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Homological properties of quantum permutation algebras

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Abstract. We show that $A_s(n)$, the coordinate algebra of Wang's quantum permutation group, is Calabi–Yau of dimension 3 when $n \geq 4$, and compute its Hochschild cohomology with trivial coefficients. We also show that, for a larger class of quantum permutation algebras, including those representing quantum symmetry groups of finite graphs, the second Hochschild cohomology group with trivial coefficients vanishes, and hence these algebras have the AC property considered in quantum probability: all cocycles can be completed to a Schürmann triple.

Contents

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1. Introduction

Let $n \geq 1$, and let $A_s(n)$ be the universal quantum permutation algebra: this means that $A_s(n)$ is the algebra presented by generators u_{ij}, $1 \leq i, j \leq n$, subject to the relations of permutation matrices

$$u_{ij}u_{ik} = \delta_{jk}u_{ij}, \quad u_{ji}u_{ki} = \delta_{jk}u_{ji}, \quad \sum_{j=1}^{n} u_{ij} = 1 = \sum_{j=1}^{n} u_{ji}.$$

It has a natural Hopf algebra structure given by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}.$$

The Hopf algebra $A_s(n)$ arose in Wang's work on quantum group actions on finite-dimensional algebras [40], and represents the quantum permutation group S_n^+, the largest compact quantum group acting on a set consisting of n points [40]. More generally, the Hopf algebra $A_s(n)$ is the universal cosemisimple Hopf algebra coacting on the algebra $\mathbb{C}^n$ [9].

The algebra $A_s(n)$ is infinite-dimensional if $n \geq 4$ [40], so that the quantum permutation group S_n^+ is infinite in that case, although it acts faithfully on a classical finite set. This intriguing property has brought a lot of attention on $A_s(n)$ among researchers in quantum group theory in the last twenty years, and a number of important contributions elucidate its structure: representation theory [2], operator-theoretic approximation properties [13], operator K-theory [38]. On the homological algebra side, the cohomological dimension of $A_s(n)$ has been computed recently in [11]. In this paper we go deeper in the homological study, proving the following result (the cohomology in the statement being Hochschild cohomology with trivial coefficients).

Theorem 1.1. *Assume that $n \geq 4$. The algebra $A_s(n)$ is Calabi–Yau of dimension 3, and we have*

$$H^p(A_s(n), \mathbb{C}) \simeq \begin{cases} \mathbb{C} & \text{if } p = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

The Calabi–Yau property, which has been the subject of numerous investigations in recent years, was named in [23], and is the algebraic generalization of the notion of orientable Poincaré duality group ([12], see [14, VIII.10]). One of its interests is that it provides a duality between Hochschild homology and cohomology [37], similar to Poincaré duality in algebraic topology.

In fact we will prove (Theorem 4.4) that the quantum symmetry algebra of a finite-dimensional semisimple normalizable measured algebra of dimension ≥ 4 is always twisted Calabi–Yau of dimension 3, see Section 4 for details. To prove this result, our main tools will be a new result that gives a condition that ensures that a Hopf subalgebra of a twisted Calabi–Yau algebra of dimension d still is a twisted Calabi–Yau of dimension d (see Theorem 3.1

in Section 3), techniques from recent papers of the first author [10, 11], and crucial results of Mrozinski [29] and of Wang–Yu–Zhang [41].

Notice that Theorem 1.1 states in particular that the second cohomology group $H^2(A_s(n), \mathbb{C})$ vanishes, and hence in particular $A_s(n)$ has the property called AC in [20]. The AC property is of particular interest in quantum probability and the study of Lévy processes on quantum groups: it means that all cocycles can be completed to a Schürmann triple. See [20] for details, and the recent survey [21] on these questions, where the AC property was shown for $A_s(n)$. In fact we will show that the vanishing result for the second cohomology (and hence the AC property) holds for a large class of quantum permutation algebras (quotients of $A_s(n)$), including those representing quantum symmetry groups of finite graphs, in the sense of [4]: see Theorem 5.2.

The paper is organized as follows. Section 2 consists of preliminaries. In Section 3 we discuss Hopf subalgebras of Hopf algebras that are twisted Calabi–Yau. In Section 4 we prove our results regarding the Calabi–Yau property for quantum symmetry algebras and Theorem 1.1. The final Section 5 is devoted to prove the vanishing of the second cohomology (with trivial coefficients) of a large class of quantum permutation algebras.

2. Preliminaries

2.1. Notations and conventions. We work over $\mathbb{C}$ (or over any algebraically closed field of characteristic zero). We assume that the reader is familiar with the theory of Hopf algebras and their tensor categories of comodules, as, e.g., in [25, 26, 28]. If A is a Hopf algebra, as usual, Δ, ε and S stand respectively for the comultiplication, counit and antipode of A. As usual, the augmentation ideal $\mathrm{Ker}(\varepsilon)$ is denoted A^+. We use Sweedler notation in the standard way.

The category of right A-comodules is denoted $\mathcal{M}^A$, the category of right A-modules is denoted $\mathcal{M}_A$, etc... The trivial right A-module is denoted $\mathbb{C}_\varepsilon$, the trivial left A-module is denoted ${}_\varepsilon\mathbb{C}$, and the trivial A-bimodule ${}_\varepsilon\mathbb{C}_\varepsilon$ is simply denoted $\mathbb{C}$. The set of A-module morphisms (resp. A-comodule morphisms) between two A-modules (resp. two A-comodules) V and W is denoted $\mathrm{Hom}_A(V, W)$ (resp. $\mathrm{Hom}^A(V, W)$). We will mostly consider right A-modules, and the corresponding Ext-spaces, denoted $\mathrm{Ext}^*_A(-, -)$, will always be considered in the category of right A-modules

We also assume some familiarity with homological algebra [42], but for the reader's convenience, we recall the definition of Hochschild (co)homology: if A is an algebra and M is an A-bimodule, the Hochschild cohomology spaces $H^*(A, M)$ are the cohomology spaces of the complex

$$0 \longrightarrow \mathrm{Hom}(\mathbb{C}, M) \xrightarrow{\delta} \mathrm{Hom}(A, M) \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{\delta} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{\delta} \cdots$$

where the differential $\delta\colon \operatorname{Hom}(A^{\otimes n}, M) \longrightarrow \operatorname{Hom}(A^{\otimes n+1}, M)$ is given by

$$\begin{aligned}\delta(f)(a_1 \otimes \cdots \otimes a_{n+1}) = {} & a_1 \cdot f(a_2 \otimes \cdots \otimes a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\ & + (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) \cdot a_{n+1}.\end{aligned}$$

The Hochschild cohomological dimension of A, which serves as a noncommutative analogue of the dimension of an algebraic variety, is defined by

$$\begin{aligned}\operatorname{cd}(A) &= \sup\{n : \ H^n(A, M) \neq 0 \text{ for some } A\text{-bimodule } M\} \in \mathbb{N} \cup \{\infty\} \\ &= \min\{n : \ H^{n+1}(A, M) = 0 \text{ for any } A\text{-bimodule } M\} \\ &= \operatorname{pd}_{{}_A\mathcal{M}_A}(A)\end{aligned}$$

where $\operatorname{pd}_{{}_A\mathcal{M}_A}(A)$ is the projective dimension of A in the category of A-bimodules.

The Hochschild homology spaces $H_*(A, M)$ are the homology spaces of the complex

$$\cdots \longrightarrow M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M \longrightarrow 0$$

where the differential $b\colon M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$ is given by

$$\begin{aligned}b(x \otimes a_1 \otimes \cdots \otimes a_n) = {} & x \cdot a_1 \otimes \cdots \otimes a_n \\ & + \sum_{i=1}^{n-1} (-1)^i x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n \cdot x \otimes a_1 \otimes \cdots \otimes a_{n-1}.\end{aligned}$$

As said in the introduction, the Calabi–Yau property is a condition that ensures a Poincaré type duality between Hochschild cohomology and homology. We will not give the definition here, since we will consider this property for Hopf algebras only, and that in this framework there is a simpler and more useful one (at least to us), to be given in the next subsection.

