$

which implies the estimate in (i).

Next, we show (ii). According to the construction of $\widetilde{D}^{\sharp}_k$ and $\widetilde{\widetilde{D}}^{\sharp}_k$, we have,

$$\langle f-f_N,g\rangle$$

$$=\Big\langle\Big(\sum_{|j|>N}\sum_{|k|>N}+\sum_{|j|\le N}\sum_{|k|>N}+\sum_{|j|>N}\sum_{|k|\le N}\Big)M_{b_1}\widetilde{D}^{\sharp}_kM_{b_2}\widetilde{\dot{D}}^{\sharp}_j$$

$$M_{b_1}D_kM_{b_2}\dot{D}_jf(x,y),\ g(x,y)\Big\rangle$$

$$=\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big(\sum_{|j|>N}\sum_{|k|>N}+\sum_{|j|\le N}\sum_{|k|>N}+\sum_{|j|>N}\sum_{|k|\le N}\Big)\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x)$$

$$\times\widetilde{D}^{\sharp}_k(x,x_1)b_2(y)\widetilde{\dot{D}}^{\sharp}_j(y,x_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x_1)D_k(x_1,y_1)b_2(x_2)$$

$$\times\dot{D}_j(x_2,y_2)f(y_1,y_2)dy_1dy_2dx_1dx_2\ g(x,y)dxdy$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^m} \Big(\sum_{|j|>N}\sum_{|k|>N} + \sum_{|j|\le N}\sum_{|k|>N} + \sum_{|j|>N}\sum_{|k|\le N} \Big) \iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(y_1,x_1)$$
$$\times\, b_1(x_1)\dot{D}_j(y_2,x_2)b_2(x_2) \iint_{\mathbb{R}^n\times\mathbb{R}^m} \widetilde{\widetilde{D}}_k(x_1,x)b_1(x)\widetilde{\widetilde{\dot{D}}}_j(x_2,y)b_2(y)g(x,y)$$
$$\times\, dxdydx_1dx_2 f(y_1,y_2)dy_1dy_2$$
$$= \iint_{\mathbb{R}^n\times\mathbb{R}^m} \Big(\sum_{|j|>N}\sum_{|k|>N} + \sum_{|j|\le N}\sum_{|k|>N} + \sum_{|j|>N}\sum_{|k|\le N} \Big) D_k M_{b_1}$$
$$\dot{D}_j M_{b_2} \widetilde{\widetilde{D}}_k M_{b_1} \widetilde{\widetilde{\dot{D}}}_j M_{b_2} g(y_1,y_2) f(y_1,y_2)dy_1dy_2.$$

By the Calderón type reproducing formula (2.1), we denote $g_N(x,y)$ by $\sum_{|j|<N}\sum_{|k|<N} D_k M_{b_1}\dot{D}_j M_{b_2}\widetilde{\widetilde{D}}_k M_{b_1}\widetilde{\widetilde{\dot{D}}}_j M_{b_2} g(x,y)$, and have

$$\big\langle f - f_N,\ g\big\rangle = \big\langle f, g - g_N\big\rangle.$$

The argument in Lemma 3.5 together with the fact that for each $g \in b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}$ $(\mathbb{R}^n\times\mathbb{R}^m)$, $\|g-g_N\|_{b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}} \to 0$ as $N\to\infty$ implies that

$$\big|\big\langle f - f_N, g\big\rangle\big| \le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\ \|g-g_N\|_{b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}} \to 0, \text{ as } N\to\infty,$$

which shows (ii) and hence the proof of Lemma 3.6 is complete. □

Lemma 3.7. *Let T be a generalized singular integral operator defined in (1.9) and bounded on $L^2(\mathbb{R}^{n+m})$. If $T_1(b_1) = T_2(b_2) = 0$, for $0<\varepsilon'<\varepsilon$, there exists a constant $C>0$ such that $D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)$, the kernel of $D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}$, satisfies the following almost orthogonality estimate:*

$$\begin{aligned}
&\big|D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\big| \\
&= \Big| \iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m} K(u_1,u_2, \\
&\qquad v_1,v_2)b_1(v_1)\widetilde{D}^{\sharp}_{k'}(v_1,y_1)b_2(v_2)\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)dv_1dv_2du_1du_2\Big| \\
&\le C(1+|k-k'|)(1+|j-j'|)(2^{-(k-k')\varepsilon'}\wedge 1)(2^{-(j-j')\varepsilon'}\wedge 1) \\
&\quad\times \frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x_2-y_2|)^{m+\varepsilon'}}.
\end{aligned} \tag{3.6}$$

Proof. In order to estimate $D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)$, for the different k and k', j and j', we will consider four cases: (I) $k>k'$, $j>j'$, (II) $k>k'$, $j\le j'$, (III) $k\le k'$, $j>j'$, (IV) $k\le k'$, $j\le j'$. We only give the details of the first case (I), the other cases are similar. For $x_1,y_1\in\mathbb{R}^n$, $x_2,y_2\in\mathbb{R}^m$, we also split (I) into four cases: (I_1) $|x_1-y_1|\ge 4C2^{-k'}$

and $|x_2-y_2|\geq 4C2^{-j'}$, (I_2) $|x_1-y_1|\geq 4C2^{-k'}$ and $|x_2-y_2|<4C2^{-j'}$, (I_3) $|x_1-y_1|<4C2^{-k'}$ and $|x_2-y_2|\geq 4C2^{-j'}$, (I_4) $|x_1-y_1|<4C2^{-k'}$ and $|x_2-y_2|<4C2^{-j'}$, where the C can be referred in Remark 1.7.

