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Equivalences from tilting theory and commutative algebra from the adjoint functor point of view

Olgur Celikbas and Henrik Holm

Abstract. We give a category theoretic approach to several known equivalences from (classic) tilting theory and commutative algebra. Furthermore, we apply our main results to establish a duality theory for relative Cohen–Macaulay modules in the sense of Hellus, Schenzel, and Zargar.

Contents

1. Introduction

In this paper, we consider an adjunction $\mathrm{F}\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon \mathrm{G}$ between abelian categories. Even though the pair $(\mathrm{L}_\ell\mathrm{F}, \mathrm{R}^\ell\mathrm{G})$ of $\ell^{\,\mathrm{th}}$ (left/right) derived functors is generally not an adjunction $\mathcal{A} \rightleftarrows \mathcal{B}$, one can obtain an adjunction, and even an adjoint equivalence, from these functors by restricting them appropriately. More precisely, in Definition 3.7 we introduce two subcategories $\mathsf{Fix}_\ell(\mathcal{A})$, the category of *$\ell$-fixed* objects in $\mathcal{A}$, and $\mathsf{coFix}_\ell(\mathcal{B})$, the category of

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ℓ-cofixed objects in $\mathcal{B}$, and show in Theorem 3.8 that one gets an adjoint equivalence:

$$\mathsf{Fix}_{\ell}(\mathcal{A}) \underset{\mathrm{R}^{\ell}\mathrm{G}}{\overset{\mathrm{L}_{\ell}\mathrm{F}}{\rightleftarrows}} \mathsf{coFix}_{\ell}(\mathcal{B})\,. \tag{1}$$

When the adjunction (F,G) is suitably nice—more precisely, when it is a *tilting adjunction* in the sense of Definition 3.11—the adjoint equivalence (1) takes the simpler form:

$$\{A \in \mathcal{A} \mid \mathrm{L}_i\mathrm{F}(A) = 0 \text{ for } i \neq \ell\} \underset{\mathrm{R}^{\ell}\mathrm{G}}{\overset{\mathrm{L}_{\ell}\mathrm{F}}{\rightleftarrows}} \{B \in \mathcal{B} \mid \mathrm{R}^i\mathrm{G}(B) = 0 \text{ for } i \neq \ell\}\,, \tag{2}$$

as shown in Theorem 3.14. These equivalences, which are our main results, are proved in Section 3. In Section 4 we apply them to various situations and recover a number of known results from tilting theory and commutative algebra, such as the Brenner–Butler and Happel theorem [5, 17], Wakamatsu's duality [34], and Foxby equivalence [4, 11]. Details can be found in Corollaries 4.2, 4.3, and 4.4.

In Section 5 we investigate the equivalence (1) further in the special case where $\ell = 0$. Under suitable hypotheses, we show in Theorem 5.8 that for any $X \in \mathsf{Fix}_0(\mathcal{A})$ and $d \geqslant 0$, (1) restricts to an equivalence:

$$\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{gen}_d^{\mathcal{A}}(X) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{coFix}_0(\mathcal{B}) \cap \mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)\,, \tag{3}$$

where $\mathsf{gen}_d^{\mathcal{A}}(X)$ is the full subcategory of $\mathcal{A}$ consisting of objects that are finitely built from X in the sense of Definition 5.1. Although (3) looks more technical than (1) and (2), it too has useful applications, for example, it contains as a special case Matlis duality [23]:

$$\{\text{Finitely generated } R\text{-modules}\} \underset{\mathrm{Hom}_R(-,E_R(k))}{\overset{\mathrm{Hom}_R(-,E_R(k))}{\rightleftarrows}} \{\text{Artinian } R\text{-modules}\}\,,$$

where R is a commutative noetherian local complete ring; see Corollary 5.9. Theorem 5.10 is a variant of (3) which yields Sharp's equivalence [28] for finitely generated modules of finite projective/injective dimension over Cohen–Macaulay rings; see Corollary 5.11.

In Section 6 we apply the equivalence (1) to study relative Cohen–Macaulay modules. To explain what this is about, recall that for a (nonzero) finitely generated module M over a commutative noetherian local ring $(R,\mathfrak{m},k)$, which we assume is complete, one has

$$\mathrm{depth}_R M = \min\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\} \quad \text{and} \quad \dim_R M = \max\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\},$$

where $\mathrm{H}^i_{\mathfrak{m}}$ denotes the i^{th} local cohomology module w.r.t. $\mathfrak{m}$. Hence M is Cohen–Macaulay (CM) of dimension t if and only if $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i \neq t$.

When R itself is CM, the most important and useful fact about the category of t-dimensional CM modules is the duality

$$\left\{ M \in \mathsf{mod}(R) \,\middle|\, \begin{array}{l} \mathrm{H}^i_{\mathfrak{m}}(M) = 0 \\ \text{for all } i \neq t \end{array} \right\} \underset{\mathrm{Ext}^{c-t}_R(-,\Omega)}{\overset{\mathrm{Ext}^{c-t}_R(-,\Omega)}{\rightleftarrows}} \left\{ M \in \mathsf{mod}(R) \,\middle|\, \begin{array}{l} \mathrm{H}^i_{\mathfrak{m}}(M) = 0 \\ \text{for all } i \neq t \end{array} \right\},$$

where c is the Krull dimension of R and Ω is the dualizing module. The theory of CM modules over CM rings is an active research area and in recent papers by, e.g., Hellus and Schenzel [20] and Zargar [35], it was suggested to investigate this theory relative to an ideal $\mathfrak{a} \subset R$. That is, in the case where R is *relative CM w.r.t.* $\mathfrak{a}$, meaning that $\mathrm{H}^i_{\mathfrak{a}}(R) = 0$ for $i \neq c$ where $\mathrm{depth}_R(\mathfrak{a}, R) = c = \mathrm{cd}_R(\mathfrak{a}, R)$, one wishes to study the category

$$\{M \in \mathsf{mod}(R) \,|\, \mathrm{H}^i_{\mathfrak{a}}(M) = 0 \text{ for all } i \neq t\} \qquad \text{(for any } t\text{)} \tag{4}$$

of finitely generated *relative CM R-modules of cohomological dimension t w.r.t.* $\mathfrak{a}$. Towards a relative CM theory, the first thing one should start looking for is a duality on the category (4). Unfortunately such a duality does not exist in general; indeed for $\mathfrak{a} = 0$ (the zero ideal) and $t = 0$ the category in (4) is the category $\mathsf{mod}(R)$ of all finitely generated R-modules, which is self-dual only in very special cases (if R is Artinian). To fix this problem, we introduce in Definition 6.7 another category, $\mathsf{CM}^t_{\mathfrak{a}}(R)$, of (not necessarily finitely generated) R-modules; it is an extension of the category (4) in the sense that:

$$\mathsf{CM}^t_{\mathfrak{a}}(R) \cap \mathsf{mod}(R) \;=\; \{M \in \mathsf{mod}(R) \,|\, \mathrm{H}^i_{\mathfrak{a}}(M) = 0 \text{ for all } i \neq t\}.$$

Our main result about this (larger) category is that it is self-dual. We show in Theorem 6.16 that if R is relative CM w.r.t. $\mathfrak{a}$ with

$$\mathrm{depth}_R(\mathfrak{a}, R) = c = \mathrm{cd}_R(\mathfrak{a}, R),$$

then there is a duality:

$$\mathsf{CM}^t_{\mathfrak{a}}(R) \underset{\mathrm{Ext}^{c-t}_R(-,\Omega_{\mathfrak{a}})}{\overset{\mathrm{Ext}^{c-t}_R(-,\Omega_{\mathfrak{a}})}{\rightleftarrows}} \mathsf{CM}^t_{\mathfrak{a}}(R)\,, \tag{5}$$

where $\Omega_{\mathfrak{a}}$ is the module from Definition 6.13. It is worth pointing out two extreme cases of this duality: For $\mathfrak{a} = \mathfrak{m}$ a ring is relative CM w.r.t. $\mathfrak{a}$ if and only if it is CM in the ordinary sense, and in this case c is the Krull dimension of R and $\Omega_{\mathfrak{a}} = \Omega$ is a dualizing module; see Example 6.14. Thus (5) extends the classic duality for CM modules of Krull dimension t mentioned above. For $\mathfrak{a} = 0$ any ring is relative CM w.r.t. $\mathfrak{a}$, and (5) specializes, in view of Examples 6.9 and 6.14, to the (well-known and almost trivial) duality:

$$\{\text{Matlis reflexive } R\text{-modules}\} \underset{\mathrm{Hom}_R(-,E_R(k))}{\overset{\mathrm{Hom}_R(-,E_R(k))}{\rightleftarrows}} \{\text{Matlis reflexive } R\text{-modules}\}\,.$$

Hence (5) is a family of dualities, parameterized by ideals $\mathfrak{a} \subset R$, that connects the known dualities for (classic) CM modules and Matlis reflexive modules.

We end this introduction by explaining how our work is related to the literature:

For $\ell = 0$ the equivalence (1) follows from Frankild and Jørgensen [13, Thm. (1.1)] as $(\mathrm{L}_0\mathrm{F}, \mathrm{R}^0\mathrm{G}) = (\mathrm{F}, \mathrm{G})$ is an adjunction $\mathcal{A} \rightleftarrows \mathcal{B}$ to begin with. For $\ell > 0$ it requires some more work as the pair $(\mathrm{L}_\ell\mathrm{F}, \mathrm{R}^\ell\mathrm{G})$ is not an adjunction. Nevertheless, having made the necessary preparations, the proof of the adjoint equivalence (1) is completely formal.

The idea of reproving and extending known equivalences/dualities from commutative algebra via an abstract approach, like we do, is certainly not new. In fact, this is the main idea in, for example, [13, 14] by Frankild and Jørgensen, however, these papers focus on the derived category setting, whereas we are interested in the the abelian category setting.

Concerning our work on relative CM modules in Section 6: The duality (5) is new but related results, again in the derived category setting, can be found in [14], Porta, Shaul, and Yekutieli [26, Sect. 7], and Vyas and Yekutieli [32, Sect. 8] (MGM equivalence).

2. Preliminaries and technical lemmas

For an abelian category $\mathcal{A}$, we write $\mathsf{K}(\mathcal{A})$ for its homotopy category.

2.1. *A chain map $\alpha\colon X \to Y$ between complexes X and Y in an abelian category is called a* quasi-isomorphism *if $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism for every $n \in \mathbb{Z}$.*

For a complex X and an integer ℓ we write $\Sigma^\ell X$ for the ℓ^{th} translate *of X; this complex is defined by $(\Sigma^\ell X)_n = X_{n-\ell}$ and $\partial_n^{\Sigma^\ell X} = (-1)^\ell \partial_{n-\ell}^X$ for $n \in \mathbb{Z}$.*

2.2. *If $\mathcal{A}$ is an abelian category with enough projectives, then we write $\mathrm{P}(A)$ for any projective resolution of $A \in \mathcal{A}$. By the unique, up to homotopy, lifting property of projective resolutions one gets a well-defined functor $\mathrm{P}\colon \mathcal{A} \to \mathsf{K}(\mathcal{A})$, and we write $\pi_A\colon \mathrm{P}(A) \to A$ for the canonical quasi-isomorphism.*

Dually, if $\mathcal{B}$ is an abelian category with enough injectives, then we write $\mathrm{I}(B)$ for any injective resolution of $B \in \mathcal{B}$. This yields a well-defined functor $\mathrm{I}\colon \mathcal{B} \to \mathsf{K}(\mathcal{B})$ and we write $\iota_B\colon B \to \mathrm{I}(B)$ for the canonical quasi-isomorphism.