2.2. Hochschild cohomology, Hopf algebras, and the Calabi–Yau property. Let us first recall the well-known fact that for a Hopf algebra, the previous Hochschild (co)homology spaces can described by using suitable Ext and Tor on the category of left or right modules. Indeed, let A be Hopf algebra and let M be an A-bimodule. We then have

$$H_*(A, M) \simeq \operatorname{Tor}_*^A(\mathbb{C}_\varepsilon, M'), \quad H^*(A, M) \simeq \operatorname{Ext}_A^*(\mathbb{C}_\varepsilon, M'')$$

where M' and M'' are M having the respective left A-module and right A-module structures defined by

$$a \rightarrow x = a_{(2)} \cdot x \cdot S(a_{(1)}), \quad x \leftarrow a = S(a_{(1)}) \cdot x \cdot a_{(2)}.$$

See for example [10, 15], and the references therein.

Definition 2.1. A Hopf algebra A is said to be *twisted Calabi–Yau of dimension* $d \geq 0$ if it satisfies the following conditions.

(1) A is homologically smooth, i.e., the trivial A-module $\mathbb{C}_\varepsilon$ admits a finite resolution by finitely generated projective A-modules.
(2) $\mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, A_A) = 0$ if $i \neq d$, and $\mathrm{Ext}^d_A(\mathbb{C}_\varepsilon, A_A)$ is one-dimensional, so that there exists an algebra map $\alpha : A \to \mathbb{C}$ such that, as left A-module, $\mathrm{Ext}^d_A(\mathbb{C}_\varepsilon, A_A) \simeq {}_\alpha\mathbb{C}$. The algebra is said to be *Calabi–Yau* if $\alpha = \varepsilon$.

The equivalence of this definition with the usual one for twisted Calabi–Yau algebras has been discussed in several papers, we refer the reader to [41]. Notice that if A is twisted Calabi–Yau of dimension d, then $d = \mathrm{cd}(A)$.

The induced homological duality is as follows, see, e.g., [15]. For α such that $\mathrm{Ext}^d_A(\mathbb{C}, A) \simeq {}_\alpha\mathbb{C}$, consider the algebra anti-morphism $\theta\colon A \to A$ defined by $\theta(a) = S(a_{(1)})\alpha(a_{(2)})$. We then have, for any right A-module M,

$$\mathrm{Ext}^i_A(\mathbb{C}_\varepsilon, M) \simeq \mathrm{Tor}^A_{d-i}(\mathbb{C}_\varepsilon, {}_\theta M)$$

where ${}_\theta M$ is M having the left A-module structure given by $a \cdot x = x \cdot \theta(a)$. At the level of Hochschild cohomology, letting $\sigma = S\theta$, we have, for any A-bimodule M,

$$H^i(A, M) \simeq H_{d-i}(A, {}_\sigma M)$$

where ${}_\sigma M$ has the A-bimodule structure given by $a \cdot x \cdot b = \sigma(a) \cdot x \cdot b$.

The following easy lemma will be used in Section 4.

Lemma 2.2. *Let A be Hopf algebra, and assume that A is twisted Calabi–Yau of dimension d. If $\mathrm{Ext}^d_A(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \neq (0)$, then A is Calabi–Yau.*

Proof. Let $\alpha\colon A \to \mathbb{C}$ be the algebra map such that $\mathrm{Ext}^d_A(\mathbb{C}_\varepsilon, A_A) \simeq {}_\alpha\mathbb{C}$. By the previous homological duality, we have

$$\mathrm{Ext}^d_A(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \simeq \mathrm{Tor}^A_0(\mathbb{C}_\varepsilon, {}_\alpha\mathbb{C}) \simeq \begin{cases} 0 & \text{if } \alpha \neq \varepsilon \\ \mathbb{C} & \text{if } \alpha = \varepsilon. \end{cases}$$

Hence the assumption ensures that $\alpha = \varepsilon$, and A is Calabi–Yau, as claimed. □

As examples of Hopf algebras that are twisted Calabi–Yau, let us mention, for example, the group algebras of Poincaré duality groups [14], the coordinate algebras of the q-deformed quantum groups [15], the coordinate algebras of the free orthogonal quantum groups [17], and more generally the coordinate algebras of quantum symmetry groups of nondegenerate bilinear forms [10, 39, 43].

2.3. Exact sequences of Hopf algebras. A sequence of Hopf algebra maps

$$\mathbb{C} \to B \xrightarrow{i} A \xrightarrow{p} L \to \mathbb{C}$$

is said to be exact [1] if the following conditions hold:

(1) i is injective and p is surjective.
(2) $\mathrm{Ker}(p) = Ai(B)^+ = i(B)^+A$, where $i(B)^+ = i(B) \cap \mathrm{Ker}(\varepsilon)$.
(3)
$$\begin{aligned} i(B) = A^{\mathrm{co}L} &= \{a \in A : \ (\mathrm{id} \otimes p)\Delta(a) = a \otimes 1\} \\ &= {}^{\mathrm{co}L}A = \{a \in A : \ (p \otimes \mathrm{id})\Delta(a) = 1 \otimes a\}. \end{aligned}$$

Note that condition (2) implies $pi = \varepsilon 1$. We do not lose generality in assuming that $B \subset A$ is a Hopf subalgebra and that i is the inclusion map.

An exact sequence as above and such that A is faithfully flat as a right B-module is called strict [34]. If L is cosemisimple, then an exact sequence is automatically strict, by [35, Theorem 2], since $p \colon A \to L$ is then faithfully coflat.

The example we are interested in is the following one. Let $\mathcal{O}(\mathrm{SL}_q(2))$ be the coordinate algebra on quantum SL(2), with its standard generators a, b, c, d. Let $\mathcal{O}(\mathrm{PSL}_q(2))$ be the subalgebra generated by the elements xy, $x, y \in \{a, b, c, d\}$, and let $p \colon \mathcal{O}(\mathrm{SL}_q(2)) \to \mathbb{C}\mathbb{Z}_2$ be the Hopf algebra map defined by $\pi(a) = \pi(d) = g$ and $\pi(b) = \pi(c) = 0$, where g is the generator of the the cyclic group $\mathbb{Z}_2$. Then it is a direct verification that the sequence
$$\mathbb{C} \to \mathcal{O}(\mathrm{PSL}_q(2)) \to \mathcal{O}(\mathrm{SL}_q(2)) \xrightarrow{p} \mathbb{C}\mathbb{Z}_2 \to \mathbb{C}$$
is exact. This exact sequence is strict since $\mathbb{C}\mathbb{Z}_2$ is cosemisimple.

3. Hopf subalgebras and the Calabi–Yau property

Let $B \subset A$ be a Hopf subalgebra. The aim of this section is to provide a condition that ensures that if A is twisted Calabi–Yau of dimension d, then so is B. Our result is the following one.

Theorem 3.1. *Let $\mathbb{C} \to B \to A \to L \to \mathbb{C}$ be a strict exact sequence of Hopf algebras with bijective antipodes, with L finite-dimensional. The following assertions are equivalent.*

(1) *A is a twisted Calabi–Yau algebra of dimension d.*
(2) *$\mathrm{cd}(A)$ is finite and B is a twisted Calabi–Yau algebra of dimension d.*

The proof will be given at the end of the section. The proof of (1) $\Rightarrow$ (2) will be a consequence of two results of independent interest (Propositions 3.5 and 3.6), while the proof of (2) $\Rightarrow$ (1) will follow from Proposition 3.6 together with some considerations on resolutions by finitely generated projective modules from [14, Chapter VIII].

3.1. Relative integrals.

We will use the following notion.

Definition 3.2. Let $B \subset A$ be a Hopf subalgebra. *A right A/B-integral* is an element $t \in A$ such that for any $a \in A$, we have $ta - \varepsilon(a)t \in B^+A$. A right A/B-integral t is said to be nontrivial if $t \notin B^+A$.

Examples 3.3.

(1) Let A be a Hopf algebra. A right $A/\mathbb{C}$-integral is a right integral in A, in the usual sense [28, 31].
(2) Let $B \subset A$ be a Hopf subalgebra such that $B^+A = AB^+$, so that $L = A/B^+A$ is a quotient Hopf algebra. If $\tau \in L$ is a right integral, an element $t \in A$ such that $p(t) = \tau$ (where $p\colon A \to L$ is the canonical surjection) is a right A/B-integral.
(3) Let G be a discrete group, let $H \subset G$ be a subgroup of finite index and $g_1, \dots, g_n \in G$ be such that $G = \cup_{i=1}^n Hg_i$ (disjoint union). Then $t = \sum_{i=1}^n g_i$ is a right $\mathbb{C}G/\mathbb{C}H$-integral.

