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# On the relative $K$-group in the ETNC Part III

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Abstract. The previous papers in this series were restricted to regular orders. In particular, we could not handle integral group rings, one of the most interesting cases of the ETNC. We resolve this issue. We obtain versions of our main results valid for arbitrary non-commutative Gorenstein orders. This encompasses the case of group rings. The only change we make is using a smaller subcategory inside all locally compact modules.

Contents



This paper is concerned with the non-commutative equivariant Tamagawa number conjecture (ETNC) in the formulation of Burns and Flach [BF01]. We assume some familiarity with this framework and use the same notation. Let $A$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ an order. Using the Burns–Flach theory, a Tamagawa number is an element

$$T\Omega \in K_0(\mathfrak{A}, \mathbb{R})$$

in the relative $K$-group $K_0(\mathfrak{A}, \mathbb{R})$. In our previous paper [Bra19b] we have proposed the following viewpoint: Originally Tamagawa numbers were defined as volumes in terms of the Haar measure. Then we argued that the universal determinant functor of the category of locally compact abelian (LCA) groups is the Haar measure in a suitable sense. Thus, when wanting

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to define an *equivariant* Tamagawa number, one should work with an equivariant Haar measure. This led us to consider the category of $\mathfrak{A}$-*equivariant* LCA groups, denoted by $\mathsf{LCA}_{\mathfrak{A}}$. The universal determinant functor of this category should be a reasonable approach to an 'equivariant Haar measure', and thus to equivariant Tamagawa numbers.

Unfortunately, the above picture turned out to be true only for *regular* orders. However, in this case it works perfectly: We proved

$$K_0(\mathfrak{A}, \mathbb{R}) \cong K_1(\mathsf{LCA}_{\mathfrak{A}}),$$

showing that our Haar measure based philosophy leads to exactly the same group as in the original Burns–Flach formulation. One of the most attractive cases of the ETNC is for integral group rings $\mathfrak{A} = \mathbb{Z}[G]$, where $G$ is a finite group. These orders are regular only for the trivial group, so [Bra19b] fails to deliver in this interesting case.

In the present paper, we introduce a full subcategory

$$\mathsf{LCA}_{\mathfrak{A}}^{*} \subseteq \mathsf{LCA}_{\mathfrak{A}}$$

which fulfills the above picture for arbitrary Gorenstein orders $\mathfrak{A}$. This encompasses hereditary orders (which we could also handle previously), but more importantly group rings. Besides switching to this smaller category, the formulation of the results remains the same:

**Theorem 1.** *Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and let $\mathfrak{A} \subset A$ be a Gorenstein order. There is a canonical long exact sequence of algebraic $K$-groups*

$$\cdots \to K_n(\mathfrak{A}) \to K_n(A_{\mathbb{R}}) \to K_n(\mathsf{LCA}_{\mathfrak{A}}^{*}) \to K_{n-1}(\mathfrak{A}) \to \cdots$$

*for positive $n$, ending in*

$$\cdots \to K_0(\mathfrak{A}) \to K_0(A_{\mathbb{R}}) \to K_0(\mathsf{LCA}_{\mathfrak{A}}^{*}) \to \mathbb{K}_{-1}(\mathfrak{A}) \to 0.$$

*Here $\mathbb{K}_{-1}$ denotes non-connective $K$-theory. There is a canonical isomorphism*

$$K_1(\mathsf{LCA}_{\mathfrak{A}}^{*}) \cong K_0(\mathfrak{A}, \mathbb{R}),$$

*where $K_0(\mathfrak{A}, \mathbb{R})$ is the relative $K$-group appearing in the Burns–Flach formulation of the non-commutative ETNC in* [BF01].

This will be Theorem 5.3. If $\mathfrak{A}$ is additionally a regular order (e.g., hereditary), this sequence agrees with the one of [Bra19b, Theorem 11.2], and moreover $\mathbb{K}_n(\mathsf{LCA}_{\mathfrak{A}}^{*}) = 0$ for $n \leq -1$ in this case. Although they have the same $K$-theory, the category $\mathsf{LCA}_{\mathfrak{A}}^{*}$ will be strictly smaller than $\mathsf{LCA}_{\mathfrak{A}}$ also in this case. As before, in the case $\mathfrak{A} = \mathbb{Z}$ the universal determinant functor is the ordinary Haar measure. This remains true also for our smaller category $\mathsf{LCA}_{\mathbb{Z}}^{*} \subset \mathsf{LCA}_{\mathbb{Z}}$.

**Theorem 2.** *The Haar functor $Ha : \mathsf{LCA}_{\mathbb{Z}}^{*\times} \to \mathsf{Tors}(\mathbb{R}_{>0}^{\times})$ is the universal determinant functor of the category $\mathsf{LCA}_{\mathbb{Z}}^{*}$. Here*

(1) *for any LCA group $G$, $Ha(G)$ denotes the $\mathbb{R}^{\times}_{>0}$-torsor of all Haar measures on $G$, and*
(2) *Deligne's Picard groupoid of virtual objects for $\mathsf{LCA}^{*}_{\mathbb{Z}}$ turns out to be isomorphic to the Picard groupoid of $\mathbb{R}^{\times}_{>0}$-torsors.*

This is exactly as [Bra19b, Theorem 12.8], which was for the bigger category $\mathsf{LCA}_{\mathbb{Z}}$. In Part II of this series [Bra18], we had introduced double exact sequences $\langle\langle P, \varphi, Q\rangle\rangle$.

**Theorem 3.** *Let $A$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ an order. Then the map*

$$K_0(\mathfrak{A}, \mathbb{R}) \longrightarrow K_1(\mathsf{LCA}^{*}_{\mathfrak{A}}) \tag{0.1}$$

*sending $[P, \varphi, Q]$ to the double exact sequence $\langle\langle P, \varphi, Q\rangle\rangle$ is a well-defined morphism from the Bass–Swan to the Nenashev presentation. If $\mathfrak{A}$ is a Gorenstein order, then this map is an isomorphism.*

See Theorem 5.6. Again, the same statement holds for the bigger category $\mathsf{LCA}_{\mathfrak{A}}$ if $\mathfrak{A}$ is regular, as we had shown in [Bra18].

All this fits into a bigger picture, which we will not recall in this text. Instead, in the manuscript [Bra19a] we explain an alternative construction of the non-commutative Tamagawa numbers based on our viewpoint. It defines the same Tamagawa numbers as Burns–Flach [BF01], i.e. leads to a fully equivalent formulation, but the way the Tamagawa number is defined is quite different.

The category $\mathsf{LCA}^{*}_{\mathfrak{A}}$ as well as the bigger $\mathsf{LCA}_{\mathfrak{A}}$ are closely connected to firstly Clausen's work on a $K$-theoretic enrichment of the Artin map [Cla17], as well as the Clausen–Scholze theory of condensed mathematics [Sch19] as well as the pyknotic mathematics of Barwick–Haine [BH19].

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## 1. Conventions

In this text the word *ring* refers to a unital associative (not necessarily commutative) ring. Ring homomorphisms preserve the unit of the ring. Unless said otherwise, modules are right modules.

Given an exact category $\mathsf{C}$, we write $\mathsf{C}^{ic}$ for the idempotent completion, "$\hookrightarrow$" for admissible monics, "$\twoheadrightarrow$" for admissible epics, and we generally follow the conventions of Bühler [Büh10].

Differing from any convention, we call objects $X \in \mathsf{C}$ in a cocomplete category $\mathsf{C}$ *categorically compact* if $\mathrm{Hom}_{\mathsf{C}}(X, -)$ commutes with filtered colimits.

Usually, such objects are merely called compact, but since this potentially conflicts with the topological meaning of compact, which plays a far bigger rôle in this text, it seems best to be careful. These objects are also called 'finitely presented', but again this could potentially cause confusion, so it is best only to refer to the ring-theoretic concept by these terms.

## 2. PI-presentations

**Definition 2.1.** *Suppose* $\mathsf{C}$ *is an exact category. Let*

(1) $\mathsf{P}$ *be a full subcategory of projective objects in* $\mathsf{C}$ *which is closed under finite direct sums,*
(2) $\mathsf{I}$ *be a full subcategory of injective objects in* $\mathsf{C}$ *which is closed under finite direct sums.*

*We write* $\mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$ *for the full subcategory of objects* $X \in \mathsf{C}$ *such that an exact sequence*

$$P \hookrightarrow X \twoheadrightarrow I$$

*with* $P \in \mathsf{P}$ *and* $I \in \mathsf{I}$ *exists in* $\mathsf{C}$*. We call any such exact sequence a* PI-presentation *for* $X$*.*

If $\mathsf{C}$ denotes a category, a morphism $r : X \to Y$ is called a *retraction* if there exists a morphism $s : Y \to X$ (then called *section*) such that $rs = \mathrm{id}_Y$. An exact category is called *weakly idempotent complete* if every retraction has a kernel. Note that $sr : X \to X$ is an idempotent, i.e. every idempotent complete category is also weakly idempotent complete (check that the kernel of the idempotent also provides a kernel for the retraction itself). We refer to [Büh10, §7] for a thorough review of these concepts.

**Lemma 2.2.** *Suppose we are in the situation of Definition* 2.1*. Assume* $\mathsf{C}$ *is weakly idempotent complete. Suppose*

$$X' \hookrightarrow X \twoheadrightarrow X'' \tag{2.1}$$

*is an exact sequence in* $\mathsf{C}$ *such that* $X', X'' \in \mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$*. Suppose we have chosen any PI-presentations for* $X'$ *and* $X''$ *(where we denote the objects accordingly with a single prime or double prime superscript). Then one can extend Sequence* 2.1 *to a commutative diagram*

$$\begin{tikzcd}
P' \arrow[r, hook] \arrow[d, hook] & P' \oplus P'' \arrow[r, two heads] \arrow[d, hook] & P'' \arrow[d, hook] \\
X' \arrow[r, hook] \arrow[d, two heads] & X \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\
I' \arrow[r, hook] & I' \oplus I'' \arrow[r, two heads] & I''
\end{tikzcd} \tag{2.2}$$

*with exact rows and exact columns. In particular, the middle column is a PI-presentation for* $X$*.*

**Proof.** First, use the PI-presentation of $X'$. We get a commutative diagram

$$\begin{tikzcd}
P' \arrow[d, hook] \arrow[dr, hook] & \\
X' \arrow[r, hook] \arrow[d, two heads] & X \\
I' &
\end{tikzcd}$$

and thus the admissible filtration $P' \hookrightarrow X' \hookrightarrow X$ with $P' \in \mathsf{P}$. Noether's Lemma ([Büh10, Lemma 3.5]) yields the exact sequence $X'/P' \hookrightarrow X/P' \twoheadrightarrow X/X'$, which after unravelling the outer terms, is isomorphic to

$$I' \hookrightarrow X/P' \twoheadrightarrow X''.$$

Since $I' \in \mathsf{I}$ is injective, the sequence splits. We get

$$X/P' \cong I' \oplus X''. \tag{2.3}$$

Next, use the PI-presentation of $X''$. The direct sum of the exact sequences

$$P'' \hookrightarrow X'' \overset{q''}{\twoheadrightarrow} I'' \qquad \text{and} \qquad 0 \hookrightarrow I' \overset{1}{\twoheadrightarrow} I' \tag{2.4}$$

is again exact. As a composition of admissible epics is an admissible epic, the kernel $Y$ in the following commutative diagram exists.

$$\begin{tikzcd}
Y \arrow[dr, hook] & & P'' \arrow[d, hook] \\
P' \arrow[u, dotted] \arrow[r, hook] & X \arrow[r, two heads] \arrow[dr, two heads] & X/P' \arrow[d, two heads, "1\oplus q''"] \\
& & I' \oplus I''
\end{tikzcd} \tag{2.5}$$

The right column comes from the sum of sequences in Equation 2.4 and the isomorphism of Equation 2.3 in the middle term of the right column. By the universal property of kernels, we obtain a unique arrow $P' \to Y$. Since $\mathsf{C}$ is weakly idempotent complete, we may apply the dual of [Büh10, Corollary 7.7] and deduce that this arrow must be an admissible monic. Thus, we obtain the admissible filtration $P' \hookrightarrow Y \hookrightarrow X$ and again by Noether's Lemma the exact sequence $Y/P' \hookrightarrow X/P' \overset{a}{\twoheadrightarrow} X/Y$. Unravelling the right term, this exact sequence is isomorphic to

$$Y/P' \hookrightarrow X/P' \twoheadrightarrow I' \oplus I''.$$

Inspecting Diagram 2.5 note that under the isomorphism of Equation 2.3 the map $a$ is identified with $1 \oplus q''$. Thus, $Y/P'$ is a kernel of this, and thus isomorphic to $P''$. Hence, $P' \hookrightarrow Y \twoheadrightarrow Y/P'$ is isomorphic to $P' \hookrightarrow Y \twoheadrightarrow P''$, which splits since $P'' \in \mathsf{P}$ is projective, and thus $Y \cong P' \oplus P''$. Then the diagonal exact sequence of Diagram 2.5 is a PI-presentation, and moreover the one in our claim. Going through the maps which we have constructed, we obtain all the arrows in Diagram 2.2. $\square$

**Corollary 2.3.** *Suppose we are in the situation of Definition* 2.1 *and* $\mathsf{C}$ *is weakly idempotent complete. Then* $\mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$ *is extension-closed in* $\mathsf{C}$*. In particular, it is a fully exact subcategory of* $\mathsf{C}$*.*

**Proof.** The lemma shows that $X$ also has a PI-presentation, so $X \in \mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$. $\square$

**Lemma 2.4.** *If* $X \in \mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$ *is injective (resp. projective) as an object in* $\mathsf{C}$*, it is also injective (resp. projective) as an object in* $\mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$*.*

**Proof.** Immediate. $\square$

In particular, all objects of $\mathsf{P}$ are still projective in $\mathsf{C}\langle\mathsf{P},\mathsf{I}\rangle$ and correspondingly for the injectives in $\mathsf{I}$.

