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Invariants, bitangents, and matrix representations of plane quartics with 3-cyclic automorphisms

Dun Liang

Abstract. In this work we compute the Dixmier invariants and bitangents of the plane quartics with 3,6 or 9-cyclic automorphisms. We find that a quartic curve with 6-cyclic automorphism will have 3 horizontal bitangents which form an asyzygetic triple. We also discuss the linear matrix representation problem of such curves, and find a degree 6 equation of 1 variable which solves the symbolic solution of the linear matrix representation problem for the curve with 6-cyclic automorphism.

Contents

1. Introduction

The study of the geometry of plane quartics is one of the most beautiful achievements in classical algebraic geometry. Back to the late 19^{th} and early 20^{th} century, there were many studies on the existence and configurations of the 28 bitangents of a plane quartic such as [11], [8], [9], [23], and so on. For the invariants of plane quartics, Shioda computed the ring of invariants in [22]. However, the algebraic invariants of plane quartics were found much later by [1], [16] and [3]. In this work we compute the invariants and bitangents of plane quartics with 3-cyclic automorphism, and discuss the linear matrix representation problem (see [6],[24],[25]) of such curves. The

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classification of automorphism was given by [12] and [26]. There are many places one can see the full list, for example, in Section 6.5 of [2].

Explicitly, we consider the curves

$$C_3 = C_3(r,s): \quad y^3 = x(x-1)(x-r)(x-s) \text{ for } r,s \neq 0,1 \text{ and } r \neq s \quad (1)$$

$$C_6 = C_6(r): \quad y^3 = x(x-1)(x-r)(x-1+r) \text{ for } r \neq 0,1 \quad (2)$$

$$C_9: \quad y^3 = x(x^3-1) \quad (3)$$

with automorphism group $\mathbb{Z}/3$, $\mathbb{Z}/6$ and $\mathbb{Z}/9$ respectively. The family C_3 is the famous Picard family of quartics (see [10], [17], [18]). Let $R_3 = \mathrm{End}(J_3)$ be the endomorphism ring of the Jacobian variety J_3 of C_3. Let ζ_3 be the cubic root of the unity. The main property of the family C_3 is that $R_3 \simeq O_K$, the ring of integers of some number field K which contains $\mathbb{Q}(\zeta_3)$.

We compute the invariants of these curves. The curves C_6 and C_9 are special cases of C_3. Thus we also compute the cutting equations of the invariants of C_6 and C_9 as special cases of C_3. In modern point of view, a smooth plane quartic is the canonical model of a smooth projective non-hyperelliptic curve of genus 3. Let $\mathcal{M}_3$ be the moduli space of projective curves of genus 3, and let $\mathcal{M}_3^{\mathrm{non}}$ be the non-hyperelliptic locus of $\mathcal{M}_3$. Then the weight zero ratios of the Dixmier invariants $I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{27}$, as functions of the coefficients of a given ternary quartic are like an analog of the j-invariant of a given cubic curve, and thus could be regarded as the coordinates of $\mathcal{M}_3^{\mathrm{non}}$. Let G be a finite group. If we write X^G as the subvariety of $\mathcal{M}_3^{\mathrm{non}}$ parametrizing curves with automorphism group containing X^G, then we have $X^{\mathbb{Z}/9} \subset X^{\mathbb{Z}/3} \subset \mathcal{M}_3^{\mathrm{non}}$ and $X^{\mathbb{Z}/6} \subset X^{\mathbb{Z}/3} \subset \mathcal{M}_3^{\mathrm{non}}$. In this point of view, we are trying to find the "defining equation" of $X^{\mathbb{Z}/3}$, $X^{\mathbb{Z}/6}$ and $X^{\mathbb{Z}/9}$ in $\mathcal{M}_3^{\mathrm{non}}$.

The explicit formulae of the Dixmier invariants are listed in Section 3.1. We use `Maxima` to compute the Dixmier invariants. Summarizing Section 3.2, we have the following.

Theorem 1.1. *The curve C_3 satisfies that*

$$I_3 = I_6 = I_{12} = I_{15} = 0,$$

and I_9, I_{18} are algebraically independent. For the curve C_6, the invariants I_9 and I_{18} satisfy a degree 8 affine equation. Furthermore, the curve C_9 is the curve on which all Dixmier invariants vanish.

The algebraic conditions between the invariants of C_6 are computed by `Macaulay2` [5].

We use the idea in [19] to compute the bitangents of a plane quartic. This program is also realized by `Macaulay2`. We summarize Section 4.2 as the following theorem.

Theorem 1.2. *The curve C_9 has all 28 explicit equations for the bitangents whose coefficients are radical expressions over $\mathbb{Q}$. The curve C_6 has 3 horizontal explicit bitangents which form a triple of asyzygetic sets.*

The definition of asyzygetic sets comes from the theory of theta characteristics. For details one can see [15]. Our definition in Section 4.1 is a geometric description as in [19].

For the linear matrix determinant representation problem of such curves, we use the idea in [20]. The problem asks whether the equation of a plane curve C could be written of the form

$$\det(xA + yB + zC)$$

for some symmetric matrices A, B, C of constants. Our result is Theorem 5.2 as follows:

Theorem 1.3. *The matrix representation of C_6 could be explicitly written over an extension field of $K(r,s) = \overline{\mathbb{Q}}(r,s)$ defined by a degree 6 polynomial $f(z) \in K(r,s)[z]$.*

2. Automorphisms of plane quartics

We consider the algebraic varieties over the algebraic closure $K = \overline{\mathbb{Q}}$ of the rational field $\mathbb{Q}$ in the complex numbers $\mathbb{C}$ since we are interested in the geometric properties of such varieties. However, some of the algorithms we use later in this work will be realized over $\mathbb{Q}$ only. In this section, let $K = \overline{\mathbb{Q}}$.

Let C be a smooth projective curve over K. If the genus $g(C)$ of C is 3, and C is non-hyperelliptic, then the canonical model of C is a plane quartic and is isomorphic to C. Let x, y, z be the coordinates of the projective plane $\mathbb{P}^2$. If we want to emphasize the coordinates, we also write $\mathbb{P}^2$ as $\mathbb{P}^2_{(x,y,z)}$. Let $k[x,y,z]_d$ be the homogeneous degree d-part of the polynomial ring $K[x,y,z]$. Thus $k[x,y,z]_d \simeq \mathrm{Sym}^d((K^\vee)^3)$, the 3rd symmetric product of $K^\vee = \mathrm{Hom}_K(K,K) \simeq K$. We write $\mathcal{P}^d_n := \mathrm{Sym}^d((K^\vee)^n)$. Thus, let $F_C = F_c(x,y,z)$ be the equation of C, we say both $F_C \in \mathcal{P}^4_3$ and $F_C \in K[x,y,z]_4$.