Firstly, for the case (I_1): $k>k'$, $j>j'$, $|x_1-y_1|\geq 4C2^{-k'}$ and $|x_2-y_2|\geq 4C2^{-j'}$, by the cancellation condition $\int_{\mathbb{R}^n}D_k(x_1,u_1)b_1(u_1)du_1=0$, $\int_{\mathbb{R}^m}\dot{D}_j(x_2,\,u_2)b_2(u_2)du_2=0$, we have

$$\begin{aligned}&\big|D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\big|\\&=\bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}\big[K(u_1,u_2,v_1,v_2)\\&\qquad-K(x_1,u_2,v_1,v_2)-K(u_1,x_2,v_1,v_2)+K(x_1,x_2,v_1,v_2)\big]b_1(v_1)\\&\qquad\times\widetilde{D}^{\sharp}_{k'}(v_1,y_1)b_2(v_2)\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)dv_1dv_2du_1du_2\bigg|.\end{aligned}$$

Noticing the Remark 1.7, we have $|u_1-x_1|\leq C2^{-k}$, $|u_2-x_2|\leq C2^{-j}$, $|v_1-y_1|\leq C2^{-k'}$ and $|v_2-y_2|\leq C2^{-j'}$. It is easy to get

$$\begin{aligned}|u_1-v_1|&=|u_1-x_1+x_1-y_1+y_1-v_1|\\&\geq|x_1-y_1|-|u_1-x_1|-|y_1-v_1|\\&\geq 4C2^{-k'}-C2^{-k}-C2^{-k'}\\&\geq 2C2^{-k}\geq 2|u_1-x_1|.\end{aligned}$$

So, by the the smooth condition of kernel K (see (1.7)), it follows that

$$\begin{aligned}&\big|D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\big|\\&\leq C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_k(x_1,u_1)|\,|b_1(u_1)||\dot{D}_j(x_2,u_2)||b_2(u_2)|\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}\\&\qquad\times\frac{|u_2-x_2|^{\varepsilon}}{|x_2-v_2|^{m+\varepsilon}}du_1du_2\;|b_1(v_1)||\widetilde{D}^{\sharp}_{k'}(v_1,y_1)||b_2(v_2)||\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)|dv_1dv_2.\end{aligned}$$

We also note that $|v_1-y_1|\leq C2^{-k'}\leq\frac14|x_1-y_1|$, $|x_1-v_1|\geq|x_1-y_1|-|y_1-v_1|\geq\frac34|x_1-y_1|$ and $\varepsilon'<\varepsilon$, which yields

$$\begin{aligned}\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}&\leq C\frac{2^{-k\varepsilon}}{|x_1-y_1|^{n+\varepsilon}}\leq C\frac{2^{-k\varepsilon}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon}}\\&=C2^{-(k-k')\varepsilon}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-k'(\varepsilon-\varepsilon')}}{(2^{-k'}+|x_1-y_1|)^{\varepsilon-\varepsilon'}}\\&\leq C2^{-(k-k')\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}.\end{aligned}\tag{3.7}$$

Similarly, $\frac{|u_2-x_2|^\varepsilon}{|x_2-v_2|^{m+\varepsilon}} \leq C2^{-(j-j')\varepsilon'}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}$. According to the above estimates, we get

$$\begin{aligned}
&\left|D_k M_{b_1}\dot{D}_j M_{b_2} T M_{b_1}\widetilde{D}^\sharp_{k'} M_{b_2}\widetilde{\dot{D}}^\sharp_{j'}(x_1,x_2,y_1,y_2)\right| \\
&\leq C2^{-(k-k')\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}2^{-(j-j')\varepsilon'}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}} \\
&\qquad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_k(x_1,u_1)||b_1(u_1)||\dot{D}_j(x_2,u_2)||b_2(u_2)|du_1du_2 \\
&\qquad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}|b_1(v_1)||\widetilde{D}^\sharp_{k'}(v_1,y_1)||b_2(v_2)||\widetilde{\dot{D}}^\sharp_{j'}(v_2,y_2)|dv_1dv_2 \\
&\leq C2^{-(k-k')\varepsilon'}\ 2^{-(j-j')\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}.
\end{aligned}$$

Secondly, for the case (I_2): $k'<k$, $j'<j$, $|x_1-y_1|\geq 4C2^{-k'}$ and $|x_2-y_2|<4C2^{-j'}$. By $T_2(b_2)=0$, we can get that

$$\begin{aligned}
&D_k M_{b_1}\dot{D}_j M_{b_2} T M_{b_1}\widetilde{D}^\sharp_{k'} M_{b_2}\widetilde{\dot{D}}^\sharp_{j'}(x_1,x_2,y_1,y_2) \\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2) \\
&\qquad\times b_1(v_1)\widetilde{D}^\sharp_{k'}(v_1,y_1)b_2(v_2)\widetilde{\dot{D}}^\sharp_{j'}(v_2,y_2)dv_1dv_2du_1du_2 \\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2) \\
&\qquad\times b_1(v_1)\widetilde{D}^\sharp_{k'}(v_1,y_1)b_2(v_2)\left(\widetilde{\dot{D}}^\sharp_{j'}(v_2,y_2)-\widetilde{\dot{D}}^\sharp_{j'}(x_2,y_2)\right)dv_1dv_2du_1du_2.
\end{aligned}$$

