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Hopf Galois structures on symmetric and alternating extensions

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Abstract. By using a recent theorem by Koch, Kohl, Truman and Underwood on normality, we determine that some types of Hopf Galois structures do not occur on Galois extensions with Galois group isomorphic to alternating or symmetric groups. Our theory of induced Hopf Galois structures allows us to obtain the whole picture of types of Hopf Galois structures on A_4-extensions, S_4-extensions, and S_5-extensions. Combining it with a result of Carnahan and Childs, we obtain a complete count of the Hopf Galois structures on S_5-extensions.

Contents

1. Introduction

A Hopf Galois structure on a finite extension of fields K/k is a pair $(\mathcal{H}, \mu)$, where $\mathcal{H}$ is a finite cocommutative k-Hopf algebra and μ is a Hopf action of $\mathcal{H}$ on K, i.e a k-linear map $\mu : \mathcal{H} \to \mathrm{End}_k(K)$ giving K a left $\mathcal{H}$-module algebra structure and inducing a bijection $K \otimes_k \mathcal{H} \to \mathrm{End}_k(K)$. Hopf Galois structures were introduced by Chase and Sweedler in [5].

In Hopf Galois theory one has the following Galois correspondence Theorem.

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Theorem 1 ([5] Theorem 7.6)**.** *Let $(\mathcal{H}, \mu)$ be a Hopf Galois structure on the field extension L/K. For a K-sub-Hopf algebra $\mathcal{H}'$ of $\mathcal{H}$ we define*

$$L^{\mathcal{H}'} = \{x \in L \mid \mu(h)(x) = \varepsilon(h) \cdot x \text{ for all } h \in \mathcal{H}'\},$$

where ε is the counity of $\mathcal{H}$. Then, $L^{\mathcal{H}'}$ is a subfield of L, containing K, and

$$\begin{array}{rcl} \mathcal{F}_{\mathcal{H}} : \{\mathcal{H}' \subseteq \mathcal{H} \text{ sub-Hopf algebra}\} & \longrightarrow & \{\text{Fields } E \mid K \subseteq E \subseteq L\} \\ \mathcal{H}' & \to & L^{\mathcal{H}'} \end{array}$$

is injective and inclusion reversing.

For separable field extensions, Greither and Pareigis [13] give the following group-theoretic equivalent condition to the existence of a Hopf Galois structure.

Theorem 2. *Let K/k be a separable field extension of degree n, $\widetilde{K}$ its Galois closure, $G = \mathrm{Gal}(\widetilde{K}/k), G' = \mathrm{Gal}(\widetilde{K}/K)$. Then there is a bijective correspondence between the set of Hopf Galois structures on K/k and the set of regular subgroups N of the symmetric group $\mathrm{Sym}(G/G')$ normalized by $\lambda(G)$, where $\lambda : G \to S_n$ is the morphism given by the action of G on the left cosets G/G'.*

For a given Hopf Galois structure on K/k, we will refer to the isomorphism class of the corresponding group N as the type of the Hopf Galois structure. The Hopf algebra $\mathcal{H}$ corresponding to a regular subgroup N of $\mathrm{Sym}(G/G')$ normalized by $\lambda(G)$ is the Hopf subalgebra $\widetilde{K}[N]^G$ of the group algebra $\widetilde{K}[N]$ fixed under the action of G, where G acts on $\widetilde{K}$ by k-automorphisms and on N by conjugation through λ. It is known that the Hopf subalgebras of $\widetilde{K}[N]^G$ are in 1-to-1 correspondence with the subgroups of N stable under the action of G (see e.g. [8] Proposition 2.2). For N' a G-stable subgroup of N, we will denote by $K^{N'}$ the subfield $K^{\mathcal{H}'}$ of K fixed by the Hopf subalgebra $\mathcal{H}'$ of $\mathcal{H}$ corresponding to N' and refer to it as fixed by N'.

Childs [6] gives an equivalent condition to the existence of a Hopf Galois structure introducing the holomorph of the regular subgroup N of $\mathrm{Sym}(G/G')$. We state the more precise formulation of this result due to Byott [1] (see also [7] Theorem 7.3).