An element $F \in K[x,y,z]_4$ should be written as

$$F(x,y,z) = \sum_{i+j+k=4} a_{ijk} x^i y^j z^k.$$

Let C, D be two smooth non-hyperelliptic genus g curves over K. The canonical models κ_C, κ_D of C and D are closed subvarieties of degree $2g-2$ in $\mathbb{P}^{g-1}$. Since C and D are non-hyperelliptic, we have $C \simeq \kappa_C$ and $D \simeq \kappa_D$. The theory of algebraic curves says that C and D are isomorphic as algebraic varieties if and only if κ_C could be transformed to κ_D by a non-degenerated projective linear transformation on the coordinates of $\mathbb{P}^{g-1}$. In particular, an automorphism of a non-hyperelliptic curve C is a projective automorphism on the canonical model κ_C of C.

In this work we consider non-hyperelliptic genus 3 curves with cyclic automorphism groups $\mathbb{Z}/3$, $\mathbb{Z}/6$ and $\mathbb{Z}/9$.

The genus 3 non-hyperelliptic curves with $\mathbb{Z}/3$-automorphisms form a 2-dimensional family

$$C_3 = C_3(r,s): \quad y^3 = x(x-1)(x-r)(x-s).$$

This is a family of smooth quartics written on the affine chart $\{z=1\}$ of the projective plane $\mathbb{P}^2_{(x,y,z)}$ with K-parameters r and s.

Also we have the 1-dimensional family

$$C_6 = C_6(r): \quad y^3 = x(x-1)(x-r)(x-1+r)$$

of curves with automorphism group $\mathbb{Z}/6$ and the curve

$$C_9: \quad y^3 = x(x^3-1)$$

whose automorphsm group is $\mathbb{Z}/9$. Let ζ_n be the n-th root of unity in $\mathbb{C}$. According to [7] and [13], the action of $\mathbb{Z}/3$ on C_3 is given by the transformation $y \mapsto \zeta_3 \cdot y$. For C_6, the $\mathbb{Z}/6$-action is defined by $x \mapsto x - r$ and $y \mapsto \zeta_3 \cdot y$. For C_9, the $\mathbb{Z}/9$-action is given by $x \mapsto \zeta_3 \cdot x$ and $y \mapsto \zeta_9 \cdot y$.

In the following sections we will compute the invariants and bitangents of C_3, C_6 and C_9.

3. Dixmier invariants of C_3, C_6 and C_9

3.1. Dixmier invariants of plane quartics. First, we introduce some notation, following [4]. In general, let $f \in K[x_1, \ldots, x_n]$ be a polynomial, we use D_f to denote the differential operator determined by f. Explicitly, let

$$f = f(x_1, \ldots, x_n) = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}_+^n} a_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}, \tag{4}$$

where $a_{i_1,\ldots,i_n} \in K$ are coefficient of the monomial $x_1^{i_1} \cdots x_n^{i_n}$ for $(i_1, \ldots, i_n) \in \mathbb{Z}_+^n$ and (4) is a finite sum. For the rest of this paper, we will not emphasize that the powers $i_1, \ldots, i_n$ are non-negative integers again.

The map D_f means

$$\begin{aligned} D_f: \quad K[x_1, \ldots, x_n] \quad &\longrightarrow \quad K[x_1, \ldots, x_n] \\ g(x_1, \ldots, x_n) \quad &\longmapsto \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}_+^n} a_{i_1,\ldots,i_n} \frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} g(x_1, \ldots, x_n). \end{aligned}$$

If we use $D(f,g)$ to denote $D_f(g)$ for all $f, g \in K[x_1, \ldots, x_n]$, then the map

$$D: K[x_1, \ldots, x_n] \times K[x_1, \ldots, x_n] \longrightarrow K[x_1, \ldots, x_n]$$

has some obvious properties as follows:

- D is bilinear.
- Let $\deg(f)$ be the degree of f for all $f \in K[x_1, \ldots, x_n]$. Let $f, g \in K[x_1, \ldots, x_n]$. If $\deg(f) > \deg(g)$, then $D_f(g) = 0$. If $\deg(f) > \deg(g)$, then $D_f(g) \leq \deg(g) - \deg(f)$. Let $f = x_1^{i_1} \cdots x_n^{i_n}$ and

$g = x_1^{j_1} \cdots x_n^{j_n}$ be two monomials such that $\deg(f) = \deg(g)$, then $D_f(g) = i_1! \cdots i_n! \delta_{fg}$ where δ_{fg} is the Kronecker delta of f and g.

For any $f \in K[x_1, \ldots, x_n]$, let $H(f)$ be the half Hessian matrix of f. For example, if $f \in K[x, y, z]$, then

$$H(f) = \frac{1}{2} \cdot \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x \partial z} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial y \partial z} \\ \frac{\partial^2}{\partial x \partial z} & \frac{\partial^2}{\partial y \partial z} & \frac{\partial^2}{\partial z^2} \end{pmatrix}.$$

Let $H^*(f)$ be the adjoint matrix of $H(f)$.

Another notation is the dot product of two matrices. Let $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ be two $n \times n$ matrices. Then the dot product "$\langle\, , \rangle$" is defined by

$$\langle A, B \rangle := \sum_{1 \leq i,j \leq n} a_{ij} b_{ji}.$$

With these notations, we describe the Dixmier invariants of plane quartics.

Let $f, g \in K[x, y, z]_2$ be two quadratic homogeneous polynomials. Define

$$\begin{aligned} J_{1,1}(f, g) &= \langle H(f), H(g) \rangle, \\ J_{2,2}(f, g) &= \langle H^*(f), H^*(g) \rangle, \\ J_{3,0}(f, g) &= J_{3,0}(f) = \det(H(f)), \\ J_{0,3}(f, g) &= J_{0,3}(g) = \det(H(g)). \end{aligned}$$

Let $F \in K[x, y]_r$, $G \in K[x, y]_s$ be two homogeneous polynomials of degree r and s, respectively. For $k \leq \min\{r, s\}$, define $(F, G)^k$ as

$$\frac{(r-k)!(s-k)!}{r!s!} \left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2} \right)^k F(x_1, y_1) G(x_2, y_2) \Bigg|_{(x_i, y_i) = (x, y), i, 1, 2} \tag{5}$$

Let $P = P(x, y) \in K[x, y]_4$ be a quartic binary form. Let $Q = (P, P)^4$ defined as (5). Also we let

$$\begin{aligned} &\Sigma(P) = \frac{1}{2}(P, P)^4, \quad \Psi(P) = \frac{1}{6}(P, Q)^4 \\ &\Delta(P) = \Sigma(P)^3 - 27\Psi(P)^2 \end{aligned} \tag{6}$$

Then $\Delta(P)$ is the discriminant of P.