Let $\phi_0\in C^\infty(\mathbb{R}^m)$ be 1 on the unit ball and 0 outside its double. Set $\phi_1=1-\phi_0$. Then we have

$$\begin{aligned}
&D_k M_{b_1}\dot{D}_j M_{b_2} T M_{b_1}\widetilde{D}^\sharp_{k'} M_{b_2}\widetilde{\dot{D}}^\sharp_{j'}(x_1,x_2,y_1,y_2) \\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2) \\
&\qquad\times b_1(v_1)\widetilde{D}^\sharp_{k'}(v_1,y_1)b_2(v_2)\left(\widetilde{\dot{D}}^\sharp_{j'}(v_2,y_2)-\widetilde{\dot{D}}^\sharp_{j'}(x_2,y_2)\right) \\
&\qquad\times\left(\phi_0\left(\frac{x_2-v_2}{2C2^{-j}}\right)+\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)\right)dv_1dv_2du_1du_2 \\
&:=I_{21}+I_{22}.
\end{aligned}$$

Combining the cancellation $\int_{\mathbb{R}^n}D_k(x_1,u_1)b_1(u_1)du_1=0$, we have

$$I_{21}=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2)$$

$$\times\, b_1(v_1)\widetilde{D}^{\sharp}_{k'}(v_1,y_1)b_2(v_2)\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)$$
$$\times\,\phi_0\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_1dv_2du_1du_2$$
$$=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}[K(u_1,u_2,v_1,v_2)$$
$$-K(x_1,u_2,v_1,v_2)]b_1(v_1)\widetilde{D}^{\sharp}_{k'}(v_1,y_1)b_2(v_2)$$
$$\times\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\phi_0\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_1dv_2du_1du_2.$$

Now we write $\psi(u_2)=\dot{D}_j(x_2,u_2)$, $\varphi(v_2)=\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\times \phi_0\left(\frac{x_2-v_2}{2C2^{-j}}\right)$ and for each $u_1,v_1\in\mathbb{R}^n$, $x_2,y_2\in\mathbb{R}^m$,

$$\left\langle\psi b_2,\widetilde{K}_1(u_1,v_1)b_2\varphi\right\rangle$$
$$=\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\psi(u_2)b_2(u_2)K(u_1,u_2,v_1,v_2)b_2(v_2)\varphi(v_2)du_2dv_2,$$

where $\widetilde{K}_1(u_1,v_1)$ is also a Calderón-Zygmund operator acting on $\mathbb{R}^m$ with kernel $\widetilde{K}_1(u_1,v_1)\ (u_2,v_2)=K(u_1,u_2,v_1,v_2)$. So we have

$$|I_{21}|=\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}D_k(x_1,u_1)b_1(u_1)\left\langle\psi b_2,[\widetilde{K}_1(u_1,v_1)-\widetilde{K}_1(x_1,v_1)]b_2\varphi\right\rangle b_1(v_1)\right.$$
$$\left.\times\,\widetilde{D}^{\sharp}_{k'}(v_1,y_1)du_1dv_1\right|$$
$$\le\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D_k(x_1,u_1)|\left|\left\langle\psi b_2,[\widetilde{K}_1(u_1,v_1)-\widetilde{K}_1(x_1,v_1)]b_2\varphi\right\rangle\right|$$
$$\times\,|\widetilde{D}^{\sharp}_{k'}(v_1,y_1)|du_1dv_1.$$

The $L^2(\mathbb{R}^{n+m})$ boundedness of T implies the following weak boundedness property, one can see [8] (page 87) for details.

$$\left|\left\langle\psi b_2,[\widetilde{K}_1(u_1,v_1)-\widetilde{K}_1(x_1,v_1)]b_2\varphi\right\rangle\right|$$
$$\le C2^{-j(m+2\eta)}\|\varphi\|_{\mathrm{Lip}_\eta}\|\psi\|_{\mathrm{Lip}_\eta}\|\widetilde{K}_1(u_1,v_1)-\widetilde{K}_1(x_1,v_1)\|_{CZ}$$
$$\le C2^{-j(m+2\eta)}\left(2^{-(j-j')\varepsilon}2^{j'm}2^{j\eta}\right)\left(2^{jm}2^{j\eta}\right)\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}$$
$$\le C2^{-(j-j')\varepsilon}2^{j'm}\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}},$$

where $|u_1-x_1|<|x_1-v_1|/2$. Since $j>j'$, $\varepsilon>\varepsilon'$, by (3.7), it is easy to see that

$$|I_{21}|\le C2^{-(j-j')\varepsilon'}2^{j'm}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D_k(x_1,u_1)|\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}|\widetilde{D}^{\sharp}_{k'}(v_1,y_1)|du_1dv_1$$

$$\leq C2^{-(j-j')\varepsilon'}2^{j'm}2^{-(k-k')\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}$$
$$\leq C2^{-(k-k')\varepsilon'}2^{-(j-j')\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}.$$