We will discuss, at the end of the section, an existence result for an A/B-integral, beyond the basic examples above.

We begin with a lemma, which illustrates the role of such an A/B-integral.

Lemma 3.4. *Let $B \subset A$ be a Hopf subalgebra, and let $t \in A$ be a right A/B-integral. Let M, N be right A-modules, and let $f\colon M \to N$ be a right B-linear map.*

(1) *The map $\tilde{f}\colon M \to N$, defined by $\tilde{f}(x) = f(x \cdot S(t_{(1)})) \cdot t_{(2)}$, is A-linear.*
(2) *If there exists an A-linear map $i\colon N \to M$ such that $fi = \mathrm{id}_N$, then $\tilde{f}i = \varepsilon(t)\mathrm{id}_N$.*

Proof. If M is an A-module, we denote, as usual, by M^A the space of A-invariants:

$$M^A = \{x \in M \mid x \cdot a = \varepsilon(a)x,\ \forall a \in A\}.$$

It is easy to check that $M^B \cdot t \subset M^A$ (this is an equality if $\varepsilon(t) \neq 0$).

Now if M and N are (right) A-modules, recall that $\mathrm{Hom}(M, N)$ admits a right A-module structure defined by

$$f \cdot a(x) = f(x \cdot S(a_{(1)})) \cdot a_{(2)}$$

and that $\mathrm{Hom}_A(M, N) = \mathrm{Hom}(M, N)^A$. Starting with $f \in \mathrm{Hom}_B(M, N)$, the previous remark that shows that $\tilde{f} = f \cdot t \in \mathrm{Hom}_A(M, N)$, as announced. The last statement is a direct verification. $\square$

We now will need a certain category of relative Hopf modules, considered in [35], see [32] as well. Let $B \subset A$ be a Hopf subalgebra. Then B^+A is co-ideal in A, so that A/B^+A is coalgebra and the quotient map

$$p\colon A \to A/B^+A$$

is a coalgebra map, and a right A-module map. We denote by $\mathcal{M}_A^C$ the category whose objects are right C-comodules and right A-modules V satisfying the following compatibility condition

$$(v \cdot a)_{(0)} \otimes (v \cdot a)_{(1)} = v_{(0)} \cdot a_{(1)} \otimes v_{(1)} \cdot a_{(2)}, \quad v \in V,\ a \in A.$$

The morphisms are the C-colinear and A-linear maps. As examples, A and C, endowed with the obvious structures, are objects of $\mathcal{M}_A^C$. Notice that if V is an object of $\mathcal{M}_A^C$, then

$$V^{\mathrm{co}C} = \{v \in V \mid v_{(0)} \otimes v_{(1)} = v \otimes p(1)\}$$

is a sub-B-module of V, since for $b \in B$, we have $p(b) = \varepsilon(b)p(1)$.

Proposition 3.5. *Let $B \subset A$ be a Hopf subalgebra. Assume that A has bijective antipode, that A is faithfully flat as a left or right B-module, and that there exists a nontrivial right A/B-integral $t \in A$.*

(1) *The coalgebra $C = A/B^+A$ is finite dimensional and A is finitely generated and projective as a right and left B-module.*
(2) *If A is homologically smooth, then so is B, with* $\mathrm{cd}(B) \leq \mathrm{cd}(A)$.
(3) *If moreover $\varepsilon(t) \neq 0$, then* $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Proof. First notice that by [33, Corollary 1.8], A is indeed projective as a left and right B-module. Let $t \in A$ be a nontrivial A/B-integral, and let V be the right subcomodule of C generated by $p(t)$:

$$V = \mathrm{Span}\{p(t_{(1)})\psi p(t_{(2)}),\ \psi \in C^*\}.$$

For $a \in A$, we have

$$\begin{aligned}
&p(ta) = \varepsilon(a)p(t) \Rightarrow p(ta_{(1)}) \otimes a_{(2)} = p(t) \otimes a \\
&\Rightarrow\ p(t_{(1)}a_{(1)}) \otimes p(t_{(2)}a_{(2)}) \otimes a_{(3)} = p(t_{(1)}) \otimes p(t_{(2)}) \otimes a \\
&\Rightarrow\ p(t_{(1)}a_{(1)}) \otimes p(t_{(2)}a_{(2)}) \cdot S(a_{(3)}) = p(t_{(1)}) \otimes p(t_{(2)}) \cdot S(a) \\
&\Rightarrow\ p(t_{(1)}a) \otimes p(t_{(2)}) = p(t_{(1)}) \otimes p(t_{(2)} \cdot S(a)).
\end{aligned}$$

Hence for $\psi \in C^*$, we have

$$\psi p(t_{(2)})p(t_{(1)}) \cdot a = \psi p(t_{(2)})p(t_{(1)}a) = \psi p(t_{(2)} \cdot S(a))p(t_{(1)})$$

and this shows that V is a sub-A-module of C. Hence V is an object of $\mathcal{M}_A^C$, and by [33, Lemma 1.3], we can use Theorem 3.7 and its proof in [32] to claim that the natural map

$$V^{\mathrm{co}C} \otimes_B A \to V,\ v \otimes_B a \mapsto v \cdot a$$

is an isomorphism (we could also deduce this from left-right variations on [35, Theorem 1, Theorem 2]). Since $C^{\mathrm{co}C}$ is one dimensional, generated by $p(1)$, we thus see that $V^{\mathrm{co}C}$ contains $p(1)$ (since V is nonzero), and hence $V = C$. Since V is finite-dimensional, we conclude that C is.

So let $a_1, \ldots, a_n \in A$ be elements in A such that $p(a_1), \ldots, p(a_n)$ linearly generate C, let W be the (finite-dimensional) right A-subcomodule of A generated by $a_1, \ldots, a_n$, and let BW be the left B-submodule generated by W. It is clear that BW is still a subcomodule, so is an object in the category of Hopf modules ${}_B\mathcal{M}^A$. By [35, Theorem 1] and [30, Remark 1.3], the functor

$${}_B\mathcal{M}^A \longrightarrow \mathcal{M}^C,\ M \longmapsto M/B^+M$$

is a category equivalence. Since BW and A have the same image by this functor (the object C), and $BW \subset A$, we have $BW = A$ and A is finitely generated as a left B-module, as well as a right B-module, by bijectivity of the antipode. This concludes the proof of (1), and (2) follows, since the restriction of a finitely generated projective A-module to a B-module is still finitely generated and projective, and then $\mathrm{cd}(B) \leq \mathrm{cd}(A)$.

To prove the converse inequality under the assumption that $\varepsilon(t) \neq 0$, we can assume that $m = \mathrm{cd}(B)$ is finite. Consider a resolution of the trivial (right) A-module

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon$$

by projective A-modules. These are in particular projective as B-modules by (1), so since $m = \mathrm{cd}(A)$, a standard argument yields an exact sequence of A-modules

$$0 \to K \xrightarrow{i} P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon$$

together with $r\colon P_m \to K$, a B-linear map such that $ri = \mathrm{id}_K$. Lemma 3.4 yields an A-linear map $\tilde{r}\colon P_m \to K$ such that $\tilde{r}i = \varepsilon(t)\mathrm{id}_K$. We thus obtain, since a direct summand of a projective is projective, a length m resolution of $\mathbb{C}_\varepsilon$ by projective A-modules, and we conclude that $\mathrm{cd}(A) \leq m$, as required. □

Proposition 3.6. *Let $B \subset A$ be a Hopf subalgebra. Assume that the antipode of A is bijective, that A is faithfully flat as a left or right B-module, that $B^+A = AB^+$ (so that $L = A/B^+A$ is a quotient Hopf algebra), and that L is finite-dimensional. Then we have, for any $M \in \mathcal{M}_A$ and any $N \in \mathcal{M}_A^L$,*

$$\mathrm{Ext}_A^*(M, N) \simeq \mathrm{Ext}_B^*(M_{|B}, N^{\mathrm{co}L}).$$

Proof. Since L is a finite-dimensional Hopf algebra, there exists a right integral $\tau \in L$ and a left integral $h\colon L \to \mathbb{C}$ on L, such that $h(\tau) = 1$ and $hS(\tau) \neq 0$ (see, e.g., [31, Theorem 10.2.2]).