Let u, v be two K-variables. For quartic $f \in K[x, y, z]_4$, let

$$g = g(x, y) = f(x, y, -ux - vy).$$

Then $g(x, y)$ is a homogeneous polynomial of degree 4 with respect to the variables x and y, and the coefficients of g are expressions of u and v. Thus we can define $\Sigma(g)$ and $\Psi(g)$ as in (6). Since Σ and Ψ are expressions of the coefficients, we have $\Sigma(g)$ and $\Psi(g)$ are expressions of u and v. An explicit computation shows that $\Sigma(g)$ and $\Psi(g)$ are polynomials of degree 2 and 3 in the polynomial ring $K[u,v]$ respectively. Let $\sigma(u,v,w)$ and $\psi(u,v,w)$ be the homogenization of $\Sigma(g)$ for w, and $\psi(u,v,w)$ be the homogenization of $\Psi(g)$ for w. Then $\sigma(u,v,w) \in K[u,v,w]_2$ and $\psi(u,v,w) \in K[u,v,w]_3$. Finally, we substitute $u = x, v = y, w = z$ into $\sigma(u,v,w)$ and $\psi(u,v,w)$. For $f \in K[x,y,z]_4$, we define

$$\begin{aligned} \sigma(f) &= \sigma = \sigma(x,y,z) \in K[x,y,z]_2 \\ \psi(f) &= \psi = \psi(x,y,z) \in K[x,y,z]_3 \end{aligned} \tag{7}$$

Definition 3.1. *Let $f \in K[x,y,z]_4$, let σ, ψ defined as in (7). Let $\rho = D_f(\psi)$ and $\tau = D_\rho(f)$. The Dixmier invariants are defined as*

$$\begin{aligned} &I_3 = D_\sigma(f), \quad I_9 = J_{1,1}(\tau,\rho), \quad I_{15} = J_{3,0}(\tau), \\ &I_6 = D_\psi(H) - 8I_3^2, \quad I_{12} = J_{0,3}(\rho), \quad I_{18} = J_{2,2}(\tau,\rho) \\ &I_{27} = \Delta = \sigma^3 - 27\psi^2 \end{aligned} \tag{8}$$

3.2. The Dixmier invariants of C_3, C_6 and C_9. We use Maxima to compute the Dixmier invariants of C_3, C_6 and C_9. And we use elimination in Macaulay2 to compute the conditions of the invariants with certain automorphisms.

Proposition 3.2. *The Dixmier invariants of*

$$C_3(r,s) : y^3 = x(x-1)(x-r)(x-s)$$

are

$$I_3 = I_6 = I_{12} = I_{15} = 0$$

$$I_9 = -\frac{r^3 s^5}{55296} + \frac{r^2 s^5}{36864} + \frac{r s^5}{36864} - \frac{s^5}{55296} - \frac{77r^4 s^4}{331776} + \frac{169r^3 s^4}{331776} - \frac{97r^2 s^4}{110592} + \frac{169r s^4}{331776} - \frac{77s^4}{331776} - \frac{r^5 s^3}{55296} + \frac{169r^4 s^3}{331776} + \frac{r^3 s^3}{13824} + \frac{r^2 s^3}{13824} + \frac{169r s^3}{331776} - \frac{s^3}{55296} + \frac{r^5 s^2}{36864} - \frac{97r^4 s^2}{110592} + \frac{r^3 s^2}{13824} - \frac{97r^2 s^2}{110592} + \frac{r s^2}{36864} + \frac{r^5 s}{36864} + \frac{169r^4 s}{331776} + \frac{169r^3 s}{331776} + \frac{r^2 s}{36864} - \frac{r^5}{55296} - \frac{77r^4}{331776} - \frac{r^3}{55296}$$

$$I_{18} = \frac{r^6 s^{10}}{402653184} - \frac{r^5 s^{10}}{134217728} + \frac{r^4 s^{10}}{67108864} - \frac{7r^3 s^{10}}{402653184} + \frac{r^2 s^{10}}{67108864} - \frac{r s^{10}}{134217728} + \frac{s^{10}}{402653184} + \frac{r^7 s^9}{1358954496} - \frac{163r^6 s^9}{10871635968} + \frac{19r^5 s^9}{1207959552} - \frac{155r^4 s^9}{10871635968} - \frac{155r^3 s^9}{10871635968} + \frac{19r^2 s^9}{1207959552} - \frac{163r s^9}{10871635968} + \frac{s^9}{1358954496} + \frac{229r^8 s^8}{48922361856} - \frac{539r^7 s^8}{24461180928} + \frac{2711r^6 s^8}{24461180928} - \frac{13241r^5 s^8}{97844723712} + \frac{20231r^4 s^8}{97844723712} - \frac{13241r^3 s^8}{97844723712} + \frac{2711r^2 s^8}{24461180928} - \frac{539r s^8}{24461180928} + \frac{229s^8}{48922361856} + \frac{r^9 s^7}{1358954496} - \frac{539r^8 s^7}{24461180928} + \frac{1913r^7 s^7}{48922361856} - \frac{5927r^6 s^7}{32614907904} - \frac{1705r^5 s^7}{97844723712} - \frac{1705r^4 s^7}{97844723712} - \frac{5927r^3 s^7}{32614907904} + \frac{1913r^2 s^7}{48922361856} - \frac{539r s^7}{24461180928} + \frac{s^7}{1358954496} + \frac{r^{10} s^6}{402653184} - \frac{163r^9 s^6}{10871635968} + \frac{2711r^8 s^6}{24461180928} - \frac{5927r^7 s^6}{32614907904} + \frac{20383r^6 s^6}{32614907904} - \frac{35327r^5 s^6}{97844723712} + \frac{20383r^4 s^6}{32614907904} - \frac{5927r^3 s^6}{32614907904} + \frac{2711r^2 s^6}{24461180928} - \frac{163r s^6}{10871635968} + \frac{s^6}{402653184} - \frac{r^{10} s^5}{134217728} + \frac{19r^9 s^5}{1207959552} - \frac{13241r^8 s^5}{97844723712} - \frac{1705r^7 s^5}{97844723712} - \frac{35327r^6 s^5}{97844723712} - \frac{35327r^5 s^5}{97844723712} - \frac{1705r^4 s^5}{97844723712} - \frac{13241r^3 s^5}{97844723712} + \frac{19r^2 s^5}{1207959552} - \frac{r s^5}{134217728} + \frac{r^{10} s^4}{67108864} - \frac{155r^9 s^4}{10871635968} + \frac{20231r^8 s^4}{97844723712} - \frac{1705r^7 s^4}{97844723712} + \frac{20383r^6 s^4}{32614907904} - \frac{1705r^5 s^4}{97844723712} + \frac{20231r^4 s^4}{97844723712} - \frac{155r^3 s^4}{10871635968} + \frac{r^2 s^4}{67108864} - \frac{7r^{10} s^3}{402653184} - \frac{155r^9 s^3}{10871635968} -$$

$$\frac{13241r^8 s^3}{97844723712} - \frac{5927r^7 s^3}{32614907904} - \frac{5927r^6 s^3}{32614907904} - \frac{13241r^5 s^3}{97844723712} - \frac{155r^4 s^3}{10871635968} - \frac{7r^3 s^3}{402653184} + \frac{r^{10} s^2}{67108864} + \frac{19r^9 s^2}{1207959552} + \frac{2711r^8 s^2}{24461180928} + \frac{1913r^7 s^2}{48922361856} + \frac{2711r^6 s^2}{24461180928} + \frac{19r^5 s^2}{1207959552} + \frac{r^4 s^2}{67108864} - \frac{r^{10}s}{134217728} - \frac{163r^9 s}{10871635968} - \frac{539r^8 s}{24461180928} - \frac{539r^7 s}{24461180928} - \frac{163r^6 s}{10871635968} - \frac{r^5 s}{134217728} + \frac{r^{10}}{402653184} + \frac{r^9}{1358954496} + \frac{229r^8}{48922361856} + \frac{r^7}{1358954496} + \frac{r^6}{402653184}$$