## 3. Construction of the category $\mathsf{PLCA}_{\mathfrak{A}}$

Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ an order. We shall use the category $\mathsf{LCA}_{\mathfrak{A}}$ of [Bra19b]. We recall that its

(1) objects are locally compact topological right $\mathfrak{A}$-modules, and
(2) morphisms are continuous $\mathfrak{A}$-module homomorphisms.

An admissible monic is a closed injective morphism, an admissible epic is an open surjective morphism. This makes $\mathsf{LCA}_{\mathfrak{A}}$ a quasi-abelian exact category, generalizing an observation due to Hoffmann–Spitzweck [HS07].

**Proposition 3.1.** *The category* $\mathsf{LCA}_{\mathfrak{A}}$ *is a quasi-abelian exact category. There is an exact functor*

$$(-)^{\vee} : \mathsf{LCA}_{\mathfrak{A}}^{op} \longrightarrow \mathsf{LCA}_{\mathfrak{A}^{op}} \qquad\qquad M \longmapsto \operatorname{Hom}(M, \mathbb{T}),$$

*where the continuous right* $\mathfrak{A}$*-module homomorphism group* $\operatorname{Hom}(M,\mathbb{T})$ *is equipped with the compact-open topology (that is: on the level of the underlying LCA group* $(M;+)$ *this is the Pontryagin dual), and the left action*

$$(\alpha \cdot \varphi)(m) := \varphi(m \cdot \alpha) \qquad \text{for all} \qquad \alpha \in \mathfrak{A},\ m \in M. \tag{3.1}$$

*There is a natural equivalence of functors from the identity functor to double dualization,*

$$\eta : \mathrm{id} \longrightarrow (-)^{\vee} \circ \left[(-)^{\vee}\right]^{op}.$$

*In other words: For every object* $M \in \mathsf{LCA}_{\mathfrak{A}}$ *there exists a reflexivity isomorphism* $\eta(M) : M \xrightarrow{\sim} M^{\vee\vee}$*, and the isomorphisms* $\eta(M)$ *are natural in* $M$*.*

See [Bra19b, Proposition 3.5]. If $\mathfrak{A}$ is commutative, it is even an exact category with duality in the sense of [Sch10, Definition 2.1].

Let $R$ be a ring. We write $P(R)$ for the category of all projective right $R$-modules, and $P_f(R)$ for the finitely generated projective right $R$-modules.

These are both exact categories in the standard way. These categories are idempotent complete and split exact.

Write $P_{\oplus}(R)$ for the full subcategory of $P(R)$ whose objects are at most countable direct sums of objects in $P_f(R)$. This is an extension-closed full subcategory and thus itself an exact category. This category may also be realized as

$$P_{\oplus}(R) = \mathsf{Ind}^a_{\aleph_0}(P_f(R)), \tag{3.2}$$

because by [BGW16, Corollary 3.19] it is the full subcategory of countable direct sums of objects in $P_f(R)$ inside $\mathsf{Lex}(P_f(R))$ and by [BGW16, Lemma 2.21] the latter category is $\mathsf{Mod}(R)$.

The following is (in different formulation) due to Akasaki and Linnell.

**Lemma 3.2** (Akasaki–Linnell)**.** *Suppose $G$ is a finite group and $R := \mathbb{Z}[G]$. Then $P_{\oplus}(R)$ is idempotent complete if and only if $G$ is solvable.*

**Proof.** By Equation 3.2 and [BGW16, Proposition 3.25] the idempotent completion of $P_{\oplus}(R)$ is the category $P_{\aleph_0}(R)$ of at most countably generated projective $R$-modules. If $G$ is solvable, Swan [Swa63, Theorem 7] has shown that every projective $R$-module is either finitely generated or free (or both), so each such is a direct sum of finitely generated projectives, hence lies in $P_{\oplus}(R)$. On the other hand, if $G$ is non-solvable, Akasaki exhibits a non-zero countably generated projective $R$-module $P \in P_{\aleph_0}(R)$ with trace ideal $\tau(M) \subsetneqq \mathbb{Z}[G]$, see [Aka82, Theorem] (or Linnell [Lin82]). If $P$ has a non-zero finitely generated projective summand $P' \subset P$, then $\tau(P') = \mathbb{Z}[G]$ by [Aka72, Corollary 1.4], and thus we would have $\tau(P) = \mathbb{Z}[G]$ because all maps from a direct summand extend to maps of all of $P$. However, the latter is impossible by Akasaki's construction. Thus, $P$ has no finitely generated projective summands and thus $P \notin P_{\oplus}(R)$. $\square$

Note that $P_{\oplus}(\mathfrak{A})$ lies inside $\mathsf{LCA}_{\mathfrak{A}}$ when being regarded as a full subcategory of objects with the discrete topology. Define $I_{\Pi}(\mathfrak{A})$ as the Pontryagin dual of $P_{\oplus}(\mathfrak{A}^{op})$. In other words, this is the category of at most countable products $\prod P_i^{\vee}$, where $P_i \in P_f(\mathfrak{A}^{op})$. Under Pontryagin duality these projective left $\mathfrak{A}$-modules (i.e. right $\mathfrak{A}^{op}$-modules) become injective right $\mathfrak{A}$-modules in $\mathsf{LCA}_{\mathfrak{A}}$.

Define

$$\mathsf{PLCA}_{\mathfrak{A}} := \mathsf{LCA}_{\mathfrak{A}} \left\langle P_{\oplus}(\mathfrak{A}), I_{\Pi}(\mathfrak{A}) \right\rangle. \tag{3.3}$$

Since $\mathsf{LCA}_{\mathfrak{A}}$ is quasi-abelian, it is in particular weakly idempotent complete and thus $\mathsf{PLCA}_{\mathfrak{A}}$ is a fully exact subcategory of $\mathsf{LCA}_{\mathfrak{A}}$ by Corollary 2.3.

We get a natural extension of Proposition 3.1.

**Proposition 3.3.** *The category $\mathsf{PLCA}_{\mathfrak{A}}$ is an exact category. The exact Pontryagin duality functor $(-)^{\vee}$ of Proposition 3.1 restricts to an exact* *equivalence of exact categories*

$$(-)^{\vee} : \mathsf{PLCA}_{\mathfrak{A}}^{op} \longrightarrow \mathsf{PLCA}_{\mathfrak{A}^{op}}$$
$$M \longmapsto \operatorname{Hom}(M, \mathbb{T}).$$

*We usually regard the objects of* $\mathsf{PLCA}_{\mathfrak{A}^{op}}$ *as topological left* $\mathfrak{A}$*-modules. If* $\mathfrak{A}$ *is commutative,* $\mathfrak{A} = \mathfrak{A}^{op}$*, and this functor makes* $\mathsf{PLCA}_{\mathfrak{A}}$ *an exact category with duality.*

**Proof.** If $P \hookrightarrow X \twoheadrightarrow I$ is a PI-presentation for $X$, the duality functor sends it to

$$I^{\vee} \hookrightarrow X^{\vee} \twoheadrightarrow P^{\vee},$$

but by construction $I^{\vee} \in P_{\oplus}(\mathfrak{A}^{op})$ and $P^{\vee} \in I_{\Pi}(\mathfrak{A}^{op})$. Hence, this gives us a PI-presentation of $X^{\vee}$. ☐

**Lemma 3.4.** *All objects in* $I_{\Pi}(\mathfrak{A})$ *are compact*[1] *connected.*

**Proof.** We use that $I_{\Pi}(\mathfrak{A})$ is the Pontryagin dual to $P_{\oplus}(\mathfrak{A}^{op})$. Each object $P \in P_{\oplus}(\mathfrak{A}^{op})$ is discrete, so $P^{\vee} \in I_{\Pi}(\mathfrak{A})$ is compact. As $P$ is projective, it is also $\mathbb{Z}$-torsionfree, and thus $P^{\vee}$ is connected by [Mor77, Corollary 1 to Theorem 31]. ☐

The following observation is trivial.

**Lemma 3.5.** *Suppose* $P \in P_{\oplus}(\mathfrak{A})$*. If* $F$ *is a finitely generated submodule of* $P$*, then there exists a direct sum splitting*

$$P \cong P_0 \oplus P_{\infty} \tag{3.4}$$

*with* $P_0 \in P_f(\mathfrak{A})$*,* $P_{\infty} \in P_{\oplus}(\mathfrak{A})$ *and* $F \subseteq P_0$*. In other words: Every finitely generated* $\mathfrak{A}$*-submodule of* $P$ *is contained in a finitely generated projective direct summand of* $P$*.*

**Proof.** Write $P = \bigoplus_{i \in \mathcal{I}} P_i$ with $P_i \in P_f(\mathfrak{A})$. Let $m_1, \ldots, m_n$ be $\mathfrak{A}$-module generators of $F$. Since $F \subseteq P$, we can write $m_j = \sum \alpha_{j,i}$ such that $\alpha_{j,i} \in P_i$ and these are finite sums. Hence, collecting all the indices $i$ which occur in these finite sums where $j = 1, \ldots, n$, we get a finite subset $\mathcal{I}_0$ of indices within $\mathcal{I}$. Define

$$P_0 := \bigoplus_{i \in \mathcal{I}_0} P_i \qquad \text{and} \qquad P_{\infty} := \bigoplus_{i \in \mathcal{I} \setminus \mathcal{I}_0} P_i.$$

Then $P \simeq P_0 \oplus P_{\infty}$ as desired, $P_0 \in P_f(\mathfrak{A})$ because $\mathcal{I}_0$ is finite, and $F \subseteq P_0$. ☐

*Example* 3.6. The property discussed in the previous lemma would in general be false if $P$ were allowed to be an arbitrary (countably generated) projective module. For example, if $G$ is a non-solvable finite group, by Lemma 3.2 one can find a countably generated indecomposable projective. Since it admits no non-trivial direct sum decompositions at all, no splitting as in Equation 3.4 can exist.

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[1] in the sense of topology

**Lemma 3.7.** *Suppose $X \in \mathsf{PLCA}_{\mathfrak{A}}$ has the PI-presentation*

$$P \hookrightarrow X \twoheadrightarrow I. \tag{3.5}$$

*Then for any finitely generated $\mathfrak{A}$-module $F \subseteq P$ there exists*

(1) *a direct sum splitting*

$$P \cong P_0 \oplus P_\infty$$

*with $F \subseteq P_0$, $P_0 \in P_f(\mathfrak{A})$ and $P_\infty \in P_\oplus(\mathfrak{A})$, and*

(2) *a direct sum splitting*

$$X \cong M \oplus P_\infty$$

*with $M \in \mathsf{PLCA}_{\mathfrak{A}}$ such that $P_0 \hookrightarrow M \twoheadrightarrow I$ is a PI-presentation for $M$.*

It might be worth unpacking what we are saying here: Given any object $X$ and any finitely generated submodule in $P$, we can up to a direct summand from $P_\oplus(\mathfrak{A})$ isomorphically replace $X$ by an object whose PI-presentation has only a finitely generated $P$, and we can demand that the given $F$ lies entirely in this $P$.