For the term I_{22}, using the condition $T_2(b_2)=0$ and the cancellation $\int_{\mathbb{R}^n} D_k(x_1,u_1)b_1(u_1)du_1=0$, $\int_{\mathbb{R}^m}\dot{D}_j(x_2,u_2)b_2(u_2)du_2=0$, we have

$$\begin{aligned}I_{22}=\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}\big[K(u_1,u_2,v_1,v_2)\\ -K(x_1,u_2,v_1,v_2)-K(u_1,x_2,v_1,v_2)+K(x_1,x_2,v_1,v_2)\big]b_1(v_1)\\ \times\widetilde{D}^{\sharp}_{k'}(v_1,y_1)b_2(v_2)\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\\ \times\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_1dv_2du_1du_2.\end{aligned}$$

The smooth condition (1.7) can be used to deal with the kernel K in I_{22}. The similar methods in (3.2) and (3.3) could be used to estimate $(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2))$. We just give the case for $2|v_2-x_2|\leq 2^{-j'}+|v_2-y_2|$.

$$\begin{aligned}|I_{22}|&\leq C\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-k\varepsilon}}{(2^{-k}+|x_1-u_1|)^{n+\varepsilon}}\frac{2^{-j\varepsilon}}{(2^{-j}+|x_2-u_2|)^{m+\varepsilon}}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\\ &\quad\times\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}\frac{|u_2-x_2|^{\varepsilon}}{|x_2-v_2|^{m+\varepsilon}}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|v_1-y_1|)^{n+\varepsilon'}}\\ &\quad\times\left(\frac{|v_2-x_2|}{2^{-j'}+|v_2-y_2|}\right)^{\varepsilon'}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|v_2-y_2|)^{m+\varepsilon'}}\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)\\ &\quad\times dv_1dv_2du_1du_2\\ &\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{2^{-k\varepsilon}}{(2^{-k}+|x_1-u_1|)^{n+\varepsilon}}\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|v_1-y_1|)^{n+\varepsilon'}}\\ &\quad\times du_1dv_1\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{2^{-j\varepsilon}}{(2^{-j}+|x_2-u_2|)^{m+\varepsilon}}\frac{|u_2-x_2|^{\varepsilon}}{|x_2-v_2|^{m+\varepsilon}}\\ &\quad\times\left(\frac{|v_2-x_2|}{2^{-j'}+|x_2-v_2|}\right)^{\varepsilon'}2^{j'm}\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)du_2dv_2\\ &\qquad(\text{by }(3.7),|u_2-x_2|<C2^{-j},|v_2-x_2|^{\varepsilon'-\varepsilon}\leq C2^{-j(\varepsilon'-\varepsilon)})\\ &\leq C2^{(k'-k)\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\;2^{-j\varepsilon}\;2^{j'm}\;2^{-j(\varepsilon'-\varepsilon)}\\ &\quad\times\int_{\mathbb{R}^m}\frac{2^{-j\varepsilon}}{(2^{-j}+|x_2-u_2|)^{m+\varepsilon}}du_2\int_{\mathbb{R}^m}\frac{1}{|x_2-v_2|^m}\left(\frac{1}{2^{-j'}+|x_2-v_2|}\right)^{\varepsilon'}\end{aligned}$$

$$
\begin{aligned}
&\times \phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_2\\
&\le C2^{(k'-k)\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\ 2^{-j\varepsilon'}\ 2^{j'm}\\
&\quad\times\left(\int_{|x_2-v_2|\ge 2^{-j'}}\frac{1}{|x_2-v_2|^{m+\varepsilon'}}dv_2\right.\\
&\qquad\left.+\,2^{j'\varepsilon'}\int_{2^{-j'}\ge|x_2-v_2|\ge 2\times 2^{-j}}\frac{1}{|x_2-v_2|^{m}}dv_2\right)\\
&\le C2^{(k'-k)\varepsilon'}\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}2^{j'm}\ 2^{-j\varepsilon'}[2^{j'\varepsilon'}+2^{j'\varepsilon'}(j-j')]\\
&\le C2^{(k'-k)\varepsilon'}\ 2^{(j'-j)\varepsilon'}(1+(j-j'))\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\\
&\quad\times\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}.
\end{aligned}
$$

For the case I_3, using the similar method with I_2, we can obtain that

$$
\begin{aligned}
|I_3|&\le C2^{(k'-k)\varepsilon'}\ 2^{(j'-j)\varepsilon'}(1+(k-k'))\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\\
&\quad\times\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}.
\end{aligned}
$$

Finally, we should focus on the last term I_4: $k>k'$, $j>j'$, $|x_1-y_1|<4C2^{-k'}$ and $|x_2-y_2|<4C2^{-j'}$. Using the condition $T_1(b_1)=T_2(b_2)=0$, it is easy to get

$$
\begin{aligned}
&D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2)\\
&\quad\times b_1(v_1)\left(\widetilde{D}^{\sharp}_{k'}(v_1,y_1)-\widetilde{D}^{\sharp}_{k'}(x_1,y_1)\right)b_2(v_2)\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\\
&\quad\times dv_1dv_2du_1du_2.
\end{aligned}
$$