Theorem 3. *Let G be a finite group, $G' \subset G$ a subgroup and $\lambda : G \to \mathrm{Sym}(G/G')$ the morphism given by the action of G on the left cosets G/G'. Let N be a group of order $[G : G']$ with identity element e_N. Then there is a bijection between*

$$\mathcal{N} = \{\alpha : N \hookrightarrow \mathrm{Sym}(G/G') \text{ such that } \alpha(N) \text{ is regular}\}$$

and

$$\mathcal{G} = \{\beta : G \hookrightarrow \mathrm{Sym}(N) \text{ such that } \beta(G') \text{ is the stabilizer of } e_N\}.$$

Under this bijection, if $\alpha \in \mathcal{N}$ corresponds to $\beta \in \mathcal{G}$, then $\alpha(N)$ is normalized by $\lambda(G)$ if and only if $\beta(G)$ is contained in the holomorph $\mathrm{Hol}(N)$ of N.

In this paper, we consider a Galois extension K/k with Galois group G equal to the symmetric group S_n or the alternating group A_n, $n \geq 4$. We prove in Proposition 4 that if $G = A_4$, the types of Hopf Galois structures on K/k are precisely A_4 and $C_3 \times V_4$ and that, if $G = S_4$, the types of Hopf Galois structures on K/k are precisely S_4 and the split ones $A_4 \times C_2, S_3 \times V_4$ and $C_6 \times V_4$. We prove in Proposition 5 that if $G = A_5$, the only type of Hopf Galois structures on K/k is A_5, which is a particular case of a result of Byott, and that if $G = S_5$, the types of Hopf Galois structures on K/k are precisely S_5 and the split one $A_5 \times C_2$. Together with the results in [7] §10, this provides a complete count of the Hopf Galois structures on Galois extensions with Galois group S_5. Finally we prove in Proposition 7 that a Galois extension with Galois group S_n or A_n, where $n \geq 5$, has no Hopf Galois structures of cyclic type.

2. Main results

We will apply Theorem 2.9 in [14] to prove the nonappearance of some types of Hopf Galois structures on Galois extensions with given Galois group. The setting will be the following. Let G be a group of order n and G' a subgroup of G of index d such that no nontrivial subgroup of G' is normal in G. Let N be a group of order equal to n having a unique conjugation class of subgroups of index d with length 1. With these hypothesis, if K/k is a Galois extension with Galois group G and $F := K^{G'}$, then if K/k had a Hopf Galois structure of type N, we would have $F := K^{N'}$, for N' the normal subgroup of N of index d. If we know that a separable extension of degree d having normal closure with Galois group G has no Hopf Galois structure of type N/N', we may conclude that a Galois extension with Galois group G has no Hopf Galois structure of type N. We will use Theorem 3 in [10] to prove that a certain type of Hopf Galois structure does occur on a Galois extension with given Galois group. Let K/k be a Galois extension with Galois group $G = H \rtimes G'$ and let $F := K^{G'}$. Then if F/k has a Hopf Galois structure of type N_1 and K/F has a Hopf Galois structure of type N_2, the extension K/k has a Hopf Galois structure of type $N_1 \times N_2$.

2.1. Galois extensions with Galois group A_4 or S_4. Let us denote by D_{2n} the dihedral group of order $2n$ and by Dic_n the dicyclic group of order $4n$, that is,

$$\begin{aligned} D_{2n} &= \langle r, s | r^n = 1, s^2 = 1, srs = r^{-1} \rangle, \\ Dic_n &= \langle a, x | a^{2n} = 1, x^2 = a^n, xax^{-1} = a^{-1} \rangle. \end{aligned}$$

Let us assume that K/k is Galois with group $G = A_4$, the alternating group. We analyze the five possible types of Hopf Galois structures: the alternating group A_4, the dicyclic group $Dic_3 = C_3 \rtimes C_4$, the cyclic group $C_{12} = C_3 \times C_4$, the dihedral group $D_{12} = C_3 \rtimes V_4$ and the direct product $C_3 \times V_4$.

The classical Galois structure gives a Hopf Galois structure of type A_4. On the other hand, since $A_4 = V_4 \rtimes C_3$, a quartic extension having Galois closure A_4 is Hopf Galois of type V_4, hence, by [10], Theorem 3, we get induced Hopf Galois structures of type $C_3 \times V_4$. Finally, since $\mathrm{Hol}(Dic_3) = \mathrm{Hol}(D_{12})$ (see [15], Proposition 2.1) either both types of Hopf Galois structures arise or none of them does. We are left with cyclic and dicyclic types.

In both cases, we consider N' the cyclic subgroup of order 3, the 3-Sylow subgroup, and we have $N/N' \simeq C_4$. Then, the corresponding fixed field F gives a quartic extension with Galois closure K. Since $\mathrm{Hol}(C_4)$ has order 8, it cannot contain G, and this extension F/k cannot have Hopf Galois structures of type C_4. This proves that K/k has neither cyclic nor dicyclic (or dihedral) Hopf Galois structures.