**Proof.** By [Bra19b, Lemma 6.5] in the bigger category $\mathsf{LCA}_{\mathfrak{A}}$ we get an exact sequence

$$V \oplus C \hookrightarrow X \twoheadrightarrow D \tag{3.6}$$

with $V$ a vector $\mathfrak{A}$-module, $C$ a compact $\mathfrak{A}$-module and $D$ a discrete $\mathfrak{A}$-module. Define

$$J := P \cap (V \oplus C) \tag{3.7}$$

in $\mathsf{LCA}_{\mathfrak{A}}$. Note that both $P$ and $V \oplus C$ are closed in $X$. As $J$ is closed in $P$, $J$ is discrete. Further, since $P$ is a projective $\mathfrak{A}$-module, it is $\mathbb{Z}$-torsionfree, so $J$ is $\mathbb{Z}$-torsionfree as well. As $J$ is closed in $V \oplus C$, its underlying LCA group must be $\mathbb{Z}^b$ for some $b \in \mathbb{Z}_{\geq 0}$ (reason: If $J \hookrightarrow V \oplus C$, then $V^\vee \oplus C^\vee \twoheadrightarrow J^\vee$ under Pontryagin duality. Here $V^\vee \oplus C^\vee$ is a vector module plus a discrete module. All quotients of such must be $\mathbb{R}^a \oplus \mathbb{T}^b \oplus \tilde{D}$ with $\tilde{D}$ discrete as an LCA group by [Mor77, Corollary 2 to Theorem 7]. Dualizing back, the underlying LCA group of $J$ must be $\mathbb{R}^a \oplus \mathbb{Z}^b \oplus \tilde{C}$ with $\tilde{C}$ compact. As we already know that $J$ is discrete and torsionfree, we must have $a = 0$ and $\tilde{C} = 0$). Combining these facts, $J$ is a discrete $\mathfrak{A}$-module with underlying LCA group $\mathbb{Z}^b$. It follows that $J$ is a finitely generated $\mathfrak{A}$-submodule of $P$. Next, define

$$J' := J + F.$$

This is still a finitely generated $\mathfrak{A}$-submodule of $P$. Thus, by Lemma 3.5 we can find a direct sum splitting

$$P \simeq P_0 \oplus P_\infty \tag{3.8}$$

with $J' \subseteq P_0$ and $P_0 \in P_f(\mathfrak{A})$. In the category $\mathsf{LCA}_{\mathfrak{A}}$ we define

$$M := (V \oplus C) + P_0 \qquad \text{inside} \qquad X. \tag{3.9}$$

Since $D$ in Equation 3.6 was discrete, $V \oplus C$ is an open submodule of $X$. Thus, the sum defining $M$ is also an open submodule, thus clopen. It follows that the inclusion $M \hookrightarrow X$ is an open admissible monic in $\mathsf{LCA}_{\mathfrak{A}}$. Both $P_\infty$ and $M$ are closed submodules of $X$. We claim that

$$P_\infty \cap M = 0.$$

(Proof: Suppose $x \in P_\infty \cap M$. As $x$ lies in $M$, we can write $x = x_{vc} + x_0$ with $x_{vc} \in V \oplus C$ and $x_0 \in P_0$ by Equation 3.9. Hence, $x_{vc} = x - x_0$. As $x \in P_\infty \subseteq P$ and $x_0 \in P_0 \subseteq P$, we find $x_{vc} \in P$. Thus, $x_{vc} \in P \cap (V \oplus C)$ and thus $x_{vc} \in J$ by Equation 3.7. As $J \subseteq P_0$ by Equation 3.8, we obtain $x_{vc} \in P_0$. It follows that $x \in P_0$. We also have $x \in P_\infty$ by assumption and therefore $x \in P_0 \cap P_\infty = 0$, giving the claim.) Thus, $M$ and $P_\infty$ are closed submodules of $X$ with trivial intersection. We get an exact sequence

$$M \oplus P_\infty \hookrightarrow X \twoheadrightarrow Q$$

for some quotient $Q$ in $\mathsf{LCA}_{\mathfrak{A}}$. As $P \subseteq M \oplus P_\infty$, it follows that $Q$ is an admissible quotient of $I$ by Equation 3.5. Since $I$ is (compact) connected by Lemma 3.4, so must be $Q$. On the other hand, since $M$ is open (or: since it contains $V \oplus C$), $Q$ is also necessarily discrete. Being both connected and discrete, we must have $Q = 0$. We get

$$X \simeq M \oplus P_\infty \tag{3.10}$$

in $\mathsf{LCA}_{\mathfrak{A}}$. Next, by Noether's Lemma ([Büh10, Lemma 3.5]) the admissible filtration

$$P_\infty \hookrightarrow P \hookrightarrow X$$

gives rise to the exact sequence

$$P/P_\infty \hookrightarrow X/P_\infty \twoheadrightarrow X/P.$$

We have $P/P_\infty \cong P_0$ from Equation 3.8, $X/P \cong I$ from Equation 3.5, and $X/P_\infty \cong M$ by Equation 3.10. Thus, $P_0 \hookrightarrow M \twoheadrightarrow I$ is exact. Since $P_0 \in P_f(\mathfrak{A})$ and $I \in I_\Pi(\mathfrak{A})$, we deduce $M \in \mathsf{PLCA}_{\mathfrak{A}}$ from Equation 3.3. Finally, since $P_\infty \in P_\oplus(\mathfrak{A})$, Equation 3.10 is not only a direct sum splitting in $\mathsf{LCA}_{\mathfrak{A}}$, but even in the fully exact subcategory $\mathsf{PLCA}_{\mathfrak{A}}$. Finally, $F \subseteq P_0$ holds by construction. $\square$

The previous result implies that the objects of $\mathsf{PLCA}_{\mathfrak{A}}$ can, up to direct summands from $P_\oplus(\mathfrak{A})$ and $I_\Pi(\mathfrak{A})$, be reduced to such where the PI-presentation is made from finitely generated discrete projectives and their Pontryagin duals.

**Proposition 3.8.** *Every object in* $\mathsf{PLCA}_{\mathfrak{A}}$ *is isomorphic to an object of the shape*

$$X \simeq P_\infty \oplus I_\infty \oplus B$$

*with* $P_\infty \in P_\oplus(\mathfrak{A})$, $I_\infty^\vee \in P_\oplus(\mathfrak{A}^{op})$ *and* $B \in \mathsf{PLCA}_{\mathfrak{A}}$ *has a PI-presentation*

$$P_0 \hookrightarrow B \twoheadrightarrow I_0$$

*with* $P_0 \in P_f(\mathfrak{A})$, $I_0^\vee \in P_f(\mathfrak{A}^{op})$.

**Proof.** Let $X \in \mathsf{PLCA}_{\mathfrak{A}}$ be any object. Pick a PI-presentation $P \hookrightarrow X \twoheadrightarrow I$. We apply Lemma 3.7 with $F = 0$. We get a direct sum splitting $X \simeq M \oplus P_\infty$ in $\mathsf{PLCA}_{\mathfrak{A}}$, where $M$ has a PI-presentation of the shape

$$P_0 \hookrightarrow M \twoheadrightarrow I$$

such that $P_0 \in P_f(\mathfrak{A})$. Now apply Pontryagin duality, giving the exact sequence

$$I^\vee \hookrightarrow M^\vee \twoheadrightarrow P_0^\vee$$

in $\mathsf{PLCA}_{\mathfrak{A}^{op}}$. This is a PI-presentation in $\mathsf{PLCA}_{\mathfrak{A}^{op}}$. Now apply Lemma 3.7 (again with $F = 0$). Then dualize back. □

We recall the following standard concept from the theory of topological groups.

**Definition 3.9.** *A subset $U$ of a topological group $G$ is called* symmetric *if it is closed under taking inverses. A topological group $G$ is called* compactly generated *if there exists a compact symmetric neighbourhood $U \subseteq G$ of the neutral element such that $G = \bigcup_{n \geq 1} U^n$.*

*Remark* 3.10. Unfortunately, the word "compactly generated" is also used with a different meaning elsewhere. Either in a category-theoretic sense related to categorically compact objects, or in a further topological meaning, probably most familiar in the setting of compactly generated Hausdorff spaces in homotopy theory; e.g., [Sch19] uses both of these other meanings. This is most unfortunate, but all uses of these words are well-established in their respective community of mathematics.

Let $\mathsf{PLCA}_{\mathfrak{A},cg}$ be the full subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$ of compactly generated $\mathfrak{A}$-modules,

$$\mathsf{PLCA}_{\mathfrak{A},cg} := \mathsf{PLCA}_{\mathfrak{A}} \cap \mathsf{LCA}_{\mathfrak{A},cg}. \tag{3.11}$$

Since compactly generated topological modules are closed under extension in $\mathsf{LCA}_{\mathfrak{A}}$ ([Bra19b, Corollary 7.2]), this is an extension-closed subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$.

**Lemma 3.11.** *We have $\mathsf{PLCA}_{\mathfrak{A},cg} = \mathsf{LCA}_{\mathfrak{A}}\langle P_f(\mathfrak{A}), I_\Pi(\mathfrak{A})\rangle$, i.e. the same category can also be described as the full subcategory of objects in $\mathsf{PLCA}_{\mathfrak{A}}$ which admit a PI-presentation*

$$P \hookrightarrow X \twoheadrightarrow I$$

*with $P$ finitely generated projective.*

**Proof.** (Step 1) Suppose $X$ lies in $\mathsf{LCA}_{\mathfrak{A}}\langle P_f(\mathfrak{A}), I_\Pi(\mathfrak{A})\rangle$. Then

$$P \hookrightarrow X \twoheadrightarrow I$$

is exact with $P$ finitely generated projective and $I \in I_\Pi(\mathfrak{A})$. By Lemma 3.4 the module $I$ is compact, hence compactly generated, and $P$ has $\mathbb{Z}^n$ for some finite $n \geq 0$ as its underlying LCA group, so it is compactly generated, too. Thus, $X$ is an extension of compactly generated LCA groups, and thus

$X \in \mathsf{PLCA}_{\mathfrak{A},cg}$.
(Step 2) Conversely, suppose $X \in \mathsf{PLCA}_{\mathfrak{A},cg}$. Proposition 3.8 gives a direct sum splitting $X = P_\infty \oplus M \oplus I_\infty$. By Step 1 we know that $M$ is compactly generated and $I_\infty$ is compact, so $X$ is compactly generated if and only if $P_\infty$ is. However, the underlying LCA group of $P_\infty$ is $\bigoplus \mathbb{Z}$, over some index set, and this is compactly generated only if $P_\infty$ is finitely generated. □

**Proposition 3.12.** *The inclusion $P_f(\mathfrak{A}) \hookrightarrow P_\oplus(\mathfrak{A})$ is left s-filtering.*[2]

**Proof.** *(Left filtering)* Suppose we are given an arrow $g : Y \to X$ with $Y \in P_f(\mathfrak{A})$ and $X \in P_\oplus(\mathfrak{A})$. The set-theoretic image of $Y$ in $X$ is again a finitely generated module, so by Lemma 3.5 we find a direct sum decomposition

$$X \cong P_0 \oplus P_\infty$$

with $P_0 \in P_f(\mathfrak{A})$, $P_\infty \in P_\oplus(\mathfrak{A})$ and $\mathrm{im}_{\mathsf{Set}}(g) \subseteq P_0$. It follows that the arrow $g$ factors as $Y \to P_0 \hookrightarrow X$, showing the left filtering property.
*(Left special)* Suppose $e : X \twoheadrightarrow X''$ is an admissible epic with $X \in P_\oplus(\mathfrak{A})$ and $X'' \in P_f(\mathfrak{A})$. As $X''$ is projective, the epic splits. We obtain a diagram

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d] & 0 \oplus X'' \arrow[r, two heads] \arrow[d] & X'' \arrow[d, equal] \\ X' \arrow[r, hook] & X \arrow[r, two heads] & X'' \end{tikzcd}$$

showing the left special property. □

**Proposition 3.13.** *The inclusion $\mathsf{PLCA}_{\mathfrak{A},cg} \hookrightarrow \mathsf{PLCA}_{\mathfrak{A}}$ is left s-filtering.*

**Proof.** *(Left filtering)* Suppose we are given an arrow $Y \to X$ with $Y \in \mathsf{PLCA}_{\mathfrak{A},cg}$ and $X \in \mathsf{PLCA}_{\mathfrak{A}}$. We apply Proposition 3.8 to $X$ and get the diagram

$$\begin{tikzcd} & M \oplus I_\infty \arrow[d, hook] \\ Y \arrow[r] \arrow[dr, "h"'] & P_\infty \oplus M \oplus I_\infty \arrow[d, two heads] \\ & P_\infty . \end{tikzcd}$$