One can also fix a smooth cut-off function $\varrho_0\in C^\infty(\mathbb{R}^n)$ on the unit ball and 0 outside its double. Set $\varrho_1=1-\varrho_0$, so we get the following four parts:

$$
\begin{aligned}
&D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\\
&=\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\left(\varrho_0\left(\frac{x_1-v_1}{2C2^{-k}}\right)+\right.\\
&\quad\left.\varrho_1\left(\frac{x_1-v_1}{2C2^{-k}}\right)\right)\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u_1,u_2,v_1,v_2)b_1(v_1)\left(\widetilde{D}^{\sharp}_{k'}(v_1,y_1)-\right.
\end{aligned}
$$

$$\widetilde{D}^{\sharp}_{k'}(x_1,y_1)\Big)\, b_2(v_2)\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\left(\phi_0\left(\frac{x_2-v_2}{2C2^{-j}}\right)+\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)\right)dv_1dv_2du_1du_2$$
$$:= I_{41}+I_{42}+I_{43}+I_{44}.$$

We give the details of I_{44}, since I_{41}, I_{42} and I_{43} are similar to I_1 and I_2. By the cancellation $\int_{\mathbb{R}^n} D_k(x_1,u_1)b_1(u_1)du_1=0$ and $\int_{\mathbb{R}^m} \dot{D}_j(x_2,u_2)b_2(u_2)du_2=0$, it will be that

$$\begin{aligned}
I_{44} = &\iint_{\mathbb{R}^n\times\mathbb{R}^m} D_k(x_1,u_1)b_1(u_1)\dot{D}_j(x_2,u_2)b_2(u_2)\varrho_1\left(\frac{x_1-v_1}{2C2^{-k}}\right)\\
&\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\Big[K(u_1,u_2,v_1,v_2)-K(x_1,u_2,v_1,v_2)-K(u_1,x_2,v_1,v_2)\\
&+K(x_1,x_2,v_1,v_2)\Big]b_1(v_1)\left(\widetilde{D}^{\sharp}_{k'}(v_1,y_1)-\widetilde{D}^{\sharp}_{k'}(x_1,y_1)\right)b_2(v_2)\\
&\times\left(\widetilde{\dot{D}}^{\sharp}_{j'}(v_2,y_2)-\widetilde{\dot{D}}^{\sharp}_{j'}(x_2,y_2)\right)\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_1dv_2du_1du_2.
\end{aligned}$$

It is also need to consider the following different cases for I_{44}: $2|v_1-x_1|\le 2^{-k'}+|x_1-y_1|$, $2|v_1-x_1|>2^{-k'}+|x_1-y_1|$, $2|v_2-x_2|\le 2^{-j'}+|x_2-y_2|$, $2|v_2-x_2|>2^{-j'}+|x_2-y_2|$. We deal with the case $2|v_1-x_1|\le 2^{-k'}+|x_1-y_1|$ and $2|v_2-x_2|\le 2^{-j'}+|x_2-y_2|$, the others are similar.

$$\begin{aligned}
|I_{44}| \le &\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left|D_k(x_1,u_1)\dot{D}_j(x_2,u_2)\right|\varrho_1\left(\frac{x_1-v_1}{2C2^{-k}}\right)\\
&\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}\frac{|u_2-x_2|^{\varepsilon}}{|x_2-v_2|^{m+\varepsilon}}\left(\frac{|v_1-x_1|}{2^{-k'}+|v_1-x_1|}\right)^{\varepsilon'}\\
&\times\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|v_1-y_1|)^{n+\varepsilon'}}\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)\left(\frac{|v_2-x_2|}{2^{-j'}+|v_2-x_2|}\right)^{\varepsilon'}\\
&\times\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|v_2-y_2|)^{m+\varepsilon'}}dv_1dv_2du_1du_2\\
\le\ & C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D_k(x_1,u_1)|\frac{|u_1-x_1|^{\varepsilon}}{|x_1-v_1|^{n+\varepsilon}}\ 2^{k'n}\left(\frac{|v_1-x_1|}{2^{-k'}+|v_1-x_1|}\right)^{\varepsilon'}\\
&\times\varrho_1\left(\frac{x_1-v_1}{2C2^{-k}}\right)du_1dv_1\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}|\dot{D}_j(x_2,u_2)|\frac{|u_2-x_2|^{\varepsilon}}{|x_2-v_2|^{m+\varepsilon}}\\
&\times\ 2^{j'm}\left(\frac{|v_2-x_2|}{2^{-j'}+|v_2-x_2|}\right)^{\varepsilon'}\ \phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)du_2dv_2\\
\le\ & C2^{-k\varepsilon'}\ 2^{k'n}\int_{\mathbb{R}^n}\frac{2^{-k\varepsilon}}{(2^{-k}+|x_1-u_1|)^{n+\varepsilon}}du_1\int_{\mathbb{R}^n}\frac{1}{|x_1-v_1|^{n}}
\end{aligned}$$