Now let us assume that K/k is Galois with group $G = S_4$, the symmetric group. There are 15 isomorphism classes of groups of order 24. Hence there are 15 possible types for Hopf Galois structures on K/k. For a group of order 24, the number n_3 of 3-Sylow subgroups may be 1 or 4. If $n_3 = 1$, the group is a semi-direct product $C_3 \rtimes S$, where S is a group of order 8, i.e. $S = C_8, C_4 \times C_2, E_8 = C_2 \times C_2 \times C_2$, the dihedral group D_8 or the quaternion group Q_8. There are 12 groups of order 24 with $n_3 = 1$. These are precisely $C_3 \rtimes C_8, C_{24} = C_3 \times C_8, S_3 \times C_4 = C_3 \rtimes (C_2 \times C_4), Dic_3 \times C_2 = C_3 \rtimes (C_2 \times C_4), S_3 \times V_4 = C_3 \rtimes (C_2 \times C_2 \times C_2), C_6 \times V_4 = C_3 \times (C_2 \times C_2 \times C_2), D_{24} = C_3 \rtimes D_8, C_3 \rtimes_\varphi D_8$, where $\varphi : D_8 \to \mathrm{Aut}\, C_3$ has kernel $C_2 \times C_2$, $C_3 \times D_8, Dic_6 = C_3 \rtimes Q_8, C_3 \times Q_8$. We have $n_3 = 4$ for $S_4, SL(2,3)$ and $A_4 \times C_2$.

Let us consider an intermediate field F for the extension K/k such that $[F : k] = 8$. Then F/k has Galois closure K and, as a transitive group of degree 8, S_4 is the group 8T14. Hence Table 1 in [12] shows that F/k has only Hopf Galois structures of type $E_8 = C_2 \times C_2 \times C_2$. By [14] Theorem 2.9, K/k has no Galois structures of type N if N has a unique subgroup N' of order 3 (then normal and G-stable) such that N/N' is not isomorphic to E_8. This is the case for $N = C_3 \rtimes C_8, C_{24}, S_3 \times C_4, Dic_3 \times C_2, D_{24}, C_3 \rtimes_\varphi D_8, C_3 \times D_8, Dic_6, C_3 \times Q_8$.

Let us consider now the subfield F of K fixed by a transposition of S_4. Since A_4 is a normal complement of $\mathrm{Gal}(K/F)$ in S_4, the extension F/k has a Hopf Galois structure of type A_4, hence by [10], Theorem 3, K/k has an induced Hopf Galois structure of type $A_4 \times C_2$. Let us now take F to be the subfield of K fixed by a subgroup of S_4 isomorphic to S_3. Then K/F is Galois with group S_3 and has a Hopf Galois structure of type C_6 (see degree 6 table in [11]). Now F/k has a Hopf Galois structure of type $C_2 \times C_2$, since V_4 is a normal complement of S_3 in S_4 (see [13] Theorem 4.6). Hence we obtain, again by [10], Theorem 3 and taking into account $S_4 = V_4 \rtimes S_3$, that K/k has induced Hopf Galois structures of types $S_3 \times V_4$ and $C_6 \times V_4$. Finally, we check, using Magma, that $\mathrm{Hol}(\mathrm{SL}(2,3))$ has no subgroup isomorphic to S_4, hence K/k has no Galois structure of type $\mathrm{SL}(2,3)$.

We have obtained the following result.

Proposition 4. *Let K/k be a Galois extension with Galois group A_4. Then, the only types of Hopf Galois structures on K/k are A_4 and $C_3 \times V_4$. The classical Galois structure realizes type A_4 and a Hopf Galois structure of type $C_3 \times V_4$ is induced by the classical Galois structure on K/F and the Hopf Galois structure of type V_4 on F/k for F an intermediate field with $[K : F] = 3$.*

Let K/k be a Galois extension with Galois group S_4. Then, the only types of Hopf Galois structures on K/k are S_4 and the split ones $A_4 \times C_2, S_3 \times V_4$ and $C_6 \times V_4$. The classical Galois structure realizes the first type and the remaining three are realized as induced structures.