Definition 2.3. Let $\mathcal{A}$ be an abelian category and let $\ell \in \mathbb{Z}$. A complex X in $\mathcal{A}$ is said to have its *homology concentrated in degree ℓ* if one has $\mathrm{H}_i(X) = 0$ for all $i \neq \ell$.

Lemma 2.4. *Let $\mathcal{A}$ be an abelian category with enough projectives and let $\ell \in \mathbb{Z}$. Let A be an object in $\mathcal{A}$ and let X be a complex in $\mathcal{A}$ whose homology is concentrated in degree ℓ. There is an isomorphism of abelian groups, natural* *in both A and X, given by:*

$$\mathrm{Hom}_{\mathcal{A}}(A, \mathrm{H}_\ell(X)) \xrightarrow[\cong]{u^\ell_{A,X}} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{P}(A), \Sigma^{-\ell}X)\,,$$

whose inverse is induced by $\mathrm{H}_0(-)$*. Furthermore, a morphism* $\sigma\colon A \to \mathrm{H}_\ell(X)$ *in* $\mathcal{A}$ *is an isomorphism if and only if* $u^\ell_{A,X}(\sigma)\colon \mathrm{P}(A) \to \Sigma^{-\ell}X$ *is a quasi-isomorphism.*

Proof. Let $\mathsf{D}(\mathcal{A})$ be the derived category of $\mathcal{A}$. As $\mathcal{A}$ is a full subcatgory of $\mathsf{D}(\mathcal{A})$, we have $\mathrm{Hom}_{\mathcal{A}}(A, \mathrm{H}_\ell(X)) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, \mathrm{H}_\ell(X))$. In $\mathsf{D}(\mathcal{A})$ one has natural isomorphisms $A \cong \mathrm{P}(A)$ and $\mathrm{H}_\ell(X) \cong \Sigma^{-\ell}X$, as the homology of X is concentrated in degree ℓ, and consequently

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, \mathrm{H}_\ell(X)) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(\mathrm{P}(A), \Sigma^{-\ell}X).$$

It is well-known that $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(\mathrm{P}(A), Y) \cong \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{P}(A), Y)$ for any complex Y in $\mathcal{A}$ since $\mathrm{P}(A)$ is a bounded below complex of projectives. By composing these natural isomorphisms, the assertion follows. □

The next lemma is proved similarly.

Lemma 2.5. *Let $\mathcal{B}$ be an abelian category with enough injectives and let $\ell \in \mathbb{Z}$. Let B be an object in $\mathcal{B}$ and let Y be a complex in $\mathcal{B}$ whose homology is concentrated in degree ℓ. There is an isomorphism of abelian groups, natural in both B and Y, given by:*

$$\mathrm{Hom}_{\mathcal{B}}(\mathrm{H}_\ell(Y), B) \xrightarrow[\cong]{v^\ell_{Y,B}} \mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(\Sigma^{-\ell}Y, \mathrm{I}(B))\,,$$

whose inverse is induced by $\mathrm{H}_0(-)$*. Furthermore, a morphism* $\tau\colon \mathrm{H}_\ell(Y) \to B$ *in* $\mathcal{B}$ *is an isomorphism if and only if* $v^\ell_{Y,B}(\tau)\colon \Sigma^{-\ell}Y \to \mathrm{I}(B)$ *is a quasi-isomorphism.* □

2.6. *As in Mac Lane* [22, I§2]*, a functor means a covariant functor. Let* $\mathrm{T}\colon \mathcal{A} \to \mathcal{B}$ *be an additive (covariant) functor between abelian categories. Recall that if $\mathcal{A}$ has enough projectives, then the* i^{th} left derived functor *of* T *is given by* $\mathrm{L}_i\mathrm{T}(A) = \mathrm{H}_i\mathrm{T}(P)$ *where P is any projective resolution of* $A \in \mathcal{A}$*. If* T *is right exact, then* $\mathrm{L}_0\mathrm{T} = \mathrm{T}$*. Dually, if $\mathcal{A}$ has enough injectives, then the* i^{th} right derived functor *of* T *is given by* $\mathrm{R}^i\mathrm{T}(A) = \mathrm{H}_{-i}\mathrm{T}(I)$ *where I is any injective resolution of* $A \in \mathcal{A}$*. And if* T *is left exact, then* $\mathrm{R}^0\mathrm{T} = \mathrm{T}$.

Consider the opposite functor $\mathrm{T}^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$*. The category* $\mathcal{A}^{\mathrm{op}}$ *has enough projectives (resp. injectives) if and only if $\mathcal{A}$ has enough injectives (resp. projectives), and in this case one has*

$$\mathrm{L}_i(\mathrm{T}^{\mathrm{op}}) = (\mathrm{R}^i\mathrm{T})^{\mathrm{op}} \quad \textit{(resp. } \mathrm{R}^i(\mathrm{T}^{\mathrm{op}}) = (\mathrm{L}_i\mathrm{T})^{\mathrm{op}}\textit{)}.$$

If $\mathrm{S}\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon \mathrm{T}$ is an adjunction, where S is the left adjoint of T, with unit $\eta\colon \mathrm{Id}_{\mathcal{A}} \to \mathrm{TS}$ and counit $\varepsilon\colon \mathrm{ST} \to \mathrm{Id}_{\mathcal{B}}$, then the composites $\mathrm{S} \xrightarrow{\mathrm{S}\eta} \mathrm{STS} \xrightarrow{\varepsilon\mathrm{S}} \mathrm{S}$ and $\mathrm{T} \xrightarrow{\eta\mathrm{T}} \mathrm{TST} \xrightarrow{\mathrm{T}\varepsilon} \mathrm{T}$ are the identities on S and T; see e.g., [22, IV§1 Thm. 1]. In the proof of Theorem 3.8 we will need the following sligthly more careful version of this fact.

Lemma 2.7. *Let* $\mathrm{S}\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon \mathrm{T}$ *be functors (not assumed to be an adjunction), let* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *be a full subcategories of* $\mathcal{A}$ *and* $\mathcal{B}$*, and assume that there is a natural bijection*

$$\mathrm{Hom}_{\mathcal{B}}(\mathrm{S}A, B) \xrightarrow{k_{A,B}} \mathrm{Hom}_{\mathcal{A}}(A, \mathrm{T}B)$$

for $A \in \mathcal{A}_0$ *and* $B \in \mathcal{B}_0$*. (We do not assume* $\mathrm{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$ *and* $\mathrm{T}(\mathcal{B}_0) \subseteq \mathcal{A}_0$*, so it is not given the functors* S *and* T *restrict to an adjunction* $\mathcal{A}_0 \rightleftarrows \mathcal{B}_0$*.)*

For every $A \in \mathcal{A}_0$ *which satisfies* $\mathrm{S}A \in \mathcal{B}_0$ *set* $\eta_A = k_{A,\mathrm{S}A}(1_{\mathrm{S}A})\colon A \to \mathrm{TS}A$*, and for every* $B \in \mathcal{B}_0$ *which satisfies* $\mathrm{T}B \in \mathcal{A}_0$ *set*

$$\varepsilon_B = k^{-1}_{\mathrm{T}B,B}(1_{\mathrm{T}B})\colon \mathrm{ST}B \to B.$$

The following hold:

(a) *If* $A \in \mathcal{A}$ *is an object with* $A, \mathrm{TS}A \in \mathcal{A}_0$ *and* $\mathrm{S}A \in \mathcal{B}_0$*, then the composition* $\mathrm{S}A \xrightarrow{\mathrm{S}(\eta_A)} \mathrm{STS}A \xrightarrow{\varepsilon_{\mathrm{S}A}} \mathrm{S}A$ *is the identity on* $\mathrm{S}A$*.*

(b) *If* $B \in \mathcal{B}$ *is an object with* $B, \mathrm{ST}B \in \mathcal{B}_0$ *and* $\mathrm{T}B \in \mathcal{A}_0$*, then the composition* $\mathrm{T}B \xrightarrow{\eta_{\mathrm{T}B}} \mathrm{TST}B \xrightarrow{\mathrm{T}(\varepsilon_B)} \mathrm{T}B$ *is the identity on* $\mathrm{T}B$*.*

Proof. Inspect the proof of [22, IV§1 Thm. 1]. $\square$

3. Fixed and cofixed objects

In this section, we prove our main result, Theorem 3.8, which in certain situations takes the simpler form of Theorem 3.14.

Setup 3.1. Throughout, $\mathcal{A}$ is an abelian category with enough projectives and $\mathcal{B}$ is an abelian category with enough injectives. Furthermore, $\mathrm{F}\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon \mathrm{G}$ is an adjunction with F being left adjoint of G. We write $h_{A,B}\colon \mathrm{Hom}_{\mathcal{B}}(\mathrm{F}A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, \mathrm{G}B)$ for the given natural isomorphism and denote by $\eta_A\colon A \to \mathrm{GF}A$ and $\varepsilon_B\colon \mathrm{FG}B \to B$ the unit and counit.

The following examples of Setup 3.1 are useful to have in mind.

Example 3.2. Let Γ and Λ be rings and let $T = {}_{\Gamma}T_{\Lambda}$ be a (Γ, Λ)-bimodule. The functors

$$\mathsf{Mod}(\Lambda) \underset{\mathrm{G}\,=\,\mathrm{Hom}_{\Gamma}(T,-)}{\overset{\mathrm{F}\,=\,T\otimes_{\Lambda}-}{\rightleftarrows}} \mathsf{Mod}(\Gamma)$$

constitute an adjunction with unit and counit:

$$\eta_A\colon A \longrightarrow \mathrm{Hom}_{\Gamma}(T, T \otimes_{\Lambda} A) \quad \text{given by} \quad \eta_A(a)(t) = t \otimes a \quad \text{and}$$
$$\varepsilon_B\colon T \otimes_{\Lambda} \mathrm{Hom}_{\Gamma}(T, B) \longrightarrow B \quad \text{given by} \quad \varepsilon_B(t \otimes \beta) = \beta(t)\,.$$

If Γ and Λ are artin algebras and the modules ${}_{\Gamma}T$ and T_{Λ} are finitely generated, then the above restricts to an adjunction between the subcategories of finitely generated modules:

$$\mathsf{mod}(\Lambda) \underset{\mathrm{G}\,=\,\mathrm{Hom}_{\Gamma}(T,-)}{\overset{\mathrm{F}\,=\,T\otimes_{\Lambda}-}{\rightleftarrows}} \mathsf{mod}(\Gamma)\,.$$

In this case the category $\mathsf{mod}(\Lambda)$ has enough projectives and $\mathsf{mod}(\Gamma)$ has enough injectives, see e.g., [3, II.3 Cor. 3.4], so the situation satisfies Setup 3.1.

Finally, we note that $\mathrm{L}_i\mathrm{F} = \mathrm{Tor}_i^{\Lambda}(T,-)$ and $\mathrm{R}^i\mathrm{G} = \mathrm{Ext}_{\Gamma}^i(T,-)$.

For a ring Λ we write Λ^{o} for the opposite ring.