If $f \in \mathrm{Hom}_B(M_{|B}, N^{\mathrm{co}L})$, we can view f as a B-linear map $M \to N$, so using an element $t \in A$ such that $p(t) = \tau$, Lemma 3.4 gives a linear map

$$\begin{aligned} \Psi\colon \mathrm{Hom}_B(M_{|B}, N^{\mathrm{co}L}) &\longrightarrow \mathrm{Hom}_A(M, N) \\ f &\longmapsto \tilde{f},\ \tilde{f}(x) = f(x \cdot S(t_{(1)})) \cdot t_{(2)}. \end{aligned}$$

To construct the inverse, notice that we have a map

$$\begin{aligned} E_N\colon N &\longrightarrow N^{\mathrm{co}L} \\ x &\longmapsto h(x_{(1)})x_{(0)} \end{aligned}$$

which is B-linear since h is. We therefore have a linear map

$$\begin{aligned} \Phi\colon \mathrm{Hom}_A(M, N) &\longrightarrow \mathrm{Hom}_B(M_{|B}, N^{\mathrm{co}L}) \\ g &\longmapsto E_N g. \end{aligned}$$

For $f \in \mathrm{Hom}_B(M_{|B}, N^{\mathrm{co}L})$ and $x \in M$, we have

$$\begin{aligned}
E_N\tilde{f}(x) &= E_N(f(x \cdot S(t_{(1)})) \cdot t_{(2)}) \\
&= h\left((f(x \cdot S(t_{(1)})) \cdot t_{(2)})_{(1)}\right)(f(x \cdot S(t_{(1)})) \cdot t_{(2)})_{(0)} \\
&= h\left(f(x \cdot S(t_{(1)}))_{(1)} \cdot t_{(3)}\right) f(x \cdot S(t_{(1)}))_{(0)} \cdot t_{(2)} \\
&= h(p(1) \cdot t_{(3)}) f(x \cdot S(t_{(1)})) \cdot t_{(2)} \quad (\text{since } f(M) \subset N^{\mathrm{co}L}) \\
&= f(x \cdot S(t_{(1)})) \cdot t_{(2)} hp(t_{(3)}).
\end{aligned}$$

The last term on the right belongs to $A^{\mathrm{co}L} = B$ ([33, Lemma 1.3], or [35, Theorem 1]), so the B-linearity of f gives

$$\begin{aligned}
E_N\tilde{f}(x) &= f(x \cdot S(t_{(1)}) \cdot t_{(2)}) hp(t_{(3)}) \\
&= f(x)h(p(t)) = f(x)h(\tau) = f(x).
\end{aligned}$$

Hence $E_N\tilde{f} = f$, and $\Phi\Psi = \mathrm{id}$. Now let $g \in \mathrm{Hom}_A(M, N)$ and $x \in M$. We have

$$\begin{aligned}
\widetilde{E_N g}(x) &= E_N g(x \cdot S(t_{(1)})) \cdot t_{(2)} = E_N(g(x) \cdot S(t_{(1)})) \cdot t_{(2)} \\
&= h(g(x)_{(1)} \cdot S(t_{(1)}))g(x)_{(0)} \cdot S(t_{(2)})t_{(3)} = h(g(x)_{(1)} \cdot S(t))g(x)_{(0)}.
\end{aligned}$$

For $a \in A$, we have $p(ta) = \varepsilon(a)p(t)$, so

$$p(a) \cdot S(t) = p(aS(t)) = pS(tS^{-1}(a)) = \varepsilon(a)pS(t) = \varepsilon(a)Sp(t).$$

Hence we have

$$\widetilde{E_N g}(x) = hSp(t)g(x) \quad \text{and} \quad \Psi\Phi = hSp(t)\mathrm{id}.$$

Combined with the already established identity $\Phi\Psi = \mathrm{id}$, this shows that Φ and Ψ are inverse isomorphisms.

The isomorphism $\Psi\colon \mathrm{Hom}_B(M_{|B}, N^{\mathrm{co}L}) \to \mathrm{Hom}_A(M, N)$ is easily seen to be functorial in M, so we can finish the proof by using standard arguments. Let $P_. \to M$ be a projective resolution by A-modules. Since A is projective as a B-module (by the previous proposition), the resolution $P_{.|B} \to M_{|B}$ is by projective B-modules, and the isomorphism Ψ induces isomorphisms of complexes

$$H^*(\mathrm{Hom}_B(P_{.|B}, N^{\mathrm{co}L})) \simeq H^*(\mathrm{Hom}_A(P_., N))$$

with the term on the left isomorphic to $\mathrm{Ext}^*_B(M_{|B}, N^{\mathrm{co}L})$, and the one on the right isomorphic to $\mathrm{Ext}^*_A(M, N)$. This concludes the proof. $\square$

3.2. Finiteness conditions for projective resolutions. We now discuss a convenient reformulation of homological smoothness. This is probably well-known, and is an adaptation of the case of groups [14], but in lack of suitable reference, we provide some details. The following definition is from [14, Chapter VIII].

Definition 3.7. Let A be an algebra and let M be an A-module.

(1) The A-module M is said to be of *type* FP_n, for $n \geq 0$, if there exists a partial projective resolution

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i finitely generated as an A-module.

(2) The A-module M is said to be of *type* FP_∞ if it is of type FP_n, for any $n \geq 0$.

If A is a Hopf algebra, then A is said to be of *type* FP_n, for $n \in \mathbb{N} \cup \{\infty\}$, if the trivial A-module $\mathbb{C}_\varepsilon$ is of type FP_n.

We have the following useful characterizations for types FP_n and FP_∞.

Proposition 3.8. *Let A be an algebra and let M be an A-module.*

(1) *For $n \geq 0$, the A-module M is of type FP_n if and only if M is finitely generated and for every partial projective resolution*

$$P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with $k < n$ and each P_i finitely generated, it holds that the A-module $\mathrm{Ker}(P_k \to P_{k-1})$ *is finitely generated.*

(2) *The A-module M is of type FP_∞ if and only if there exists a projective resolution*

$$\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with each P_i finitely generated.

Proof. The first assertion is [14, Proposition 4.3], and the second one is [14, Proposition 4.5]. □

We arrive at the announced characterization of homological smoothness for Hopf algebras.

Proposition 3.9. *Let A be a Hopf algebra. Then A is homologically smooth if and only if* $\mathrm{cd}(A) < \infty$ *and A is of type FP_∞.*

Proof. The direct implication is obvious, while the proof of the converse assertion is the same as the one of [14, Proposition 6.1]. Assume that $\mathrm{cd}(A) = n < \infty$ and that A is of type FP_∞, consider a projective resolution of $\mathbb{C}_\varepsilon$ as in (2) of Proposition 3.8

$$\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

and use (1) of Proposition 3.8 to get a resolution

$$0 \to P_n' \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

where each term is finitely generated. Since $n = \mathrm{cd}(A)$, the module P_n' is projective as well, and A is indeed homologically smooth. □

We conclude the subsection with a last ingredient for the proof of Theorem 3.1.

Proposition 3.10. *Let $B \subset A$ be a Hopf subalgebra with A projective and finitely generated as a right B-module. Then A is of type FP_n if and only if B is, for any $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. The restriction of a finitely generated and projective A-module to B is again finitely generated and projective, so it is clear that if A is of type FP_n, so is B. The proof of the converse assertion is the same as the one in [14, Proposition 5.1]. Assume that B is of type FP_n, and consider a projective A-module partial resolution

$$P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to \mathbb{C}_\varepsilon \to 0$$

with $k < n$ and each P_i finitely generated as an A-module. Then each P_i is finitely generated and projective as a B-module, and Proposition 3.8 ensures that $\operatorname{Ker}(P_k \to P_{k-1})$ is finitely generated as a B-module, and hence as an A-module. Proposition 3.8 thus ensures that A is of type FP_n. □

3.3. Proof of Theorem 3.1. We have now all the ingredients to prove Theorem 3.1:

Let $\mathbb{C} \to B \to A \to L \to \mathbb{C}$ be a strict exact sequence of Hopf algebras with bijective antipodes, with L finite-dimensional. Then, by Example 3.3 and Proposition 3.5, A is projective and finitely generated as a left and right B-module.