We first work entirely on the level of $\mathsf{LCA}_{\mathbb{Z}}$: Since $Y$ is compactly generated, we get some isomorphism $Y \simeq C \oplus \mathbb{Z}^n \oplus \mathbb{R}^m$ for some $n, m$ and $C$ compact, [Mos67, Theorem 2.5]. As $C$ is compact, its set-theoretic image under $h$ is compact, but since $P_\infty$ is discrete and torsionfree, $h(C)$ must be zero. Moreover, the set-theoretic image of $\mathbb{R}^m$ under $h$ is connected and thus also zero. It follows that the set-theoretic image of $h$ agrees with the image $h(\mathbb{Z}^n)$, and thus must be a finitely generated $\mathbb{Z}$-submodule of $P_\infty$. Now

[2] This concept originates from the work of Schlichting [Sch04]. We use the formulation of [BGW16, §2.2.2].

return to $\mathsf{LCA}_{\mathfrak{A}}$. By the previous consideration, the image under $h$ must be a finitely generated $\mathfrak{A}$-submodule of $P_\infty$. Thus, by Lemma 3.5 we find some $P_{\infty,0} \in P_f(\mathfrak{A})$ and $P_{\infty,\infty} \in P_\oplus(\mathfrak{A})$ such that $P_\infty \simeq P_{\infty,0} \oplus P_{\infty,\infty}$ and $\mathrm{im}(h) \subseteq P_{\infty,0}$. Thus, we obtain a new diagram

$$\begin{tikzcd} & P_{\infty,0} \oplus M \oplus I_\infty \arrow[d, hook] \\ Y \arrow[ur, dotted] \arrow[r] \arrow[dr, "0"'] & P_\infty \oplus M \oplus I_\infty \arrow[d, two heads] \\ & P_{\infty,\infty}. \end{tikzcd}$$

and by the universal property of kernels, we learn that $Y \to X$ factors over $Y' := P_{\infty,0} \oplus M \oplus I_\infty$, which lies in $\mathsf{PLCA}_{\mathfrak{A},cg}$ since all summands do. This gives the required factorization to see that $\mathsf{PLCA}_{\mathfrak{A},cg} \hookrightarrow \mathsf{PLCA}_{\mathfrak{A}}$ is left filtering.

*(Left special)* (Step 1) Suppose $X \twoheadrightarrow X''$ is an admissible epic with $X \in \mathsf{PLCA}_{\mathfrak{A}}$ and $X'' \in \mathsf{PLCA}_{\mathfrak{A},cg}$. Being an epic, there exists an exact sequence

$$X' \hookrightarrow X \twoheadrightarrow X'' \tag{3.12}$$

in $\mathsf{PLCA}_{\mathfrak{A}}$. Pick PI-presentations for $X'$ and $X''$, where we denote the objects accordingly with a single prime or double prime superscript. For $P''$ we may assume $P'' \in P_f(\mathfrak{A})$ since $X'' \in \mathsf{PLCA}_{\mathfrak{A},cg}$. By Lemma 2.2 we may extend Equation 3.12 to the diagram

$$\begin{tikzcd} P' \arrow[r, hook] \arrow[d, hook] & P' \oplus P'' \arrow[r, two heads] \arrow[d, hook] & P'' \arrow[d, hook] \\ X' \arrow[r, hook] \arrow[d, two heads] & X \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\ I' \arrow[r, hook] & I' \oplus I'' \arrow[r, two heads] & I''. \end{tikzcd}$$

Next, apply Lemma 3.7 to $X$ with $F := P''$. Write $X_{new} \in \mathsf{PLCA}_{\mathfrak{A},cg}$ for its output $M$. We can now change the above diagram to

$$\begin{tikzcd} P' \arrow[r, hook] \arrow[d, hook] & P_\infty \oplus P_0 \arrow[r, two heads, "q"] \arrow[d, hook, "1 \oplus i"] & P'' \arrow[d, hook] \\ X' \arrow[r, hook] \arrow[d, two heads] & P_\infty \oplus X_{new} \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\ I' \arrow[r, hook] & 0 \oplus (I' \oplus I'') \arrow[r, two heads] & I''. \end{tikzcd}$$

As $P'' \subseteq P_0$, we have $q(P_\infty) = 0$ in $P''$. Since $q$ is an admissible epic to the projective object $P''$, the map $q$ splits, so we may decompose $P_0 \simeq \tilde{P} \oplus P''$

for some $\tilde{P} \in P_f(\mathfrak{A})$ and our diagram becomes

$$\begin{tikzcd}
P' \arrow[r, hook] \arrow[d, hook] & P_\infty \oplus (\tilde{P} \oplus P'') \arrow[r, two heads, "q"] \arrow[d, hook, "1\oplus i"] & P'' \arrow[d, hook] \\
X' \arrow[r, hook] \arrow[d, two heads] & P_\infty \oplus X_{new} \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\
I' \arrow[r, hook] & 0 \oplus (I' \oplus I'') \arrow[r, two heads] & I''.
\end{tikzcd} \tag{3.13}$$

(Step 2) Following the arrows of the diagram, we see that both $P'$ as well as $X_{new}$ are closed submodules of $X$ $(= P_\infty \oplus X_{new})$. Define

$$J := P' \cap X_{new}. \tag{3.14}$$

We claim that this is a finitely generated discrete $\mathfrak{A}$-submodule of $P'$. The argument is the same as in the proof of Lemma 3.7 (namely: write $C \oplus V \hookrightarrow X_{new} \twoheadrightarrow D$ with $C$ compact, $V$ a vector module, $D$ discrete. Then $C \cap P' = 0$ since $C$ is compact, $P'$ discrete, but $P'$ is also torsionfree. So it suffices to consider $V \cap P'$, and since this is a closed subgroup, $J$ can only be a lattice in $V$). Next, observe that the top row in Diagram 3.13 is actually split, i.e.

$$P' \cong P_\infty \oplus \tilde{P},$$

i.e. we can interpret $\tilde{P}$ as a submodule of $P'$. Now apply Lemma 3.7 to $X'$ with $F := J + \tilde{P}$. Write $X'_{new} \in \mathsf{PLCA}_{\mathfrak{A},cg}$ for its output $M$. Hence, we can rewrite the left downward column

$$P' \hookrightarrow X' \twoheadrightarrow I'$$

as

$$P'_\infty \oplus P'_0 \overset{1\oplus i'}{\hookrightarrow} P'_\infty \oplus X'_{new} \twoheadrightarrow 0 \oplus I',$$

where $J \subseteq P'_0$ and $P'_0 \in P_f(\mathfrak{A})$. By inspection of the proof of the lemma, we pick $P'_\infty \oplus P'_0$ as direct summands and we can without loss of generality assume $\tilde{P}$ to be a sub-summand appearing in $P'_0$, say $P'_0 \cong P'_{00} \oplus \tilde{P}$. We can thus rewrite Diagram 3.13 as

$$\begin{tikzcd}
P'_\infty \oplus P'_{00} \oplus \tilde{P} \arrow[r, hook, "b"] \arrow[d, hook, "1\oplus i'"] & P_\infty \oplus (\tilde{P} \oplus P'') \arrow[r, two heads, "q"] \arrow[d, hook, "1\oplus i"] & P'' \arrow[d, hook] \\
P'_\infty \oplus X'_{new} \arrow[r, hook] \arrow[d, two heads] & P_\infty \oplus X_{new} \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\
0 \oplus I' \arrow[r, hook] & 0 \oplus (I' \oplus I'') \arrow[r, two heads] & I''
\end{tikzcd} \tag{3.15}$$

such that $b$ is the inclusion of a direct summand and the identity on $\tilde{P}$. It follows that $b$ makes $P'_\infty$ a direct summand of $P_\infty$ (so that $P_\infty \cong P'_\infty \oplus P'_{00}$).

It follows that we can compatibly remove the direct summands $P'_\infty$ resp. $P_\infty$ in Diagram 3.15. We get

$$\begin{tikzcd} P'_{00} \oplus \tilde{P} \arrow[r, hook] \arrow[d, hook, "i'"] & P'_{00} \oplus \tilde{P} \oplus P'' \arrow[r, two heads, "q"] \arrow[d, hook, "1\oplus i"] & P'' \arrow[d, hook] \\ X'_{new} \arrow[r, hook] \arrow[d, two heads] & P'_{00} \oplus X_{new} \arrow[r, two heads] \arrow[d, two heads] & X'' \arrow[d, two heads] \\ I' \arrow[r, hook] & 0 \oplus (I' \oplus I'') \arrow[r, two heads] & I''. \end{tikzcd} \tag{3.16}$$

Now compare the middle row of the previous diagram with the middle row in the previous diagrams: We have merely replaced $X'$ (resp. $X$) by a direct summand of itself. Thus, we get a commutative diagram

$$\begin{tikzcd} X'_{new} \arrow[r, hook] \arrow[d] & P'_{00} \oplus X_{new} \arrow[r, two heads] \arrow[d] & X'' \arrow[d, equal] \\ X' \arrow[r, hook] & X \arrow[r, two heads] & X'', \end{tikzcd}$$

where the top row comes from the middle row in Diagram 3.16 and the downward arrows are the inclusions of the respective direct summands. All objects in the top row lie in $\mathsf{PLCA}_{\mathfrak{A},cg}$. This shows the left special property. $\square$

**Lemma 3.14.** *There is an exact equivalence of exact categories*

$$P_\oplus(\mathfrak{A})/P_f(\mathfrak{A}) \xrightarrow{\sim} \mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg},$$

*sending a projective module to itself, equipped with the discrete topology.*

**Proof.** We clearly have an exact functor $P_\oplus(\mathfrak{A}) \to \mathsf{PLCA}_{\mathfrak{A}}$, basically using that $P_\oplus$ is a full subcategory of the latter. Since every finitely generated projective $\mathfrak{A}$-module has underlying abelian group $\mathbb{Z}^n$ for some $n$, it is compactly generated, so we get the exact functor

$$P_\oplus(\mathfrak{A})/P_f(\mathfrak{A}) \longrightarrow \mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg}.$$

This functor is essentially surjective: Given any $X \in \mathsf{PLCA}_{\mathfrak{A}}$, let $P \hookrightarrow X \twoheadrightarrow I$ be a PI-presentation. Since $I \in \mathsf{PLCA}_{\mathfrak{A},cg}$ it follows that $P \hookrightarrow X$ is an isomorphism in the quotient exact category ([BGW16, Proposition 2.19, (2)]), but $P \in P_\oplus(\mathfrak{A})$. We next show that the functor is fully faithful: Morphisms $Y_1 \to Y_2$ in $\mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg}$ are roofs

$$Y_1 \overset{e}{\twoheadleftarrow} Y'_1 \to Y_2, \tag{3.17}$$

where $e$ is an admissible epic with compactly generated kernel $K$. For $Y_1, Y_2$ in the strict image of the functor, these objects carry the discrete topology.

Using the structure theorem of $\mathsf{LCA}_{\mathfrak{A}}$ for $Y_1'$, [Bra19b, Lemma 6.5], we get a decomposition

$$C \oplus V \hookrightarrow Y_1' \twoheadrightarrow D$$

with $C$ a compact $\mathfrak{A}$-module, $V$ a vector $\mathfrak{A}$-module and $D$ a discrete $\mathfrak{A}$-module. Since the image of a compactum in a discrete group is compact, it must be finite, hence torsion, but $Y_1, Y_2$ are projective $\mathfrak{A}$-modules, so the image of $C$ in both $Y_1, Y_2$ must be zero. Similarly, $V$ is connected and hence its image in $Y_1, Y_2$ must be zero. Thus, without loss of generality, the roof in Equation 3.17 can be assumed to have $Y_1'$ discrete, as any roof is equivalent to such a roof. However, if $Y_1$ is discrete, the compactly generated kernel $K$ must be finitely generated. Thus, as $Y_1$ is projective, the epic $e$ in Equation 3.17 is split and such that $Y_1' \cong Y_1 \oplus K$ with $K$ (then by necessity) a finitely generated projective $\mathfrak{A}$-module. Thus, the roofs representing morphisms in $\mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg}$ are precisely the same roofs as for morphisms in $P_{\oplus}(\mathfrak{A})/P_f(\mathfrak{A})$, and up to the same equivalence relation, proving full faithfulness. Combining all these facts, the functor in our claim is an exact equivalence. $\square$

The next proposition relies on the concept of localizing invariants in the sense of [BGT13].

