

Inverse spectral analysis for a class of finite band symmetric matrices

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ABSTRACT. In this note, we solve an inverse spectral problem for a class of finite band symmetric matrices. We provide necessary and sufficient conditions for a matrix valued function to be a spectral function of the operator corresponding to a matrix in our class and give an algorithm for recovering this matrix from the spectral function. The reconstructive algorithm is applicable to matrices which cannot be treated by known inverse block matrix methods. Our approach to the inverse problem is based on the rational interpolation theory developed in a previous paper.

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1. Introduction

This work deals with the direct and inverse spectral analysis of a class of finite symmetric band matrices with emphasis in the inverse problems of characterization and reconstruction. Inverse spectral problems for band matrices have been studied extensively in the particular case of Jacobi matrices (see for instance [CG05, dBG78, dRK12, GeS97, GrW76, Hoc67, Hoc74, Hoc79, NU97a, NU97b, Ram93] for the finite case and [dRKS12, dRKS13, GaG89, GeS97, Gus78, Hal67, SilW06, SilW08] for the infinite case). Works

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dealing with band matrices non-necessarily tridiagonal are not so abundant (see [BK81, GoK09, GoK10, Kud98, Kud99, MatH81, Zag04, Zag08] for the finite case and [BecO03, GoK08] for the infinite case).

Let \mathcal{H} be a finite dimensional Hilbert space with a fixed orthonormal basis $\{\delta_k\}_{k=1}^N$ in it. For any $j = 0, 1, \dots, n$ with $n < N$, consider the operator D_j whose matrix representation with respect to $\{\delta_k\}_{k=1}^N$ is a diagonal real matrix, i.e., $D_j \delta_k = d_k^{(j)} \delta_k$ for all $k = 1, \dots, N$, where $d_k^{(j)}$ is a real number. Also, let S be the shift operator, that is,

$$S\delta_k = \begin{cases} \delta_{k+1} & k = 1, \dots, N - 1 \\ 0 & k = N. \end{cases}$$

The object of our considerations in this note is the symmetric operator

$$A := D_0 + \sum_{j=1}^n S^j D_j + \sum_{j=1}^n D_j (S^*)^j.$$

Hence, the matrix representation of A with respect to $\{\delta_k\}_{k=1}^N$ is an Hermitian band matrix with real entries which is denoted by \mathcal{A} . Alongside the matrix \mathcal{A} , for any $j \in \{0, \dots, n\}$, we consider the diagonal matrix \mathcal{D}_j being the matrix representation with respect to $\{\delta_k\}_{k=1}^N$ of the operator D_j .

We assume that the diagonals \mathcal{D}_j ($j \in \{0, \dots, n\}$) satisfy the following conditions. The diagonal farthest from the main one, that is

$$\mathcal{D}_n = \text{diag}\{d_k^{(n)}\},$$

is such that one of the following alternatives takes place:

- For some $m_1 < N - n + 1$, all the numbers $d_1^{(n)}, \dots, d_{m_1-1}^{(n)}$ are strictly positive and $d_{m_1}^{(n)} = \dots = d_{N-n}^{(n)} = 0$.
- All the elements of the sequence \mathcal{D}_n are positive which we convene to mean that $m_1 = N - n + 1$. In this case, we define $m_j = N - n + j$ for $j = 2, \dots, n$.

If $m_1 < N - n + 1$, we say that \mathcal{D}_n undergoes a degeneration at m_1 and, then, the elements $d_{m_1+1}^{(n-1)}, \dots, d_{N-n+1}^{(n-1)}$ of \mathcal{D}_{n-1} behave in the same way as the elements of \mathcal{D}_n , that is, one of the following alternatives takes place:

- For some m_2 such that $m_1 < m_2 < N - n + 2$, $d_{m_1+1}^{(n-1)}, \dots, d_{m_2-1}^{(n-1)} > 0$ and $d_{m_2}^{(n-1)} = \dots = d_{N-n+1}^{(n-1)} = 0$.
- $d_k^{(n-1)} > 0$ for $k = m_1 + 1, \dots, N - n + 1$ and we define $m_2 = N - n + 2$ and $m_j = N - n + j$ for $j = 3, \dots, n$.

We continue applying this rule up to some $j_0 \leq n - 1$ such that $m_{j_0} < N - n + j_0$ and $m_{j_0+1} = N - n + j_0 + 1$. Finally, we define $m_j = N - n + j$ for $j = j_0 + 2, \dots, n$.

Note that if one assumes that $m_j + 1 < m_{j+1}$ for $j = 0, \dots, j_0 - 1$, i.e., there are no consecutive degenerations, then the elements of \mathcal{D}_{n-j} satisfy

$$d_{m_j+1}^{(n-j)}, \dots, d_{m_{j+1}-1}^{(n-j)} > 0,$$

$$d_{m_j+1}^{(n-j)} = \dots = d_{N-n+j}^{(n-j)} = 0$$

for $j = 0, \dots, j_0 - 1$, with $m_0 = 0$. The entries of \mathcal{D}_{n-j_0} satisfy

$$d_{m_{j_0}+1}^{(n-j_0)}, \dots, d_{N-n+j_0}^{(n-j_0)} > 0.$$

The diagonal \mathcal{D}_{n-j} undergoes a degeneration at m_{j+1} for $j \in \{0, \dots, j_0 - 1\}$. When $j_0 = 0$, there is no degeneration of the diagonal \mathcal{D}_n . Note that \mathcal{D}_2 is the innermost diagonal where a degeneration may occur. Observe also that, in all cases, one has the set of numbers $\{m_1, \dots, m_n\}$.

Definition 1. Fix the natural numbers n and N such that $n < N$. All the matrices satisfying the above properties for a given set of numbers $\{m_i\}_{i=1}^n$ are denoted by $\mathcal{M}(n, N)$. Note that in this notation, N represents the dimension and $2n + 1$ is the number of diagonals of the matrices.

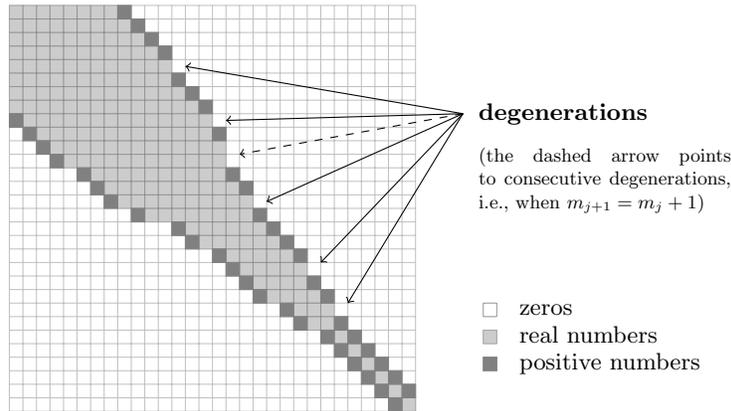


FIGURE 1. The structure of a matrix in $\mathcal{M}(n, N)$

An example of a matrix in $\mathcal{M}(3, 7)$, when $m_1 = 3$, $m_2 = 5$ and $m_3 = 7$ is the following.

$$A = \begin{pmatrix} d_1^{(0)} & d_1^{(1)} & d_1^{(2)} & d_1^{(3)} & 0 & 0 & 0 \\ d_1^{(1)} & d_2^{(0)} & d_2^{(1)} & d_2^{(2)} & d_2^{(3)} & 0 & 0 \\ d_1^{(2)} & d_2^{(1)} & d_3^{(0)} & d_3^{(1)} & d_3^{(2)} & 0 & 0 \\ d_1^{(3)} & d_2^{(2)} & d_3^{(1)} & d_4^{(0)} & d_4^{(1)} & d_4^{(2)} & 0 \\ 0 & d_2^{(3)} & d_3^{(2)} & d_4^{(1)} & d_5^{(0)} & d_5^{(1)} & 0 \\ 0 & 0 & 0 & d_4^{(2)} & d_5^{(1)} & d_6^{(0)} & d_6^{(1)} \\ 0 & 0 & 0 & 0 & 0 & d_6^{(1)} & d_7^{(0)} \end{pmatrix}.$$

Here we say that the matrix \mathcal{A} underwent a degeneration of the diagonal \mathcal{D}_3 in $m_1 = 3$ and a degeneration of \mathcal{D}_2 in $m_2 = 5$. Observe that $j_0 = 2$ and that $d_1^{(3)}, d_2^{(3)}, d_4^{(2)}, d_6^{(1)}$ are positive numbers.