$$\times\left(\frac{1}{2^{-k'}+|x_1-v_1|}\right)^{\varepsilon'}\varrho_1\left(\frac{x_1-v_1}{2C2^{-k}}\right)dv_1\ 2^{-j\varepsilon'}\ 2^{j'm}$$
$$\times\int_{\mathbb{R}^m}\frac{2^{-j\varepsilon}}{(2^{-j}+|x_2-u_2|)^{m+\varepsilon}}du_2\int_{\mathbb{R}^m}\frac{1}{|x_2-v_2|^m}$$
$$\times\left(\frac{1}{2^{-j'}+|x_2-v_2|}\right)^{\varepsilon'}\phi_1\left(\frac{x_2-v_2}{2C2^{-j}}\right)dv_2$$
$$\le C2^{-k\varepsilon'}\ 2^{k'n}\Bigg(\int_{|x_1-v_1|\ge 2^{-k'}}\frac{1}{|x_1-v_1|^{n+\varepsilon'}}dv_1$$
$$+2^{k'\varepsilon'}\int_{2^{-k'}\ge|x_1-v_1|\ge 2\times 2^{-k}}\frac{1}{|x_1-v_1|^n}dv_1\Bigg)$$
$$\times 2^{-j\varepsilon'}\ 2^{j'm}\Bigg(\int_{|x_2-v_2|\ge 2^{-j'}}\frac{1}{|x_2-v_2|^{m+\varepsilon'}}dv_2$$
$$+2^{j'\varepsilon'}\int_{2^{-j'}\ge|x_2-v_2|\ge 2\times 2^{-j}}\frac{1}{|x_2-v_2|^m}dv_2\Bigg)$$
$$\le C2^{k'n}\ 2^{-k\varepsilon'}[2^{k'\varepsilon'}+2^{k'\varepsilon'}(k-k')]\ 2^{j'm}\ 2^{-j\varepsilon'}[2^{j'\varepsilon'}+2^{j'\varepsilon'}(k-k')]$$
$$\le C2^{(k'-k)\varepsilon'}2^{(j'-j)\varepsilon'}(1+(k-k'))(1+(j-j'))$$
$$\times\frac{2^{-k'\varepsilon'}}{(2^{-k'}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-j'\varepsilon'}}{(2^{-j'}+|x_2-y_2|)^{m+\varepsilon'}}.$$

Combining the estimates of I_1, I_2, I_3 and I_4, we get the desired result in (3.6). □

Using the same method as in Lemma 3.7, we can get the following results.

Lemma 3.8. *Let T be a generalized singular integral operator defined in (1.9) and bounded on $L^2(\mathbb{R}^{n+m})$. For $0<\varepsilon'<\varepsilon$, $D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)$, the kernel of $D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}$, satisfies the following almost orthogonality estimate:*

(i) If $T_1^*(b_1)=T_2^*(b_2)=0$, there exists a constant C such that

$$\left|D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\right|$$
$$\le C(1+|k-k'|)(1+|j-j'|)(2^{-(k'-k)\varepsilon'}\wedge 1)(2^{-(j'-j)\varepsilon'}\wedge 1)$$
$$\times\frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x_2-y_2|)^{m+\varepsilon'}}.$$

(ii) If $T_1^*(b_1)=T_2^*(b_2)=T_1(b_1)=T_2(b_2)=0$, there exists a constant C such that

$$\left|D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x_1,x_2,y_1,y_2)\right|$$
$$\le C2^{-|k-k'|\varepsilon'}2^{-|j-j'|\varepsilon'}\frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x_1-y_1|)^{n+\varepsilon'}}\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|x_2-y_2|)^{m+\varepsilon'}}.$$

On the basis of Lemma 3.7 and Lemma 3.8, we will investigate the boundedness of T on the product Besov spaces.

Lemma 3.9. *Let T be a generalized singular integral operator defined in (1.9) and bounded on $L^2(\mathbb{R}^{n+m})$. Then*

(i) T is bounded from $\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ to $b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ if $\alpha_1,\alpha_2\in(-\varepsilon,0)$ and $T_1^*(b_1)=T_2^*(b_2)=0$.

(ii) T is bounded from $\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ to $b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ if $\alpha_1,\alpha_2\in(0,\varepsilon)$ and $T_1(b_1)=T_2(b_2)=0$.

(iii) T is bounded from $\dot{B}^{0,0}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ to $b^{-1}\dot{B}^{0,0}_{1,1,b}(\mathbb{R}^n\times\mathbb{R}^m)$ if $T_1^*(b_1)=T_2^*(b_2)=T_1(b_1)=T_2(b_2)=0$.

Proof. We just give the details of the first item, the others are similar. By Lemma 3.4, $b\mathcal{M}(\varepsilon',\varepsilon';\varepsilon',\varepsilon')$ is dense in $\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}$ for $-\varepsilon'<\alpha_1,\alpha_2<\varepsilon'$. Thus, it suffices to show that for $f\in b\mathcal{M}(\varepsilon',\varepsilon';\varepsilon',\varepsilon')\cap\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}$ with $-\varepsilon'<\alpha_1,\alpha_2<0$,

$$\|Tf\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\le C\|f\|_{\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}.$$

From the definition of product Besov spaces, we have

$$\|Tf\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_k\dot{D}_j(bTf)(x,y)|dxdy.$$