Example 3.3. Let Γ and Λ be rings and let $T = {}_{\Gamma}T_{\Lambda}$ be a (Γ,Λ)-bimodule. The functors

$$\mathsf{Mod}(\Gamma) \underset{\mathrm{G}=\mathrm{Hom}_{\Lambda^{\mathrm{o}}}(-,T)}{\overset{\mathrm{F}=\mathrm{Hom}_{\Gamma}(-,T)^{\mathrm{op}}}{\rightleftarrows}} \mathsf{Mod}(\Lambda^{\mathrm{o}})^{\mathrm{op}}$$

constitute an adjunction whose unit and counit are the so-called biduality homomorphisms:

$$\begin{aligned} &\eta_A\colon A \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{o}}}(\mathrm{Hom}_{\Gamma}(A,T),T) &&\text{given by} &&\eta_A(a)(\alpha) = \alpha(a) \quad\text{and}\\ &\varepsilon_B\colon B \longrightarrow \mathrm{Hom}_{\Gamma}(\mathrm{Hom}_{\Lambda^{\mathrm{o}}}(B,T),T) &&\text{given by} &&\varepsilon_B(b)(\beta) = \beta(b)\,. \end{aligned}$$

(Note that, a priori, the counit is a morphism $\mathrm{FG}B \to B$ in $\mathsf{Mod}(\Lambda^{\mathrm{o}})^{\mathrm{op}}$, but that corresponds to the morphism $B \to \mathrm{FG}B$ in $\mathsf{Mod}(\Lambda^{\mathrm{o}})$ displayed above.)

If Γ is left coherent and Λ is right coherent, then the categories $\mathsf{mod}(\Gamma)$ and $\mathsf{mod}(\Lambda^{\mathrm{o}})$ of finitely presented Γ- and Λ^{o}-modules are abelian with enough projectives (and hence the category $\mathsf{mod}(\Lambda^{\mathrm{o}})^{\mathrm{op}}$ is abelian with enough injectives). In this case, and if the modules ${}_{\Gamma}T$ and T_{Λ} are finitely presented, the above restricts to an adjunction:

$$\mathsf{mod}(\Gamma) \underset{\mathrm{G}=\mathrm{Hom}_{\Lambda^{\mathrm{o}}}(-,T)}{\overset{\mathrm{F}=\mathrm{Hom}_{\Gamma}(-,T)^{\mathrm{op}}}{\rightleftarrows}} \mathsf{mod}(\Lambda^{\mathrm{o}})^{\mathrm{op}}\,.$$

Finally, we note that $\mathrm{L}_i\mathrm{F} = \mathrm{Ext}_{\Gamma}^i(-,T)^{\mathrm{op}}$ and $\mathrm{R}^i\mathrm{G} = \mathrm{Ext}_{\Lambda^{\mathrm{o}}}^i(-,T)$ by 2.6.

Proposition 3.4. *Let ℓ be an integer. For $A \in \mathcal{A}$ that satisfies $\mathrm{L}_i\mathrm{F}(A) = 0$ for all $i \neq \ell$, and for $B \in \mathcal{B}$ that satisfies $\mathrm{R}^i\mathrm{G}(B) = 0$ for all $i \neq \ell$, there is a natural isomorphism:*

$$\mathrm{Hom}_{\mathcal{B}}(\mathrm{L}_{\ell}\mathrm{F}(A),B) \xrightarrow[\cong]{h^{\ell}_{A,B}} \mathrm{Hom}_{\mathcal{A}}(A,\mathrm{R}^{\ell}\mathrm{G}(B))\,.$$

Proof. The assumptions mean that the homology of the complex $\mathrm{F}(\mathrm{P}(A))$ is concentrated in degree ℓ and that the homology of $\mathrm{G}(\mathrm{I}(B))$ is concentrated in degree $-\ell$. We now define $h^{\ell}_{A,B}$ to be the unique homomorphism (which is forced to be an isomorphism) that makes the following diagram commutative:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(\mathrm{L}_\ell\mathrm{F}(A),B) \arrow[r, dotted, "h^\ell_{A,B}"] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{A}}(A,\mathrm{R}^\ell\mathrm{G}(B)) \arrow[d, equal] \\
\mathrm{Hom}_{\mathcal{B}}(\mathrm{H}_\ell\mathrm{F}(\mathrm{P}(A)),B) \arrow[d, "v^\ell_{\mathrm{F}(\mathrm{P}(A)),B}"', "\cong"] & \mathrm{Hom}_{\mathcal{A}}(A,\mathrm{H}_{-\ell}\mathrm{G}(\mathrm{I}(B))) \arrow[d, "\cong"', "u^{-\ell}_{A,\mathrm{G}(\mathrm{I}(B))}"] \\
\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(\Sigma^{-\ell}\mathrm{F}(\mathrm{P}(A)),\mathrm{I}(B)) \arrow[d, "\Sigma^\ell(-)"', "\cong"] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{P}(A),\Sigma^\ell\mathrm{G}(\mathrm{I}(B))) \arrow[d, equal] \\
\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(\mathrm{F}(\mathrm{P}(A)),\Sigma^\ell\mathrm{I}(B)) \arrow[r, "\text{adjunction}", "\cong"'] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\mathrm{P}(A),\mathrm{G}(\Sigma^\ell\mathrm{I}(B)))\,.
\end{tikzcd}
\tag{6}
$$

The vertical isomorphisms come from Lemmas 2.4 and 2.5. The given adjunction $\mathrm{F}\colon\mathcal{A}\rightleftarrows\mathcal{B}\colon\mathrm{G}$ induces an adjunction $\mathsf{K}(\mathcal{A})\rightleftarrows\mathsf{K}(\mathcal{B})$ by degreewise application of the functors F and G; this explains the lower vertical isomorphism in the diagram. Finally, we note that all the displayed isomorphisms are natural in A and B. $\square$

Definition 3.5. Let ℓ be an integer. If $A\in\mathcal{A}$ satisfies

$$\mathrm{L}_i\mathrm{F}(A)=0=\mathrm{R}^i\mathrm{G}(\mathrm{L}_\ell\mathrm{F}(A))$$

for all $i\neq\ell$, then we can apply Proposition 3.4 to $B=\mathrm{L}_\ell\mathrm{F}(A)$, and thereby obtain a morphism:

$$\eta^\ell_A\colon A\longrightarrow\mathrm{R}^\ell\mathrm{G}(\mathrm{L}_\ell\mathrm{F}(A))\quad\text{defined by}\quad\eta^\ell_A=h^\ell_{A,\mathrm{L}_\ell\mathrm{F}(A)}(1_{\mathrm{L}_\ell\mathrm{F}(A)}).$$

Similarly, if $B\in\mathcal{B}$ has $\mathrm{R}^i\mathrm{G}(B)=0=\mathrm{L}_i\mathrm{F}(\mathrm{R}^\ell\mathrm{G}(B))$ for all $i\neq\ell$, then we get a morphism

$$\varepsilon^\ell_B\colon\mathrm{L}_\ell\mathrm{F}(\mathrm{R}^\ell\mathrm{G}(B))\longrightarrow B\quad\text{defined by}\quad\varepsilon^\ell_B=(h^\ell_{\mathrm{R}^\ell\mathrm{G}(B),B})^{-1}(1_{\mathrm{R}^\ell\mathrm{G}(B)}).$$

Remark 3.6. The proofs of Lemmas 2.4 and 2.5 show how the maps $u^\ell_{A,X}$ and $v^\ell_{Y,B}$ act, and the diagram (6) shows how $h^\ell_{A,B}$ is a composition of these maps and the given adjunction. This tells us how $h^\ell_{A,B}$ acts. It can verified that for $\ell=0$ the isomorphism $h^\ell_{A,B}=h^0_{A,B}$ coincides with the given natural isomorphism $h_{A,B}$ from Setup 3.1, and hence η^0_A and ε^0_B from Definition 3.5 coincide with the unit η_A and the counit ε_B of the adjunction (F,G).

The following is the key definition in this paper.

Definition 3.7. Let ℓ be an integer. An object $A\in\mathcal{A}$ is called *ℓ-fixed* with respect to the adjunction (F,G) if it satisfies the following three conditions:

(i) $\mathrm{L}_i\mathrm{F}(A)=0$ for all $i\neq\ell$.
(ii) $\mathrm{R}^i\mathrm{G}(\mathrm{L}_\ell\mathrm{F}(A))=0$ for all $i\neq\ell$.
(iii) The morphism $\eta^\ell_A\colon A\to\mathrm{R}^\ell\mathrm{G}(\mathrm{L}_\ell\mathrm{F}(A))$ is an isomorphism.

The full subcategory of $\mathcal{A}$ whose objects are the ℓ-fixed ones is denoted by $\mathsf{Fix}_\ell(\mathcal{A})$.

Dually, an object $B\in\mathcal{B}$ is *ℓ-cofixed* with respect to (F,G) if it satisfies:

(i′) $\mathrm{R}^i\mathrm{G}(B) = 0$ for all $i \neq \ell$.
(ii′) $\mathrm{L}_i\mathrm{F}(\mathrm{R}^\ell\mathrm{G}(B)) = 0$ for all $i \neq \ell$.
(iii′) The morphism $\varepsilon^\ell_B\colon \mathrm{L}_\ell\mathrm{F}(\mathrm{R}^\ell\mathrm{G}(B)) \to B$ is an isomorphism.

The full subcategory of $\mathcal{B}$ whose objects are the ℓ-cofixed ones is denoted $\mathsf{coFix}_\ell(\mathcal{B})$.

The categories of ℓ-fixed objects in $\mathcal{A}$ and ℓ-cofixed objects in $\mathcal{B}$ are, in fact, equivalent:

Theorem 3.8. *In the notation from Setup* 3.1 *and Definition* 3.7 *there is for every integer ℓ an adjoint equivalence of categories:*

$$\mathsf{Fix}_\ell(\mathcal{A}) \underset{\mathrm{R}^\ell\mathrm{G}}{\overset{\mathrm{L}_\ell\mathrm{F}}{\rightleftarrows}} \mathsf{coFix}_\ell(\mathcal{B}).$$

Proof. Let $\mathcal{A}_0$, respectively, $\mathcal{B}_0$, be the full subcategory of $\mathcal{A}$, respectively, $\mathcal{B}$, whose objects satisfy condition (i), respectively, (i′), in Definition 3.7. By Proposition 3.4 we may apply Lemma 2.7 to these choices of $\mathcal{A}_0$ and $\mathcal{B}_0$ and to $\mathrm{S} = \mathrm{L}_\ell\mathrm{F}$ and $\mathrm{T} = \mathrm{R}^\ell\mathrm{G}$. From part (a) of that lemma (and from Definition 3.5) we conclude that if $A \in \mathcal{A}$ satisfies the conditions

(1°) $A \in \mathcal{A}_0$, that is, A satisfies 3.7(i),
(2°) $\mathrm{S}A \in \mathcal{B}_0$, that is, A satisfies 3.7(ii), and
(3°) $\mathrm{TS}A \in \mathcal{A}_0$, that is, $B = \mathrm{L}_\ell\mathrm{F}(A)$ satisfies 3.7(ii′),

then one has $\varepsilon^\ell_{\mathrm{L}_\ell\mathrm{F}(A)} \circ \mathrm{L}_\ell\mathrm{F}(\eta^\ell_A) = 1_{\mathrm{L}_\ell\mathrm{F}(A)}$. We now see that the functor $\mathrm{L}_\ell\mathrm{F}$ maps $\mathsf{Fix}_\ell(\mathcal{A})$ to $\mathsf{coFix}_\ell(\mathcal{B})$, indeed, if A belongs to $\mathsf{Fix}_\ell(\mathcal{A})$, then $B := \mathrm{L}_\ell\mathrm{F}(A)$ satisfies (i′) as A satisfies (ii), and B satisfies (ii′) since A satisfies (iii) and (i). In particular, conditions (1°)–(3°) above hold, and hence $\varepsilon^\ell_B \circ \mathrm{L}_\ell\mathrm{F}(\eta^\ell_A) = 1_B$. Since η^ℓ_A is an isomorphism by (iii), it follows that ε^ℓ_B is an isomorphism as well, that is, B satisfies condition (iii′).