It is known that the dynamics of a finite linear mass-spring system is characterized by the spectral properties of a finite Jacobi matrix [Gla04, Mar05] (see Figure 2) when the system is within the regime of validity of the Hooke law. The entries of the Jacobi matrix are determined by the masses and spring constants of the system [dRK12, dRKS12, dRKS13, Gla04, Mar05]. The movement of the mechanical system of Figure 2 is a superposition of harmonic oscillations whose frequencies are given by the spectrum of the Jacobi operator. Analogously, one can deduce that a matrix in $\mathcal{M}(n, N)$

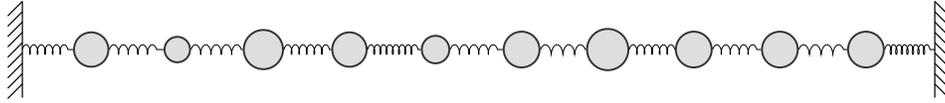


FIGURE 2. Mass-spring system corresponding to a Jacobi matrix

models a linear mass-spring system where the interaction extends to all the n neighbors of each mass (see Appendix A). For instance, if the matrix is in $\mathcal{M}(2, 10)$ and no degeneration of the diagonals occurs, viz. $m_1 = 9$, the corresponding mass-spring system is given in Figure 3. If for another matrix

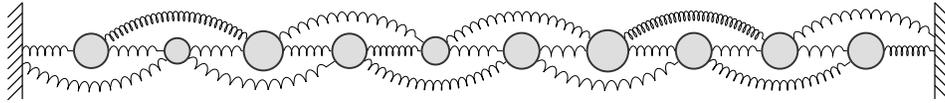


FIGURE 3. Mass-spring system of a matrix in $\mathcal{M}(2, 10)$: non-degenerated case

in $\mathcal{M}(2, 10)$, one has degeneration of the diagonals, for instance $m_1 = 4$, the corresponding mass-spring system is given in Figure 4.

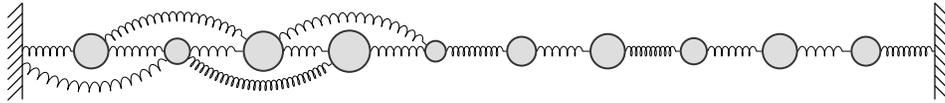


FIGURE 4. Mass-spring system of a matrix in $\mathcal{M}(2, 10)$: degenerated case

In this work, the approach to the inverse spectral analysis of the operators whose matrix representation belongs to $\mathcal{M}(n, N)$ is based on the one used in [Kud98, Kud99], but it allows to treat the case of arbitrary

n . An important ingredient of the method used here is the linear interpolation of n -dimensional vector polynomials, recently developed in [KPS15]. The linear interpolation theory of [KPS15] is a nontrivial generalization of the rational interpolation theory developed in [GoK09] from ideas given in [Kud98, Kud99]. It is on the basis of the results of [KPS15] that the inverse spectral theory developed in [Kud98, Kud99] is extended here to band matrices with $2n + 1$ diagonals ($n \in \mathbb{N}$). This generalization required some new developments, in particular, we modified the technique used in the reconstruction of the matrices in the class $\mathcal{M}(n, N)$ which was concocted having uniqueness of the reconstruction in mind. Indeed, the class $\mathcal{M}(2, N)$ turns out to be a subclass of the class studied in [Kud98, Kud99] for which uniqueness of the reconstruction is guaranteed.

The main results of this paper are:

- (A) A complete characterization of the spectral functions (measures) of operators whose matrix representation with respect to some orthonormal basis is an element of $\mathcal{M}(n, N)$ (Proposition 2.1).
- (B) A characterization for degenerations of the diagonals of \mathcal{A} in terms of the presence of polynomials of zero norm in $L_2(\mathbb{R}, \sigma)$, where σ is the spectral measure of the operator A .
- (C) An algorithm for reconstructing the matrix from the corresponding spectral measure (Section 4).
- (D) Uniqueness of reconstruction (see Theorem 4.1).

After finishing the first version of this paper, we learned about [MarS15] (see Acknowledgments). This book considers inverse spectral problems for a class of band symmetric finite matrices and introduces the notion of *restricted spectral data* for which any matrix-valued spectral function in (A) is a particular case. Some of the results of [MarS15] concerning this matter are based on [LM13]. Theorem 8 in [MarS15, Sec. 15] gives necessary and sufficient conditions for the matrix to have restricted spectral data from which the matrix can be recovered. Moreover, from the results of [MarS15, Sec. 14], one knows when the reconstruction is unique. Theorem 9 in [MarS15, Sec. 15] characterizes the restricted spectral data when the matrix can be reconstructed from them. The class $\mathcal{M}(n, N)$ satisfies [MarS15, Thm. 8 Sec. 15] and the distribution of positive numbers in Figure 1 guarantees that the matrix is not “decomposable” [MarS15, Sec. 12] and satisfies the conditions for a unique reconstruction [MarS15, Sec. 14]. Thus, some of our results, and particularly (D), are actually contained in [MarS15]. It is worth remarking that (A) is a different characterization of a particular case of restricted spectral data from which the matrix can be reconstructed. The algorithm of reconstruction (C) differs from the one of [MarS15] and, remarkably, it can be applied without essential modifications to spectral functions with infinitely many points of increase (see [KPS17]). Finally, (B) has no analogue in [MarS15] and introduces *inner* boundary conditions (see Section 2).

It is known that if there exists a natural number K such that $N = Kn$, then a band matrix with $2n + 1$ diagonals can be reduced to a tridiagonal block matrix. However, the spectral theory for tridiagonal block matrices requires that the off-diagonal block matrices be invertible. The matrices in $\mathcal{M}(n, N)$ do not satisfy this requirement when there is a degeneration of the diagonals. The technique for recovering a matrix from its spectral function developed in this paper is applicable to any element in $\mathcal{M}(n, N)$ even when N is not an integer multiple of n .

This paper is organized as follows. The next section deals with the direct spectral analysis of the operators under consideration. In this section, a family of spectral functions is constructed for each element in $\mathcal{M}(n, N)$. In Section 3, the connection of the spectral analysis and the interpolation problem is established. Section 4 treats the problem of reconstruction and characterization. In Section 5, we discuss alternative approaches to the inverse spectral problem and give a comparative analysis with the method given in Section 4. Appendix A gives a brief account of how to deduce the band symmetric matrix associated with a mass-spring system from the dynamical equations.

2. The spectral function

Consider $\varphi = \sum_{k=1}^N \varphi_k \delta_k \in \mathcal{H}$ and the equation

$$(1) \quad (A - zI)\varphi = 0, \quad z \in \mathbb{C}.$$

We know that the equation has nontrivial solutions only for a finite set of z .

From (1) one obtains a system of N equations, where each equation, given by a fixed $k \in \{1, \dots, N\}$, is of the form

$$(2) \quad \sum_{i=0}^{n-1} d_{k-n+i}^{(n-i)} \varphi_{k-n+i} + d_k^{(0)} \varphi_k + \sum_{i=1}^n d_k^{(i)} \varphi_{k+i} = z\varphi_k,$$

where it has been assumed that

$$(3a) \quad \varphi_k = 0, \quad \text{for } k < 1,$$

$$(3b) \quad \varphi_k = 0, \quad \text{for } k > N.$$

One can consider (3) as boundary conditions where (3a) is the condition at the left endpoint and (3b) is the condition at the right endpoint.