Since T is bounded on $L^2(\mathbb{R}^{n+m})$, by the Calderón type reproducing formula (2.2) and Lemma 3.8 (i), we have

$$\begin{aligned}
&\|Tf\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}\\
&=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x,u)b_1(u)\dot{D}_j(y,v)b_2(v)Tf(u,v)\\
&\qquad\times dudv\bigg|dxdy\\
&=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}D_k(x,u)b_1(u)\dot{D}_j(y,v)b_2(v)\\
&\qquad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}K(u,u_1,v,v_1)\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}b_1(u_1)\widetilde{D}^{\sharp}_{k'}(u_1,x_1)b_2(v_1)\\
&\qquad\times\widetilde{\dot{D}}^{\sharp}_{j'}(v_1,x_2)\iint_{\mathbb{R}^n\times\mathbb{R}^m}b_1(x_1)D_{k'}(x_1,y_1)b_2(x_2)\dot{D}_{j'}(x_2,y_2)f(y_1,y_2)dy_1\\
&\qquad\times dy_2dx_1dx_2du_1dv_1dudv\bigg|dxdy\\
&=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}D_kM_{b_1}\dot{D}_jM_{b_2}TM_{b_1}\widetilde{D}^{\sharp}_{k'}
\end{aligned}$$

$$M_{b_2}\widetilde{\dot{D}}^{\sharp}_{j'}(x,y,x_1,x_2)b_1(x_1)b_2(x_2)D_{k'}\dot{D}_{j'}f(x_1,x_2)dx_1dx_2\Big|dxdy$$

$$\begin{aligned}
&\leq C\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}(1+|k-k'|)(1+|j-j'|)(2^{-(k'-k)\varepsilon'}\wedge 1)\\
&\quad\times(2^{-(j'-j)\varepsilon'}\wedge 1)\iint_{\mathbb{R}^n\times\mathbb{R}^m}\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x-x_1|)^{n+\varepsilon'}}\\
&\quad\times\frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|y-x_2|)^{m+\varepsilon'}}dxdy\ |b_1(x_1)||b_2(x_2)|\\
&\quad\times|D_{k'}\dot{D}_{j'}f(x_1,x_2)|dx_1dx_2\\
&\leq C\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}2^{k\alpha_1}2^{j\alpha_2}(1+|k-k'|)(1+|j-j'|)(2^{-(k'-k)\varepsilon'}\wedge 1)\\
&\quad\times(2^{-(j'-j)\varepsilon'}\wedge 1)\ 2^{-k'\alpha_1}2^{-j'\alpha_2}\ 2^{k'\alpha_1}2^{j'\alpha_2}\\
&\quad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_{k'}\dot{D}_{j'}f(x_1,x_2)|dx_1dx_2\\
&\leq C\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}2^{(k-k')\alpha_1}(1+|k-k'|)(2^{-(k'-k)\varepsilon'}\wedge 1)\sum_{j\in\mathbb{Z}}2^{(j-j')\alpha_2}\\
&\quad\times(1+|j-j'|)(2^{-(j'-j)\varepsilon'}\wedge 1)\ 2^{k'\alpha_1}2^{j'\alpha_2}\\
&\quad\times\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_{k'}\dot{D}_{j'}f(x_1,x_2)|dx_1dx_2.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}}2^{(k-k')\alpha_1}(1+|k-k'|)(2^{-(k'-k)\varepsilon'}\wedge 1)\\
&\qquad\times\sum_{j\in\mathbb{Z}}2^{(j-j')\alpha_2}(1+|j-j'|)(2^{-(j'-j)\varepsilon'}\wedge 1)\\
&=\left(\sum_{k\geq k'}2^{(k-k')\alpha_1}(1+k-k')+\sum_{k<k'}2^{(k-k')\alpha_1}(1+k'-k)2^{-(k'-k)\varepsilon'}\right)\\
&\qquad\times\left(\sum_{j\geq j'}2^{(j-j')\alpha_2}(1+j-j')+\sum_{j<j'}2^{(j-j')\alpha_2}(1+j'-j)2^{-(j'-j)\varepsilon'}\right)\\
&\leq C,
\end{aligned}$$

so we obtain

$$\begin{aligned}
\|Tf\|_{b^{-1}\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}&\leq C\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}2^{k'\alpha_1}2^{j'\alpha_2}\iint_{\mathbb{R}^n\times\mathbb{R}^m}|D_{k'}\dot{D}_{j'}f(x_1,x_2)|dx_1dx_2\\
&\leq C\|f\|_{\dot{B}^{\alpha_1,\alpha_2}_{1,1,b}}.
\end{aligned}$$

□

We now prove Theorem 1.9. The necessary conditions of Theorem 1.9. For all $f_2 \in C_0^\eta(\mathbb{R}^m)$, we may note that $\|1 \otimes f_2\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} = 0$. By the boundedness of T on Lipschitz spaces, we have

$$\|TM_b 1 \otimes f_2\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \le C\|M_b 1 \otimes f_2\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)} = C\|1 \otimes f_2\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} = 0.$$

So we get $TM_b 1 \otimes f_2 = 0$ and

$$\langle M_b T M_b 1 \otimes f_2, g_1 \otimes g_2\rangle = 0$$

for all $g_1 \in C_{b_1,0}^\eta(\mathbb{R}^n)$ and $g_2, f_2 \in C_0^\eta(\mathbb{R}^m)$. By our definition (1.10), we get $T_1(b_1) = 0$. Similarly, we can obtain $T_2(b_2) = 0$.