Assume that A is twisted Calabi–Yau of dimension d. Then $\operatorname{cd}(A) = d$ is finite, and B is homologically smooth by Proposition 3.5, with

$$\operatorname{cd}(B) \leq \operatorname{cd}(A) = d.$$

Moreover, Proposition 3.6, applied to $M = \mathbb{C}_\varepsilon$ and $N = A$, yields

$$\operatorname{Ext}^*_A(\mathbb{C}_\varepsilon, A_A) \simeq \operatorname{Ext}^*_B(\mathbb{C}_\varepsilon, A^{\operatorname{co}L}) = \operatorname{Ext}^*_B(\mathbb{C}_\varepsilon, B_B)$$

so we see simultaneously that $\operatorname{cd}(B) = d$ and that B is twisted Calabi–Yau of dimension d.

Assume now that $\operatorname{cd}(A)$ is finite and that B is twisted Calabi–Yau of dimension d. Then B is of type FP_∞, and by Proposition 3.10, so is A, and A is homologically smooth by Proposition 3.9. Then, similarly as above, we conclude from Proposition 3.6 that A is twisted Calabi–Yau of dimension d. □

Theorem 3.1, together with the known fact that $\mathcal{O}(\mathrm{SL}_q(2))$ is twisted Calabi–Yau of dimension 3 (see [15]) and the discussion at the end of the previous section, yield the following result.

Corollary 3.11. *The algebra $\mathcal{O}(\mathrm{PSL}_q(2))$ is twisted Calabi–Yau of dimension 3.*

3.4. On the existence of relative integrals. We finish the section with a discussion on the existence of an A/B-integral (these results will not be used in the latter sections). Let $B \subset A$ be a Hopf subalgebra. We have seen in Proposition 3.5 that, under some mild conditions, the existence of a nontrivial right A/B-integral implies that A is finitely generated as a B-module. The converse is not true, as shown by the following example.

Example 3.12. Let

$$A = \mathbb{C}\langle x, g, g^{-1} \mid gg^{-1} = 1 = g^{-1}g,\ x^2 = 0,\ xg = -gx\rangle$$

with the Hopf algebra structure defined by

$$\Delta(g) = g \otimes g,\ \Delta(x) = 1 \otimes x + x \otimes g,\ \varepsilon(g) = 1,\ \varepsilon(x) = 0,$$
$$S(g) = g^{-1},\ S(x) = -xg^{-1}$$

and let $B = \mathbb{C}[g, g^{-1}] \simeq \mathbb{C}\mathbb{Z}$. Then A is finitely generated as a B-module, the vector space A/B^+A has dimension 2 (with $\{p(1), p(x)\}$ as a linear basis, where $p : A \to A/B^+A$ is, as before, the canonical surjection), but there does not exist a nontrivial right A/B-integral.

Proof. Let $t \in A$ be a right A/B-integral, with

$$t = \sum_{i\in\mathbb{Z}} \lambda_i g^i + \sum_{j\in\mathbb{Z}} \mu_j g^j x.$$

Since $p(g^i) = p(1)$ and $p(g^j x) = p(x)$, for any i, j, we have

$$p(t) = \Big(\sum_i \lambda_i\Big)p(1) + \Big(\sum_j \mu_j\Big)p(x).$$

Since t is a right A/B-integral, we have $tg - t \in B^+A$ and $tx \in B^+A$, hence $p(tg) = p(t)$ and $p(tx) = 0$. We then have

$$p(tx) = p\Big(\sum_i \lambda_i g^i x\Big) = \Big(\sum_i \lambda_i\Big)p(x), \text{ hence } \sum_i \lambda_i = 0,$$

and

$$\begin{aligned} p(tg) &= p\Big(\sum_i \lambda_i g^{i+1} - \sum_j \mu_j g^{j+1} x\Big) = \Big(\sum_i \lambda_i\Big)p(1) - \Big(\sum_j \mu_j\Big)p(x) \\ &= -\Big(\sum_j \mu_j\Big)p(x). \end{aligned}$$

Hence $\sum_j \mu_j = 0$, and $p(t) = 0$: our integral is trivial. $\square$

On the positive side, we have the following result, inspired by considerations in [16, Section 3].

Proposition 3.13. *Let $B \subset A$ be a Hopf subalgebra, with A finitely generated as a left B-module. Assume that one of the following conditions holds.*

(1) *A is of Kac type, i.e., the square of the antipode is the identity.*

(2) *A is a compact Hopf algebra, i.e., A is the canonical dense Hopf $*$-algebra associated to a compact quantum group.*

Then there exists an A/B-integral t with $\varepsilon(t) \neq 0$.

Proof. First notice that S^2 induces an endomorphism of $C = A/B^+A$, that we denote S_C^2. Since A is finitely generated as a left B-module, the coalgebra C is finite-dimensional. Let x_i, $i \in I$, be a (finite) basis of C, with dual basis x_i^*. A natural candidate to provide a nontrivial A/B-integral is then

$$\tau = \sum_{i \in I} x_i^* \left(S_C^2(x_{i(1)})\right) x_{i(2)} \in C.$$

Indeed, one checks, very similarly to the Hopf algebra case (see [36, Proposition 1.1]), that $\tau \cdot a = \varepsilon(a)\tau$ for any $a \in A$ (recall that the right A-module structure on C is given by $p(a) \cdot a' = p(aa')$). Hence choosing $t \in A$ such that $p(t) = \tau$ provides an A/B-integral. We then have

$$\varepsilon(t) = \varepsilon(\tau) = \mathrm{Tr}(S_C^2).$$

If $S^2 = \mathrm{id}$, then $S_C^2 = \mathrm{id}$, and $\mathrm{Tr}(S_C^2) = \dim(C) \neq 0$ (since $C \neq 0$), so we have the announced result. In the compact case, recall, see, e.g., [26, Chapter 11], that $S^2 \colon A \to A$ preserves every subcoalgebra of A, and its restriction to such a coalgebra is a diagonalizable endomorphism with positive eigenvalues. Since a finite-dimensional subspace of a coalgebra is contained in a finite-dimensional subcoalgebra, there exists a finite-dimensional subcoalgebra $D \subset A$ such that $p_{|D} \colon D \to C$ is surjective. From the above properties of S^2, we see that $\mathrm{Tr}(S_C^2) > 0$, which concludes the proof. □

Corollary 3.14. *Let $B \subset A$ be a Hopf subalgebra, with A finitely generated as a left B-module. Assume that one of the following conditions holds.*

(1) *A is of Kac type, and faithfully flat as a left or right B-module.*
(2) *A is cosemisimple of Kac type.*
(3) *A is a compact Hopf algebra.*

Then $\mathrm{cd}(A) = \mathrm{cd}(B)$.

Proof. The proof follows by combining Propositions 3.5 and 3.13, together with the fact that a cosemisimple Hopf algebra is faithfully flat over its Hopf subalgebras [16, Theorem 2.1]. □

We believe that the conclusion of Corollary 3.14 holds whenever A is cosemisimple, but we have no proof of this.

4. Quantum symmetry algebras

We now will apply Corollary 3.11, combined with crucial results from [29] and [41], to show that $A_s(n)$ is Calabi–Yau of dimension 3 when $n \geq 4$. In fact our result concerns a more general family of algebras, that we recall now.

Let (R, φ) be a *finite-dimensional measured algebra*, i.e., R is a finite-dimensional algebra and $\varphi\colon R \to \mathbb{C}$ is a linear map (a measure on R) such that the associated bilinear map $R \times R \to \mathbb{C}$, $(x, y) \mapsto \varphi(xy)$ is nondegenerate. Thus a finite-dimensional measured algebra is a Frobenius algebra together with a fixed measure. A coaction of a Hopf algebra A on a finite-dimensional measured algebra (R, φ) is an A-comodule algebra structure on R such that $\varphi\colon R \to \mathbb{C}$ is A-colinear.

Definition 4.1. The universal Hopf algebra coacting on a finite-dimensional measured algebra (R, φ) is denoted $A_{\mathrm{aut}}(R, \varphi)$, and is called the *quantum symmetry algebra of* (R, φ).

The existence of $A_{\mathrm{aut}}(R, \varphi)$ (see [40] in the compact case with R semisimple and [7] in general) follows from standard Tannaka–Krein duality arguments. The following particular cases are of special interest.