Similar arguments show that the functor $\mathrm{R}^\ell\mathrm{G}$ maps $\mathsf{coFix}_\ell(\mathcal{B})$ to $\mathsf{Fix}_\ell(\mathcal{A})$. Now Proposition 3.4 and Definition 3.5 show that $(\mathrm{L}_\ell\mathrm{F}, \mathrm{R}^\ell\mathrm{G})$ gives an adjunction between the categories $\mathsf{Fix}_\ell(\mathcal{A})$ to $\mathsf{coFix}_\ell(\mathcal{B})$ with unit η^ℓ and counit ε^ℓ. Finally, conditions 3.7(iii) and (iii′) show that $(\mathrm{L}_\ell\mathrm{F}, \mathrm{R}^\ell\mathrm{G})$ yields an adjoint equivalence between $\mathsf{Fix}_\ell(\mathcal{A})$ and $\mathsf{coFix}_\ell(\mathcal{B})$. □

Lemma 3.9. *The categories* $\mathsf{Fix}_\ell(\mathcal{A})$ *and* $\mathsf{coFix}_\ell(\mathcal{B})$ *are closed under direct summands and extensions in $\mathcal{A}$ and $\mathcal{B}$, respectively.*

Proof. Straightforward from the definitions. □

The next lemma (which does not use that G is a right adjoint, but only that it is left exact) is variant of Hartshorne [19, III§1 Prop. 1.2A]. Recall that $B \in \mathcal{B}$ is called G-*acyclic* if $\mathrm{R}^i\mathrm{G}(B) = 0$ for all $i > 0$. Similarly, $A \in \mathcal{A}$ is called F-*acyclic* if $\mathrm{L}_i\mathrm{F}(A) = 0$ for all $i > 0$.

Also recall that an additive functor T between abelian categories is said to have *finite homological dimension*, respectively, *finite cohomological dimension*, if one has $\mathrm{L}_d\mathrm{T} = 0$, respectively, $\mathrm{R}^d\mathrm{T} = 0$, for some integer $d \geqslant 0$.

Lemma 3.10. *Let $\gamma\colon X \to Y$ be a quasi-isomorphism between complexes in $\mathcal{B}$ that consist of G-acyclic objects. If G has finite cohomological dimension, then $\mathrm{G}\gamma\colon \mathrm{G}X \to \mathrm{G}Y$ is a quasi-isomorphism.*

Proof. This is left as an exercise to the reader. □

Under suitable assumptions we obtain in Propositions 3.12 and 3.13 below simplified descriptions of the categories $\mathsf{Fix}_\ell(\mathcal{A})$ and $\mathsf{coFix}_\ell(\mathcal{B})$.

Definition 3.11. The adjunction (F,G) from Setup 3.1 is called a *tilting adjunction* if it satisfies the following four conditions:

(TA1) For every projective object $P \in \mathcal{A}$ the object $\mathrm{F}(P)$ is G-acyclic and the unit of adjunction $\eta_P\colon P \to \mathrm{GF}(P)$ is an isomorphism. I.e., $\mathrm{Prj}(\mathcal{A}) \subseteq \mathsf{Fix}_0(\mathcal{A})$.

(TA2) The functor G has finite cohomological dimension.

(TA3) For every injective object $I \in \mathcal{B}$ the object $\mathrm{G}(I)$ is F-acyclic and the counit of adjunction $\varepsilon_I\colon \mathrm{FG}(I) \to I$ is an isomorphism. I.e., $\mathrm{Inj}(\mathcal{B}) \subseteq \mathsf{coFix}_0(\mathcal{B})$.

(TA4) The functor F has finite homological dimension.

Proposition 3.12. *If the adjunction* (F,G) *satisfies conditions* (TA1) *and* (TA2) *in Definition* 3.11*, then for every integer ℓ and every $A \in \mathcal{A}$ one has:*

$$A \in \mathsf{Fix}_\ell(\mathcal{A}) \quad \Longleftrightarrow \quad \mathrm{L}_i\mathrm{F}(A) = 0 \text{ for all } i \neq \ell\,.$$

In other words, in this case, conditions (ii) *and* (iii) *in Definition* 3.7 *are automatic.*

Proof. The implication "$\Rightarrow$" holds by Definition 3.7(i). Conversely, assume that $\mathrm{L}_i\mathrm{F}(A) = 0$ for all $i \neq \ell$. We must argue that conditions (ii) and (iii) in Definition 3.7 hold as well. Let P be a projective resolution of A and let I be a injective resolution of $\mathrm{L}_\ell\mathrm{F}(A) = \mathrm{H}_\ell\mathrm{F}(P)$. Our assumption means that the homology of the complex $\mathrm{F}(P)$ is concentrated in degree ℓ. With $B = \mathrm{L}_\ell\mathrm{F}(A)$ we now consider the following part of the diagram (6):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(\mathrm{L}_\ell\mathrm{F}(A),\mathrm{L}_\ell\mathrm{F}(A)) \arrow[d, equal] & \\
\mathrm{Hom}_{\mathcal{B}}(\mathrm{H}_\ell\mathrm{F}(P),\mathrm{L}_\ell\mathrm{F}(A)) \arrow[d, "\nu := \nu^\ell_{\mathrm{F}(P),\mathrm{L}_\ell\mathrm{F}(A)}"', "\cong"] & \\
\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(\Sigma^{-\ell}\mathrm{F}(P),I) \arrow[d, "\Sigma^\ell(-)"', "\cong"] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P,\Sigma^\ell\mathrm{G}(I)) \arrow[d, equal] \\
\mathrm{Hom}_{\mathsf{K}(\mathcal{B})}(\mathrm{F}(P),\Sigma^\ell I) \arrow[r, "\text{adjunction}", "\cong"'] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(P,\mathrm{G}(\Sigma^\ell I))\,.
\end{tikzcd} \tag{7}$$

Set $\gamma = \nu(1_{\mathrm{L}_\ell\mathrm{F}(A)})\colon \Sigma^{-\ell}\mathrm{F}(P) \to I$ in $\mathsf{K}(\mathcal{B})$ and note that γ is a quasi-isomorphism by Lemma 2.5. Under the maps in (7), the identity morphism $1_{\mathrm{L}_\ell\mathrm{F}(A)}$

is mapped to $\theta \in \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(P, \Sigma^\ell \mathrm{G}(I))$ given by $\theta = \mathrm{G}(\Sigma^\ell \gamma) \circ \eta_P$, that is, θ is the composite:

$$P \xrightarrow{\eta_P} \mathrm{G}(\mathrm{F}(P)) \xrightarrow{\mathrm{G}(\Sigma^\ell\gamma)} \mathrm{G}(\Sigma^\ell I) = \Sigma^\ell \mathrm{G}(I)\,. \tag{8}$$

Here η_P is an isomorphism by assumption (TA1). Since $\mathrm{F}(P)$ and $\Sigma^\ell I$ consist of G-acyclic objects—again by (TA1)—the other assumption (TA2) together with Lemma 3.10 imply that the quasi-isomorphism $\Sigma^\ell\gamma\colon \mathrm{F}(P) \to \Sigma^\ell I$ remains to be a quasi-isomorphism after application of G. Consequently, $\theta\colon P \to \Sigma^\ell \mathrm{G}(I)$ is a quasi-isomorphism. As the homology of P is concentrated in degree 0 we get

$$\mathrm{R}^i\mathrm{G}(\mathrm{L}_\ell\mathrm{F}(A)) = \mathrm{H}_{-i}\mathrm{G}(I) \cong \mathrm{H}_{-i}(\Sigma^{-\ell}P) = \mathrm{H}_{-i+\ell}(P) = 0 \quad \text{for all} \quad i \neq \ell,$$

which proves condition 3.7(ii). It now makes sense to consider the remaining part of the diagram (6) (still with $B = \mathrm{L}_\ell\mathrm{F}(A)$), which gives the middle equality below:

$$\eta_A^\ell = h^\ell_{A,\mathrm{L}_\ell\mathrm{F}(A)}(1_{\mathrm{L}_\ell\mathrm{F}(A)}) = (u^{-\ell}_{A,\mathrm{G}(I)})^{-1}(\theta) = \mathrm{H}_0(\theta).$$

Here the first equality is by Definition 3.5 and the last equality is by Lemma 2.4. As θ is a quasi-isomorphism, $\eta_A^\ell = \mathrm{H}_0(\theta)$ is an isomorphism, and hence condition 3.7(iii) holds. □

Proposition 3.13. *If the adjunction* (F,G) *satisfies conditions* (TA3) *and* (TA4) *in Definition* 3.11*, then for every integer ℓ and every $B \in \mathcal{B}$ one has:*

$$B \in \mathsf{coFix}_\ell(\mathcal{B}) \quad \Longleftrightarrow \quad \mathrm{R}^i\mathrm{G}(B) = 0 \text{ for all } i \neq \ell\,.$$

That is, in this case, conditions (ii′) *and* (iii′) *in Definition* 3.7 *are automatic.*

Proof. Similar to the proof of Proposition 3.12. □

Theorem 3.14. *If* (F,G) *is a tilting adjunction then there is an adjoint equivalence:*

$$\{A \in \mathcal{A} \mid \mathrm{L}_i\mathrm{F}(A) = 0 \textit{ for all } i \neq \ell\} \underset{\mathrm{R}^\ell\mathrm{G}}{\overset{\mathrm{L}_\ell\mathrm{F}}{\rightleftarrows}} \{B \in \mathcal{B} \mid \mathrm{R}^i\mathrm{G}(B) = 0 \textit{ for all } i \neq \ell\}\,.$$

Proof. In view of Propositions 3.12 and 3.13 this is immediate from Theorem 3.8. □

4. Applications to tilting theory and commutative algebra

In this section, we show how some classic equivalences of categories from tilting theory and commutative algebra are special cases of Theorems 3.8 and 3.14.

Tilting modules of projective dimension $\leqslant 1$ over artin algebras were originally considered by Brenner and Butler [5] (although the term "tilting" first appeared in [18] by Happel and Ringel). Later people, such as Happel [17, III§3] and Miyashita [25], studied tilting modules of arbitrary finite projective dimension over general rings. If Γ is an artin algebra with duality $\mathrm{D}\colon \mathsf{mod}(\Gamma) \to \mathsf{mod}(\Gamma^{\mathrm{o}})$, then a finitely generated Γ-module C is called cotilting if the Γ^{o}-module $\mathrm{D}(C)$ is tilting.

The so-called Wakamatsu tilting modules constitute a good common generalization of both tilting and cotilting modules. In [33] Wakamatsu introduced such modules over artin algebras; the following more general definition can be found in Wakamatsu [34, Sec. 3].

Definition 4.1 (Wakamatsu)**.** Let Γ and Λ be rings. A *Wakamatsu tilting module* for the pair (Γ,Λ) is a (Γ,Λ)-bimodule $T = {}_{\Gamma}T_{\Lambda}$ that satisfies the next conditions:

(W1) ${}_{\Gamma}T$ and T_{Λ} admit resolutions by finitely generated projective modules.
(W2) $\mathrm{Ext}^i_{\Gamma}(T,T)=0$ and $\mathrm{Ext}^i_{\Lambda^{\mathrm{o}}}(T,T)=0$ for all $i>0$.
(W3) The canonical map $\Lambda \to \mathrm{Hom}_{\Gamma}(T,T)$ is an isomorphism of (Λ,Λ)-bimodules, and $\Gamma \to \mathrm{Hom}_{\Lambda^{\mathrm{o}}}(T,T)$ is an isomorphism of (Γ,Γ)-bimodules.