We now prove the sufficiency. Assuming $\alpha_1, \alpha_2 \in (0, \varepsilon)$, we will show that for any $f \in L^2(\mathbb{R}^{n+m}) \cap \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$,

$$\|Tf\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}, \tag{3.8}$$

and then extend to the case for $\mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$. To do this, by Theorem 1.8,

$$\|Tf\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} \approx \sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n, y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j(bTf)(x,y)|.$$

Since T is bounded on $L^2(\mathbb{R}^{n+m})$, $f \in L^2(\mathbb{R}^{n+m}) \cap \mathrm{Lip}_b(\alpha_1,\alpha_2)(\mathbb{R}^n \times \mathbb{R}^m)$, the Calderón type reproducing formula (2.2) and Lemma 3.7 give us that

$$\begin{aligned}
&2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_j(bTf)(x,y)|\\
&= 2^{k\alpha_1}2^{j\alpha_2}\bigg|\iint_{\mathbb{R}^n\times\mathbb{R}^m}\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}} D_k M_{b_1}\dot{D}_j M_{b_2} T M_{b_1}\widetilde{D}_{k'}^\sharp M_{b_2}\widetilde{\dot{D}}_{j'}^\sharp(x,y,x_1,x_2)\\
&\qquad\qquad \times b_1(x_1)b_2(x_2)D_{k'}\dot{D}_{j'}f(x_1,x_2)dx_1dx_2\bigg|\\
&\le C2^{k\alpha_1}2^{j\alpha_2}\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}(1+|k-k'|)(1+|j-j'|)(2^{-(k-k')\varepsilon'}\wedge 1)\\
&\qquad \times (2^{-(j-j')\varepsilon'}\wedge 1)\iint_{\mathbb{R}^n\times\mathbb{R}^m}\frac{2^{-(k'\wedge k)\varepsilon'}}{(2^{-(k'\wedge k)}+|x-x_1|)^{n+\varepsilon'}}\\
&\qquad \times \frac{2^{-(j'\wedge j)\varepsilon'}}{(2^{-(j'\wedge j)}+|y-x_2|)^{m+\varepsilon'}}dx_1dx_2 \sup_{x_1\in\mathbb{R}^n, x_2\in\mathbb{R}^m}|D_{k'}\dot{D}_{j'}f(x_1,x_2)|\\
&\le \sup_{\substack{k',j'\in\mathbb{Z}\\ x_1\in\mathbb{R}^n, x_2\in\mathbb{R}^m}} 2^{k'\alpha_1}2^{j'\alpha_2}|D_{k'}\dot{D}_{j'}f(x_1,x_2)|\sum_{k'\in\mathbb{Z}}\sum_{j'\in\mathbb{Z}}2^{(k-k')\alpha_1}2^{(j-j')\alpha_2}\\
&\qquad \times (1+|k-k'|)(1+|j-j'|)(2^{-(k-k')\varepsilon'}\wedge 1)(2^{-(j-j')\varepsilon'}\wedge 1)\\
&\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\\
&\qquad \times \left(\sum_{k'>k}(1+k'-k)2^{(k-k')\alpha_1}+\sum_{k'\le k}(1+k'-k)2^{(k'-k)(\varepsilon'-\alpha_1)}\right)
\end{aligned}$$

$$\times\left(\sum_{j'>j}(1+j'-j)2^{(j-j')\alpha_2}+\sum_{j'\le j}(1+j'-j)2^{(j'-j)(\varepsilon'-\alpha_2)}\right).$$

We can chose ε' such that $\max\{\alpha_1,\alpha_2\}<\varepsilon'<\varepsilon$, so we get (3.8).

Next, we extend T to the case for $f\in \mathrm{Lip}_b(\alpha_1,\alpha_2)$ as follows. If $f\in \mathrm{Lip}_b(\alpha_1,\alpha_2)$, by Lemma 3.6, there exists a sequence

$\{f_N\}\subset L^2(\mathbb{R}^{n+m})\cap \mathrm{Lip}_b(\alpha_1,\alpha_2)$ such that $\|f_N\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}$. From (3.8), we have

$$\|T(f_N-f_{N'})\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}\le C\|f_N-f_{N'}\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\le 2C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}.$$

Thus, we can define

$$\langle Tf,g\rangle=\lim_{N\to\infty}\langle Tf_N,g\rangle,\ g\in \dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}. \tag{3.9}$$

To see the existence of the limit in (3.9), applying Lemma 3.9 (i), T^* is bounded from $\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}$ to $b^{-1}\dot{B}^{-\alpha_1,-\alpha_2}_{1,1,b}$. We can also write

$$\langle T(f_N-f_{N'}),g\rangle=\langle f_N-f_{N'},T^*g\rangle.$$

By Lemma 3.6 (ii), $\langle f_N-f_{N'},T^*g\rangle$ tends to zero as $N,N'\to\infty$.

Applying Theorem 1.8 and (3.8), we have

$$\begin{aligned}
\|Tf\|_{\mathrm{Lip}(\alpha_1,\alpha_2)} &\approx \sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n,y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_jb(Tf)(x,y)|\\
&= \sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n,y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|\lim_{N\to\infty} D_k\dot{D}_jb(Tf_N)(x,y)|\\
&\le \liminf_{N\to\infty}\sup_{\substack{k,j\in\mathbb{Z},\\ x\in\mathbb{R}^n,y\in\mathbb{R}^m}} 2^{k\alpha_1}2^{j\alpha_2}|D_k\dot{D}_jb(Tf_N)(x,y)|\\
&\approx \liminf_{N\to\infty}\|Tf_N\|_{\mathrm{Lip}(\alpha_1,\alpha_2)}\\
&\le C\liminf_{N\to\infty}\|f_N\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}\\
&\le C\|f\|_{\mathrm{Lip}_b(\alpha_1,\alpha_2)}.
\end{aligned}$$

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