The elimination of the ideal generated by I_9 and I_{18} with respect to r and s is the 0 ideal, which shows that I_9 and I_{18} are algebraically independent.

We can compute the invariants of C_6 by substitute $s = 1 - r$ into the invariants of C_3.

Proposition 3.3. *The Dixmier invariants of*

$$C_6(r) : y^3 = x(x-1)(x-r)(x-1+r)$$

are

$$I_3 = I_6 = I_{12} = I_{15} = 0$$

$$I_9 = -\frac{65r^8 - 260r^7 + 1150r^6 - 2540r^5 + 3959r^4 - 3988r^3 + 2326r^2 - 712r + 89}{331776}$$

$$I_{18} = \frac{25r^{16}}{3057647616} - \frac{25r^{15}}{382205952} + \frac{1325r^{14}}{3057647616} - \frac{1925r^{13}}{1019215872} + \frac{79229r^{12}}{12230590464} - \frac{105737r^{11}}{6115295232} + \frac{447307r^{10}}{12230590464} - \frac{31385r^9}{509607936} + \frac{998905r^8}{12230590464} - \frac{57233r^7}{679477248} + \frac{817465r^6}{12230590464} - \frac{123275r^5}{3057647616} + \frac{221939r^4}{12230590464} - \frac{1337r^3}{226492416} + \frac{16037r^2}{12230590464} - \frac{17r}{95551488} + \frac{17}{1528823808}$$

The elimination of the ideal generated by I_9 and I_{18} with respect to r is irreducible and generated by

$$4000000I_9^8 - 1998092052000I_9^7 - 676000000I_9^6 I_{18} - 715090537681178\,31I_9^6 + 224328787434000I_9^5 I_{18} + 428415000000I_9^4 I_{18}^2 - 3953613122539196273\,46I_9^5 + 846024860024321274\,0I_9^4 I_{18} - 837233565155325\,0I_9^3 I_{18}^2 - 1206702250000I_9^2 I_{18}^3 + 363921043179975076114\,65I_9^4 + 319148801927571534424\,92I_9^3 I_{18} - 332936970436116610650I_9^2 I_{18}^2 + 1038506377261275\,00I_9 I_{18}^3 + 12745792515625I_{18}^4 - 826890695963630262273456I_9^3 - 987543996424727566300344\,0I_9^2 I_{18} - 644187721569909674246640I_9 I_{18}^2 + 436275239454979198200\,0I_{18}^3 - 168880832609781468337056I_9^2 + 308264209077872446483720\,32I_9 I_{18} + 474410438868202394564990304I_{18}^2 + 254553912947483480448\,0I_9 + 693921318828231679754112\,0I_{18} + 960605665900794374400.$$

For C_9, we have

Proposition 3.4. *The Dixmier invariants of*

$$C_9 : y^3 = x(x^3 - 1)$$

are all zero.

4. The Bitangents of C_3, C_6 and C_9

4.1. The bitangents of plane quartics. The classical theory of plane quartics says that it has 28 bitangents. Recall that a line L is a **bitangent** of a plane curve C if it tangents C at two points p_1, p_2 where p_1 and p_2 could be coincide. Recall that a point is called an undulation point (see [21]) of a plane curve if a tangent line at that point meets the curve with multiplicity four or higher, this time the tangent line is called an undulation line of the curve. Thus, if p_1 and p_2 are coincide, then this point is an undulation point of C and L is an undulation line.

Explicitly, let $f = f(x,y,z) \in K[x,y,z]_4$ be the equation of a plane quartic C. Let $L : ax+by+cz = 0$, $a,b,c \in K$ be a line in $\mathbb{P}^2_{(x,y,z)}$. Thus the point $(a,b,c) \in \mathbb{P}^2_{(a,b,c)}$ determines the line L. Thus, in order to find all bitangents, we should consider all the affine charts $a \neq 0$, $b \neq 0$ and $c \neq 0$. For example, if we consider $c \neq 0$, and say $c = 1$. This time $L : ax+by+z = 0$ gives the condition $z = -ax-by$. Substitute this relation into $f(x,y,z)$ we have a quadratic form $f(x,y,-ax-by) \in R[x,y,z]_2$ where $R = K[a,b]$. If L is a bitangent for some $a,b \in K$, then there exist $\lambda_0, \lambda_1, \lambda_2 \in K$ such that

$$f(x,y,-ax-by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2. \tag{9}$$

The other two affine charts $a \neq 0$, $b \neq 0$ should be considered in a similar way to find bitangents of the form $ax+by = 0$. From now on let us consider the equation (9).

Definition 4.1. *For any quartic $f \in K[x,y,z]_4$, let $I(f)$ be the ideal of $K[a,b,\lambda_0,\lambda_1,\lambda_2]$ generated by comparing the coefficients of both sides of the monomials of x,y in the expansion of (9). Let $J(f)$ be elimination ideal of I with respect to $\lambda_0,\lambda_1,\lambda_2$ in $K[a,b]$.*

The ideal $J(f)$ gives the conditions of L being a bitangent of C. In general one cannot solve a,b over $\mathbb{Q}$, and even there exists L such that $a,b \in \mathbb{Q}$, the tangency points p_1, p_2 are not $\mathbb{Q}$-rational points of C.

There is a description of the relative positions of the bitangents of C. Let $L_1, \ldots, L_{28}$ be the bitangents of C, be careful that the number 28 counts the overlaps of the bitangents. Let L_i, L_j, L_k, where $i,j,k = 1,\ldots,28$ are distinct, be a triple of bitangents. For each L_ν, $\nu = 1,\ldots,28$, let p_{ν_1}, p_{ν_2} be the two tangency points of L_ν and C. Then L_i, L_j, L_k determine 6 points on C. Generically a plane conic is determined by 5 points.