The original version of the next result is a classic theorem by Brenner and Butler [5]; it was later improved by Happel [17, III§3] and Miyashita [25, Thm. 1.16]. All of these results are covered by following corollary of Theorem 3.14.

Corollary 4.2 (Brenner–Butler and Happel)**.** *Let Γ and Λ be rings. If $T = {}_{\Gamma}T_{\Lambda}$ is a Wakamatsu tilting module for which $\mathrm{pd}_{\Gamma}(T)$ and $\mathrm{pd}_{\Lambda^{\mathrm{o}}}(T)$ are finite, then there is for every $\ell \in \mathbb{Z}$ an adjoint equivalence:*

$$\left\{ M \in \mathsf{Mod}(\Lambda) \;\middle|\; \begin{array}{c} \mathrm{Tor}^{\Lambda}_i(T,M)=0 \\ \textit{for all } i \neq \ell \end{array} \right\} \underset{\mathrm{Ext}^{\ell}_{\Gamma}(T,-)}{\overset{\mathrm{Tor}^{\Lambda}_{\ell}(T,-)}{\rightleftarrows}} \left\{ N \in \mathsf{Mod}(\Gamma) \;\middle|\; \begin{array}{c} \mathrm{Ext}^{i}_{\Gamma}(T,N)=0 \\ \textit{for all } i \neq \ell \end{array} \right\}.$$

If Γ and Λ are artian algebras and the modules ${}_{\Gamma}T$ and T_{Λ} are finitely generated, then the categories $\mathsf{Mod}(\Lambda)$ and $\mathsf{Mod}(\Gamma)$ may be replaced by $\mathsf{mod}(\Lambda)$ and $\mathsf{mod}(\Gamma)$.

Proof. Look at the adjunction

$$T \otimes_{\Lambda} -\colon \mathsf{Mod}(\Lambda) \rightleftarrows \mathsf{Mod}(\Gamma) : \mathrm{Hom}_{\Gamma}(T,-)$$

from Example 3.2. Under the given assumptions on T, it is easy to verify that this is a tilting adjunction in the sense of Definition 3.11. Now apply Theorem 3.14. □

The next corollary of Theorem 3.14 recovers [34, Prop. 8.1] by Wakamatsu.

Corollary 4.3 (Wakamatsu)**.** *Assume that Γ is a left coherent ring and that Λ is right coherent ring. If $T = {}_{\Gamma}T_{\Lambda}$ is a Wakamatsu tilting module for which $\mathrm{id}_{\Gamma}(T)$ and $\mathrm{id}_{\Lambda^{\mathrm{o}}}(T)$ are finite, then there is for every $\ell \in \mathbb{Z}$ an adjoint* *equivalence:*

$$\left\{ M \in \mathsf{mod}(\Gamma) \;\middle|\; \begin{array}{c} \mathrm{Ext}^i_\Gamma(M,T)=0 \\ \textit{for all } i \neq \ell \end{array} \right\} \underset{\mathrm{Ext}^\ell_{\Lambda^{\mathrm{o}}}(-,T)}{\overset{\mathrm{Ext}^\ell_\Gamma(-,T)^{\mathrm{op}}}{\rightleftarrows}} \left\{ N \in \mathsf{mod}(\Lambda^{\mathrm{o}}) \;\middle|\; \begin{array}{c} \mathrm{Ext}^i_{\Lambda^{\mathrm{o}}}(N,T)=0 \\ \textit{for all } i \neq \ell \end{array} \right\}^{\mathrm{op}}.$$

Proof. Look at $\mathrm{Hom}_\Gamma(-,T)^{\mathrm{op}} \colon \mathsf{mod}(\Gamma) \rightleftarrows \mathsf{mod}(\Lambda^{\mathrm{o}})^{\mathrm{op}} : \mathrm{Hom}_{\Lambda^{\mathrm{o}}}(-,T)$ from Example 3.3. Under the given assumptions on T, it is easy to verify that this is a tilting adjunction in the sense of Definition 3.11. Now apply Theorem 3.14. $\square$

Recall that a *semidualizing* module over a commutative noetherian ring R is nothing but a (balanced) Wakamatsu tilting module for the pair (R,R).

The following consequence of Theorem 3.8 seems to be new in the case where $\ell > 0$. For $\ell = 0$ it is a classic result, sometimes called *Foxby equivalence*, of Foxby [11, Sect. 1]; see also Avramov and Foxby [4, Thm. (3.2) and Prop. (3.4)] and Christensen [8, Obs. (4.10)].

Corollary 4.4 (Foxby)**.** *Let R be a commutative noetherian ring. If C is a semidualizing R-module, then there is for every $\ell \in \mathbb{Z}$ an adjoint equivalence:*

$$\left\{ M \in \mathsf{Mod}(R) \;\middle|\; \begin{array}{l} \mathrm{Tor}^R_i(C,M)=0 \textit{ for all } i \neq \ell, \\ \mathrm{Ext}^i_R(C,\mathrm{Tor}^R_\ell(C,M))=0 \textit{ for all } i \neq \ell, \\ \eta^\ell_M \colon M \to \mathrm{Ext}^\ell_R(C,\mathrm{Tor}^R_\ell(C,M)) \textit{ is an isomorphism} \end{array} \right\}$$

$$\mathrm{Tor}^R_\ell(C,-) \downarrow \uparrow \mathrm{Ext}^\ell_R(C,-)$$

$$\left\{ N \in \mathsf{Mod}(R) \;\middle|\; \begin{array}{l} \mathrm{Ext}^i_R(C,N)=0 \textit{ for all } i \neq \ell, \\ \mathrm{Tor}^R_i(C,\mathrm{Ext}^\ell_R(C,N))=0 \textit{ for all } i \neq \ell \\ \varepsilon^\ell_N \colon \mathrm{Tor}^R_\ell(C,\mathrm{Ext}^\ell_R(C,N)) \to B \textit{ is an isomorphism} \end{array} \right\}.$$

Proof. Apply Theorem 3.8 to Example 3.2 with $\Gamma = R = \Lambda$ and $T = C$. $\square$

Example 4.5. Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring. Recall that an R-module M is *Matlis reflexive* if the map

$$M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,E_R(k)),E_R(k))$$

is an isomorphism. By applying Theorem 3.8 with $\ell = 0$ to the adjunction from Example 3.3 with $\Gamma = R = \Lambda$ and $T = E_R(k)$, one gets the (trivial) equivalence:

$$\{\text{Matlis reflexive } R\text{-modules}\} \underset{\mathrm{Hom}_R(-,E_R(k))}{\overset{\mathrm{Hom}_R(-,E_R(k))^{\mathrm{op}}}{\rightleftarrows}} \{\text{Matlis reflexive } R\text{-modules}\}^{\mathrm{op}}.$$

5. Derivatives of the main result in the case $\ell = 0$

In this section, we consider the equivalence from Theorem 3.8 with $\ell = 0$ and show that sometimes it restricts to an equivalence between certain "finite" objects in $\mathsf{Fix}_0(\mathcal{A})$ and $\mathsf{coFix}_0(\mathcal{B})$. The precise statements can be found in Theorems 5.8 and 5.10.

For an object X in an abelian category $\mathcal{C}$ we use the standard notation $\mathrm{add}_{\mathcal{C}}(X)$ for the class of objects in $\mathcal{C}$ that are direct summands in finite direct sums of copies of X.

Definition 5.1. Let $\mathcal{C}$ be an abelian category, let $X \in \mathcal{C}$, and let $d \in \mathbb{N}_0$.

An object $C \in \mathcal{C}$ is said to be *d-generated by X*, respectively, *d-cogenerated by X*, if there is an exact sequence $X_d \to \cdots \to X_0 \to C \to 0$, respectively, $0 \to C \to X^0 \to \cdots \to X^d$, where $X_0, \ldots, X_d$, respectively, $X^0, \ldots, X^d$, belong to $\mathrm{add}_{\mathcal{C}}(X)$. The full subcategory of $\mathcal{C}$ consisting of all such objects is denoted by $\mathsf{gen}_d^{\mathcal{C}}(X)$, respectively, $\mathsf{cogen}_{\mathcal{C}}^d(X)$.

We say that $C \in \mathcal{C}$ *has an* $\mathrm{add}_{\mathcal{C}}(X)$*-resolution of length d*, respectively, *has an* $\mathrm{add}_{\mathcal{C}}(X)$*-coresolution of length d*, if there exists an exact sequence $0 \to X_d \to \cdots \to X_0 \to C \to 0$, respectively, $0 \to C \to X^0 \to \cdots \to X^d \to 0$, where $X_0, \ldots, X_d$, respectively, $X^0, \ldots, X^d$, belong to $\mathrm{add}_{\mathcal{C}}(X)$. The full subcategory of $\mathcal{C}$ consisting of all such objects is denoted by $\mathsf{res}_d^{\mathcal{C}}(X)$, respectively, $\mathsf{cores}_{\mathcal{C}}^d(X)$.

Remark 5.2. Note that as subcategories of $\mathcal{C}^{\mathrm{op}}$ one has

$$\mathsf{gen}_d^{\mathcal{C}^{\mathrm{op}}}(X) = \mathsf{cogen}_{\mathcal{C}}^d(X)^{\mathrm{op}} \quad \text{and} \quad \mathsf{res}_d^{\mathcal{C}^{\mathrm{op}}}(X) = \mathsf{cores}_{\mathcal{C}}^d(X)^{\mathrm{op}}.$$

Also note that $\mathsf{res}_0^{\mathcal{C}}(X) = \mathrm{add}_{\mathcal{C}}(X) = \mathsf{cores}_{\mathcal{C}}^0(X)$.

Example 5.3. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. One has:

$$\begin{aligned} \mathsf{gen}_0^{\mathsf{Mod}(R)}(R) &= \{\text{Finitely generated } R\text{-modules}\} \\ \mathsf{cogen}^0_{\mathsf{Mod}(R)}(E_R(k)) &= \{\text{Artinian } R\text{-modules}\}, \end{aligned}$$

where the first one is trivial and the second one is well-known; see [10, Thm. 3.4.3].

If R is Cohen–Macaulay with dimension d and a dualizing module Ω, then:

$$\begin{aligned} \mathsf{res}_d^{\mathsf{Mod}(R)}(R) &= \{\text{Finitely generated } R\text{-modules with finite} \\ &\qquad \text{projective dimension}\} \\ \mathsf{res}_d^{\mathsf{Mod}(R)}(\Omega) &= \{\text{Finitely generated } R\text{-modules with finite} \\ &\qquad \text{injective dimension}\}. \end{aligned}$$

Here the first equality is well-known and the second one follows easily from the existence of maximal Cohen–Macaulay approximations [2, Thm. A]; see also [7, Exer. 3.3.28].

Lemma 5.4. *For $\ell = 0$ the categories from Definition* 3.7 *have the properties:*

(a) *The category $\mathsf{Fix}_0(\mathcal{A})$ is closed under direct summands, extensions, and kernels of epimorphisms in $\mathcal{A}$.*

(b) *The category* $\mathsf{coFix}_0(\mathcal{B})$ *is closed under direct summands, extensions, and cokernels of monomorphisms in* $\mathcal{B}$.

Proof. The closure under direct summands and extensions comes from Lemma 3.9. The remaining assertions are proved by using similar methods. $\square$

Lemma 5.5. *For* $\ell = 0$ *the categories from Definition* 3.7 *have the properties:*

(a) *If the kernel of* G *is trivial, that is, if* $\mathrm{G}(B) = 0$ *implies* $B = 0$ *(for any* $B \in \mathcal{B}$*), then* $\mathsf{Fix}_0(\mathcal{A})$ *is closed under cokernels of monomorphisms in* $\mathcal{A}$.