**Proposition 3.15.** *Let $A$ be any finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subseteq A$ an order. Let $\mathsf{A}$ be a stable $\infty$-category. Suppose $K : \mathrm{Cat}_{\infty}^{\mathrm{ex}} \to \mathsf{A}$ is a localizing invariant with values in $\mathsf{A}$.*

(1) *There is a fiber sequence*

$$K(\mathfrak{A}) \xrightarrow{g} K(\mathsf{PLCA}_{\mathfrak{A},cg}) \xrightarrow{h} K(\mathsf{PLCA}_{\mathfrak{A}}) \tag{3.18}$$

*in $\mathsf{A}$. Here the map $g$ is induced from the exact functor sending a finitely generated projective right $\mathfrak{A}$-module to itself, equipped with the discrete topology. The map $h$ is induced from the inclusion $\mathsf{PLCA}_{\mathfrak{A},cg} \hookrightarrow \mathsf{PLCA}_{\mathfrak{A}}$.*

(2) *There is a morphism of fiber sequences*[3] *from Sequence* 3.18 *to*

$$K(\mathsf{Mod}_{\mathfrak{A},fg}) \xrightarrow{g} K(\mathsf{LCA}_{\mathfrak{A},cg}) \xrightarrow{h} K(\mathsf{LCA}_{\mathfrak{A}}),$$

*based on the fully exact inclusions*

$$P_f(\mathfrak{A}) \subseteq \mathsf{Mod}_{\mathfrak{A},fg} \qquad \textit{and} \qquad \mathsf{PLCA}_{\mathfrak{A}} \subseteq \mathsf{LCA}_{\mathfrak{A}}$$

*and the compactly generated modules respectively.*

**Proof.** The proof is a mild variation of [Bra19b, Proposition 11.1], but using the fully exact subcategory $\mathsf{PLCA}_{\mathfrak{A}}$ instead of $\mathsf{LCA}_{\mathfrak{A}}$. However, especially

[3]that is: when we write the fiber sequences as their underlying bi-Cartesian square along with a null homotopy for the fourth vertex, then we have a morphism of bi-Cartesian squares, in particular the null homotopies are compatible

since the proofs are compatible otherwise, the second claim is automatically true. For the first claim, we set up the diagram

$$\begin{tikzcd} K(P_f(\mathfrak{A})) \arrow[r] \arrow[d, "g"'] & K(P_{\oplus}(\mathfrak{A})) \arrow[r] \arrow[d] & K(P_{\oplus}(\mathfrak{A})/P_f(\mathfrak{A})) \arrow[d, "\Phi"] \\ K(\mathsf{PLCA}_{\mathfrak{A},cg}) \arrow[r] & K(\mathsf{PLCA}_{\mathfrak{A}}) \arrow[r] & K(\mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg}) \end{tikzcd} \tag{3.19}$$

as follows: By Proposition 3.12 and 3.13 we get fiber sequences in $K$, forming the rows. The equivalence $\Phi$ stems from the equivalence of the underlying exact categories, coming from Lemma 3.14. The downward arrows come from the exact functors sending the respective $\mathfrak{A}$-modules to themselves, equipped with the discrete topology. As $P_{\oplus}(\mathfrak{A})$ is closed under countable direct sums, $K(P_{\oplus}(\mathfrak{A})) = 0$ by the Eilenberg swindle. $\square$

## 4. Gorenstein orders

For any order $\mathfrak{A} \subset A$ define

$$\mathfrak{A}^* := \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{A}, \mathbb{Z}). \tag{4.1}$$

The left $\mathfrak{A}$-module structure on this is given by

$$(\alpha \cdot \varphi)(q) := \varphi(q\alpha) \tag{4.2}$$

(and correspondingly for the right module structure, for which we however have no need).

*Example* 4.1. A general order is far from being reflexive, i.e. $\mathfrak{A}^{**}$ is usually strictly bigger than $\mathfrak{A}$ under the natural inclusion $\mathfrak{A} \to \mathfrak{A}^{**}$ (view both as submodules of $A \overset{\sim}{\to} A^{**}$). If $\mathfrak{A}$ is a maximal order, the inclusion is the identity $\mathfrak{A} \overset{=}{\to} \mathfrak{A}^{**}$, and in our situation over the ring $\mathbb{Z}$ this is an equivalent characterization of maximality by Auslander–Goldman [Rei03, (11.4) Theorem].

**Definition 4.2.** *An order $\mathfrak{A} \subset A$ is called a* Gorenstein order *if one (then all) of the following properties hold:*

(1) *$A/\mathfrak{A}$ is an injective left $\mathfrak{A}$-module,*
(2) left-injdim$_{\mathfrak{A}}(\mathfrak{A}) = 1$,
(3) *$\mathfrak{A}^*$ is a categorically compact projective generator*[4] *for the category of left $\mathfrak{A}$-modules,*
(4) *or any of (1), (2), (3) as a right module.*

The concept was introduced in [DKR67]. Most of the equivalence of these conditions is proven in [DKR67, Proposition 6.1], [Rog70, Chapter IX, §4, §5], while the characterization (1) is due to Roggenkamp [Rog73, Lemma 5].

[4] sometimes this is also called a *progenerator*. In the situation at hand being categorically compact is equivalent to being a finitely presented $\mathfrak{A}$-module.

Non-commutative Gorenstein rings are rings with finite left and right injective dimension, so Gorenstein orders are in particular Gorenstein rings.

We collect a few well-known facts, only in order to exhibit the usefulness of the concept.

**Lemma 4.3.** *For any finite group $G$, $\mathbb{Z}[G] \subset \mathbb{Q}[G]$ is a Gorenstein order.*

**Proof.** ([Rog73, Corollary 6]) For any $g \in G \setminus \{e\}$ the action of $g$ is a fixed-point free permutation of the $\mathbb{Z}$-module generators $G$, so $\operatorname{tr}(g) = 0$, while for $g = e$ we have $\operatorname{tr}(e) = |G|$. It follows that $\mathfrak{A}^* = \frac{1}{|G|}\mathfrak{A}$ inside $\mathbb{Q}[G]$. $\square$

*Remark* 4.4. If we want to work with group rings $\mathbb{Z}[G] \subset \mathbb{Q}[G]$ we are basically forced to work at least in the generality of Gorenstein orders. The slightly more specialized class of Bass orders is in general not sufficient, [Kle90]. A group ring $\mathbb{Z}[G]$ has finite global dimension if and only if $G = 1$, so the even more specialized classes of regular or hereditary (let alone maximal) orders are hopeless.

**Lemma 4.5.** *Any hereditary order is Gorenstein.*

**Proof.** Consider $\mathfrak{A} \hookrightarrow A \twoheadrightarrow A/\mathfrak{A}$. As $A$ is semisimple, $A$ is an injective $\mathfrak{A}$-module, but since $\mathfrak{A}$ is hereditary, quotients of injectives are injective, so $A/\mathfrak{A}$ is injective. An order is left hereditary if and only if it is right hereditary, so there is no question about left or right here. $\square$

**Lemma 4.6** ([JT15, Prop. 3.6])**.** *If $A$ is a number field, then any order of the shape $\mathbb{Z}[\alpha]$ with $\alpha \in A$ is Gorenstein.*

The paper [JT15] also provides some examples of non-Gorenstein orders.

Recall that $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} A$ denotes the base change to the reals.

**Proposition 4.7.** *Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra. If $\mathfrak{A} \subset A$ is a Gorenstein order, then*

$$\mathfrak{A} \hookrightarrow A_{\mathbb{R}} \twoheadrightarrow A_{\mathbb{R}}/\mathfrak{A} \tag{4.3}$$

*is a PI-presentation for $A_{\mathbb{R}}$. In particular, $A_{\mathbb{R}} \in \mathsf{PLCA}_{\mathfrak{A}}$.*

**Proof.** It is clear that $\mathfrak{A}$ is a projective right $\mathfrak{A}$-module, so we only need to show that $(A_{\mathbb{R}}/\mathfrak{A})^{\vee}$ is a projective left $\mathfrak{A}$-module.
(Step 1) First of all, we recall that there is a non-degenerate symmetric trace pairing

$$\operatorname{tr} : A \times A \longrightarrow \mathbb{Q}$$

on any finite-dimensional separable $\mathbb{Q}$-algebra, [Rei03, (9.26) Theorem]. Now define

$$\widetilde{\mathfrak{A}} := \{p \in A_{\mathbb{R}} \mid \operatorname{tr}(pq) \in \mathbb{Z} \text{ for all } q \in \mathfrak{A}\}. \tag{4.4}$$

This is a subset of $A_{\mathbb{R}}$ (it corresponds to the *inverse different*, [Rei03, p. 150]). We give it the natural left $\mathfrak{A}$-module structure induced from $A_{\mathbb{R}}$. We claim that there is an isomorphism of left $\mathfrak{A}$-modules

$$\begin{aligned} h : \widetilde{\mathfrak{A}} &\longrightarrow (A_{\mathbb{R}}/\mathfrak{A})^{\vee} \\ p &\longmapsto \left(q \mapsto e^{2\pi i \operatorname{tr}(pq)}\right), \end{aligned}$$

where the term on the right refers to the corresponding character on $A_{\mathbb{R}}/\mathfrak{A}$. For the left scalar action we compute

$$h(\alpha p) = \left(q \mapsto e^{2\pi i \operatorname{tr}(\alpha pq)}\right) = \left(q \mapsto e^{2\pi i \operatorname{tr}(pq\alpha)}\right)$$

by using that $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for all $x, y$ (the symmetry of the trace pairing). However, the left scalar action on characters amounts to pre-composing with the right scalar action in the argument, see Equation 3.1, so the character on the right agrees with $\alpha \cdot h(p)$ as required. Next, $h$ is an isomorphism because really $\widetilde{\mathfrak{A}}$ is just the orthogonal complement under the Pontryagin duality pairing,

$$\widetilde{\mathfrak{A}} = \{p \in A_{\mathbb{R}} \mid e^{2\pi i \operatorname{tr}(pq)} = 1 \text{ for all } q \in \mathfrak{A}\} = \mathfrak{A}^{\perp},$$

so that $h$ being an isomorphism of groups is just the standard fact $\mathfrak{A}^{\perp} \cong (A_{\mathbb{R}}/\mathfrak{A})^{\vee}$ [Fol16, (4.39) Theorem].
(Step 2) Next, we claim that there is an isomorphism of left $\mathfrak{A}$-modules

$$\begin{aligned} g : \widetilde{\mathfrak{A}} &\longrightarrow \mathfrak{A}^{*} \\ p &\longmapsto (q \mapsto \operatorname{tr}(pq)) \end{aligned}$$

(with $\mathfrak{A}^{*}$ as in Equation 4.1). Firstly, for the left scalar action we find

$$g(\alpha p) = (q \mapsto \operatorname{tr}(\alpha pq)) = (q \mapsto \operatorname{tr}(pq\alpha))$$

using the same argument as before and this is in line with the natural left action as we had recalled in Equation 4.2. The map $g$ is injective. If not, we find a $p \neq 0$ such that $q \mapsto \operatorname{tr}(pq)$ is the zero pairing, contradicting the non-degeneracy of the trace pairing. Surjective: Given any functional $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{A}, \mathbb{Z})$, by the non-degeneracy of the trace pairing, we find some $p \in A_{\mathbb{Q}}$ such that $\varphi(q) = \operatorname{tr}(pq)$. Since we know that for all $q \in \mathfrak{A}$ we have $\varphi(q) \in \mathbb{Z}$, we literally get that $p$ meets the condition to lie in $\widetilde{\mathfrak{A}}$.
(Step 3) Combining $h$ and $g$, we obtain an isomorphism of left $\mathfrak{A}$-modules,

$$(A_{\mathbb{R}}/\mathfrak{A})^{\vee} \cong \mathfrak{A}^{*},$$

but by Definition 4.2 one of the characterizations of Gorenstein orders implies that $\mathfrak{A}^{*}$ is a projective left module. This is what we had to show. $\square$

**Definition 4.8.** *Let $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$ be the full subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$ of objects which are also vector $\mathfrak{A}$-modules. In other words, this is the full subcategory whose objects have the underlying LCA group $\mathbb{R}^n$ for some $n$.*

**Lemma 4.9.** *If $\mathfrak{A} \subset A$ is a Gorenstein order, there is an exact equivalence of exact categories*

$$P_f(A_{\mathbb{R}}) \xrightarrow{\sim} \mathsf{PLCA}^{ic}_{\mathfrak{A},\mathbb{R}},$$