The system (2) with (3), restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$, can be solved recursively whenever the first n entries of the vector φ are given. Let $\varphi^{(j)}(z)$ ($j \in \{1, \dots, n\}$) be a solution of (2) for all $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$ such that

$$(4) \quad \langle \delta_i, \varphi^{(j)}(z) \rangle = t_{ji}, \quad \text{for } i = 1, \dots, n,$$

where $\mathcal{T} = \{t_{ji}\}_{j,i=1}^n$ is an upper triangular real matrix and $t_{jj} \neq 0$ for all $j \in \{1, \dots, n\}$. In (4) and in the sequel, we consider the inner product in \mathcal{H} to be antilinear in its first argument.

The condition given by (4) can be seen as the initial conditions for the system (2) and (3a). We emphasize that given the boundary condition at the left endpoint (3a) and the initial condition (4), the system restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$ has a unique solution for any fixed $j \in \{1, \dots, n\}$ and $z \in \mathbb{C}$.

Remark 2.1. Note that the properties of the matrix \mathcal{J} guarantee that the collection of vectors $\{\varphi^{(j)}(z)\}_{j=1}^n$ is a fundamental system of solutions of (2) restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$ with the boundary condition (3a).

The entries of the vector $\varphi^{(j)}(z)$ are polynomials, so we denote $P_k^{(j)}(z) := \varphi_k^{(j)}(z)$, for all $k \in \{1, \dots, N\}$. And, define

$$Q_i^{(j)}(z) := (z - d_{m_i}^{(0)})P_{m_i}^{(j)}(z) - \sum_{k=0}^{n-1} d_{m_i-n+k}^{(n-k)}P_{m_i-n+k}^{(j)}(z) - \sum_{k=1}^{n-i} d_{m_i}^{(k)}P_{m_i+k}^{(j)}(z)$$

for $i \in \{1, \dots, n\}$ (it is assumed that the last sum is zero when $i = n$).

It is worth remarking that the polynomials $\{P_k^{(j)}(z)\}_{k=1}^N$ and $\{Q_i^{(j)}(z)\}_{i=1}^n$ depend on the initial conditions given by the matrix \mathcal{J} .

Define the matrix

$$\mathcal{Q}(z) := \begin{pmatrix} Q_1^{(1)}(z) & \dots & Q_1^{(n)}(z) \\ \vdots & \ddots & \vdots \\ Q_n^{(1)}(z) & \dots & Q_n^{(n)}(z) \end{pmatrix}, \quad \text{for } z \in \mathbb{C}.$$

There exists a solution of (1) for a given $z \in \mathbb{C}$ if and only if there is a vector $(\beta_1(z), \dots, \beta_n(z))^t$ such that

$$(5) \quad \mathcal{Q}(z) \begin{pmatrix} \beta_1(z) \\ \vdots \\ \beta_n(z) \end{pmatrix} = 0.$$

Indeed, since $\{\varphi^{(j)}(z)\}_{j=1}^n$ is a fundamental system of solutions of (2) restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$ with the boundary condition (3a), the vector $\beta(z)$, given by

$$(6) \quad \beta(z) = \sum_{j=1}^n \beta_j(z)\varphi^{(j)}(z),$$

is a solution of (2) restricted to $k \in \{1, 2, \dots, N\} \setminus \{m_i\}_{i=1}^n$, satisfying (3a), for any collection of complex numbers $\{\beta_j(z)\}_{j=1}^n$. Thus, it follows from (2) and (3b) that

$$(A - zI)\beta(z) = \sum_{k=1}^N c_k(z)\delta_k,$$

where

$$c_k(z) := \begin{cases} \sum_{j=1}^n \beta_j(z) Q_i^{(j)}(z) & \text{if } k = m_i, \text{ for all } i = 1, \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (6) is a solution of (1) if and only if

$$(7) \quad \sum_{j=1}^n \beta_j(z) Q_i^{(j)}(z) = 0$$

for all $i \in \{1, \dots, n\}$, which is equivalent to (5).

Lemma 2.1. *Let $\tilde{n}(z) := \dim \ker(A - zI)$. Then,*

$$\text{rank}(\mathcal{Q}(z)) = n - \tilde{n}(z).$$

Observe that $\tilde{n}(z) \leq n$ for all $z \in \mathbb{C}$.

Proof. This is straightforward. Having fixed $z \in \mathbb{C}$, the Kronecker–Capelli–Rouché Theorem (see [IP86, Chap.3 Secs.1-2]) implies that the dimension of the space of solutions of (5) is equal to $n - \text{rank}(\mathcal{Q}(z))$. \square

Immediately from Lemma 2.1 it follows that

$$\text{spec}(A) = \{z \in \mathbb{C} : \det \mathcal{Q}(z) = 0\}.$$

Fix $j \in \{1, \dots, n\}$. For $\varphi^{(j)}(z_0)$ to be a solution of (1), the equation

$$(8) \quad Q_i^{(j)}(z_0) = 0$$

should be satisfied for any $i \in \{1, \dots, n\}$. The conditions (8) can be seen as *inner* boundary conditions (of the right endpoint type) for the difference equation (2). Note that the degeneration of diagonals gives rise to inner boundary conditions.

Let $\{x_k\}_{k=1}^N$ be such that $x_k \in \text{spec}(A)$ for $k = 1, \dots, N$. Note that the elements of this sequence have been enumerated taking into account the multiplicity of eigenvalues. Also, let $\alpha(x_k)$ be the corresponding eigenvectors such that

$$\langle \alpha(x_k), \alpha(x_l) \rangle = \delta_{kl}, \quad \text{with } k, l \in \{1, \dots, N\}.$$

It follows from Remark 2.1 that, for any fixed $k = 1, \dots, N$, there are complex numbers $\alpha_j(x_k)$ ($j = 1, \dots, n$) such that

$$(9) \quad \alpha(x_k) = \sum_{j=1}^n \alpha_j(x_k) \varphi^{(j)}(x_k)$$

for any $k \in \{1, \dots, N\}$. Clearly, by construction

$$(10) \quad \sum_{j=1}^n |\alpha_j(x_k)| > 0 \quad \text{for all } k \in \{1, \dots, N\}.$$

Additionally, since $\{\alpha(x_k)\}_{k=1}^N$ is a basis of \mathcal{H} , it follows from (4) that

$$(11) \quad \sum_{k=1}^N |\alpha_j(x_k)| > 0 \quad \text{for all } j \in \{1, \dots, n\}.$$

By (7) and the fact that $\alpha(x_k) \in \ker(A - x_k I)$, it follows that

$$(12) \quad \sum_{j=1}^n \alpha_j(x_k) Q_i^{(j)}(x_k) = 0 \quad \text{for all } i \in \{1, \dots, n\}$$

is true.

Now, define the matrix valued function

$$(13) \quad \sigma(t) := \sum_{x_k < t} \sigma_k,$$

where

$$(14) \quad \sigma_k = \begin{pmatrix} |\alpha_1(x_k)|^2 & \alpha_1(x_k)\overline{\alpha_2(x_k)} & \dots & \alpha_1(x_k)\overline{\alpha_n(x_k)} \\ \alpha_2(x_k)\overline{\alpha_1(x_k)} & |\alpha_2(x_k)|^2 & \dots & \alpha_2(x_k)\overline{\alpha_n(x_k)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n(x_k)\overline{\alpha_1(x_k)} & \alpha_n(x_k)\overline{\alpha_2(x_k)} & \dots & |\alpha_n(x_k)|^2 \end{pmatrix}$$

is a rank-one, nonnegative matrix (cf. [Kud98, Sec. 1]). Note that

$$\sigma(t) = \sigma^{\mathcal{J}}(t)$$

depends on the initial conditions given by \mathcal{J} .