(b) *If the kernel of* F *is trivial, that is, if* $\mathrm{F}(A) = 0$ *implies* $A = 0$ *(for any* $A \in \mathcal{A}$*), then* $\mathsf{coFix}_0(\mathcal{B})$ *is closed under kernels of epimorphisms in* $\mathcal{B}$.

Proof. (a) Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{A}$ with $A', A \in \mathsf{Fix}_0(\mathcal{A})$. As $\mathrm{L}_1\mathrm{F}(A) = 0$ we get the exact sequence

$$0 \to \mathrm{L}_1\mathrm{F}(A'') \to \mathrm{F}(A') \to \mathrm{F}(A),$$

and as G is left exact we get exactness of

$$0 \to \mathrm{G}(\mathrm{L}_1\mathrm{F}(A'')) \to \mathrm{GF}(A') \to \mathrm{GF}(A).$$

Since $\eta_{A'}$ and η_A are isomorphisms, the morphism $\mathrm{GF}(A') \to \mathrm{GF}(A)$ may be identified with $A' \to A$, which is mono. It follows that $\mathrm{G}(\mathrm{L}_1\mathrm{F}(A'')) = 0$, and consequently $\mathrm{L}_1\mathrm{F}(A'') = 0$. Having established this, arguments as in the proof of Lemma 3.9 show that $A'' \in \mathsf{Fix}_0(\mathcal{A})$.

(b) Similar to the proof of part (a). $\square$

We give a few examples of adjunctions that satisfy the hypotheses in Lemma 5.5.

Example 5.6. Let R be a commutative ring and let E be a faithfully injective R-module, that is, the functor $\mathrm{Hom}_R(-,E)$ is faithfully exact. In this case, the adjunction $(\mathrm{F},\mathrm{G}) = (\mathrm{Hom}_R(-,E)^{\mathrm{op}}, \mathrm{Hom}_R(-,E))$ from Example 3.3 has the property that either of the conditions $\mathrm{F}(M) = 0$ or $\mathrm{G}(M) = 0$ imply $M = 0$ (for any R-module M).

Example 5.7. Let R be a commutative noetherian ring and let C be a finitely generated R-module with $\mathrm{Supp}_R C = \mathrm{Spec}\, R$. In this case, the adjunction $(\mathrm{F},\mathrm{G}) = (C \otimes_R -, \mathrm{Hom}_R(C,-))$ from Example 3.2 has the property that either of the conditions $\mathrm{F}(M) = 0$ or $\mathrm{G}(M) = 0$ imply $M = 0$ (for any R-module M). This follows from basic results in commutative algebra; cf. [21, §3.3].

Theorem 5.8. *Assume that* $\mathrm{F}(A) = 0$ *implies* $A = 0$ *(for any* $A \in \mathcal{A}$*). For any* $X \in \mathsf{Fix}_0(\mathcal{A})$ *and* $d \geqslant 0$ *the equivalence from Theorem* 3.8 *with* $\ell = 0$

restricts to an equivalence:

$$\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{gen}_d^{\mathcal{A}}(X) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{coFix}_0(\mathcal{B}) \cap \mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)\,.$$

Proof. By Theorem 3.8 it suffices to argue that F maps $\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{gen}_d^{\mathcal{A}}(X)$ to $\mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)$ and that G maps $\mathsf{coFix}_0(\mathcal{B}) \cap \mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)$ to $\mathsf{gen}_d^{\mathcal{A}}(X)$.

First assume A belongs to $\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{gen}_d^{\mathcal{A}}(X)$. Since $A \in \mathsf{gen}_d^{\mathcal{A}}(X)$ there is an exact sequence $X_d \to \cdots \to X_0 \to A \to 0$ with $X_0, \dots, X_d \in \mathrm{add}_{\mathcal{A}}(X)$. Since $A, X \in \mathsf{Fix}_0(\mathcal{A})$ one has, in particular, $\mathrm{L}_i\mathrm{F}(A) = 0 = \mathrm{L}_i\mathrm{F}(X_n)$ for all $i > 0$ and $n = 0, \dots, d$, so it follows that the sequence $\mathrm{F}X_d \to \cdots \to \mathrm{F}X_0 \to \mathrm{F}A \to 0$ is exact, and hence $\mathrm{F}A$ belongs to $\mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)$.

Next assume that B is in $\mathsf{coFix}_0(\mathcal{B}) \cap \mathsf{gen}_d^{\mathcal{B}}(\mathrm{F}X)$ and let

$$Y_d \to \cdots \to Y_0 \to B \to 0$$

be an exact sequence in $\mathcal{B}$ with $Y_0, \dots, Y_d \in \mathrm{add}_{\mathcal{B}}(\mathrm{F}X)$. As $X \in \mathsf{Fix}_0(\mathcal{A})$ we have $\mathrm{F}X \in \mathsf{coFix}_0(\mathcal{B})$ and hence $Y_0, \dots, Y_d \in \mathsf{coFix}_0(\mathcal{B})$. The assumption on F and Lemma 5.5(b) imply that $\mathsf{coFix}_0(\mathcal{B})$ is closed under kernels of epimorphisms in $\mathcal{B}$, so all kernels

$$K_0 = \mathrm{Ker}(Y_0 \twoheadrightarrow B),\ K_1 = \mathrm{Ker}(Y_1 \twoheadrightarrow K_0), \dots,\ K_d = \mathrm{Ker}(Y_d \twoheadrightarrow K_{d-1})$$

belong to $\mathsf{coFix}_0(\mathcal{B})$. In particular,

$$\mathrm{R}^i\mathrm{G}(K_0) = \mathrm{R}^i\mathrm{G}(K_1) = \cdots = \mathrm{R}^i\mathrm{G}(K_d) = 0$$

for all $i > 0$, hence $\mathrm{G}Y_d \to \cdots \to \mathrm{G}Y_0 \to \mathrm{G}B \to 0$ is exact. Since $\mathrm{GF}X \cong X$ and $Y_0, \dots, Y_d \in \mathrm{add}_{\mathcal{B}}(\mathrm{F}X)$, it follows that $\mathrm{G}Y_0, \dots, \mathrm{G}Y_d \in \mathrm{add}_{\mathcal{A}}(X)$, and thus $\mathrm{G}B \in \mathsf{gen}_d^{\mathcal{A}}(X)$. $\square$

The next corollary of Theorem 5.8 is a classic result of Matlis [23, Cor. 4.3].

Corollary 5.9 (Matlis)**.** *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local $\mathfrak{m}$-adically complete ring. There is an adjoint equivalence:*

$$\{\text{Finitely generated } R\text{-modules}\} \underset{\mathrm{Hom}_R(-,E_R(k))}{\overset{\mathrm{Hom}_R(-,E_R(k))^{\mathrm{op}}}{\rightleftarrows}} \{\text{Artinian } R\text{-modules}\}^{\mathrm{op}}\,.$$

Proof. Consider the situation from Example 4.5. The assumption that R is $\mathfrak{m}$-adically complete yields that R (viewed as an R-module) is Matlis reflexive; see, e.g., [10, Thm. 3.4.1(8)]. The asserted equivalence now follows directly from Theorem 5.8 with $X = R$ and $d = 0$ in view of Example 5.6 and of Remark 5.2 and Example 5.3 (first half). $\square$

Theorem 5.10. *For any $X \in \mathsf{Fix}_0(\mathcal{A})$ and $d \geqslant 0$ the equivalence from Theorem 3.8 with $\ell = 0$ restricts to an equivalence:*

$$\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{res}_d^{\mathcal{A}}(X) \underset{\mathrm{G}}{\overset{\mathrm{F}}{\rightleftarrows}} \mathsf{res}_d^{\mathcal{B}}(\mathrm{F}X)\,.$$

If $\mathrm{G}(B) = 0$ *implies* $B = 0$ *(for any* $B \in \mathcal{B}$*), then* $\mathsf{res}_d^{\mathcal{A}}(X) \subseteq \mathsf{Fix}_0(\mathcal{A})$ *and hence the equivalence takes the simpler form* $\mathsf{res}_d^{\mathcal{A}}(X) \leftrightarrows \mathsf{res}_d^{\mathcal{B}}(\mathrm{F}X)$*.*

Proof. By Lemma 5.4(b) the class $\mathsf{coFix}_0(\mathcal{B})$ is closed under cokernels of monomorphisms in $\mathcal{B}$, and therefore $\mathsf{res}_d^{\mathcal{B}}(\mathrm{F}X) \subseteq \mathsf{coFix}_0(\mathcal{B})$. So in view of Theorem 3.8 we only have to show that F maps $\mathsf{Fix}_0(\mathcal{A}) \cap \mathsf{res}_d^{\mathcal{A}}(X)$ to $\mathsf{res}_d^{\mathcal{B}}(\mathrm{F}X)$ and that G maps $\mathsf{res}_d^{\mathcal{B}}(\mathrm{F}X)$ to $\mathsf{res}_d^{\mathcal{A}}(X)$. This follows from arguments similar to the ones found in the proof of Theorem 5.8. The last assertion follows from Lemma 5.5(a). □

The next corollary of Theorem 5.10 is a classic result of Sharp [28, Theorem (2.9)].

Corollary 5.11 (Sharp)**.** *Let* $(R, \mathfrak{m}, k)$ *be a commutative noetherian local Cohen–Macaulay ring with a dualizing module* Ω*. There is an adjoint equivalence:*

$$\left\{\begin{matrix}\text{Finitely generated } R\text{-modules}\\ \text{with finite projective dimension}\end{matrix}\right\} \underset{\mathrm{Hom}_R(\Omega,-)}{\overset{\Omega\otimes_R -}{\rightleftarrows}} \left\{\begin{matrix}\text{Finitely generated } R\text{-modules}\\ \text{with finite injective dimension}\end{matrix}\right\}$$

Proof. Immediate from Example 5.7, Theorem 5.10 with $X = R$, and Example 5.3. □

6. Applications to relative Cohen–Macaulay modules

Throughout this section, $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring and $\mathfrak{a} \subset R$ is a proper ideal. We apply Theorem 3.8 to study the category of (not necessarily finitely generated) relative Cohen–Macaulay modules. Our main result is Theorem 6.16. We begin by recalling a few well-known defintions and facts about local (co)homology.

6.1. *The* $\mathfrak{a}$-torsion functor *and the* $\mathfrak{a}$-adic completion functor *are defined by*

$$\Gamma_{\mathfrak{a}} = \varinjlim_{n\in\mathbb{N}} \mathrm{Hom}_R(R/\mathfrak{a}^n, -) \quad \textit{and} \quad \Lambda^{\mathfrak{a}} = \varprojlim_{n\in\mathbb{N}} (R/\mathfrak{a}^n \otimes_R -).$$

The i^{th} *right derived functor of* $\Gamma_{\mathfrak{a}}$ *is written* $\mathrm{H}_{\mathfrak{a}}^i$ *and called the* i^{th} local cohomology w.r.t. $\mathfrak{a}$*. The* i^{th} *left derived functor of* $\Lambda^{\mathfrak{a}}$ *is written* $\mathrm{H}_i^{\mathfrak{a}}$ *and called the* i^{th} local homology w.r.t. $\mathfrak{a}$.