(1) For $R = \mathbb{C}^n$ and $\varphi = \varphi_n$ the canonical integration map (with $\varphi_n(e_i) = 1$ for $e_1, \dots, e_n$ the canonical basis of $\mathbb{C}^n$), we have

$$A_{\mathrm{aut}}(\mathbb{C}^n, \varphi_n) = A_s(n),$$

the coordinate algebra of the quantum permutation group [40], discussed in the introduction.

(2) For $R = M_2(\mathbb{C})$ and $q \in \mathbb{C}^*$, let $\mathrm{tr}_q\colon M_2(\mathbb{C}) \to \mathbb{C}$ be the q-trace, i.e., for $g = (g_{ij}) \in M_2(\mathbb{C})$, $\mathrm{tr}_q(g) = qg_{11} + q^{-1}g_{22}$. Then we have $A_{\mathrm{aut}}(M_2(\mathbb{C}), \mathrm{tr}_q) \simeq \mathcal{O}(\mathrm{PSL}_q(2))$, the isomorphism being constructed using the universal property of $A_{\mathrm{aut}}(M_2(\mathbb{C}), \mathrm{tr}_q)$ (and the verification that it is indeed an isomorphism being a long and tedious computation, as in [19]).

Let (R, φ) be a finite-dimensional measured algebra. Since $\varphi \circ m$ is nondegenerate, where m is the multiplication of R, there exists a linear map $\delta\colon \mathbb{C} \to R \otimes R$ such that $(R, \varphi \circ m, \delta)$ is a left dual for R, i.e.,

$$((\varphi \circ m) \otimes \mathrm{id}_R) \circ (\mathrm{id}_R \otimes \delta) = \mathrm{id}_R = (\mathrm{id}_R \otimes (\varphi \circ m)) \circ (\delta \otimes \mathrm{id}_R).$$

Following [29], we put

$$\tilde{\varphi} = \varphi \circ m \circ (m \otimes \mathrm{id}_R) \circ (\mathrm{id}_R \otimes \delta)\colon R \to \mathbb{C}.$$

Using the definition of Frobenius algebra in terms of coalgebras, the coproduct of R is $\Delta = (m \otimes \mathrm{id}_R) \circ (\mathrm{id}_R \otimes \delta) = (\mathrm{id}_R \otimes m) \circ (\delta \otimes \mathrm{id}_R)$, and we have $\tilde{\varphi} = \varphi \circ m \circ \Delta$.

Definition 4.2. We say that (R, φ) (or φ) is *normalizable* if $\varphi(1) \neq 0$ and if there exists $\lambda \in \mathbb{C}^*$ such that $\tilde{\varphi} = \lambda\varphi$.

The condition that φ is normalizable is equivalent to require, in the language of [24, Definition 3.1], that $R/\mathbb{C}$ is a strongly separable extension with Frobenius system (φ, x_i, y_i), where $\delta(1) = \sum_i x_i \otimes y_i$. Hence it follows that if φ is normalizable, then R is necessarily a separable (semisimple) algebra.

Conversely, if R is semisimple, writing R as a direct product of matrix algebras, one easily sees the conditions that ensure that φ is normalizable, see [29].

The following result is [29, Corollary 4.9], generalizing earlier results from [2, 3, 18]:

Theorem 4.3. *Let (R, φ) is a finite-dimensional semisimple measured algebra with $\dim(R) \geq 4$ and φ normalizable. Then there exists an equivalence of $\mathbb{C}$-linear tensor categories*

$$\mathcal{M}^{A_{\text{aut}}(R,\varphi)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(\text{PSL}_q(2))}$$

for some $q \in \mathbb{C}^*$ with $q + q^{-1} \neq 0$.

The above parameter q is determined as follows. First consider $\lambda \in \mathbb{C}^*$ such that $\tilde{\varphi} = \lambda\varphi$ and choose $\mu \in \mathbb{C}^*$ such that $\mu^2 = \lambda\varphi(1)$. Then q is any solution of the equation $q+q^{-1} = \mu$ (recall that $\mathcal{O}(\text{PSL}_q(2)) = \mathcal{O}(\text{PSL}_{-q}(2))$, so the choice of μ does not play any role).

As an example, for $(\mathbb{C}^n, \varphi_n)$ as above (and $n \geq 4$), it is immediate that φ_n is normalizable with the corresponding λ equal to 1, and q is any solution of the equation $q + q^{-1} = \sqrt{n}$.

It was shown in [11] (Theorem 6.5 and the comments after) that if (R, φ) is a finite-dimensional semisimple measured algebra with $\dim(R) \geq 4$ and φ normalizable and such that $A_{\text{aut}}(R, \varphi))$ is cosemisimple, then

$$\text{cd}(A_{\text{aut}}(R, \varphi)) \leq 3,$$

with equality if φ is a trace. We generalize this result here, as follows.

Theorem 4.4. *Let (R, φ) be a finite-dimensional semisimple measured algebra with $\dim(R) \geq 4$ and φ normalizable. Then $A_{\text{aut}}(R, \varphi)$ is a twisted Calabi–Yau algebra of dimension 3, and is Calabi–Yau if φ is a trace.*

Proof. We have already pointed out that, by [29, Corollary 4.9], there exists $q \in \mathbb{C}^*$, with $q + q^{-1} \neq 0$, such that

$$\mathcal{M}^{A_{\text{aut}}(R,\varphi)} \simeq^{\otimes} \mathcal{M}^{\mathcal{O}(\text{PSL}_q(2))}$$

Put, for notational simplicity, $A = A_{\text{aut}}(R, \varphi)$. By the proof of Theorem 6.4 in [11] and the comments after [11, Theorem 6.5], the trivial Yetter–Drinfeld module over A has a finite resolution by finitely generated relative projective Yetter–Drinfeld modules over A, so since $\mathcal{O}(\text{PSL}_q(2))$ is twisted Calabi–Yau of dimension 3 by Corollary 3.11, it follows from [41, Theorem 4] that A is twisted Calabi–Yau of dimension 3.

If φ is a trace, it follows from the normalizability and from the analysis in Section 2 of [29] that μ as above is a real number such that $\mu^2 \geq 4$, so q as above is such that $\mathcal{O}(\text{PSL}_q(2))$ is cosemisimple, and hence A is cosemisimple. Thus by [11, Theorem 6.5] we have $H_b^3(A) \simeq \mathbb{C}$, where H_b^* denotes bialgebra cohomology [22], so by [11, Proposition 5.9] we have $H^3(A, \mathbb{C}) \simeq \text{Ext}_A^3(\mathbb{C}_\varepsilon, \mathbb{C}_\varepsilon) \neq (0)$. We conclude that A is Calabi–Yau by Lemma 2.2. $\square$

Proof of Theorem 1.1. We already know from the previous result that $A_s(n)$ is Calabi–Yau of dimension 3 for $n \geq 4$, and it remains to compute the cohomology spaces with trivial coefficients. We have, as usual, $H^0(A_s(n), \mathbb{C}) \simeq \mathbb{C}$, so the homological duality gives $H^3(A_s(n), \mathbb{C}) \simeq \mathbb{C}$. It is easy to check, using the fact that $A_s(n)$ is generated by projections, that $H^1(A_s(n), \mathbb{C}) = (0)$. We also have

$$H^2(A_s(n), \mathbb{C}) \simeq H_1(A_s(n), \mathbb{C}) \simeq \mathrm{Tor}_1^{A_s(n)}(\mathbb{C}_\varepsilon, {}_\varepsilon\mathbb{C})$$

where the latter space is the quotient of $A_s(n)$ by the subspace generated by the elements $ab - \varepsilon(a)b - \varepsilon(b)a,\ a, b \in A_s(n)$, and is easily seen to be zero. □

5. Second cohomology of a quantum permutation algebra

In this section we show that the vanishing of the second cohomology with trivial coefficients of $A_s(n)$ holds for a much more general class of quantum permutation algebras.

Given a matrix $d = (d_{ij}) \in M_n(\mathbb{C})$, we denote by $A_s(n, d)$ the quotient of $A_s(n)$ by the relations

$$\sum_{k=1}^n d_{ik} u_{kj} = \sum_{k=1}^n u_{ik} d_{kj}.$$

The algebras $A_s(n, d)$ are Hopf algebras, with structure maps induced by those of $A_s(n)$. When d is the adjacency matrix of a finite graph X, the Hopf algebra $A_s(n, d)$ represents the quantum automorphism group of the graph X, in the sense of [4].