We have thus arrived at the following:

Proposition 2.1. *The matrix valued function $\sigma(t)$ has the following properties:*

- (i) *It is a nondecreasing monotone step function.*
- (ii) *Each jump of the function is a matrix whose rank is not greater than n .*
- (iii) *The sum of the ranks of all jumps is equal to N (the dimension of the space \mathcal{H}).*

Remark 2.2. A nondecreasing matrix valued function satisfying (ii)–(iii) uniquely determines a Borel matrix valued measure with finite support (see [AG93, Sec. 72]).

For any matrix valued function $\sigma(t)$ satisfying properties (i)–(iii), there is a collection of vectors $\{\alpha(x_k)\}_{k=1}^N$ satisfying (10) and (11) such that $\sigma(t)$ is given by (13) and (14) (cf. [Kud98, Thm. 2.2]).

If $\mathcal{J} = I$, then $\sigma_{ij}(t) = \langle \delta_i, E(t)\delta_j \rangle$ ($i, j \in \{1, \dots, n\}$), where $E(t)$ is the spectral resolution of A . Indeed,

$$\begin{aligned} \langle \delta_i, E(t)\delta_j \rangle &= \left\langle \delta_i, \sum_{x_l < t} \langle \alpha(x_l), \delta_j \rangle \alpha(x_l) \right\rangle = \sum_{x_l < t} \langle \alpha(x_l), \delta_j \rangle \langle \delta_i, \alpha(x_l) \rangle \\ &= \sum_{x_l < t} \overline{\alpha_j(x_l)} \alpha_i(x_l) = \sigma_{ij}(t). \end{aligned}$$

Therefore, in this case, the matrix valued function $\sigma(t)$ is the spectral function of the operator A with respect to $\{\delta_k\}_{k=1}^n$.

Definition 2. The set of all matrix valued functions $\sigma(t)$ given by (13) and (14), where the collection of vectors $\{\alpha(x_k)\}_{k=1}^N$ satisfies (10) and (11), is denoted by $\mathfrak{M}(n, N)$.

Note that any matrix valued function in $\mathfrak{M}(n, N)$ satisfies (i)–(iii) of Proposition 2.1. On the basis of what has been discussed we refer to the matrix valued functions in $\mathfrak{M}(n, N)$ as spectral functions. Alternatively, by Remark 2.2, one can consider the corresponding matrix valued measures (spectral measures).

Consider the Hilbert space $L_2(\mathbb{R}, \sigma)$, where σ is the spectral function corresponding to the operator A given by (13) and (14) (see [AG93, Sec. 72]). We agree that the inner product $\langle \cdot, \cdot \rangle$ is antilinear in its first argument. Clearly, the property (iii) implies that $L_2(\mathbb{R}, \sigma)$ is an N -dimensional space and in each equivalence class there is an n -dimensional vector polynomial.

Define the vector polynomials in $L_2(\mathbb{R}, \sigma)$

$$(15) \quad \mathbf{q}_i(z) := (Q_i^{(1)}(z), \dots, Q_i^{(n)}(z))^t$$

for all $i \in \{1, \dots, n\}$, and

$$(16) \quad \mathbf{p}_k(z) := (P_k^{(1)}(z), \dots, P_k^{(n)}(z))^t$$

for all $k \in \{1, \dots, N\}$.

Lemma 2.2. *The vector polynomials $\{\mathbf{p}_k(z)\}_{k=1}^N$ defined by (16) satisfy*

$$\langle \mathbf{p}_j, \mathbf{p}_k \rangle_{L_2(\mathbb{R}, \sigma)} = \delta_{jk}$$

for $j, k \in \{1, \dots, N\}$.

Proof.

$$\begin{aligned} \langle \mathbf{p}_j, \mathbf{p}_k \rangle_{L_2(\mathbb{R}, \sigma)} &= \sum_{l=1}^N \langle \mathbf{p}_j(x_l), \sigma_l \mathbf{p}_k(x_l) \rangle \\ &= \sum_{l=1}^N \left(\sum_{s=1}^n \alpha_s(x_l) \overline{P_j^{(s)}(x_l)} \right) \sum_{s=1}^n \overline{\alpha_s(x_l)} P_k^{(s)}(x_l) \\ &= \sum_{l=1}^N \langle \delta_j, \alpha(x_l) \rangle \langle \alpha(x_l), \delta_k \rangle = \delta_{jk}, \end{aligned}$$

where it has been used that $\delta_k = \sum_{i=1}^n \langle \alpha(x_i), \delta_k \rangle \alpha(x_i)$. □

Let $U : \mathcal{H} \rightarrow L_2(\mathbb{R}, \sigma)$ be the isometry given by $U\delta_k \mapsto \mathbf{p}_k$ for all $k \in \{1, \dots, N\}$. Under this isometry, the operator A becomes the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \sigma)$. Indeed,

$$\begin{aligned} \langle \delta_j, A\delta_k \rangle &= \left\langle \sum_{l=1}^N \langle \alpha(x_l), \delta_j \rangle \alpha(x_l), A \sum_{s=1}^N \langle \alpha(x_s), \delta_k \rangle \alpha(x_s) \right\rangle \\ &= \sum_{l=1}^N \langle \delta_j, \alpha(x_l) \rangle \left\langle \alpha(x_l), \sum_{s=1}^N x_s \langle \alpha(x_s), \delta_k \rangle \alpha(x_s) \right\rangle \\ &= \sum_{l=1}^N \langle \delta_j, \alpha(x_l) \rangle x_l \langle \alpha(x_l), \delta_k \rangle \\ &= \langle \mathbf{p}_j, t\mathbf{p}_k \rangle_{L_2(\mathbb{R}, \sigma)}. \end{aligned}$$

If the matrix \mathcal{T} in (4) turns out to be the identity matrix, i.e., $\mathcal{T} = I$, then it can be shown that U^{-1} is the isomorphism corresponding to the canonical representation of the operator A [AG93, Sec. 75], that is,

$$\delta_k = U^{-1}\mathbf{p}_k = \sum_{j=1}^n P_k^{(j)}(A)\delta_j$$

for all $k \in \{1, \dots, N\}$.

Remark 2.3. The matrix representation of the multiplication operator in $L_2(\mathbb{R}, \sigma)$ with respect to the basis $\{\mathbf{p}_1(z), \dots, \mathbf{p}_N(z)\}$ is again the matrix \mathcal{A} . Thus,

$$(17) \quad \sum_{i=0}^{n-1} d_{k-n+i}^{(n-i)} \mathbf{p}_{k-n+i}(z) + d_k^{(0)} \mathbf{p}_k(z) + \sum_{i=1}^n d_k^{(i)} \mathbf{p}_{k+i}(z) = z\mathbf{p}_k(z)$$

for $k = 1, \dots, N$, where it is assumed that $\mathbf{p}_l = 0$ whenever $l < 1$. Also, one verifies that

$$(18) \quad \mathbf{q}_j(z) = (z - d_{m_j}^{(0)})\mathbf{p}_{m_j}(z) - \sum_{i=0}^{n-1} d_{m_j-n+i}^{(n-i)} \mathbf{p}_{m_j-n+i}(z) - \sum_{i=1}^{n-j} d_{m_j}^{(i)} \mathbf{p}_{m_j+i}(z)$$

for all $j \in \{1, \dots, n\}$, where the last sum vanishes when $j = n$.

The relationship between the spectral functions $\sigma^{\mathcal{T}} = \sigma$ and σ^I for an arbitrary \mathcal{T} is given by the following lemma.