*sending a right $A_{\mathbb{R}}$-module to itself, equipped with the real vector space topology. Moreover, the fully exact subcategory inclusion* $\mathsf{PLCA}_{\mathfrak{A}} \hookrightarrow \mathsf{LCA}_{\mathfrak{A}}$ *induces the equality*

$$\mathsf{PLCA}^{ic}_{\mathfrak{A},\mathbb{R}} \xrightarrow{\sim} \mathsf{LCA}_{\mathfrak{A},\mathbb{R}}$$

*with the category of all vector* $\mathfrak{A}$*-modules in* $\mathsf{LCA}_{\mathfrak{A}}$*.*

**Proof.** Suppose $F(A_{\mathbb{R}})$ denotes the category of finitely generated free right $A_{\mathbb{R}}$-modules. We have an exact functor

$$F(A_{\mathbb{R}}) \longrightarrow \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$$

sending $A_{\mathbb{R}}$ to itself, equipped with the real topology. We have $A_{\mathbb{R}} \in \mathsf{PLCA}_{\mathfrak{A}}$ thanks to Proposition 4.7. By the 2-functoriality of idempotent completion [Büh10, §6], we get a unique induced exact functor $C : P_f(A_{\mathbb{R}}) \longrightarrow \mathsf{PLCA}^{ic}_{\mathfrak{A},\mathbb{R}}$. By the same argument, the inclusion

$$\mathsf{PLCA}_{\mathfrak{A}} \hookrightarrow \mathsf{LCA}_{\mathfrak{A}}$$

functorially induces an exact functor $C' : \mathsf{PLCA}^{ic}_{\mathfrak{A},\mathbb{R}} \longrightarrow \mathsf{LCA}_{\mathfrak{A},\mathbb{R}}$ since $\mathsf{LCA}_{\mathfrak{A}}$ is already idempotent complete (as it is quasi-abelian), and moreover the image consists only of vector modules. We show that $C$ is essentially surjective: Every vector module $X$ is a right $A_{\mathbb{R}}$-module, necessarily finitely generated since it must be finite-dimensional as a real vector space. Since $A_{\mathbb{R}}$ is semisimple, all its modules are projective and therefore $X$ is a finitely generated projective right $A_{\mathbb{R}}$-module. Hence, $X$ is a direct summand of some $A^n_{\mathbb{R}}$. However, by Proposition 4.7 we have $A_{\mathbb{R}} \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$, so the idempotent completion settles the claim. Note that this argument did not use $X \in \mathsf{PLCA}_{\mathfrak{A}}$, so it also settles essential surjectivity of $C'$. For $C'$ it is clear that the functor is fully faithful. For $C$ it follows from continuity. (More precisely: Any $A_{\mathbb{R}}$-module homomorphism is also an $\mathbb{R}$-linear map and all linear maps between real vector spaces are continuous in the real topology. Conversely, any abelian group homomorphism between uniquely divisible groups must be a $\mathbb{Q}$-vector space map. By continuity, it then must be an $\mathbb{R}$-linear map using the density of $\mathbb{Q} \subset \mathbb{R}$. Finally, this means that the $\mathfrak{A}$-module homomorphisms are even $\mathfrak{A} \otimes_{\mathbb{Z}} \mathbb{R} = A_{\mathbb{R}}$ module homomorphisms) □

*Example* 4.10. We point out that this lemma would not hold without the idempotent completion. Take $A := \mathbb{Q}[\sqrt{2}]$, a number field. Then $\mathfrak{A} := \mathbb{Z}[\sqrt{2}]$ is the ring of integers, and thus a maximal order. We have $A_{\mathbb{R}} \simeq \mathbb{R}_{\sigma} \oplus \mathbb{R}_{\sigma'}$, where $\sigma, \sigma'$ correspond to the two real embeddings $\sqrt{2} \mapsto \pm\sqrt{2}$, giving the two possible $\mathfrak{A}$-module structures on the reals. While $\mathbb{R}_{\sigma}$ is a vector module, we have $\mathbb{R}_{\sigma} \notin \mathsf{PLCA}_{\mathfrak{A}}$, for otherwise there would be a PI-presentation

$$P \hookrightarrow \mathbb{R}_{\sigma} \twoheadrightarrow I.$$

Here $P \in P_f(\mathfrak{A})$. As $A$ has class number one, $\mathfrak{A}$ is a principal ideal domain, so all projective $\mathfrak{A}$-modules are free. As the underlying abelian group of $\mathfrak{A}$ is $\mathbb{Z}^2$, it follows that the underlying LCA group of $P$ can only be $\mathbb{Z}^{2n}$. On the other hand, $I$ is compact (Lemma 3.4). However, all cocompact closed subgroups of $\mathbb{R}$ are isomorphic to $\mathbb{Z}$. Thus, no PI-presentation can exist.

**Corollary 4.11.** *If $\mathfrak{A} \subset A$ is a Gorenstein order, all vector right $\mathfrak{A}$-modules lie in $\mathsf{PLCA}_{\mathfrak{A}}^{ic}$, and they are both injective and projective objects in this category.*

**Proof.** As vector $\mathfrak{A}$-modules are projective (resp. injective) objects in $\mathsf{LCA}_{\mathfrak{A}}$ by [Bra19b, Proposition 8.1], they remain so in $\mathsf{PLCA}_{\mathfrak{A}}$ (Lemma 2.4). ☐

**Proposition 4.12.** *Suppose $\mathfrak{A} \subset A$ is a Gorenstein order.*

(1) *Then for every finitely generated projective right $\mathfrak{A}$-module $P$ the sequence*
$$P \hookrightarrow P_{\mathbb{R}} \twoheadrightarrow P_{\mathbb{R}}/P \tag{4.5}$$
*is a PI-presentation, where $P_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} P$ is regarded as equipped with the real vector space topology. In particular, $P_{\mathbb{R}}/P \in I_{\Pi}(\mathfrak{A})$.*

(2) *Moreover, this is a projective resolution of $P_{\mathbb{R}}/P$ in $\mathsf{PLCA}_{\mathfrak{A}}$.*

(3) *Moreover, this is an injective resolution of $P$ in $\mathsf{PLCA}_{\mathfrak{A}}$.*

**Proof.** (1) Since $P$ is projective, there exists some $n \geq 0$ and idempotent $e$ with $P = e\mathfrak{A}^n$. After tensoring with the reals, this cuts out the exact sequence of Equation 4.5 as a direct summand of a direct sum of sequences of Proposition 4.7. Thus, $(P_{\mathbb{R}}/P)^{\vee}$ is a direct summand of $(A_{\mathbb{R}}/\mathfrak{A})^{\vee}$ and thus injective, and $P_f(\mathfrak{A})$ is closed under direct summands in all right $\mathfrak{A}$-modules as well. We arrive at the said PI-presentation. (2) As $P$ and $P_{\mathbb{R}}$ are projective objects in $\mathsf{LCA}_{\mathfrak{A}}$ by [Bra19b, Proposition 8.1], they remain projective in $\mathsf{PLCA}_{\mathfrak{A}}$ by Lemma 2.4, and the claim follows. (3) Use [Bra19b, Proposition 8.1] analogously. ☐

*Remark* 4.13. Note that all discrete modules in the above proof are finitely generated, so we do not run into the issue that $P_{\oplus}(\mathfrak{A})$ itself need not be idempotent complete in general (Lemma 3.2).

**Definition 4.14.** *Let $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ be the full subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$ of objects which can be written as a direct sum*
$$X \simeq P \oplus V$$
*with $P \in P_{\oplus}(\mathfrak{A})$ and $V$ a vector right $\mathfrak{A}$-module.*

**Lemma 4.15.** *$\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ is an extension-closed subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$ (and even in $\mathsf{LCA}_{\mathfrak{A}}$).*

**Proof.** Take $\mathsf{C} := \mathsf{LCA}_{\mathfrak{A}}$, which is weakly idempotent complete. We want to apply Lemma 2.2 to $\mathsf{C}$ with $\mathsf{P} := P_{\oplus}(\mathfrak{A})$ and $\mathsf{I}$ the full subcategory of vector $\mathfrak{A}$-modules. This works since vector modules are injective in $\mathsf{LCA}_{\mathfrak{A}}$ [Bra19b, Proposition 8.1]. Every object $X \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ has the PI-presentation
$$P \hookrightarrow X \twoheadrightarrow V$$

with respect to this choice of $\mathsf{P}$ and $\mathsf{I}$. Now let

$$X' \hookrightarrow X \twoheadrightarrow X''$$

be an exact sequence with $X', X'' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ and $X \in \mathsf{LCA}_{\mathfrak{A}}$. Use Lemma 2.2. It provides a PI-presentation for $X$ of the shape

$$P \hookrightarrow X \twoheadrightarrow V,$$

with $P \in \mathsf{P}$, $V \in \mathsf{I}$, but since vector modules are also projective [Bra19b, Proposition 8.1], this splits, giving $X \simeq P \oplus V$, proving the claim. $\square$

It follows that $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ is a fully exact subcategory of $\mathsf{LCA}_{\mathfrak{A}}$.

**Lemma 4.16.** *The category* $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$ *is left s-filtering in* $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$.

**Proof.** *(Left filtering)* If $f : V' \to P \oplus V$ is any morphism with $V' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$, then since $V'$ is connected, we get a factorization $V' \to V \hookrightarrow P \oplus V$ of $f$. *(Left special)* If

$$X' \hookrightarrow X \twoheadrightarrow V$$

is an exact sequence with $V \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}$, then since $V$ is projective, we get a splitting, providing us with the commutative diagram

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d] & V \arrow[r, two heads, "1"] \arrow[d] & V \arrow[d, equal] \\ X' \arrow[r, hook] & X \arrow[r, two heads] & V \end{tikzcd}$$

settling left specialness. $\square$

**Lemma 4.17.** *There is an exact equivalence of exact categories*

$$P_{\oplus}(\mathfrak{A}) \xrightarrow{\sim} \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}/\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}.$$

**Proof.** Send a module $P \in P_{\oplus}(\mathfrak{A})$ to itself, equipped with the discrete topology. This is an exact functor. It is essentially surjective, directly by the definition of $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$. Homomorphisms $X \to X'$ on the right between objects in the strict image correspond to roofs

$$X \overset{e}{\twoheadleftarrow} V \oplus P \to X'$$

with $V$ a vector module and $e$ having vector module kernel. However, since $V$ is connected but $X, X'$ discrete, any such roof is trivially equivalent to one with $V = 0$. But for these the vector module kernel of $e$ must be trivial, i.e. $e$ must be an isomorphism in $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$. Thus, any roof is equivalent to $X \overset{1}{\twoheadleftarrow} X \to X'$, i.e. we get just ordinary right $\mathfrak{A}$-module homomorphisms. This shows that the functor in our claim is fully faithful. $\square$

**Lemma 4.18.** *Suppose*

$$X' \hookrightarrow V \oplus P \twoheadrightarrow V'' \oplus P''$$

*is an exact sequence in* $\mathsf{PLCA}_{\mathfrak{A}}$ *whose middle and right object lie in the subcategory* $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$. *Then* $X' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$.

**Proof.** (Step 1) Let us work in the category $\mathsf{LCA}_{\mathfrak{A}}$. First of all, we show that it suffices to handle the case where $V''=0$ and $P''\in P_f(\mathfrak{A})$. Consider

$$X' \hookrightarrow V\oplus P \twoheadrightarrow V''\oplus P''.$$

Note that $V''$ is a projective object in $\mathsf{LCA}_{\mathfrak{A}}$. Hence, there is a section $g: V''\hookrightarrow V\oplus P$ to the epic, and since $V''$ is connected, the image of $g$ must lie in $V$. We split off this direct summand, giving

$$X' \hookrightarrow V\oplus P \twoheadrightarrow P'' \tag{4.6}$$

after having changed the definition of $V$. Next, $P''$ is projective, so we get a section $h: P''\hookrightarrow V\oplus P$. The intersection $V\cap h(P'')$ must be a discrete finitely generated $\mathfrak{A}$-module (we refer to Equation 3.7 for a completely analogous construction, where we give a detailed argument). Thus, by Lemma 3.5 and since $P''\in P_{\oplus}(\mathfrak{A})$ we can find a direct sum splitting $P''\cong P_0''\oplus P_\infty''$ such that $h(P_\infty'')$ lies entirely in $P$ and $P_0''\in P_f(\mathfrak{A})$. Thus, Sequence 4.6 becomes

$$X' \hookrightarrow V\oplus P \twoheadrightarrow P_0''\oplus P_\infty''$$

and $h\mid_{P_\infty''}$ is a section for $P_\infty''$, giving

$$X' \hookrightarrow V\oplus P_0\oplus P_\infty'' \twoheadrightarrow P_0''\oplus P_\infty'',$$

(where $P_0$ denotes a complement of the image of the section) and after we split off the summand $P_\infty''$, we obtain $X'\hookrightarrow V\oplus P_0\twoheadrightarrow P_0''$ with $P_0''\in P_f(\mathfrak{A})$. It follows that if we prove the claim of the lemma for this special case, it implies the general case.