Definition 4.2. *If the 6 points $p_{i_1}, p_{i_2}, p_{j_1}, p_{j_2}, p_{k_1}, p_{k_2}$ lie on a plane conic, then we say the triple L_i, L_j, L_k are* **syzygetic***, or else we say they are* **asyzygetic***.*

4.2. The Bitangents of C_3, C_6 and C_9. Before we use the computer to comply the algorithm above, let us observe an obvious bitangent of

$$C_3(r,s) : y^3 = x(x-1)(x-r)(x-s).$$

In the algorithm above, we considered the generic case on the affine chart $z \neq 0$. But if we expand C_3 and homogenize it with respect to z, then we have

$$F_3(r,s) : rsx\,z^3 - rs\,x^2\,z^2 - s\,x^2\,z^2 - r\,x^2\,z^2 + y^3 z + s\,x^3 z + r\,x^3 z + x^3 z - x^4. \tag{10}$$

Substitute $z = 0$ into (10) we get $x^4 = (x^2)^2$, which is a square. Thus $z = 0$ is a bitangent of C_3. To compute the tangent point, we observe that $x^2 = 0$ implies that $x = 0$. Substitute $x = 0, z = 0$ into (10) we get 0. This means that the intersection of C_3 and the line $z = 0$ is the point $(0, y, 0)$, or $(0, 1, 0) \in \mathbb{P}^2_{(x,y,z)}$. This is the only undulation point of C_3.

In [21], the invariants of a generic plane quartic is constructed in order to determine if it has an undulation point. The expression is the determinant of a 21×21 matrix. On the other hand, a quartic curve with homogeneous equation $F(x, y, z) = 0$ has an undualtion point if and only if it could be written as the form

$$F(x, y, z) = U_1(x, y, z)^4 + V_3(x, y, z)W_1(x, y, z)$$

where U_1 and W_1 are linear forms and V_3 is a cubic form. But according to (10), let $U_1 = x$, $W_1 = z$, and $V_3 = x(x - z)(x - rz)(x - sz) - x^4 - y^3$, then

$$F_3 = U_1^4 + V_3 W_1.$$

So $z = 0$ is an undulation line of C_3.

Beyond this undulation line, there are another 27 bitangents of C_3. Let $J(C_3)$ be the ideal defined as Definition 4.1. This time the coefficient list becomes $K[r, s]$, but we still can define $J(C_3)$ by the same analogos. We can compute the primary decomposition of $J(C_3)$ using Macaulay2. The inputs are as the following.

```
R = QQ[r,a,b,k_0,k_1,k_2][x,y,z]
f = -r^2*x*z^3+r*x*z^3+r^2*x^2*z^2-r*x^2*z^2-x^2*z^2+y^3*z
     +2*x^3*z-x^4
g = (k_0*x^2+k_1*x*y+k_2*y^2)^2
h = substitute(f,{z => -a*x-b*y})
H= h-g
Coe = coefficients H
L = flatten entries Coe#1
S = QQ[r,a,b,k_0,k_1,k_2]
I = ideal L
psi=map(S,R)
phi=map(R,S)
J = psi I
E=eliminate(J,{k_0,k_1,k_2})
T = QQ[r,a,b]
xi=map(T,S)
U = xi E
```

```
primaryDecomposition U
```

The primary decomposition of $J(C_3)$ has two components, one of them is the ideal $< a = 0, b = 0 >$, which gives the undulation line $z = 0$. Another component is irreducible in general. Let J' be this component, and let J'_a be the elimination of J' with respect to b. Then one can see that a satisfies the degree 9 equation

$$r^4s^4a^9 - 12r^4s^3a^7 - 12r^3s^4a^7 - 8r^4s^3a^6 - 8r^3s^4a^6 - 12r^3s^3a^7 - 8r^4s^2a^6 - 120r^3s^3a^6 - 8r^2s^4a^6 + 30r^4s^2a^5 - 156r^3s^3a^5 + 30r^2s^4a^5 - 8r^3s^2a^6 - 8r^2s^3a^6 + 48r^4s^2a^4 - 96r^3s^3a^4 + 48r^2s^4a^4 - 156r^3s^2a^5 - 156r^2s^3a^5 + 16r^4s^2a^3 - 32r^3s^3a^3 + 16r^2s^4a^3 + 48r^4sa^4 - 168r^3s^2a^4 - 168r^2s^3a^4 + 48rs^4a^4 + 30r^2s^2a^5 + 68r^4sa^3 - 68r^3s^2a^3 - 68r^2s^3a^3 + 68rs^4a^3 - 96r^3sa^4 - 168r^2s^2a^4 - 96rs^3a^4 + 24r^4sa^2 - 24r^3s^2a^2 - 24r^2s^3a^2 + 24rs^4a^2 + 16r^4a^3 - 68r^3sa^3 - 216r^2s^2a^3 - 68rs^3a^3 + 16s^4a^3 + 48r^2sa^4 + 48rs^2a^4 + 24r^4a^2 + 24r^3sa^2 - 96r^2s^2a^2 + 24rs^3a^2 + 24s^4a^2 - 32r^3a^3 - 68r^2sa^3 - 68rs^2a^3 - 32s^3a^3 + 9r^4a + 12r^3sa - 42r^2s^2a + 12rs^3a + 9s^4a - 24r^3a^2 - 96r^2sa^2 - 96rs^2a^2 - 24s^3a^2 + 16r^2a^3 + 68rsa^3 + 16s^2a^3 + 12r^3a - 12r^2sa - 12rs^2a + 12s^3a - 24r^2a^2 + 24rsa^2 - 24s^2a^2 + 8r^3 - 8r^2s - 8rs^2 + 8s^3 - 42r^2a - 12rsa - 42s^2a + 24ra^2 + 24sa^2 - 8r^2 + 16rs - 8s^2 + 12ra + 12sa - 8r - 8s + 9a + 8 = 0$$

which is able to be output by `Macaulay2`. This equation is irreducible over $\mathbb{Q}$. In the following cases, we try to find explicit bitangents for special cases of $C_3(r, s)$.

Theorem 4.3. *The curve*

$$C_9 : y^3 = x(x^3 - 1) \tag{11}$$

has all 28 explicit equations for the bitangents whose coefficients are radical expressions over $\mathbb{Q}$, *the group* $\mathbb{Z}/9$ *acts on the configuration of the bitangents.*

Proof. Let $J(C_9)$ be the ideal of $K[a, b]$ defined as Definition 4.1. Let J' be the component of $J(C_9)$ beyond $< a = 0, b = 0 >$. Let J'_a be the elimination of J' with respect to b. Then a satisfies the following equation.

$$a^9 - 96a^6 + 48a^3 + 64. \tag{12}$$

Let $u = a^3$, then u satisfies the cubic equation

$$u^3 - 96u^2 + 48u + 64. \tag{13}$$

This equation is solvable. For example, using Maxima, we have

$$u_1 = -\frac{(\sqrt{3}\,\mathrm{i}+1)\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{2}{3}} - 64\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}} - 112\cdot 3^{\frac{5}{2}}\,\mathrm{i}+1008}{2\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}}},$$

$$u_2 = \frac{(\sqrt{3}\,\mathrm{i}-1)\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{2}{3}} + 64\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}} - 112\cdot 3^{\frac{5}{2}}\,\mathrm{i}-1008}{2\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}}},$$

$$u_3 = \frac{\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{2}{3}} + 32\cdot\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}} + 1008}{\left(32\cdot 3^{\frac{7}{2}}\,\mathrm{i}+31968\right)^{\frac{1}{3}}}$$

Taking the cube root of each u_i we can get all 9 solutions of a.