Example 5.1. As interesting examples of Hopf algebras of the type $A_s(n, d)$, let us mention the coordinate algebras of the so-called quantum reflection groups [8, 6]. Let $p \geq 1$ and let $A_h^p(n)$ be the algebra presented by generators u_{ij}, $1 \leq i, j \leq n$, subject to the relations

$$u_{ij} u_{ik} = 0 = u_{ji} u_{ki},\ k \neq j, \quad \sum_{j=1}^n u_{ij}^p = 1 = \sum_{j=1}^n u_{ji}^p.$$

This is a Hopf algebra again [8], with $S(u_{ij}) = u_{ji}^{p-1}$, and we have

$$A_h^p(n) \simeq A_s(np, d),$$

where d is the adjacency matrix of the graph formed by n disjoint copies of an oriented p-cycle [5, Corollary 7.6].

Theorem 5.2. *We have $H^2(A_s(n, d), \mathbb{C}) = (0)$ for any matrix $d \in M_n(\mathbb{C})$.*

Remarks 5.3.

(1) It is an immediate verification that $H^1(A_s(n,d),\mathbb{C}) = (0)$. On the other hand, it is not possible to extend the vanishing result of Theorem 5.2 to degree 3, since $H^3(A_s(n),\mathbb{C}) \neq (0)$ for $n \geq 4$, by Theorem 1.1.

(2) It follows from Theorem 5.2 and Proposition 5.9 in [11] that
$$H_b^2(A_s(n,d)) = (0)$$
as well, where H_b^* denotes the bialgebra cohomology of Gerstenhaber–Schack [22].

(3) The recent L^2-Betti numbers computations in [27] show that the algebra $A_h^p(n)$ is not Calabi–Yau for $p \geq 2$, hence Theorem 1.1 cannot be generalized to the class of Hopf algebras $A_s(n,d)$.

5.1. A general lemma. In this subsection we present a general lemma, useful to show when a 2-cocycle of an augmented algebra is a coboundary.

We fix an augmented algebra (A,ε), and put as usual $A^+ = \mathrm{Ker}(\varepsilon)$. We denote by $T_3(A)$ the subspace of endomorphisms of $\mathbb{C}\oplus A^+\oplus\mathbb{C}$ represented by matrices
$$\begin{pmatrix}\lambda & f & \mu\\ 0 & a & b\\ 0 & 0 & \gamma\end{pmatrix}$$
where $\lambda,\mu,\gamma\in\mathbb{C}$, $f\in(A^+)^*$, $a\in A$, $b\in A^+$, and such a matrix acts on $(\alpha,c,\beta)\in\mathbb{C}\oplus A^+\oplus\mathbb{C}$ by
$$\begin{pmatrix}\lambda & f & \mu\\ 0 & a & b\\ 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}\alpha\\ c\\ \beta\end{pmatrix} = \begin{pmatrix}\lambda\alpha + f(c) + \mu\beta\\ ac+\beta b\\ \gamma\beta\end{pmatrix}.$$
It is straightforward to check that the composition of two such operators is again of the same type, with the composition corresponding to the naive matrix multiplication
$$\begin{pmatrix}\lambda & f & \mu\\ 0 & a & b\\ 0 & 0 & \gamma\end{pmatrix}\begin{pmatrix}\lambda' & f' & \mu'\\ 0 & a' & b'\\ 0 & 0 & \gamma'\end{pmatrix} = \begin{pmatrix}\lambda\lambda' & \lambda f' + f(a'-) & \lambda\mu' + f(b') + \mu\gamma'\\ 0 & aa' & ab'+b\gamma'\\ 0 & 0 & \gamma\gamma'\end{pmatrix}$$
and thus $T_3(A)$ is a subalgebra of $\mathrm{End}(\mathbb{C}\oplus A^+\oplus\mathbb{C})$.

Now let $\psi\colon A\to\mathbb{C}$ and $c\colon A\otimes A\to\mathbb{C}$ be normalized linear functionals, which means that $\psi(1)=0$ and $c(1\otimes 1)=0$. Recall (see Section 2) that c is a Hochschild 2-cocycle for the trivial bimodule $\mathbb{C}$ if and only if for any $a,b\in A$, we have
$$(\star)\qquad \varepsilon(a)c(b\otimes -) - c(ab\otimes -) + c(a\otimes b-) - c(a\otimes b)\varepsilon(-) = 0.$$
It is easy to see that a normalized 2-cocycle satisfies $c(a\otimes 1) = 0 = c(1\otimes a)$ for any $a\in A$.

Lemma 5.4. *Let $\psi\colon A \to \mathbb{C}$ be a normalized linear functional, and let $c \in Z^2(A,\mathbb{C})$ be a normalized 2-cocycle. The map*

$$\begin{aligned} \rho\colon A &\longrightarrow T_3(A) \\ a &\longmapsto \begin{pmatrix} \varepsilon(a) & c(a\otimes -) & \psi(a) \\ 0 & a & a-\varepsilon(a) \\ 0 & 0 & \varepsilon(a) \end{pmatrix} \end{aligned}$$

is an algebra map if and only if $c = \delta(-\psi)$

Proof. We have $\rho(1) = 1$ since c and ψ are normalized. For $a, b \in A$, the 2-cocycle condition gives

$$c(ab\otimes -)_{|A^+} = (\varepsilon(a)c(b\otimes -) + c(a\otimes b-))_{|A^+}.$$

It thus follows that $\rho(ab) = \rho(a)\rho(b)$ if and only if

$$c(a\otimes(b-\varepsilon(b))) = c(a\otimes b) = -\varepsilon(a)\psi(b) + \psi(ab) - \psi(a)\varepsilon(b) = \delta(-\psi)(a\otimes b)$$

and this gives the announced result. □

5.2. Proof of Theorem 5.2. We now let $A = A_s(n,d)$ as in the beginning of the section. The following lemma provides the main properties of 2-cocycles on A.

Lemma 5.5. *Let $c \in Z^2(A,\mathbb{C})$ be a normalized 2-cocycle.*

(1) *For $j \neq k$, we have*
$$\delta_{ij}c(u_{ik}\otimes -) + c(u_{ij}\otimes u_{ik}-) - c(u_{ij}\otimes u_{ik})\varepsilon(-) = 0.$$
(2) *For $j \neq k$, we have*
$$\delta_{ij}c(u_{ki}\otimes -) + c(u_{ji}\otimes u_{ki}-) - c(u_{ji}\otimes u_{ki})\varepsilon(-) = 0.$$
(3) *For $i \neq j$, $k \neq i$, $k \neq j$, we have $c(u_{ij}\otimes u_{ik}) = 0 = c(u_{ji}\otimes u_{ki})$.*
(4) *We have $c(u_{ij}\otimes u_{ii}) = c(u_{ii}\otimes u_{ij}) = c(u_{ij}\otimes u_{jj}) = c(u_{jj}\otimes u_{ij})$.*
(5) *We have*
$$\delta_{ij}c(u_{ij}\otimes -) - c(u_{ij}\otimes -) + c(u_{ij}\otimes u_{ij}-) - \varepsilon(-)c(u_{ij}\otimes u_{ij}) = 0.$$
(6) $c(u_{ii}\otimes u_{ii}) = \frac{1}{2}\sum_j c(u_{ij}\otimes u_{ij}) = \frac{1}{2}\sum_j c(u_{ji}\otimes u_{ji})$.
(7) *We have for $i \neq j$, $c(u_{ji}\otimes u_{ji}) = -c(u_{ii}\otimes u_{ji})$.*
(8) *Put $X_{ij} = \sum_k d_{ik}u_{kj} = \sum_k u_{ik}d_{kj}$. Then*
$$c(u_{ii}\otimes X_{ij}) = c(X_{ij}\otimes u_{ii}) = c(X_{ij}\otimes u_{jj}) = c(u_{jj}\otimes X_{ij}).$$

Proof. (1) (resp. (2)) follows from the cocycle identity $(\star)$ applied to

$$a = u_{ij} \quad \text{and} \quad b = u_{ik}$$

(resp. to $a = u_{ji}$ and $b = u_{ki}$). The first (resp. second) identity in (3) follows from (1) (resp. (2)) applied to u_{ii}.