Lemma 2.3. *Fix a natural number $N > n$. For any $n \times n$ upper triangular real matrix with no zeros in the main diagonal \mathcal{T} , the spectral function $\sigma^{\mathcal{T}}$ given in (13) satisfies*

$$\mathcal{T}^* \sigma^{\mathcal{T}} \mathcal{T} = \sigma^I.$$

Proof. Let \mathcal{T} be $n \times n$ upper triangular real matrix with no zeros in the main diagonal. Then, by (9) one has

$$\langle \delta_j, \alpha(x_l) \rangle = \sum_{i=1}^j \alpha_i(x_l) t_{ij}, \quad \forall j \in \{1, \dots, n\}.$$

Now, for the particular case, when $\mathcal{T} = I$, one considers

$$\sigma^I(t; i, j) = \sum_{x_l < t} \alpha'_i(x_l) \overline{\alpha'_j(x_l)}.$$

Therefore, $\langle \delta_j, \alpha(x_l) \rangle_{\mathbb{C}^N} = \alpha'_j(x_l)$ and

$$(19) \quad \sigma^I(t; i, j) = \sum_{x_l < t} \sum_{k=1}^i \alpha_k(x_l) t_{ki} \sum_{s=1}^j \overline{\alpha_s(x_l) t_{ks}}.$$

Observe that

$$\mathcal{T}^* \begin{pmatrix} \alpha_1(x_l) \\ \vdots \\ \alpha_n(x_l) \end{pmatrix} = \begin{pmatrix} \alpha_1(x_l) t_{11} \\ \sum_{k=1}^2 \alpha_k(x_l) t_{k2} \\ \vdots \\ \sum_{k=1}^n \alpha_k(x_l) t_{kn} \end{pmatrix},$$

and by (19)

$$\sigma^I(t; i, j) = \left(\mathcal{T}^* \sigma^{\mathcal{T}} \mathcal{T} \right) (t; i, j). \quad \square$$

An immediate consequence of the previous lemma is the following assertion:

Corollary 2.1. *Fix $N > n$. For any $n \times n$ upper triangular matrix \mathcal{T} with no zeros in the main diagonal, one has*

$$\mathcal{T}^* \int_{\mathbb{R}} d\sigma^{\mathcal{T}} \mathcal{T} = \int_{\mathbb{R}} d\sigma^I = I.$$

3. Connection with a linear interpolation problem

Motivated by (12), we consider the following interpolation problem. Given a collection of complex numbers $\{z_k\}_{k=1}^N$ and $\{\alpha_j(k)\}_{j=1}^n$ ($k = 1, \dots, N$), find the scalar polynomials $R_j(z)$ ($j = 1, \dots, n$) which satisfy the equation

$$(20) \quad \sum_{j=1}^n \alpha_j(k) R_j(z_k) = 0, \quad \forall k \in \{1, \dots, N\}.$$

The polynomials satisfying (20) are the solutions to the interpolation problem and the numbers $\{z_k\}_{k=1}^N$ are called the interpolation nodes.

In [KPS15], this interpolation problem is studied in detail. Let us introduce some of the notions and results given in [KPS15].

Definition 3. For a collection of complex numbers z_1, \dots, z_N , and matrices $\sigma_k := \left\{ \alpha_i(k) \overline{\alpha_j(k)} \right\}_{i,j=1}^n$ ($k \in \{1, \dots, N\}$), let us consider the equations

$$(21) \quad \langle \mathbf{r}(z_k), \sigma_k \mathbf{r}(z_k) \rangle_{\mathbb{C}^n} = 0$$

for $k = 1, \dots, N$, where $\mathbf{r}(z)$ is a nonzero n -dimensional vector polynomial. We denote by $\mathbb{S} = \mathbb{S}(\{\sigma_k\}_{k=1}^N, \{z_k\}_{k=1}^N)$ the set of all vector polynomials $\mathbf{r}(z)$ which satisfy (21) (c.f. [KPS15, Def. 3]).

It is worth remarking that solving (21) is equivalent to solving the linear interpolation problem (20) whenever $\mathbf{r}(z) = (R_1(z), \dots, R_n(z))^t$.

Definition 4. Let $\mathbf{r}(z) = (R_1(z), R_2(z), \dots, R_n(z))^t$ be an n -dimensional vector polynomial. The height of $\mathbf{r}(z)$ is the number

$$h(\mathbf{r}) := \max_{j \in \{1, \dots, n\}} \{n \deg(R_j) + j - 1\},$$

where it is assumed that $\deg 0 := -\infty$ and $h(\mathbf{0}) := -\infty$.

In [KPS15, Thm. 2.1] the following proposition is proven.

Proposition 3.1. *Let $\{\mathbf{g}_1(z), \dots, \mathbf{g}_{m+1}(z)\}$ be a sequence of vector polynomials such that $h(\mathbf{g}_i) = i - 1$ for all $i \in \{1, \dots, m + 1\}$. Any vector polynomial $\mathbf{r}(z)$ with height $m \neq -\infty$ can be written as follows*

$$\mathbf{r}(z) = \sum_{i=1}^{m+1} c_i \mathbf{g}_i(z),$$

where $c_i \in \mathbb{C}$ for all $i \in \{1, \dots, n\}$ and $c_{m+1} \neq 0$.

Definition 5. Let \mathcal{S} be an arbitrary subset of the set of all n -dimensional vector polynomials. We define the height of \mathcal{S} by

$$h(\mathcal{S}) := \min \{h(\mathbf{r}) : \mathbf{r} \in \mathcal{S}, \mathbf{r} \neq \mathbf{0}\}.$$

We say that $\mathbf{r}(z)$ in the set \mathbb{S} is a first generator of \mathbb{S} when

$$h(\mathbf{r}) = h(\mathbb{S}).$$

Definition 6. For any fixed arbitrary vector polynomial \mathbf{r} , let $\mathbb{M}(\mathbf{r})$ be the subset of vector polynomials given by

$$\mathbb{M}(\mathbf{r}) := \{\mathbf{s}(z) : \mathbf{s}(z) = S(z)\mathbf{r}(z), S(z) \text{ is an arbitrary scalar polynomial}\}.$$

Note that for all $\mathbf{s}(z) \in \mathbb{M}(\mathbf{r})$, there is a $k \in \mathbb{N} \cup \{0\}$ such that

$$h(\mathbf{s}) = nk + h(\mathbf{r}).$$

In this case $k = \deg S$, where $\mathbf{s}(z) = S(z)\mathbf{r}(z)$.

Proposition 3.2 ([KPS15, Lem. 4.3]). *Fix a natural number m such that $1 \leq m < n$. If the vector polynomials $\mathbf{r}_1(z), \dots, \mathbf{r}_m(z)$ are arbitrary elements of \mathbb{S} , then*

$$h(\mathbb{S} \setminus [\mathbb{M}(\mathbf{r}_1) + \dots + \mathbb{M}(\mathbf{r}_m)]) \neq h(\mathbf{r}_j) + nk$$

for any $j \in \{1, \dots, m\}$ and $k \in \mathbb{N} \cup \{0\}$. In other words,

$$h(\mathbb{S} \setminus [\mathbb{M}(\mathbf{r}_1) + \dots + \mathbb{M}(\mathbf{r}_m)]) \quad \text{and} \quad h(\mathbf{r}_j)$$

are different elements of the factor space $\mathbb{Z}/n\mathbb{Z}$ for any $j \in \{1, \dots, m\}$.

Due to Proposition 3.2, the following definition makes sense.

Definition 7. One defines recursively the j -th generator of \mathbb{S} as the vector polynomial $\mathbf{r}_j(z)$ in $\mathbb{S} \setminus [\mathbb{M}(\mathbf{r}_1) \dot{+} \dots \dot{+} \mathbb{M}(\mathbf{r}_{j-1})]$ such that

$$h(\mathbf{r}_j) = h(\mathbb{S} \setminus [\mathbb{M}(\mathbf{r}_1) \dot{+} \dots \dot{+} \mathbb{M}(\mathbf{r}_{j-1})]).$$

In [KPS15, Thm. 5.3 and Rem. 3], the following results were obtained.

Proposition 3.3. *There are exactly n generators of \mathbb{S} . Moreover, if the vector polynomials $\mathbf{r}_1(z), \dots, \mathbf{r}_n(z)$ are the generators of \mathbb{S} , then*

$$\mathbb{S} = \mathbb{M}(\mathbf{r}_1) \dot{+} \dots \dot{+} \mathbb{M}(\mathbf{r}_n)$$

and the heights of the generators of \mathbb{S} are different elements of the factor space $\mathbb{Z}/n\mathbb{Z}$.