(Step 2) Since the underlying LCA group of $V\oplus P_0$ has the shape $\mathbb{R}^n\oplus$ (discrete), the closed subgroup $X'$ must also have the shape $\mathbb{R}^\ell\oplus$ (discrete) by [Mor77, Corollary 2 to Theorem 7 and Remark]. This direct sum splitting on the level of LCA groups lifts to a direct sum splitting in $\mathsf{LCA}_{\mathfrak{A}}$ by [Bra19b, Lemma 6.1, (1)], so we can write

$$X'\cong V'\oplus D' \tag{4.7}$$

with $V'$ a vector $\mathfrak{A}$-module and $D'$ discrete in the category $\mathsf{LCA}_{\mathfrak{A}}$. Next, we apply Proposition 3.8 to $X'$, giving a further direct sum decomposition $X'\simeq P_\infty'\oplus I_\infty'\oplus B'$. We note that $I_\infty'$ is compact connected by Lemma 3.4, but by Equation 4.7 $X'$ has no non-trivial compact connected subgroup at all, so we must have $I_\infty'=0$. Hence, $X'\simeq P_\infty'\oplus B'$. Since $P_\infty'\in\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$, we conclude that the lemma is proven if we can prove $B'\in\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$.

(Step 3) Thus, we may prove the claim of the lemma in the special case where $X'$ has a PI-presentation $P'\hookrightarrow X'\twoheadrightarrow I'$ with $P'\in P_f(\mathfrak{A})$ and $I'^{\vee}\in P_f(\mathfrak{A}^{op})$. In the isomorphism $X'\cong V'\oplus D'$ of Equation 4.7 this implies that $D'$ must be a finitely generated $\mathfrak{A}$-module. Then our sequence reads (thanks to the simplification in Step 1)

$$V'\oplus D' \hookrightarrow V\oplus P \twoheadrightarrow P'' \tag{4.8}$$

with $P'' \in P_f(\mathfrak{A})$. We get an admissible filtration $V' \hookrightarrow V' \oplus D' \hookrightarrow V \oplus P$ and Noether's Lemma yields the exact sequence

$$D' \hookrightarrow \frac{V \oplus P}{V'} \twoheadrightarrow \frac{V \oplus P}{V' \oplus D'}$$

in $\mathsf{LCA}_{\mathfrak{A}}$. We note that the term on the right is $P''$ in view of Equation 4.8. Moreover, the image of the connected $V'$ inside $V \oplus P$ will again be connected, so it must lie in $V$. Thus, we get the exact sequence

$$D' \hookrightarrow \frac{V}{V'} \oplus P \twoheadrightarrow P''.$$

Since $D'$ and $P''$ are discrete, so must be the group in the middle. This forces $V/V' = 0$. We get $D' \hookrightarrow P \twoheadrightarrow P''$. As both $P''$ and $D'$ are finitely generated $\mathfrak{A}$-modules, so must be $P$, i.e. $P \in P_f(\mathfrak{A})$. Since $P''$ is projective, the sequence must split, i.e. $P \cong D' \oplus P''$. Since $P \in P_f(\mathfrak{A})$ and this category is idempotent complete, we deduce that $D' \in P_f(\mathfrak{A})$. Since $X' \cong V' \oplus D'$ this implies $X' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$ as desired. $\square$

*Remark* 4.19. The intermediate reduction to finitely generated modules in the proof was necessary because we used idempotent completeness and this holds for $P_f(\mathfrak{A})$, but not necessarily for $P_{\oplus}(\mathfrak{A})$.

**Lemma 4.20.** *Suppose $\mathfrak{A} \subset A$ is a Gorenstein order. Suppose $X \in \mathsf{PLCA}_{\mathfrak{A}}$ has a PI-presentation*

$$P \hookrightarrow X \twoheadrightarrow I$$

*with $P \in P_f(\mathfrak{A})$ and $I^{\vee} \in P_f(\mathfrak{A}^{op})$. Then there exists a projective resolution*

$$P_1' \hookrightarrow P_0' \twoheadrightarrow X$$

*with $P_1', P_0' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$.*

**Proof.** (Step 1) Since $I^{\vee} \in P_f(\mathfrak{A}^{op})$, we apply Proposition 4.12 to get an injective resolution in $\mathsf{PLCA}_{\mathfrak{A}^{op}}$. Under Pontryagin duality, this gives us a projective resolution

$$P_1 \hookrightarrow P_0 \overset{q}{\twoheadrightarrow} I,$$

where $P_0$ is a vector right $\mathfrak{A}$-module and $P_1 \in P_f(\mathfrak{A})$. We consider the commutative diagram

$$\begin{tikzcd} & & P_1 \arrow[d, hook] \\ & & P_0 \arrow[dl, dashed, "f"'] \arrow[d, two heads, "q"] \\ P \arrow[r, hook, "i"] & X \arrow[r, two heads] & I, \end{tikzcd} \tag{4.9}$$

where we obtain the lift $f$ by exploiting that $P_0$ is a projective object. Now consider the morphism $i + f : P \oplus P_0 \to X$. Since $i, f$ are continuous, so is $i + f$. Moreover, the map is clearly surjective. Next, since $P$ is finitely generated and $P_0$ a vector module, the underlying LCA group of $P \oplus P_0$ is of the shape $\mathbb{Z}^n \oplus \mathbb{R}^m$ for suitable $n, m \geq 0$, and thus $P \oplus P_0$ is $\sigma$-compact. Thus, by Pontryagin's Open Mapping Theorem [Mor77, Theorem 3] $i + f$ must be an open map. Hence, $i + f$ is an admissible epic in $\mathsf{LCA}_{\mathfrak{A}}$. Let $K$ be its kernel in $\mathsf{LCA}_{\mathfrak{A}}$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r, hook] \arrow[d, hook] & P \arrow[r, two heads, "1"] \arrow[d, hook] & P \arrow[d, hook, "i"] \\
K \arrow[r, hook] \arrow[d, "\sim"] & P \oplus P_0 \arrow[r, two heads, "i+f"'] \arrow[d, two heads] & X \arrow[d, two heads] \\
K \arrow[r, hook] & P_0 \arrow[r, two heads, "\overline{i+f}"'] & I
\end{tikzcd} \tag{4.10}$$

in $\mathsf{LCA}_{\mathfrak{A}}$. It can be constructed by first setting up the top two rows, which obviously commute, and which then gives rise to the bottom row by a naïve version of the snake lemma. We note that the quotient map $\overline{i+f}$ agrees with $q$ because any $p \in P_0$ can be lifted to $(0, p)$ in $P \oplus P_0$ and then the remaining arrows to $I$ agree with $q$ in Diagram 4.9. Thus, $K$ is a kernel for $q$, which provides us with an isomorphism $K \cong P_0$. It follows that $K \in \mathsf{PLCA}_{\mathfrak{A}}$. It follows that Diagram 4.10 is actually a diagram in the category $\mathsf{PLCA}_{\mathfrak{A}}$. Note that the middle row now provides a projective resolution of $X$. $\square$

Define the full subcategory of modules with *no small subgroups*,

$$\mathsf{PLCA}_{\mathfrak{A},nss} := \mathsf{PLCA}_{\mathfrak{A}} \cap \mathsf{LCA}_{\mathfrak{A},nss},$$

much in the spirit of Equation 3.11. As Pontryagin duality exchanges groups without small subgroups with compactly generated ones, we can also define $\mathsf{PLCA}_{\mathfrak{A},nss}$ as the Pontryagin dual of the full subcategory $\mathsf{PLCA}_{\mathfrak{A}^{op},cg}$ of $\mathsf{PLCA}_{\mathfrak{A}^{op}}$. In particular, it is clear that $\mathsf{PLCA}_{\mathfrak{A},nss}$ is a fully exact subcategory of $\mathsf{PLCA}_{\mathfrak{A}}$.

**Corollary 4.21.** *Suppose that $\mathfrak{A} \subset A$ is a Gorenstein order. Then every object $X \in \mathsf{PLCA}_{\mathfrak{A},nss}$ has a projective resolution*

$$P_1' \hookrightarrow P_0' \twoheadrightarrow X$$

*with $P_1', P_0' \in \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$.*

**Proof.** Use Proposition 3.8 to write $X$ as $X \simeq P_\infty \oplus M \oplus I_\infty$ such that $M$ satisfies the conditions of Lemma 4.20 (and therefore has a projective resolution as required). Further, $P_\infty \in P_\oplus(\mathfrak{A})$ is projective and lies in $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}$, so this also satisfies our claim. Finally, the underlying LCA group of $I_\infty$ is $\prod_{i \in \mathcal{I}} \mathbb{T}$ for some index set, but this has no small subgroups if and only if $\mathcal{I}$ is finite ([Mos67, Theorem 2.4]). In that case, and since we know that $I_\infty^\vee \in P_f(\mathfrak{A}^{op})$, it follows that $I_\infty$ also satisfies the conditions of Lemma 4.20. $\square$

**Theorem 4.22.** *Let $\mathfrak{A} \subset A$ be a Gorenstein order. Let $\mathsf{A}$ be a stable $\infty$-category. Suppose $K : \mathrm{Cat}_{\infty}^{\mathrm{ex}} \to \mathsf{A}$ is a localizing invariant with values in $\mathsf{A}$. Then there is an equivalence*

$$K(A_{\mathbb{R}}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A},nss}),$$

*induced from the exact functor sending a right $A_{\mathbb{R}}$-module to itself, equipped with the real vector space topology.*

**Proof.** (Step 1) First of all, we show that the inclusion of the fully exact subcategory $\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D} \hookrightarrow \mathsf{PLCA}_{\mathfrak{A},nss}$ induces an equivalence

$$K(\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A},nss}),$$

because this exact functor induces a derived equivalence [Kel96, §12, Theorem 12.1]. The assumptions of the cited theorem are met, because the inclusion functor satisfies (the categorical opposite of) the axiom C1 by Corollary 4.21. Further, it satisfies the stronger condition implying C2 by Lemma 4.18.

(Step 2) Next, by Lemma 4.16 and 4.17 we have the localization fiber sequence

$$K(\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}) \longrightarrow K(\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}D}) \longrightarrow K(P_{\oplus}(\mathfrak{A})),$$

where $K(P_{\oplus}(\mathfrak{A})) = 0$ since $P_{\oplus}(\mathfrak{A})$ is closed under countable coproducts and we may thus apply the Eilenberg swindle. Next, since $K$ is localizing, it is invariant under going to idempotent completion, so the exact equivalence of exact categories $P_f(A_{\mathbb{R}}) \xrightarrow{\sim} \mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}^{ic}$ of Lemma 4.9 induces an equivalence

$$K(A_{\mathbb{R}}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}}^{ic}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A},\mathbb{R}})$$

in $\mathsf{A}$. Combine both results and check that the equivalence is indeed induced by the functor claimed. $\square$

**Theorem 4.23.** *Let $\mathfrak{A} \subset A$ be a Gorenstein order. Let $\mathsf{A}$ be a stable $\infty$-category. Suppose $K : \mathrm{Cat}_{\infty}^{\mathrm{ex}} \to \mathsf{A}$ is a localizing invariant with values in $\mathsf{A}$. Then there is an equivalence*

$$K(A_{\mathbb{R}}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A},cg}),$$

*induced from the exact functor sending a right $A_{\mathbb{R}}$-module to itself, equipped with the real vector space topology.*

**Proof.** Pontryagin duality is an exact functor exchanging the full subcategories of compactly generated modules with those without small subgroups. Thus, Proposition 3.3 restricts to an exact equivalence of exact categories

$$\mathsf{PLCA}_{\mathfrak{A},cg} \overset{\sim}{\to} \mathsf{PLCA}_{\mathfrak{A}^{op},nss}^{op}.$$

Along with Theorem 4.22 applied to $\mathfrak{A}^{op}$, we get the two equivalences

$$K(\mathsf{PLCA}_{\mathfrak{A},cg}) \xrightarrow{\sim} K(\mathsf{PLCA}_{\mathfrak{A}^{op},nss}^{op}) \xrightarrow{\sim} K(P_f(A_{\mathbb{R}}^{op})^{op}).$$

Note that if $\mathfrak{A} \subset A$ is a Gorenstein order in a semisimple algebra, so is its opposite $\mathfrak{A}^{op} \subset A^{op}$, see Definition 4.2, so using Theorem 4.22 was legitimate. Next, for any ring $R$ the functor $P \mapsto \mathrm{Hom}_R(P, R)$ induces an exact equivalence $P_f(R^{op}) \xrightarrow{\sim} P_f(R)^{op}$, relating the opposite ring with the opposite category. Applied to $R := A_{\mathbb{R}}$ this yields $K(P_f(A_{\mathbb{R}}^{op})^{op}) \xrightarrow{\sim} K(P_f(A_{\mathbb{R}}))$. This proves our claim. $\square$

## 5. Main theorems

We may now collect all our results to obtain a locally compact topological analogue of the relative $K$-group appearing in the Burns–Flach formulation of the ETNC with non-commutative coefficients [BF01].