In (1), for $i = j \neq k$, we get

$$c(u_{ik}\otimes -) + c(u_{ii}\otimes u_{ik}-) - c(u_{ii}\otimes u_{ik})\varepsilon(-) = 0,$$

which gives, applied to u_{ii} and to u_{kk},

$$c(u_{ii} \otimes u_{ik}) = c(u_{ik} \otimes u_{ii}), \quad c(u_{ik} \otimes u_{kk}) = c(u_{ii} \otimes u_{ik}).$$

We get the remaining relation in (4) using (2) at $i = j$. (5) follows from the cocycle identity ($\star$) applied to $a = u_{ij} = b$, and to get (6), we sum (5) over j to get

$$c(u_{ii} \otimes -) + \sum_j c(u_{ij} \otimes u_{ij}-) - \sum_j c(u_{ij} \otimes u_{ij})\varepsilon(-) = 0,$$

which gives in particular

$$c(u_{ii} \otimes u_{ii}) + c(u_{ii} \otimes u_{ii}^2) = 2c(u_{ii} \otimes u_{ii}) = \sum_j c(u_{ij} \otimes u_{ij}).$$

The proof that

$$c(u_{ii} \otimes u_{ii}) = \frac{1}{2} \sum_j c(u_{ji} \otimes u_{ji})$$

is similar. (7) follows from (1) at $i = j \neq k$, applied at u_{ik}. To prove (8), first notice that

$$u_{ii}X_{ij} = d_{ij}u_{ii} = X_{ij}u_{ii}, \quad u_{jj}X_{ij} = d_{ij}u_{jj} = X_{ij}u_{jj}.$$

The cocycle identity ($\star$), applied to $a = u_{ii}$, $b = X_{ij}$, gives

$$c(X_{ij} \otimes -) - c(u_{ii}X_{ij} \otimes -) + c(u_{ii} \otimes X_{ij}-) - c(u_{ii} \otimes X_{ij})\varepsilon(-) = 0$$

and hence

$$c(X_{ij} \otimes u_{jj}) - c(u_{ii}X_{ij} \otimes u_{jj}) + c(u_{ii} \otimes X_{ij}u_{jj}) - c(u_{ii} \otimes X_{ij}) = 0$$

which gives $c(X_{ij} \otimes u_{jj}) = c(u_{ii} \otimes X_{ij})$. Using (4), we also have

$$c(X_{ij} \otimes u_{jj}) = \sum_k d_{ik}c(u_{kj} \otimes u_{jj}) = \sum_k d_{ik}c(u_{jj} \otimes u_{kj}) = c(u_{jj} \otimes X_{ij})$$

and similarly

$$c(X_{ij} \otimes u_{ii}) = c(u_{ii} \otimes X_{ij}).$$

This concludes the proof of the lemma. $\square$

Lemma 5.6. *Let $c \in Z^2(A, \mathbb{C})$ be a normalized 2-cocycle. Put, for any i, j, $\lambda_{ij} = -c(u_{ii} \otimes u_{ij})$. Then there exists an algebra map $\rho_c \colon A \to T_3(A)$ such that*

$$\rho_c(u_{ij}) = \begin{pmatrix} \delta_{ij} & c(u_{ij} \otimes -) & \lambda_{ij} \\ 0 & u_{ij} & u_{ij} - \delta_{ij} \\ 0 & 0 & \delta_{ij} \end{pmatrix}.$$

Proof. Let t_{ij} be the element of $T_3(A)$ corresponding to the above matrix. We have to check that the elements t_{ij} satisfy the defining relations of A. Using the fact that c is normalized, we easily see that $\sum_{j=1}^n t_{ij} = 1$, and similarly, using Relations (4) in Lemma 5.5, we see that $\sum_{j=1}^n t_{ji} = 1$. We have

$$t_{ij}t_{ik}$$
$$= \begin{pmatrix} \delta_{ij}\delta_{ik} & \delta_{ij}c(u_{ik}\otimes -) + c(u_{ij}\otimes u_{ik}-) & \delta_{ij}\lambda_{ik} + c(u_{ij}\otimes (u_{ik}-\delta_{ik})) + \lambda_{ij}\delta_{ik} \\ 0 & u_{ij}u_{ik} & u_{ij}u_{ik} - \delta_{ik}u_{ij} + \delta_{ik}u_{ij} - \delta_{ij}\delta_{ik} \\ 0 & 0 & \delta_{ij}\delta_{ik} \end{pmatrix}$$
$$= \begin{pmatrix} \delta_{ij}\delta_{ik} & \delta_{ij}c(u_{ik}\otimes -) + c(u_{ij}\otimes u_{ik}-) & \delta_{ij}\lambda_{ik} + c(u_{ij}\otimes u_{ik}) + \lambda_{ij}\delta_{ik} \\ 0 & u_{ij}u_{ik} & u_{ij}u_{ik} - \delta_{ij}\delta_{ik} \\ 0 & 0 & \delta_{ij}\delta_{ik} \end{pmatrix}.$$

Hence for $j \neq k$ we have

$$t_{ij}t_{ik} = \begin{pmatrix} 0 & \delta_{ij}c(u_{ik}\otimes -) + c(u_{ij}\otimes u_{ik}-) & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $x = -\delta_{ij}c(u_{ii}\otimes u_{ik}) + c(u_{ij}\otimes u_{ik}) - c(u_{ii}\otimes u_{ij})\delta_{ik}$. By (1) in Lemma 5.5, we have $(\delta_{ij}c(u_{ik}\otimes -) + c(u_{ij}\otimes u_{ik}-))_{|A^+} = 0$. Moreover by (3) and (4) in Lemma 5.5 we have

$$-\delta_{ij}c(u_{ii}\otimes u_{ik}) + c(u_{ij}\otimes u_{ik}) - c(u_{ii}\otimes u_{ij})\delta_{ik} = 0.$$

Hence $t_{ij}t_{ik} = 0$ for $j \neq k$. We have also

$$t_{ij}^2 = \begin{pmatrix} \delta_{ij} & \delta_{ij}c(u_{ij}\otimes -) + c(u_{ij}\otimes u_{ij}-) & 2\delta_{ij}\lambda_{ij} + c(u_{ij}\otimes u_{ij}) \\ 0 & u_{ij} & u_{ij} - \delta_{ij} \\ 0 & 0 & \delta_{ij} \end{pmatrix}.$$

We have

$$(\delta_{ij}c(u_{ij}\otimes -) + c(u_{ij}\otimes u_{ij}-))_{|A^+} = c(u_{ij}\otimes -)_{|A^+}$$

by (5) in Lemma 5.5, and $-2\delta_{ij}c(u_{ii}\otimes u_{ij}) + c(u_{ij}\otimes u_{ij}) = -c(u_{ii}\otimes u_{ij})$ by (7) and (4) in Lemma 5.5, hence $t_{ij}^2 = t_{ij}$.

The proof that for $t_{ji}t_{ki} = 0$ for $j \neq k$ is similar, using (2), (3) and (4). Finally that $\sum_k d_{ik}t_{kj} = \sum_k t_{ik}d_{kj}$ follows from (4) and (8) in Lemma 5.5. □

We are now ready to prove Theorem 5.2. Let $c \in Z^2(A, \mathbb{C})$ be a Hochschild 2-cocycle, that we can assume to be normalized without changing its cohomology class.

A representation $\rho\colon A \to T_3(A)$ is necessarily of the form

$$a \longmapsto \begin{pmatrix} \varepsilon_1(a) & C(a) & \psi(a) \\ 0 & f(a) & g(a) \\ 0 & 0 & \varepsilon_2(a) \end{pmatrix}$$

for algebra maps

$$\varepsilon_1, \varepsilon_2\colon A \to \mathbb{C}, \quad f\colon A \to A$$

and linear maps

$$C\colon A \to (A^+)^*, \quad g\colon A \to A^+, \quad \psi\colon A \to \mathbb{C}.$$

Using the definition of the representation ρ_c of Lemma 5.6, we see that for any $a \in A$ we have

$$\rho_c(a) = \begin{pmatrix} \varepsilon(a) & C(a) & \psi(a) \\ 0 & a & a - \varepsilon(a) \\ 0 & 0 & \varepsilon(a) \end{pmatrix}$$

with moreover, for any a, b,

$$C(ab) = \varepsilon(a)C(b) + C(a)(b-).$$

Since $C(u_{ij}) = c(u_{ij} \otimes -)$, an easy induction shows that $C(a) = c(a \otimes -)$ for any a. We conclude from Lemma 5.4 that c is a coboundary, as needed.

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