Similarly we have an equation

$$b^{27} - 29496b^{18} + 401808b^9 - 64 = 0 \tag{14}$$

and let $v = b^9$ we have a cubic equation

$$v^3 - 29496v^2 + 401808v - 64 = 0.$$

This time one has to take the ninth root of all the three solutions v_i's , $i = 1, 2, 3$ of this equation. At the end, one has to judge which pairs (a, b) among the solutions give a bitangent $ax + by + z = 0$ of the original curve. We list the `Macaulay2` input as the following.

```
R = QQ[r,s,b,k_0,k_1,k_2][x,y,z]
f = r*s*x*z^3-r*s*x^2*z^2-s*x^2*z^2-r*x^2*z^2+y^3*z+s*x^3*z
    +r*x^3*z+x^3*z-x^4
g = (k_0*x^2+k_1*x*y+k_2*y^2)^2
h = substitute(f,{z => -b*y})
H= h-g
Coe = coefficients H
L = flatten entries Coe#1
S = QQ[r,s,b,k_0,k_1,k_2]
I = ideal L
psi=map(S,R)
phi=map(R,S)
J = psi I
E=eliminate(J,{k_0,k_1,k_2})
T = QQ[r,s,b]
xi=map(T,S)
U = xi E
primaryDecomposition U
```

The equations (12) and (14) contain terms of degree $3n$ and $9n$ for a and b, respectively. Thus if $ax+by+z = 0$ is a bitangent, so is $\zeta_3\cdot ax+\zeta_9\cdot by+z =$

0. But this means $a(\zeta_3 \cdot x) + b(\zeta_9 \cdot y) + z = 0$, which means this bitangent is in the orbit of the $\mathbb{Z}/9$-action. This means the group $\mathbb{Z}/9$ acts on the configuration of the bitangents. □

There is no canonical method to find explicit bitangents for special cases. Our observation is that we can try to find $r, s \in K$ such that the bitangent is "horizontal", that is, for those bitangents such that $a = 0$. The equation of the bitangent becomes $bx + z = 0$. Repeat the same idea in Section 4.1, we get the following result.

Theorem 4.4. *The family C_3 has a horizontal bitangent when $r-s = \pm 1$ or $r+s = 1$. In each of these cases, the slope b satisfies a cubic equation whose coefficients are polynomials of s, thus there are 3 horizontal bitangents.*

Proof. Let F_3 be the polynomial defined in (10). Generically, $a = 0$ is not a solution to the degree 9 equation of a. However, when $a = 0$, we have $L : by + z = 0$. Then $z = -by$. Using the same idea as in Section 4.1, we have the equation

$$F_3(x, y, -by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2. \tag{15}$$

Let $I(F_3)$ be the ideal of $R[b, \lambda_0, \lambda_1, \lambda_2]$ generated by comparing the coefficients of both sides of the monomials of x, y in the expansion of (15). Let $J(F_3)$ be elimination ideal of $I(F_3)$ with respect to $\lambda_0, \lambda_1, \lambda_2$ in $R[b]$. Then the primary decomposition of $J(F_3)$ as an ideal in $K[r, s, b]$ is

$$\begin{gathered}\langle b\rangle, \quad \langle r-s-1, s^2b^3-4\rangle \\ \langle r+s-1, s^4b^3-2s^3b^3+s^2b^3-4\rangle \\ \langle r-s+1, s^2b^3-2sb^3+b^3-4\rangle.\end{gathered} \tag{16}$$

□

The first ideal of (16) corresponds to the bitangent $z = 0$. The third ideal of (16) gives $r + s - 1 = 0$, which implies $s = r - 1$, this is the family C_6. Furthermore, we have a result on the positions of the horizontal bitangents of C_6.

Theorem 4.5. *The three horizontal bitangents of C_6 form an asyzygetic triple. Furthermore, the automprhism group $\mathbb{Z}/6$ acts on this asyzygetic triple.*

Proof. Let

$$F_6 : -r^2x\,z^3 + rx\,z^3 + r^2\,x^2\,z^2 - r\,x^2\,z^2 - x^2\,z^2 + y^3z + 2x^3z - x^4 \in R[x, y, z]_4$$

be the homogenization of C_6 with respect to z where $R = K[r]$. As before, we have the equation

$$F_6(x, y, -by) = (\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2)^2 \tag{17}$$

Let $I(F_6)$ be the ideal of $R[b, \lambda_0, \lambda_1, \lambda_2]$ generated by comparing the coefficients of both sides of the monomials of x, y in the expansion of (17). In

Theorem 4.4 we have proved that for C_6 the condition of being a horizontal bitangent for the line $bx + z = 0$ is given by the ideal

$$\langle\, r + s - 1,\ s^4 b^3 - 2s^3 b^3 + s^2 b^3 - 4 \,\rangle.$$

Substitute $s = 1-r$ into the second generator of this ideal, we have a relation

$$p(r, b) = b^3\, r^4 - 2b^3\, r^3 + b^3\, r^2 - 4$$

This time, let $\mathscr{J}(F_6)$ be intersection of the elimination ideal of $I(F_6)$ with respect to r, b in $K[\lambda_0, \lambda_1, \lambda_2]$ and the ideal $\langle p(r, b)\rangle$. Macaulay2 outputs

$$\mathscr{J}(F_6) = \langle\ \rangle,$$

which means that generically there is no conic $\lambda_0 x^2 + \lambda_1 xy + \lambda_2 y^2$ satisfies the conditions of passing through the 6 tangent points at the same time.