**Theorem 5.1.** *Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and let $\mathfrak{A} \subset A$ be a Gorenstein order. Let $\mathsf{A}$ be a stable $\infty$-category. Suppose $K : \mathrm{Cat}_{\infty}^{\mathrm{ex}} \to \mathsf{A}$ is a localizing invariant with values in $\mathsf{A}$. Then there is a fiber sequence*

$$K(\mathfrak{A}) \longrightarrow K(A_{\mathbb{R}}) \longrightarrow K(\mathsf{PLCA}_{\mathfrak{A}})$$

*in $\mathsf{A}$. If $\mathfrak{A}$ is regular, there is a morphism of fiber sequences to the one of* [Bra19b, Theorem 11.2]

$$K(\mathsf{Mod}_{\mathfrak{A},fg}) \longrightarrow K(A_{\mathbb{R}}) \longrightarrow K(\mathsf{LCA}_{\mathfrak{A}}),$$

*coming from the inclusion $P_f(\mathfrak{A}) \subseteq \mathsf{Mod}_{\mathfrak{A},fg}$ and $\mathsf{PLCA}_{\mathfrak{A}} \subseteq \mathsf{LCA}_{\mathfrak{A}}$. This morphism is an equivalence of fiber sequences.*

**Proof.** Use Proposition 3.15 and Theorem 4.23. Unravelling the maps gives all the claims about the compatibility with [Bra19b]. $\square$

Finally, we may apply this to usual algebraic $K$-theory.

**Definition 5.2.** *Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and let $\mathfrak{A} \subset A$ be an order. Define*

$$\mathsf{LCA}_{\mathfrak{A}}^{*} := \mathsf{PLCA}_{\mathfrak{A}}^{ic},$$

*i.e. as the idempotent completion of $\mathsf{PLCA}_{\mathfrak{A}}$.*

**Theorem 5.3.** *Suppose $A$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and let $\mathfrak{A} \subset A$ be a Gorenstein order. There is a long exact sequence of algebraic $K$-groups*

$$\cdots \to K_n(\mathfrak{A}) \to K_n(A_{\mathbb{R}}) \to K_n(\mathsf{LCA}_{\mathfrak{A}}^{*}) \to K_{n-1}(\mathfrak{A}) \to \cdots$$

*for positive $n$, ending in*

$$\cdots \to K_0(\mathfrak{A}) \to K_0(A_{\mathbb{R}}) \to K_0(\mathsf{LCA}_{\mathfrak{A}}^{*}) \to \mathbb{K}_{-1}(\mathfrak{A}) \to 0.$$

*Here $\mathbb{K}_{-1}$ denotes non-connective $K$-theory. Classically, these groups are simply called the "negative $K$-groups". Moreover,*

$$\mathbb{K}_n(\mathsf{LCA}_{\mathfrak{A}}^{*}) \cong \mathbb{K}_{n-1}(\mathfrak{A})$$

*for all* $n \leq -1$. *If* $\mathfrak{A}$ *is additionally a regular order (e.g. hereditary), this sequence agrees with the one of* [Bra19b, Theorem 11.2], *and moreover* $\mathbb{K}_n(\mathsf{LCA}_{\mathfrak{A}}^*) = 0$ *for* $n \leq -1$ *in this case.*

**Proof.** Connective $K$-theory is not a localizing invariant, so we first need to work with non-connective $K$-theory, which we shall denote by $\mathbb{K}$, instead. It takes values in $\mathsf{A} := \mathsf{Sp}$, the stable $\infty$-category of spectra. From the fiber sequence of spectra provided by Theorem 5.1, we obtain the long exact sequence of homotopy groups (i.e. non-connective $K$-groups)

$$\cdots \to \mathbb{K}_n(\mathfrak{A}) \to \mathbb{K}_n(A_{\mathbb{R}}) \to \mathbb{K}_n(\mathsf{PLCA}_{\mathfrak{A}}) \to \mathbb{K}_{n-1}(\mathfrak{A}) \to \cdots.$$

Next, for $K$ denoting connective $K$-theory, recall that $K_n(\mathsf{C}^{ic}) \cong \mathbb{K}_n(\mathsf{C})$ for all $n \geq 0$ and any exact category [Sch06]. The underlying category of $\mathbb{K}_n(\mathfrak{A})$ is $P_f(\mathfrak{A})$, which is idempotent complete, so we deduce

$$K_n(\mathfrak{A}) = \mathbb{K}_n(\mathfrak{A})$$

for all $n \geq 0$. The ring $A_{\mathbb{R}}$ is semisimple and in particular any module is projective, so $K_n(A_{\mathbb{R}}) = \mathbb{K}_n(A_{\mathbb{R}})$ for $n \geq 0$, but moreover since this is a regular ring, $\mathbb{K}_n(A_{\mathbb{R}}) = 0$ for all $n < 0$. Thus, our sequence can be rewritten as

$$\cdots \to K_1(\mathsf{PLCA}_{\mathfrak{A}}) \to K_0(\mathfrak{A}) \to K_0(A_{\mathbb{R}}) \to K_0(\mathsf{PLCA}_{\mathfrak{A}}^{ic}) \to \mathbb{K}_{-1}(\mathfrak{A}) \to 0$$

as well as $\mathbb{K}_n(\mathsf{PLCA}_{\mathfrak{A}}) \cong \mathbb{K}_{n-1}(\mathfrak{A})$ for $n \leq -1$. $\square$

The case of group rings is of particular relevance.

**Corollary 5.4.** *Suppose $G$ is a finite group. Take $A = \mathbb{Q}[G]$ and $\mathfrak{A} := \mathbb{Z}[G]$. There is a long exact sequence of algebraic $K$-groups*

$$\cdots \to K_n(\mathbb{Z}[G]) \to K_n(\mathbb{R}[G]) \to K_n(\mathsf{LCA}_{\mathbb{Z}[G]}^*) \to K_{n-1}(\mathbb{Z}[G]) \to \cdots$$

*for positive $n$, ending in*

$$\cdots \to K_0(\mathbb{Z}[G]) \to K_0(\mathbb{R}[G]) \to K_0(\mathsf{LCA}_{\mathbb{Z}[G]}^*) \to \mathbb{K}_{-1}(\mathbb{Z}[G]) \to 0$$

*and* $\mathbb{K}_n(\mathsf{LCA}_{\mathbb{Z}[G]}^*) = 0$ *for* $n < 0$.

The group $\mathbb{K}_{-1}(\mathbb{Z}[G])$ is well-understood by the work of Carter. He has shown that

$$\mathbb{K}_{-1}(\mathbb{Z}[G]) \cong \mathbb{Z}^a \oplus (\mathbb{Z}/2)^b$$

for suitable $a, b \in \mathbb{Z}_{\geq 0}$, which are a little involved to describe explicitly, [Car80b, Theorem 1].

A lot of explicit computations can be found for example in [LMO10], [Mag13]. Although this shows that some literature and research exists, it appears that in general the study of negative $K$-groups of orders in semisimple algebras is not very developed.

**Proof.** Apply Theorem 5.3. This is possible because $\mathbb{Z}[G]$ is a Gorenstein order by Lemma 4.3. Moreover, $\mathbb{K}_n(\mathbb{Z}[G]) = 0$ for $n \leq -2$ by work of Carter [Car80b], [Car80a].[5] $\square$

We also observe the following consequence.

**Corollary 5.5.** *The non-connective K-theory spectrum $\mathbb{K}(\mathsf{LCA}^*_{\mathbb{Z}[G]})$ for the integral group ring of any finite group is actually connective.*

Finally, let us discuss the analogue of the comparison map in [Bra18]. We refer to that paper for background on the terms and notation we employ.

**Theorem 5.6.** *Let $A$ be a finite-dimensional semisimple $\mathbb{Q}$-algebra and $\mathfrak{A} \subset A$ any order. Then the map*

$$\vartheta : K_0(\mathfrak{A}, \mathbb{R}) \longrightarrow K_1(\mathsf{LCA}^*_{\mathfrak{A}}) \tag{5.1}$$

*which sends the Bass–Swan representative $[P, \varphi, Q]$ to the double exact sequence $\langle\langle P, \varphi, Q\rangle\rangle$ (as defined in* [Bra18]*) is a well-defined morphism from the Bass–Swan to the Nenashev presentation. If $\mathfrak{A}$ is a Gorenstein order, then this map is an isomorphism.*

**Proof.** One can adapt the proof of [Bra18] with only a few changes. First of all, note that all objects which occur in the Nenashev representative $\langle\langle P, \varphi, Q\rangle\rangle$ lie in the full subcategory $\mathsf{PLCA}_{\mathfrak{A}} \subset \mathsf{LCA}_{\mathfrak{A}}$, so the map naturally lands in $K_1(\mathsf{LCA}^*_{\mathfrak{A}}) = K_1(\mathsf{PLCA}_{\mathfrak{A}})$. Moreover, all the proofs that the map is well-defined carry over verbatim. This already suffices to show that the map exists. It only remains to prove that it is an isomorphism if $\mathfrak{A}$ is Gorenstein. To this end, we also copy the proof of [Bra18]. Replace the diagram in the statement of [Bra18, Theorem 3.2] by

$$\begin{tikzcd}
\cdots \arrow[r] & K_1(\mathfrak{A}, \mathbb{R}) \arrow[r] \arrow[d] & K_1(\mathfrak{A}) \arrow[r] \arrow[d, equal] & K_1(A_{\mathbb{R}}) \arrow[r, "\delta"] \arrow[d, equal] & K_0(\mathfrak{A}, \mathbb{R}) \arrow[r] \arrow[d, "\vartheta"] & K_0(\mathfrak{A}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & K_2(\mathsf{LCA}_{\mathfrak{A}}) \arrow[r] & K_1(\mathfrak{A}) \arrow[r] & K_1(A_{\mathbb{R}}) \arrow[r] & K_1(\mathsf{LCA}^*_{\mathfrak{A}}) \arrow[r, "\partial"'] & K_0(\mathfrak{A}) \arrow[r] & \cdots,
\end{tikzcd}$$

where $\vartheta$ is the map of Equation 5.1 and the bottom row is the one coming from Theorem 5.3. Then proceed in the proof exactly as loc. cit., except for the following changes: The diagram

$$\begin{tikzcd}
\mathsf{Mod}_{\mathfrak{A},fg} \arrow[r] \arrow[d, "g"'] & \mathsf{Mod}_{\mathfrak{A}} \arrow[r] \arrow[d] & \mathsf{Mod}_{\mathfrak{A}}/\mathsf{Mod}_{\mathfrak{A},fg} \arrow[d, "\Phi"] \\
\mathsf{LCA}_{\mathfrak{A},cg} \arrow[r] & \mathsf{LCA}_{\mathfrak{A}} \arrow[r] & \mathsf{LCA}_{\mathfrak{A}}/\mathsf{LCA}_{\mathfrak{A},cg}
\end{tikzcd}$$

[5] We remark that Hsiang has conjectured that $\mathbb{K}_n(\mathbb{Z}[G]) = 0$ for $n \leq -2$ for any finitely presented group. This remains open.

needs to be replaced by the one of categories underlying Diagram 3.19. The exact equivalence of exact categories

$$\mathsf{Mod}_{\mathfrak{A}}/\mathsf{Mod}_{\mathfrak{A},fg} \xrightarrow{\sim} \mathsf{LCA}_{\mathfrak{A}}/\mathsf{LCA}_{\mathfrak{A},cg}$$

needs to be replaced by $P_{\oplus}(\mathfrak{A})/P_f(\mathfrak{A}) \xrightarrow{\sim} \mathsf{PLCA}_{\mathfrak{A}}/\mathsf{PLCA}_{\mathfrak{A},cg}$ of Lemma 3.14. The rest works verbatim, always just using that all the objects which the proof uses already lie in the full subcategory $\mathsf{LCA}_{\mathfrak{A}}^*$ of $\mathsf{LCA}_{\mathfrak{A}}$. □

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