Proposition 3.4. *Let $\mathbf{r}_j(z)$ be the j -th generator of $\mathbb{S}(n, N)$. It holds true that*

$$(22) \quad \sum_{j=1}^n h(\mathbf{r}_j) = Nn + \frac{n(n-1)}{2}.$$

Now, let us apply these results to the spectral analysis of the operator A . To this end, consider the solution of (12) as elements of $\mathbb{S}(\{\sigma_k\}_{k=1}^N, \{x_k\}_{k=1}^N)$, where σ_k is given by (14).

Lemma 3.1. *Fix $j \in \{1, \dots, n\}$ and let $\{x_k\}_{k=1}^N = \text{spec}(A)$, where $\{x_k\}_{k=1}^N$ is enumerated taking into account the multiplicity of eigenvalues. If $\mathbf{q}_j(z)$ is the vector polynomial given in (15), then*

$$\mathbf{q}_j(z) \in \mathbb{S}(\{\sigma_k\}_{k=1}^N, \{x_k\}_{k=1}^N).$$

Proof. The assertion follows by comparing (7) with (12). □

From this lemma, taking into account the definition of the inner product in $L_2(\mathbb{R}, \sigma)$ (see the proof of Lemma 2.2) and Definition 3, one arrives at the following assertion.

Corollary 3.1. *For all $j \in \{1, \dots, n\}$ the vector polynomial $\mathbf{q}_j(z)$ is in the equivalence class of the zero in $L_2(\mathbb{R}, \sigma)$, that is,*

$$\langle \mathbf{q}_j, \mathbf{q}_j \rangle_{L_2(\mathbb{R}, \sigma)} = 0$$

and, for all $\mathbf{r} \in L_2(\mathbb{R}, \sigma)$,

$$(23) \quad \langle \mathbf{r}, \mathbf{q}_j \rangle_{L_2(\mathbb{R}, \sigma)} = 0.$$

Lemma 3.2. Fix $k \in \{1, \dots, N\}$.

(i) If $m_j < k < m_{j+1}$, with $j = 0, \dots, n - 1$ and $m_0 = 0$, then

$$h(\mathbf{p}_{k+n-j}) = n + h(\mathbf{p}_k).$$

(ii) If there are no degenerations of the diagonals, then

$$h(\mathbf{p}_k) = k - 1, \quad \text{for all } k \in \{1, \dots, N\}.$$

(iii) For any $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, n\}$, the following holds

$$h(\mathbf{p}_i) \neq h(\mathbf{p}_{m_j}) + n = h(\mathbf{q}_j).$$

Proof. (i) The heights of the vector polynomials $\{\mathbf{p}_k\}_{k=n+1}^N$ are determined recursively by means of the system (17). For any $m_j < k < m_{j+1}$, with $j = 0, \dots, n - 1$, one has the equation

$$\dots + d_k^{(0)} \mathbf{p}_k + d_k^{(1)} \mathbf{p}_{k+1} + \dots + d_k^{(n-j)} \mathbf{p}_{k+n-j} = z\mathbf{p}_k.$$

Since $d_k^{(n-j)}$ never vanishes, the height of \mathbf{p}_{k+n-j} coincides with the one of $z\mathbf{p}_k$, this implies the assertion.

(ii) If there are no degenerations of the diagonals, then $m_1 = N - n + 1$. So, the heights of the vector polynomials $\mathbf{p}_1, \dots, \mathbf{p}_N$ are determined by (i) with $j = 0$.

(iii) follows from the recurrence equations (17) and (18). □

Lemma 3.3. For any nonnegative integer s , there exist $k \in \{1, \dots, N\}$ or a pair $j \in \{1, \dots, n\}$ and $l \in \mathbb{N} \cup \{0\}$ such that either $s = h(\mathbf{p}_k)$ or $s = h(\mathbf{q}_j) + nl$.

Proof. Due to Lemma 3.2(i), it follows from (4) and (16) that

$$(24) \quad h(\mathbf{p}_k) = k - 1 \quad \text{for } k = 1, \dots, h(\mathbf{q}_1).$$

Suppose that there is $s \in \mathbb{N}$ ($s > n$) such that $s \neq h(\mathbf{p}_k)$ for all $k \in \{1, \dots, N\}$ and $s \neq h(\mathbf{q}_j) + nl$ for all $j \in \{1, \dots, n\}$ and $l \in \mathbb{N} \cup \{0\}$. Let \hat{l} be an integer such that $s - n\hat{l} \in \{h(\mathbf{p}_k)\}_{k=1}^N \cup \{h(\mathbf{q}_j) + nl\}$ ($j \in \{1, \dots, n\}$ and $l \in \mathbb{N} \cup \{0\}$). There is always such an integer due to (24) and the fact that $h(\mathbf{q}_1) > n$ (see Lemma 3.2(iii)). We take \hat{l}_0 to be the minimum of all \hat{l} 's. Thus, there is $k_0 \in \{1, \dots, N\}$ or $j_0 \in \{1, \dots, n\}$, respectively, such that either

- (a) $s - n\hat{l}_0 = h(\mathbf{p}_{k_0})$ or
- (b) $s - n\hat{l}_0 = h(\mathbf{q}_{j_0}) + nl$, with $l \in \mathbb{N} \cup \{0\}$.

In the case (a), we prove that \hat{l}_0 is not the minimum integer, this implies the assertion of the lemma. Indeed, if there is $j \in \{1, \dots, n\}$ such that $k_0 = m_j$, then $s - n\hat{l}_0 + n = h(\mathbf{p}_{m_{j_0}}) + n = h(\mathbf{q}_{j_0})$ due to Lemma 3.2(iii). If there is not such j , then $m_j < k_0 < m_{j+1}$, and Lemma 3.2(i) implies $s - n\hat{l}_0 + n = h(\mathbf{p}_{k_0}) + n = h(\mathbf{p}_{k_0+n-j})$.

For the case (b), if $s - n\hat{l}_0 = h(\mathbf{q}_{j_0}) + nl$, then $s = h(\mathbf{q}_{j_0}) + n(l + \hat{l}_0)$ which is a contradiction. \square

As a consequence of Proposition 3.1, the above lemma yields the following result.

Corollary 3.2. *Any vector polynomial $\mathbf{r}(z)$ is a finite linear combination of*

$$\{\mathbf{p}_k(z) : k \in \{1, \dots, N\}\} \cup \{z^l \mathbf{q}_j(z) : l \in \mathbb{N}, j \in \{1, \dots, n\}\}.$$

Theorem 3.1. *For $j \in \{1, \dots, n\}$, the vector polynomial $\mathbf{q}_j(z)$ is a j -th generator of*

$$\mathbb{S}(\{\sigma_k\}_{k=1}^N, \{x_k\}_{k=1}^N).$$

Proof. For any fixed $j \in \{1, \dots, n\}$, suppose that there is an element $\mathbf{r}(z) \in \mathbb{S}(\{\sigma_k\}_{k=1}^N, \{x_k\}_{k=1}^N) \setminus (\mathbb{M}(\mathbf{q}_1) \dot{+} \dots \dot{+} \mathbb{M}(\mathbf{q}_{j-1}))$, where $\mathbf{q}_0(z) := 0$ such that $h(\mathbf{q}_{j-1}) < h(\mathbf{r}) < h(\mathbf{q}_j)$. Write \mathbf{r} as Corollary 3.2, then by Corollary 3.1

$$0 = \langle \mathbf{r}, \mathbf{r} \rangle_{L_2(\mathbb{R}, \sigma)} = \left\langle \sum_{k=1}^N c_k \mathbf{p}_k, \sum_{k=1}^N c_k \mathbf{p}_k \right\rangle_{L_2(\mathbb{R}, \sigma)} = \sum_{k=1}^N |c_k|^2.$$

This implies that $c_k = 0$ for all $k \in \{1, \dots, N\}$. In turn, again by Corollary 3.2, one has

$$\mathbf{r}(z) \in \mathbb{M}(\mathbf{q}_1) \dot{+} \dots \dot{+} \mathbb{M}(\mathbf{q}_{j-1})$$

for $j > 1$, and $\mathbf{r}(z) \equiv 0$ for $j = 1$. This contradiction yields that $\mathbf{q}_j(z)$ satisfies the definition of j -generator for any $j \in \{1, \dots, n\}$. \square

The following assertion is a direct consequence of Theorem 3.1, Proposition 3.3, and Proposition 3.4.