2.2. Galois extensions with Galois group A_5 or S_5. Let us assume that K/k is Galois with Galois group $G = A_5$, the alternating group. There are 13 possible types of Hopf Galois structures. If we take $N \not\simeq A_5$ a group of order 60, then N has a unique 5−Sylow subgroup that we can take for N'. Since none of the groups of order 12 has holomorph of order divisible by 60, we know that $K^{N'}/k$ is not a Hopf Galois extension and therefore Theorem 2.9 in [14] implies than N is not a Hopf Galois type for K/k.

Let us assume that K/k is Galois with group $G = S_5$, the symmetric group. Now there are 47 possible types of Hopf Galois structures.

The classical Galois structure gives a Hopf Galois structure of type S_5. On the other hand, since $S_5 = A_5 \rtimes C_2$, again by [10], Theorem 3, we get induced Hopf Galois structures of type $A_5 \times C_2$. (Recall that an extension F/k of degree 60 with Galois closure K/k has an almost classical Hopf Galois structure of type A_5, since A_5 is a normal complement of $\mathrm{Gal}(K/F)$ in $\mathrm{Gal}(K/k) \simeq S_5$.)

Checking on the remaining 45 types, we see that all but one, namely $N = \mathrm{SL}(2, 5)$, have a normal p−Sylow subgroup. For a given $N \not\simeq \mathrm{SL}(2, 5)$, we choose N' a normal p−Sylow subgroup. Therefore, N' is a normal G-stable subgroup of N. On the other hand, the fixed field $F = K^{N'}$ provides an extension F/k with Galois closure K/k (K/F has degree 8, 5 or 3 and S_5 has no nontrivial normal subgroup of order dividing any of these numbers). In the proofs of Propositions 3.2 and 4.11 in [9], mostly arguing on solvability of holomorphs of groups of order 15, 24 and 40, respectively, we proved that F/k is not Hopf Galois. In this way, Theorem 2.9 in [14] rules out all these 44 Hopf Galois types. We perform a computation with Magma to check that the holomorph of $\mathrm{SL}(2, 5)$ does not contain S_5 as a transitive subgroup and then we have the following result.

Proposition 5. *Let K/k be a Galois extension with Galois group A_5. Then, the only type of Hopf Galois structures on K/k is A_5. The classical Galois structure realizes this type.*

Let K/k be a Galois extension with Galois group $S_5 = A_5 \rtimes C_2$. Then, the only types of Hopf Galois structures on K/k are S_5 and the split one

$A_5 \times C_2$. The classical Galois structure realizes the first type and the second type is realized as the induced Hopf Galois structure by an almost classical Hopf Galois structure on $K^{<\tau>}/k$, where τ denotes a transposition in S_5.

In [4] (see also [7] §10), the authors compute the number of Hopf Galois structures of types S_n and $A_n \times C_2$ on a Galois extension with Galois group S_n. They obtain that those of type S_n amount to twice the number of even permutations in S_n of order at most 2 and those of type $A_n \times C_2$ amount to twice the number of odd permutations in S_n of order 2. In S_5, there are 16 even permutations of order at most 2 (15 of the form $(ab)(cd)$, plus the identity) and 10 odd permutations of order 2 (of the form (ab)). Together with Proposition 5 this implies the following corollary which gives the total count of Hopf Galois structures on a Galois extension K/k with Galois group S_5.

Corollary 6. *A Galois field extension with Galois group S_5 has precisely 52 Hopf Galois structures: 32 of type S_5 and 20 of type $A_5 \times C_2$.*

2.3. Galois extensions with Galois group A_n or S_n, $n \geq 5$. Let K/k be a Galois extension with Galois group $G = S_n$ or A_n, $n \geq 5$. Let us assume that K/k is Hopf Galois of cyclic type. If N is a cyclic group corresponding to this Hopf Galois structure, its holomorph $\mathrm{Hol}(N)$ is a solvable group, hence cannot have a subgroup isomorphic to A_n or S_n, for $n \geq 5$, and Theorem 3 implies that K/k cannot be Hopf Galois of cyclic type. We have then obtained the following result.

Proposition 7. *Let K/k be a Galois extension with Galois group S_n or A_n, where $n \geq 5$. Then K/k has no Hopf Galois structures of cyclic type.*

Let us note that the results for the alternating group in sections 2.2 and 2.3 are special cases of Byott's main result in [2] where the author proves that a Galois extension K/k with Galois group a non-abelian simple group G has exactly two Hopf Galois structures: the Galois one and the classical non-Galois one. Propositions 5 and 7 support a query of Byott in [3] where he states that we do not have any examples where an extension with nonsolvable Galois group admits a Hopf Galois structure of solvable type.

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