The functor $\Lambda^{\mathfrak{a}}$ *is not right exact on the category of all* R*-modules, so its zeroth left derived functor* $\mathrm{H}_0^{\mathfrak{a}}$ *is, in general, not naturally isomorphic to* $\Lambda^{\mathfrak{a}}$*. For every* R*-module* M *there are canonical maps* $\psi_M \colon M \to \mathrm{H}_0^{\mathfrak{a}}(M)$ *and* $\varphi_M \colon \mathrm{H}_0^{\mathfrak{a}}(M) \twoheadrightarrow \Lambda^{\mathfrak{a}}M$ *whose composite* $\varphi_M \circ \psi_M$ *is the* $\mathfrak{a}$*-adic completion map* $\tau_M \colon M \to \Lambda^{\mathfrak{a}}M$*; see Simon* [29, §5.1]*. On the category of finitely generated* R*-modules, the functor* $\Lambda^{\mathfrak{a}}$ *is exact, as it is naturally isomorphic to* $- \otimes_R \widehat{R}^{\mathfrak{a}}$*; see* [24, Thms. 8.7 and 8.8]*. Hence, if* M *is a finitely generated* R*-module,* φ_M *is an isomorphism,* ψ_M *may be identified with* τ_M*, and* $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ *for* $i > 0$*.*

On the derived category $\mathsf{D}(R)$ *one can consider the total right derived functor* $\mathbf{R}\Gamma_{\mathfrak{a}}$ *of* $\Gamma_{\mathfrak{a}}$*. A classic result due to Grothendieck* [16, Prop. 1.4.1][1] *asserts that* $\mathbf{R}\Gamma_{\mathfrak{a}} \cong \mathrm{C}(\mathfrak{a}) \otimes_R^{\mathbf{L}} -$, *where* $\mathrm{C}(\mathfrak{a})$ *is the Čech complex on any set of generators of* $\mathfrak{a}$*. Similarly,* $\mathbf{L}\Lambda^{\mathfrak{a}} \cong \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\mathfrak{a}), -)$ *by Greenlees and May* [15, Sect. 2] *(with corrections by Schenzel* [27]*)*[2]*. For any* R*-module* M *one has by definition* $\mathrm{H}^i_{\mathfrak{a}}(M) = \mathrm{H}_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}M)$ *and* $\mathrm{H}_i^{\mathfrak{a}}(M) = \mathrm{H}_i(\mathbf{L}\Lambda^{\mathfrak{a}}M)$*.*

6.2. *Recall that for any* R*-module* M*, its* depth (or grade) w.r.t. $\mathfrak{a}$ *is the number*

$$\mathrm{depth}_R(\mathfrak{a}, M) = \inf\{i \mid \mathrm{Ext}^i_R(R/\mathfrak{a}, M) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}.$$

If M *is finitely generated, then this number is the common length all maximal* M*-sequences contained in* $\mathfrak{a}$*; see* [7, §1.2]*. Strooker* [31, Prop. 5.3.15] *shows that for every* M *one has:*

$$\inf\{i \,|\, \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\} = \mathrm{depth}_R(\mathfrak{a}, M).$$

Thus, if M *is finitely generated, then* $\inf\{i \,|\, \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\} = \mathrm{depth}_R M$*.*

The number $\sup\{i \,|\, \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$ *is less well understood; it is often called the* cohomological dimension *of* M *w.r.t.* $\mathfrak{a}$ *and denoted by* $\mathrm{cd}_R(\mathfrak{a}, M)$*. If* M *is finitely generated, then* $\mathrm{cd}_R(\mathfrak{m}, M) = \dim_R M$ *by* [6, Thms. 6.1.2 and 6.1.4]*.*

From 6.2 one gets the well-known fact that a (non-zero) finitely generated module M is Cohen–Macaulay with $t = \mathrm{depth}_R M = \dim_R M$ if and only if $\mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for $i \neq t$. In view of this, the next definition due to Zargar [35, Def. 2.1] is natural.

Definition 6.3 (Zargar)**.** A finitely generated R-module M is said to be *relative Cohen–Macaulay of cohomological dimension* t *w.r.t.* $\mathfrak{a}$ if $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i \neq t$.

The ring R is said to be *relative Cohen–Macaulay w.r.t.* $\mathfrak{a}$ if it is so when viewed as a module over itself, that is, if $c(\mathfrak{a}) := \mathrm{grade}_R(\mathfrak{a}, R) = \mathrm{cd}_R(\mathfrak{a}, R)$. In the terminology of Hellus and Schenzel [20], this means that $\mathfrak{a}$ is a *cohomologically complete intersection ideal.*

Example 6.4. Let $x_1, \dots, x_n \in R$ be a sequence of elements. It follows from [6, Thm. 3.3.1] (and 6.2) that any finitely generated R-module M for which $x_1, \dots, x_n$ is an M-sequence is relative Cohen–Macaulay of cohomological dimension n with respect to $\mathfrak{a} = (x_1, \dots, x_n)$. In particular, if $x_1, \dots, x_n$ is an R-sequence, then R is relative Cohen–Macaulay with respect to $\mathfrak{a} = (x_1, \dots, x_n)$ and one has $c(\mathfrak{a}) = n$.

---

[1] See also Brodmann and Sharp [6, Thm. 5.1.19], Alonso Tarrío, Jeremías López, and Lipman [1, Lem. 3.1.1] (with corrections by Schenzel [27]), and Porta, Shaul, and Yekutieli [26, Prop. 5.8].

[2] See also Porta, Shaul, and Yekutieli [26, Cor. 7.13] for a very clear exposition.

For a ring R that is relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ we now set out to study the category

$$\{M \in \mathsf{mod}(R) \mid \mathrm{H}^i_{\mathfrak{a}}(M) = 0 \text{ for all } i \neq t\} \quad (\text{for any } t)$$

of finitely generated relative Cohen–Macaulay of cohomological dimension t w.r.t. the ideal $\mathfrak{a}$. But first we extend the notion of relative Cohen–Macaulayness to the realm of all modules.

Definition 6.5. An R-module M is said to be $\mathfrak{a}$-*trivial* if $\mathrm{H}^{\mathfrak{a}}_i(M) = 0$ for all $i \in \mathbb{Z}$.

Remark 6.6. By Strooker [31, Prop. 5.3.15] and Simon [30, Thm. 2.4 and Cor. p. 970 part (ii)] $\mathfrak{a}$-trivialness of a module M is equivalent to any of the conditions:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i \in \mathbb{Z}$.
(ii) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M) = 0$ for all $i \in \mathbb{Z}$.
(iii) $\mathrm{Tor}^R_i(R/\mathfrak{a}, M) = 0$ for all $i \in \mathbb{Z}$.

We denote the Matlis duality functor $\mathrm{Hom}_R(-, E_R(k))$ by $(-)^\nu$, and for an R-module M we write $\delta_M \colon M \to M^{\nu\nu}$ for the canonical monomorphism.

Definition 6.7. An R-module M (not necessarily finitely generated) is said to be *relative Cohen–Macaulay of cohomological dimension t w.r.t.* $\mathfrak{a}$ if it satisfies the conditions:

(CM1) $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$ for all $i \neq t$.
(CM2) The canonical map $\psi_M \colon M \to \mathrm{H}^{\mathfrak{a}}_0(M)$ is an isomorphism.
(CM3) The cokernel of $\delta_M \colon M \to M^{\nu\nu}$ is $\mathfrak{a}$-trivial.

The category of all such R-modules is denoted $\mathsf{CM}^t_{\mathfrak{a}}(R)$.

Observation 6.8. Assume that R is $\mathfrak{m}$-adically complete, and hence also $\mathfrak{a}$-adically complete by [31, Cor. 2.2.6]. In this case, conditions (CM2) and (CM3) automatically hold for all finitely generated R-modules, see 6.1 and [10, Thm. 3.4.1(8)], so there is an equality,

$$\mathsf{CM}^t_{\mathfrak{a}}(R) \cap \mathsf{mod}(R) \;=\; \{M \in \mathsf{mod}(R) \mid \mathrm{H}^i_{\mathfrak{a}}(M) = 0 \text{ for all } i \neq t\}.$$

Thus, in this case, a finitely generated module is relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ in the sense of Definition 6.7 if and only if it is so in the sense of Zargar (Definition 6.3).

Example 6.9. For $\mathfrak{a} = 0$ we have $\Gamma_{\mathfrak{a}} = \mathrm{Id}_{\mathsf{Mod}(R)} = \Lambda^{\mathfrak{a}}$, and the only $\mathfrak{a}$-trivial module is the zero module. So for $\mathfrak{a} = 0$ one has

$$\mathsf{CM}^0_{\mathfrak{a}}(R) = \{\text{Matlis reflexive } R\text{-modules}\}.$$

Lemma 6.10. *Assume that R is relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ in the sense of Definition* 6.3 *and set $c = c(\mathfrak{a})$. In this case, the R-module $\mathrm{H}^c_{\mathfrak{a}}(R)$ has the following properties:*

(a) $\mathrm{H}^c_{\mathfrak{a}}(R)$ *has finite projective dimension.*

(b) $\operatorname{Ext}_R^i(\mathrm{H}_{\mathfrak{a}}^c(R), \mathrm{H}_{\mathfrak{a}}^c(R)) = 0$ *for all* $i > 0$.
(c) $\operatorname{Hom}_R(\mathrm{H}_{\mathfrak{a}}^c(R), \mathrm{H}_{\mathfrak{a}}^c(R))$ *is isomorphic to the* $\mathfrak{a}$*-adic completion* $\widehat{R}^{\mathfrak{a}}$*.*
(d) *One has* $\mathbf{R}\Gamma_{\mathfrak{a}} \cong \Sigma^{-c}(\mathrm{H}_{\mathfrak{a}}^c(R) \otimes_R^{\mathbf{L}} -)$ *and* $\mathrm{H}_{\mathfrak{a}}^i \cong \operatorname{Tor}_{c-i}^R(\mathrm{H}_{\mathfrak{a}}^c(R), -)$*.*
(e) *One has* $\mathbf{L}\Lambda^{\mathfrak{a}} \cong \Sigma^c \mathbf{R}\operatorname{Hom}_R(\mathrm{H}_{\mathfrak{a}}^c(R), -)$ *and* $\mathrm{H}_i^{\mathfrak{a}} \cong \operatorname{Ext}_R^{c-i}(\mathrm{H}_{\mathfrak{a}}^c(R), -)$*.*

Proof. Since $\mathrm{H}_{\mathfrak{a}}^i(R) \cong \mathrm{H}_{-i}(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \cong \mathrm{H}_{-i}(\mathrm{C}(\mathfrak{a}))$ by 6.1, the assumption that R is relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ means that the homology of $\mathrm{C}(\mathfrak{a})$ is concentrated in degree $-c$. Thus there are isomorphisms

$$\mathrm{H}_{\mathfrak{a}}^c(R) \cong \mathrm{H}_{-c}(\mathrm{C}(\mathfrak{a})) \cong \Sigma^c \mathrm{C}(\mathfrak{a})$$

in $\mathsf{D}(R)$. In view of this, part (a) follows as $\mathrm{C}(\mathfrak{a})$ has finite projective dimension [9, §5.8], parts (b) and (c) follow from [14, Lem. 1.9], and (d) and (e)) follow from 6.1. ☐

Definition 6.11. Naturality of ψ from 6.1 shows that for any R-module M there is an equality

$$\psi_{M^{\vee\vee}} \circ \delta_M = \mathrm{H}_0^{\mathfrak{a}}(\delta_M) \circ \psi_M$$

of homomorphisms $M \to \mathrm{H}_0^{\mathfrak{a}}(M^{\vee\vee})$; we write θ_M for this map.