Corollary 3.3. *Let $\{\mathbf{q}_1(z), \dots, \mathbf{q}_n(z)\}$ be the n -dimensional vector polynomials defined by (15). Then, $h(\mathbf{q}_1), \dots, h(\mathbf{q}_n)$ are different elements of the equivalence class of the factor space $\mathbb{Z}/n\mathbb{Z}$. Also,*

$$\sum_{j=1}^n h(\mathbf{q}_j) = Nn + \frac{n(n-1)}{2}.$$

4. Reconstruction

In this section, we take as a starting point a matrix valued function $\tilde{\sigma} \in \mathfrak{M}(n, N)$ and construct a matrix \mathcal{A} in $\mathcal{M}(n, N)$ from this function. Moreover, we verify that, for some matrix \mathcal{T} giving the initial conditions, the function σ generated by the matrix \mathcal{A} (see Section 2) coincides with $\tilde{\sigma}$.

Thus, the results of this section show that any matrix in $\mathcal{M}(n, N)$ can be reconstructed from its function in $\mathfrak{M}(n, N)$.

Let $\tilde{\sigma}(t)$ be a matrix valued function in $\mathfrak{M}(n, N)$. Thus, one can associate an interpolation problem (20) which is equivalent to (21) (with $\tilde{\sigma}_k$ instead of σ_k). Then, by Proposition 3.3 there are n generators $\tilde{\mathbf{q}}_1(z), \dots, \tilde{\mathbf{q}}_n(z)$ of $\mathbb{S}(\{\tilde{\sigma}_k\}_{k=1}^N, \{\tilde{z}_k\}_{k=1}^N)$.

Let $\{\mathbf{e}_i(z)\}_{i \in \mathbb{N}}$ be a sequence of n -dimensional vector polynomials defined by

$$(25) \quad \mathbf{e}_{nk+1}(z) := \begin{pmatrix} z^k \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{e}_{nk+2}(z) := \begin{pmatrix} 0 \\ z^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e}_{n(k+1)}(z) := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ z^k \end{pmatrix}.$$

Clearly, $h(\mathbf{e}_i) = i - 1$. In the Hilbert space $L_2(\mathbb{R}, \tilde{\sigma})$, let us apply the Gram–Schmidt procedure of orthonormalization to the sequence $\{\mathbf{e}_i(z)\}_{i \in \mathbb{N}}$. Suppose that $\tilde{\mathbf{q}}_1$ is the first generator of the corresponding interpolation problem and let $\{\tilde{\mathbf{p}}_k\}_{k=1}^{h(\tilde{\mathbf{q}}_1)}$ be the orthonormalized vector polynomials obtained by the first $h(\tilde{\mathbf{q}}_1)$ iterations of the Gram–Schmidt procedure. Hence, if one defines

$$\mathbf{s} := \mathbf{e}_{h(\tilde{\mathbf{q}}_1)+1} - \sum_{i=1}^{h(\tilde{\mathbf{q}}_1)} \langle \tilde{\mathbf{p}}_i, \mathbf{e}_{h(\tilde{\mathbf{q}}_1)+1} \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \tilde{\mathbf{p}}_i,$$

then, in view of the fact that $h(\tilde{\mathbf{p}}_k) = k - 1$ for $k = 1, \dots, h(\tilde{\mathbf{q}}_1)$, one has $\mathbf{e}_{h(\tilde{\mathbf{q}}_1)+1} = c\tilde{\mathbf{q}}_1 + \sum_{i=1}^{h(\tilde{\mathbf{q}}_1)} c_i \tilde{\mathbf{p}}_i$ which in turn leads to

$$(26) \quad \mathbf{s} = c\tilde{\mathbf{q}}_1 + \sum_{k=1}^{h(\tilde{\mathbf{q}}_1)} \tilde{c}_k \tilde{\mathbf{p}}_k.$$

This implies that $\|\mathbf{s}\|_{L_2(\mathbb{R}, \tilde{\sigma})} = 0$. One continues with the procedure by taking the next vector of the sequence (25). Note that if $\tilde{\mathbf{p}}_k$ is a normalized element given by the Gram–Schmidt procedure, then the next *completed* iteration yields a normalized vector $\tilde{\mathbf{p}}_{k+1}$. Observe that if the Gram–Schmidt technique has produced a vector polynomial \mathbf{q} of zero norm and height h , then for any integer number l , the vector polynomial \mathbf{t} which is obtained at the $h + 1 + nl$ -th iteration of the Gram–Schmidt process, that is,

$$\mathbf{t} = \mathbf{e}_{h+1+nl} - \sum_{h(\tilde{\mathbf{p}}_i) < h+nl} \langle \tilde{\mathbf{p}}_i, \mathbf{e}_{h+1+nl} \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \tilde{\mathbf{p}}_i,$$

satisfies that $\|\mathbf{t}\|_{L_2(\mathbb{R}, \tilde{\sigma})} = 0$ (for all $l \in \mathbb{N}$) due to the fact that

$$\mathbf{e}_{h+1+nl} = R_l \mathbf{q} + \sum_{h(\tilde{\mathbf{p}}_i) < h+1+nl} c_i \tilde{\mathbf{p}}_i + \sum_{h(\mathbf{r}_i) < h+1+nl} \mathbf{r}_i,$$

where R_l is a scalar polynomial with degree k and each \mathbf{r}_i is a vector polynomial of zero norm with $h(\mathbf{r}_i) \neq h + nk$.

Since $L_2(\mathbb{R}, \tilde{\sigma})$ has dimension N , then one obtains from the Gram–Schmidt algorithm the orthonormalized sequence $\{\tilde{\mathbf{p}}_k\}_{k=1}^N$. Furthermore, the zeros found by the unsuccessful iterations yields all the generators $\{\tilde{\mathbf{q}}_i\}_{i=1}^n$ of the interpolation problem given by $\tilde{\sigma}$ and also polynomials in $\mathbb{S}(\{\tilde{\sigma}_k\}_{k=1}^N, \{\tilde{z}_k\}_{k=1}^N)$ such that their height are of the form $h(\tilde{\mathbf{q}}_i) + nl$ with $i \in \{1, \dots, n\}$ and $l \in \mathbb{N}$.

Remark 4.1. If one is interested only in the heights of the vector polynomials $\tilde{\mathbf{p}}_N$ and $\tilde{\mathbf{q}}_n$, then one can stop the Gram–Schmidt procedure when $\tilde{\mathbf{q}}_{n-1}$ appears and use Corollary 3.3.

Due to the fact that

$$(27) \quad h \left(\mathbf{e}_k - \sum_{h(\mathbf{p}_i) < k-1} \langle \tilde{\mathbf{p}}_i, \mathbf{e}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \tilde{\mathbf{p}}_i \right) = h(\mathbf{e}_k),$$

the heights of the set $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N \cup \{z^l \tilde{\mathbf{q}}_i(z)\}_{i=1}^n$ are in one-to-one correspondence with the set $\{0\} \cup \mathbb{N}$. Thus, in view of Proposition 3.1, one can write any n -dimensional vector polynomial \mathbf{r} as

$$\mathbf{r}(z) = \sum_{k=1}^N c_k \tilde{\mathbf{p}}_k(z) + \sum_{j=1}^n S_j(z) \tilde{\mathbf{q}}_j(z),$$

where $c_k \in \mathbb{C}$, $S_j(z)$ are scalar polynomials. Also, $c_k = 0$, respectively $S_j(z) = 0$, if $h(\mathbf{r}) < h(\tilde{\mathbf{p}}_k)$, respectively $h(\mathbf{r}) < h(\tilde{\mathbf{q}}_j)$. In particular, for $k \in \{1, \dots, N\}$,

$$(28) \quad z \tilde{\mathbf{p}}_k(z) = \sum_{l=1}^N c_{lk} \tilde{\mathbf{p}}_l(z) + \sum_{j=1}^n S_{kj}(z) \tilde{\mathbf{q}}_j(z),$$

where $c_{kl} \in \mathbb{C}$ and $S_{kj}(z)$ is scalar polynomial.