Consider the action $x \mapsto -x - r$ and $y \mapsto \zeta_3 \cdot y$ on C_6. The equation $p(r, b)$ only contains degree $3n$ terms, so as we have seen, the transformation $y \mapsto \zeta_3 \cdot y$ will transform a bitangent to another. On the other hand, since $a = 0$, a transformation $x \mapsto -x - r$ will fix a horizontal bitangent $z = -by$. Thus $\mathbb{Z}/6$ acts on the configuration of this asyzygetic triple. □

Remark 4.6. In general, there is another way to check whether 6 points lie on a common conic in $\mathbb{P}^2$. Let $p_i = (x_i, y_i, z_i) \in \mathbb{P}^2_{(x,y,z)}, i = 1, \ldots, 6$ be 6 points in the projective plane. Let $\mathbf{V}$ be the Veronese map

$$\begin{array}{cccc} \mathbf{V}: & \mathbb{P}^2_{(x,y,z)} & \longrightarrow & \mathbb{P}^5 \\ & (x, y, z) & \longmapsto & (x^2, y^2, z^2, xy, yz, zx) \end{array}.$$

If we regard $\mathbf{V}(p)$ as a row matrix for any $p = (x, y, z) \in \mathbb{P}^2$, then for the given 6 poins $p_1, \ldots, p_6$, we have a 6×6 matrix

$$V := \begin{pmatrix} \mathbf{V}(p_1) \\ \mathbf{V}(p_2) \\ \mathbf{V}(p_3) \\ \mathbf{V}(p_4) \\ \mathbf{V}(p_5) \\ \mathbf{V}(p_6) \end{pmatrix} = \begin{pmatrix} x_1^2 & y_1^2 & z_1^2 & x_1y_1 & y_1z_1 & z_1x_1 \\ x_2^2 & y_2^2 & z_2^2 & x_2y_2 & y_2z_2 & z_2x_2 \\ x_3^2 & y_3^2 & z_3^2 & x_3y_3 & y_3z_3 & z_3x_3 \\ x_4^2 & y_4^2 & z_4^2 & x_4y_4 & y_4z_4 & z_4x_4 \\ x_5^2 & y_5^2 & z_5^2 & x_5y_5 & y_5z_5 & z_5x_5 \\ x_6^2 & y_6^2 & z_6^2 & x_6y_6 & y_6z_6 & z_6x_6 \end{pmatrix}.$$

For our problem, let $p_1, \ldots, p_6$ be the 6 points of tangency of the three horizontal bitangents in Theorem 4.5. From the proof of Theorem 4.5 we see that there is a symbolic solution of these three bitangents, and since the algorithm of finding the points of tangency is essentially solving a quadratic equation, we can find the symbolic solutions of the points of tangency. But this algorithm costs too much for a popular processor. We can compute it in special values. For example, let $r = \frac{1}{8}$, we can compute the determinant using Maxima, the result is[1]

$$V = -\frac{\sqrt{-25\sqrt{3}\,\mathrm{i} - 25}\,\sqrt{25\sqrt{3}\,\mathrm{i} - 25}\,\left(120052^{\frac{10}{3}}\,3^{\frac{7}{2}}\,4^{\frac{2}{3}}\,\mathrm{i} + 3241352^{\frac{10}{3}}\,4^{\frac{2}{3}}\right)}{2^{\frac{247}{6}}\,\sqrt{3}\,\mathrm{i} + 2^{\frac{247}{6}}}$$

[1]This value could be simplified, we put the original result from Maxima.

which is not zero.

5. Discussion on the matrix representation problem

We discuss the matrix representation problem of the curves C_3 and C_6 using the idea in [20]. In order to coincide the notations with respect to [20], we exchange y and z, and write C_3 as

$$C_3: \quad z^3 = x(x-1)(x-r)(x-s).$$

Homogenize C_3 with respect to y we have

$$C_3: \quad f(x,y,z) := F_3(r,s) = x(x-y)(x-ry)(x-sy) - yz^3 = 0. \tag{18}$$

This time we have

$$f(x,0,0) = x^4 \quad \text{and} \quad f(x,y,0) = \prod_{i=1}^{4}(x+\beta_i y) \tag{19}$$

where $\beta_1 = 0, \beta_2 = -1, \beta_3 = -r, \beta_4 = -s$. The matrix representation problem for C_3 asks whether the polynomial $f(x,y,z)$ in (18) could be written of the form

$$f(x,y,z) = \det(xA + yB + zC)$$

where A, B, C are symmetric matrices. Here the entries of the matrices A, B and C belong to the algebraic closure of the rational function field $K(r,s)$. According to Section 2 in [20], if (19) holds, then one can assume that

$$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & -r & \\ & & & -s \end{pmatrix}, \ C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{24} \\ c_{13} & c_{23} & c_{33} & c_{34} \\ c_{14} & c_{24} & c_{34} & c_{44} \end{pmatrix}.$$

and we also have that

$$c_{ii} = \beta_i \cdot \frac{\frac{\partial f}{\partial z}(-\beta_i, 1, 0)}{\frac{\partial f}{\partial y}(-\beta_i, 1, 0)}, \quad i = 1,2,3,4. \tag{20}$$

But for (18) we have $\frac{\partial f}{\partial z} = -3yz^2$, which implies if $z = 0$, then $c_{ii} = 0$ for $i = 1,2,3,4$ by (20).

For convinience we denote

$$D = \begin{pmatrix} & c_{12} & c_{13} & c_{14} \\ & & c_{23} & c_{24} \\ & & & c_{34} \\ & & & \end{pmatrix} = \begin{pmatrix} & a & b & d \\ & & c & e \\ & & & f \\ & & & \end{pmatrix},$$

then $C = D + {}^{\mathtt{t}}D$ where ${}^{\mathtt{t}}D$ is the matrix transpose of D since $c_{ii} = 0$ for $i = 1,2,3,4$.

Using Maxima, we directly compute the coefficients of

$$\det(xA+yB+zC)=\det\begin{pmatrix} x & az & bz & dz \\ az & x-y & cz & ez \\ bz & cz & x-ry & fz \\ dz & ez & fz & x-sy \end{pmatrix}$$

and compare the coefficients with $f(x,y,z)$ in (18), the output is a system of equations

$$-c^2s-b^2s-a^2s-e^2r-d^2r-a^2r-f^2-d^2-b^2=0, \tag{21}$$
$$a^2rs+b^2s+d^2r=0, \tag{22}$$
$$2abcs+2ader+2bdf-1=0, \tag{23}$$
$$f^2+e^2+d^2+c^2+b^2+a^2=0, \tag{24}$$
$$-2cef-2bdf-2ade-2abc=0, \tag{25}$$
$$-a^2f^2+2abef+2acdf-b^2e^2+2bcde-c^2d^2=0. \tag{26}$$

We add the first equation with the fourth one, and rewrite the system of as 6 equations

$$a^2rs+b^2s+d^2r=0, \tag{27}$$
$$a^2(1-r)(s-1-s)+c^2(1-s)+e^2(1-r)=0 \tag{28}$$
$$2abcs+2ader+2bdf-1=0, \tag{29}$$
$$f^2+e^2+d^2+c^2+b^2+a^2=0, \tag{30}$$
$$cef+bdf+ade+abc=0, \tag{31}$$
$$a^2f^2-2af(be+cd)+(be-cd)^2=0. \tag{32}$$

of the 6 variables a,b,c,d,e,f.

It is too complicated to solve this entire system. Our computation are proceeded under the following principle:

- We only seek for one solution to the equation system (27)-(32), thus if there is an "either-or" argument in any step, we can choose one of them as our solution.

We eliminate a,f, and get a system of 4 equations with respect to the 4 variables b,c,d,e.