Lemma 6.12. *An R-module M satisfies conditions* (CM2) *and* (CM3) *in Definition* 6.7 *if and only if it satisfies:*

(†) $\mathrm{H}_i^{\mathfrak{a}}(M^{\vee\vee}) = 0$ *for all* $i > 0$*, and*
(‡) $\theta_M \colon M \to \mathrm{H}_0^{\mathfrak{a}}(M^{\vee\vee})$ *is an isomorphism.*

Proof. "Only if": By (CM2) and [29, p. 238, second Lem., part (ii)] we get isomorphisms $\mathrm{H}_i^{\mathfrak{a}}(M) \cong \mathrm{H}_i^{\mathfrak{a}}(\mathrm{H}_0^{\mathfrak{a}}(M)) = 0$ for all $i > 0$. The short exact sequence

$$0 \to M \to M^{\vee\vee} \to C_M \to 0,$$

where the map $M \to M^{\vee\vee}$ is δ_M and $C_M = \operatorname{Coker}\delta_M$, induces a long exact sequence of local homology modules w.r.t $\mathfrak{a}$, and since C_M is $\mathfrak{a}$-trivial by (CM3), we conclude that $\mathrm{H}_i^{\mathfrak{a}}(\delta_M) \colon \mathrm{H}_i^{\mathfrak{a}}(M) \to \mathrm{H}_i^{\mathfrak{a}}(M^{\vee\vee})$ is an isomorphism for all $i \in \mathbb{Z}$. Thus (†) follows. As $\mathrm{H}_0^{\mathfrak{a}}(\delta_M)$ is an isomorphism, so is

$$\theta_M = \mathrm{H}_0^{\mathfrak{a}}(\delta_M) \circ \psi_M,$$

that is, (‡) holds.

"If": As (‡) holds, M has the form $M \cong \mathrm{H}_0^{\mathfrak{a}}(X)$ so [29, p. 238, second Lem., part (ii)] yields that $\psi_M \colon M \to \mathrm{H}_0^{\mathfrak{a}}(M)$ is an isomorphism, i.e., (CM2) holds, and $\mathrm{H}_i^{\mathfrak{a}}(M) = 0$ for $i > 0$. As $\theta_M = \mathrm{H}_0^{\mathfrak{a}}(\delta_M) \circ \psi_M$ and ψ_M are both isomorphisms, so is $\mathrm{H}_0^{\mathfrak{a}}(\delta_M)$. By (†) we also have $\mathrm{H}_i^{\mathfrak{a}}(M^{\vee\vee}) = 0$ for all $i > 0$, so the long exact sequence of local homology modules induced by $0 \to M \to M^{\vee\vee} \to C_M \to 0$ shows that $\mathrm{H}_i^{\mathfrak{a}}(C_M) = 0$ for all $i \in \mathbb{Z}$, i.e., (CM3) holds. ☐

We prove in Theorem 6.16 below that the category $\mathsf{CM}_{\mathfrak{a}}^t(R)$ is self-dual. The duality is realized via the following module which was already introduced by Zargar [36, Def. 2.3].

Definition 6.13 (Zargar)**.** Let R be relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ in the sense of Definition 6.3. With $c = c(\mathfrak{a})$ we set

$$\Omega_{\mathfrak{a}} = \mathrm{H}^c_{\mathfrak{a}}(R)^{\nu} = \mathrm{Hom}_R(\mathrm{H}^c_{\mathfrak{a}}(R), E_R(k)).$$

In the extreme cases $\mathfrak{a} = 0$ and $\mathfrak{a} = \mathfrak{m}$ the module $\Omega_{\mathfrak{a}}$ is well-understood:

Example 6.14. Any ring R is relative Cohen–Macaulay w.r.t. $\mathfrak{a} = 0$; in this case one has $c = 0$, $\mathrm{H}^0_{\mathfrak{a}}(R) = R$, and $\Omega_{\mathfrak{a}} = E_R(k)$.

Assume that R is Cohen–Macaulay (w.r.t. $\mathfrak{m}$) and $\mathfrak{m}$-adically complete. In this case, one has $c = \operatorname{depth} R = \dim R$ and $\mathrm{H}^c_{\mathfrak{m}}(R)$ is Artinian by [6, Thm. 7.1.3]. Thus $\Omega_{\mathfrak{m}} = \mathrm{H}^c_{\mathfrak{m}}(R)^{\nu}$ is finitely generated so Proposition 6.15 below shows that $\Omega_{\mathfrak{m}}$ is the dualizing module for R.

Proposition 6.15. *If R is $\mathfrak{m}$-adically complete and relative Cohen–Macaulay w.r.t. the ideal $\mathfrak{a}$, then $\Omega_{\mathfrak{a}}$ has finite injective dimension, $\mathrm{Ext}^i_R(\Omega_{\mathfrak{a}}, \Omega_{\mathfrak{a}}) = 0$ for $i > 0$, and $\mathrm{Hom}_R(\Omega_{\mathfrak{a}}, \Omega_{\mathfrak{a}}) \cong R$. Furthermore, in the derived category $\mathsf{D}(R)$ there is an isomorphism $\Omega_{\mathfrak{a}} \cong \Sigma^{-c}\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k)$.*

Proof. It is immediate from Lemma 6.10(a) that $\Omega_{\mathfrak{a}}$ has finite injective dimension. Part (e) of the same lemma shows that $\Omega_{\mathfrak{a}} \cong \Sigma^{-c}\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k)$ in $\mathsf{D}(R)$, and hence

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\Omega_{\mathfrak{a}}, \Omega_{\mathfrak{a}}) &\cong \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k), \mathbf{L}\Lambda^{\mathfrak{a}}E_R(k))\\ &\cong \mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\mathrm{Hom}_R(E_R(k), E_R(k)),\end{aligned}$$

where the last isomorphism comes from [12, (2.6)] and [26, Lem. 7.6]. As R is $\mathfrak{m}$-adically complete, we have $\mathbf{R}\mathrm{Hom}_R(E_R(k), E_R(k)) \cong R$, and thus the last expression above is the same as $\mathbf{L}\Lambda^{\mathfrak{a}}R \cong \widehat{R}^{\mathfrak{a}}$. As R is also $\mathfrak{a}$-adically complete, we get $\mathbf{R}\mathrm{Hom}_R(\Omega_{\mathfrak{a}}, \Omega_{\mathfrak{a}}) \cong R$. □

Theorem 6.16. *Assume that R is relative Cohen–Macaulay w.r.t. $\mathfrak{a}$ in the sense of Definition 6.3 and set $c = c(\mathfrak{a})$. For every integer t there is a duality:*

$$\mathsf{CM}^t_{\mathfrak{a}}(R) \underset{\mathrm{Ext}^{c-t}_R(-,\Omega_{\mathfrak{a}})}{\overset{\mathrm{Ext}^{c-t}_R(-,\Omega_{\mathfrak{a}})}{\rightleftarrows}} \mathsf{CM}^t_{\mathfrak{a}}(R)\,.$$

Proof. We consider the adjunction (F, G) from Example 3.3 with $\Gamma = R = \Lambda$ and $T = \Omega_{\mathfrak{a}}$. From Theorem 3.8 with $\ell = c - t$ we get that the functor $\mathrm{Ext}^{c-t}_R(-, \Omega_{\mathfrak{a}})$ yields a duality (a contravariant equivalence) on the category $\mathcal{F} := \mathsf{Fix}_{c-t}(\mathsf{Mod}(R))$, whose objects are those R-modules M that satisfy the following conditions:

(i) $\mathrm{Ext}^i_R(M, \Omega_{\mathfrak{a}}) = 0$ for all $i \neq c - t$.
(ii) $\mathrm{Ext}^i_R(\mathrm{Ext}^{c-t}_R(M, \Omega_{\mathfrak{a}}), \Omega_{\mathfrak{a}}) = 0$ for all $i \neq c - t$.
(iii) The canonical map $\eta^{c-t}_M \colon M \to \mathrm{Ext}^{c-t}_R(\mathrm{Ext}^{c-t}_R(M, \Omega_{\mathfrak{a}}), \Omega_{\mathfrak{a}})$ is an isomorphism.

We now show $\mathcal{F} = \mathsf{CM}^t_{\mathfrak{a}}(R)$, that is, we prove that an R-module M satisfies (i), (ii), and (iii) if and only if it satisfies (CM1), (CM2), and (CM3) in Definition 6.7. First note that

$$\operatorname{Ext}^i_R(M,\Omega_{\mathfrak{a}}) = \operatorname{Ext}^i_R(M,\mathrm{H}^c_{\mathfrak{a}}(R)^v) \cong \operatorname{Tor}^R_i(\mathrm{H}^c_{\mathfrak{a}}(R),M)^v \cong \mathrm{H}^{c-i}_{\mathfrak{a}}(M)^v,$$

where the last isomorphism is by Lemma 6.10(d). It follows that condition (i) is equivalent to (CM1). If (i) holds, then

$$\operatorname{Ext}^{c-t}_R(M,\Omega_{\mathfrak{a}}) \cong \Sigma^{c-t}\mathbf{R}\operatorname{Hom}_R(M,\Omega_{\mathfrak{a}})$$

in $\mathsf{D}(R)$, which explains the first isomorphism in the computation below. The second isomorphism below follows as $\Omega_{\mathfrak{a}} \cong \Sigma^{-c}\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k)$, see Proposition 6.15, and the third isomorphism comes from [12, (2.6)] and [26, Lem. 7.6]. The last isomorphism is by definition (see 6.1):

$$\begin{aligned}
&\operatorname{Ext}^i_R(\operatorname{Ext}^{c-t}_R(M,\Omega_{\mathfrak{a}}),\Omega_{\mathfrak{a}})\\
&\cong \mathrm{H}_{-i}\mathbf{R}\operatorname{Hom}_R(\Sigma^{c-t}\mathbf{R}\operatorname{Hom}_R(M,\Omega_{\mathfrak{a}}),\Omega_{\mathfrak{a}})\\
&\cong \mathrm{H}_{(c-t)-i}\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k)),\mathbf{L}\Lambda^{\mathfrak{a}}E_R(k))\\
&\cong \mathrm{H}_{(c-t)-i}\mathbf{L}\Lambda^{\mathfrak{a}}\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,E_R(k)),E_R(k))\\
&\cong \mathrm{H}^{\mathfrak{a}}_{(c-t)-i}(M^{vv}).
\end{aligned}$$

Thus, under assumption of (i), condition (ii) is equivalent to

$$(\dagger) \qquad \mathrm{H}^{\mathfrak{a}}_n(M^{vv}) = 0 \text{ for all } n > 0.$$

Setting $i = c - t$ in the computation above we get an isomorphism,

$$\alpha_M\colon\ \operatorname{Ext}^{c-t}_R(\operatorname{Ext}^{c-t}_R(M,\Omega_{\mathfrak{a}}),\Omega_{\mathfrak{a}}) \xrightarrow{\cong} \mathrm{H}^{\mathfrak{a}}_0(M^{vv}),$$

which identifies the map η^{c-t}_M from condition (iii) above with the map θ_M from Definition 6.11, that is, $\alpha_M \circ \eta^{c-t}_M = \theta_M$. So under assumption of (i), condition (iii) is equivalent to

$$(\ddagger) \qquad \theta_M \text{ is an isomorphism.}$$

Now apply Lemma 6.12. $\square$

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(Olgur Celikbas) Department of Mathematics, West Virginia University, Morgantown, WV 26506 U.S.A
olgur.celikbas@math.wvu.edu

(Henrik Holm) Department of Mathematical Sciences, Universitetsparken 5, University of Copenhagen, 2100 Copenhagen Ø, Denmark
holm@math.ku.dk
http://www.math.ku.dk/~holm/

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