Proposition 5.1. *The equation system*

$$\frac{b^2}{r} + \frac{c^2}{r-1} + \frac{d^2}{s} + \frac{e^2}{s-1} = 0, \tag{33}$$

$$\frac{(b-e)^4}{be} = \frac{(c+d)^4}{cd}, \tag{34}$$

$$(bc+de)(bd+ce) = \left(2(\sqrt{be}+\sqrt{cd}) \cdot \begin{vmatrix} bcs+der & bd \\ bc(1-s)+de(1-r) & ce \end{vmatrix}\right)^2, \tag{35}$$

$$\frac{(bd+ce)^2+(bc+de)^2}{\begin{vmatrix} bcs+der & bd \\ bc(1-s)+de(1-r) & ce \end{vmatrix}^2} + (b^2+c^2+d^2+e^2) = 0 \tag{36}$$

with respect to the variables b, c, d, e give solutions to the equation system (27)-(32) where

$$a = \frac{bd+ce}{\begin{vmatrix} bcs+der & bd \\ bc(1-s)+de(1-r) & ce \end{vmatrix}}, \quad f = -\frac{bc+de}{\begin{vmatrix} bcs+der & bd \\ bc(1-s)+de(1-r) & ce \end{vmatrix}}. \tag{37}$$

Proof. First, the equation (33) is simply from $\frac{1}{rs}(27) - \frac{1}{(1-r)(1-s)}(28)$.

Next we regard a, f as unknowns and b, c, d, e, r, s as constants. The solution (37) is the solution to the linear system (29) and (31). From (31) we also have

$$\frac{a}{f} = -\frac{bd+ce}{bc+de}, \quad \frac{f}{a} = -\frac{bc+de}{bd+ce1}. \tag{38}$$

Substitute (38) into (30) we have (36).

Let $g = af$, then (32) becomes a quadratic equation

$$g^2 - 2(be+cd)g + (be-cd)^2 = 0$$

of g whose solution is

$$af = (\sqrt{be} \pm \sqrt{cd})^2$$

As before, for "$\pm$" we choose $+$, which is

$$af = (\sqrt{be} \pm \sqrt{cd})^2 \tag{39}$$

Substitute (37) into (39) we get (35).

Last, let us prove (34). The quadratic equation (30) and the linear equation (31) have an solution

$$a = -\frac{\sqrt{-1}(ce+bd)\sqrt{e^2+d^2+c^2+b^2}\sqrt{(d^2+c^2)\,e^2+4bcde+b^2\,d^2+b^2\,c^2}}{(d^2+c^2)\,e^2+4bcde+b^2\,d^2+b^2\,c^2},$$

$$f = \frac{\sqrt{-1}(bc+de)\sqrt{e^2+d^2+c^2+b^2}\sqrt{(d^2+c^2)\,e^2+4bcde+b^2\,d^2+b^2\,c^2}}{(d^2+c^2)\,e^2+4bcde+b^2\,d^2+b^2\,c^2}. \tag{40}$$

From (40) we have

$$af = \frac{P}{(d^2+c^2)\; e^2 + 4bcde + b^2\, d^2 + b^2\, c^2}$$

where the numerator P equals to minus the product of

$$cde^4 - 4bcde^3 + 6b^2cde^2 - bd^4e - 4bcd^3e - 6bc^2d^2e - 4bc^3de - 4b^3cde - bc^4e + b^4cd \quad (41)$$

and

$$cde^4 + 4bcde^3 + 6b^2cde^2 - bd^4e + 4bcd^3e - 6bc^2d^2e + 4bc^3de + 4b^3cde - bc^4e + b^4cd.$$

Dividing (41) by $bcde$ and regrouping the terms, we prove (34). □

As we reminded, it is hard to continue solving this equation system. Our observation is that for (34), we have an obvious solution

$$e = b, \quad \text{and} \quad d = -c. \tag{42}$$

From (24) we have $b^2 + d^2 + f^2 = -a^2 - c^2 - e^2$, thus we can rewrite (21) as

$$(a^2 + b^2 + d^2)r + (a^2 + b^2 + c^2)s = a^2 + c^2 + e^2.$$

Substitute (42) into this equation we have

$$r + s = 1$$

which means the curve C_3 becomes C_6 in this situation.

Next, we subsitude (42) into the equation system (27)-(32), then (31) is trivial, and (27) is the same as (28). We have a system of 4 equations

$$a^2rs + b^2s + c^2r = 0 \tag{43}$$

$$2abc(s-r) - 2bcf - 1 = 0 \tag{44}$$

$$a^2 + f^2 + 2(b^2 + c^2) = 0 \tag{45}$$

$$a^2f^2 - 2af(b^2 - c^2) + (b^2 + c^2)^2 = 0 \tag{46}$$

of the 4 variables a, b, c, f.

Theorem 5.2. *The matrix representation of C_6 could be explicitly written over an extension field of $K(r,s) = \overline{\mathbb{Q}}(r,s)$ defined by a degree 6 polynomial $f(z) \in K(r,s)[z]$.*

Proof. From (44) we have

$$(a(s-r) - f) = \frac{1}{2bc},$$

thus we have

$$a^2(s-r)^2 - 2af(s-r) + f^2 = \frac{1}{4b^2c^2}. \tag{47}$$

From (45) we have $f^2 = -2(b^2 + c^2) - a^2$, substitute it into (47) we have

$$a^2[(s-r)^2 - 1] - 2af(s-r) - 2(b^2 + c^2) = \frac{1}{4b^2c^2}. \tag{48}$$

From (43) we have

$$a^2 = -\frac{b^2}{r} - \frac{c^2}{s} \tag{49}$$

and from (46) we have

$$af = (b + \sqrt{-1}c)^2 \tag{50}$$

if we take one of the solutions of the quadratic equation with respect to af. Substitute them into (48), we have

$$4(b^2 s + c^2 r) - 2(s - r)(b + \sqrt{-1}c)^2 - 2(b^2 + c^2) = \frac{1}{4b^2c^2}. \tag{51}$$

This is a degree 6 equation with respect to b and c. Thus, if we know $q = b/c$, then the theorem is proved. From (45) and (49) we can solve

$$f^2 = -2(b^2 + c^2) + \frac{b^2}{r} + \frac{c^2}{s}. \tag{52}$$

The trivial equation

$$(af)^2 = a^2 \cdot f^2$$

implies that $(50)^2 = (49)\cdot(52)$, which is

$$(b + \sqrt{-1}c)^4 = \left(-\frac{b^2}{r} - \frac{c^2}{s}\right) \cdot \left(-2(b^2 + c^2) + \frac{b^2}{r} + \frac{c^2}{s}\right) \tag{53}$$

This equation is homogeneous of degree 4 with respect to b and c, thus if we set $q = b/c$, it will become a degree 4 equation of q, which is solvable. □

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(Dun Liang) School of Mathematics and Statistics, Hengyang Normal University, Hengyang, Hunan 421002, China
liangdun@hynu.edu.cn

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