Remark 4.2. In (28), it holds that, for each k :

- (i) $c_{lk} = 0$ if $h(z \tilde{\mathbf{p}}_k) < h(\tilde{\mathbf{p}}_l)$.
- (ii) $S_{kj}(z) = 0$ if $h(z \tilde{\mathbf{p}}_k) < h(S_{kj}(z) \tilde{\mathbf{q}}_j)$.
- (iii) $c_{lk} > 0$ if there is $l \in \mathbb{N}$ such that $h(z \tilde{\mathbf{p}}_k) = h(\tilde{\mathbf{p}}_l)$.

Items (i) and (ii) are obtained by comparing the heights of the left- and right-hand sides of (28). In item (iii), one has to take into account that the leading coefficient of \mathbf{e}_k is positive for $k \in \mathbb{N}$ and therefore the Gram–Schmidt procedure yields the sequence $\{\tilde{\mathbf{p}}_k\}_{k=1}^N$ with its elements having positive leading coefficients.

Remark 4.3. For $k = 1, \dots, h_1$. We have that

$$h(\tilde{\mathbf{p}}_k) = k - 1,$$

and, for $k > h_1$

$$h(\tilde{\mathbf{p}}_k) = k - 1 + b_k,$$

where b_k is the number of elements in the set $\mathbb{M}(\tilde{\mathbf{q}}_1) \dot{+} \dots \dot{+} \mathbb{M}(\tilde{\mathbf{q}}_n)$ obtained by the Gram–Schmidt procedure and whose heights are less than $h(\tilde{\mathbf{p}}_k)$. Observe that $b_k < b_{n+k}$, which in turn implies

$$(29) \quad h(\tilde{\mathbf{p}}_{n+k}) > h(\tilde{\mathbf{p}}_k) + n.$$

Therefore, if we take the inner product of (28) with $\tilde{\mathbf{p}}_l(z)$ in $L_2(\mathbb{R}, \tilde{\sigma})$, we obtain

$$(30) \quad c_{lk} = \langle \tilde{\mathbf{p}}_l, z\tilde{\mathbf{p}}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \langle z\tilde{\mathbf{p}}_l, \tilde{\mathbf{p}}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = c_{kl},$$

where (23) has been used. Hence, the matrix $\{c_{lk}\}_{l,k=1}^N$ is symmetric and it is the matrix representation of the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \tilde{\sigma})$ with respect to the basis $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N$.

The following results shed light on the structure of the matrix $\{c_{lk}\}_{l,k=1}^N$.

Lemma 4.1. *If $|l - k| > n$. Then,*

$$c_{kl} = c_{lk} = 0.$$

Proof. For $l - k > n$, we obtain from (29) that

$$h(\tilde{\mathbf{p}}_l) > h(\tilde{\mathbf{p}}_{k+n}) \geq h(\tilde{\mathbf{p}}_k) + n = h(z\tilde{\mathbf{p}}_k).$$

Therefore by Remark 4.2,

$$c_{lk} = \langle \tilde{\mathbf{p}}_l, z\tilde{\mathbf{p}}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = 0.$$

And similarly for $k - l > n$. □

Lemma 4.1 shows that $\{c_{lk}\}_{l,k=1}^N$ is a band matrix. Let us turn to the question of characterizing the diagonals of $\{c_{lk}\}_{l,k=1}^N$. It will be shown that they undergo the kind of degeneration given in the introduction.

For a fixed number $i \in \{0, \dots, n\}$, we define the numbers

$$(31) \quad d_k^{(i)} := c_{k+i,k} = c_{k,k+i}$$

for $k = 1, \dots, N - i$.

Lemma 4.2. *Fix $j \in \{0, \dots, n - 1\}$.*

(i) *If k is such that $h(\tilde{\mathbf{q}}_j) < h(z\tilde{\mathbf{p}}_k) < h(\tilde{\mathbf{q}}_{j+1})$, then $d_k^{(n-j)} > 0$. Here one assumes that $h(\mathbf{q}_0) := n - 1$.*

(ii) *If k is such that $h(z\tilde{\mathbf{p}}_k) \geq h(\tilde{\mathbf{q}}_{j+1})$, one has that $d_k^{(n-j)} = 0$.*

Proof. Fix a number $j \in \{0, \dots, n - 1\}$, then any vector polynomial of the basis $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N$ satisfies either

$$(32) \quad h(\tilde{\mathbf{q}}_j) < h(z\tilde{\mathbf{p}}_k) < h(\tilde{\mathbf{q}}_{j+1})$$

or

$$(33) \quad h(z\tilde{\mathbf{p}}_k) \geq h(\tilde{\mathbf{q}}_{j+1}).$$

Suppose that $k \in \{1, \dots, N\}$ is such that (32) holds, then there is $l \in \{1, \dots, N\}$ such that

$$h(\tilde{\mathbf{p}}_l) = h(\tilde{\mathbf{p}}_k) + n = h(z\tilde{\mathbf{p}}_k).$$

Indeed, if there is no vector polynomial $\tilde{\mathbf{p}}_l(z)$ such that $h(\tilde{\mathbf{p}}_l) = h(z\tilde{\mathbf{p}}_k)$, then $h(z\tilde{\mathbf{p}}_k) = h(z^s\tilde{\mathbf{q}}_i)$ for some $i \leq j$ and $s \geq 1$. Therefore $h(\tilde{\mathbf{p}}_k) = h(z^{s-1}\tilde{\mathbf{q}}_i)$, which contradicts the fact that the heights of the set $\{\tilde{\mathbf{p}}_k\}_{k=1}^N \cup \{z^l\tilde{\mathbf{q}}_j\}_{j=1}^n$ ($l \in \mathbb{N} \cup \{0\}$) are in one-to-one correspondence with the set $\{0\} \cup \mathbb{N}$.

Let f_k be the number of elements of the sequence $\{\tilde{\mathbf{g}}_i\}_{i=1}^\infty$ in

$$\mathbb{M}(\tilde{\mathbf{q}}_1) \dot{+} \dots \dot{+} \mathbb{M}(\tilde{\mathbf{q}}_j)$$

whose heights lies between $h(\tilde{\mathbf{p}}_k)$ and $h(\tilde{\mathbf{p}}_k) + n$. If one assumes that (32) holds, then

$$(34) \quad f_k = j.$$

This is so because there are $n - 1$ “places” between $h(\tilde{\mathbf{p}}_k)$ and $h(\tilde{\mathbf{p}}_k) + n$ and, for each generator $\tilde{\mathbf{q}}_j(z)$ ($j \in \{1, \dots, n\}$), the heights of the elements of $\mathbb{M}(\tilde{\mathbf{q}}_j)$ fall into the same equivalence class of $\mathbb{Z}/n\mathbb{Z}$ (see Proposition 3.2). By (34), one has

$$h(z\tilde{\mathbf{p}}_k) = h(\tilde{\mathbf{p}}_k) + n = h(\tilde{\mathbf{p}}_{k+n-f_k}) = h(\tilde{\mathbf{p}}_{k+n-j}).$$

Therefore, Remark 4.2(iii) implies that $d_k^{(n-j)} > 0$.

Now, suppose that (33) takes place. In this case, one verifies that

$$(35) \quad f_k \geq j + 1.$$

Let \tilde{f}_k be the number of elements in $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N$ whose heights lies between $h(\tilde{\mathbf{p}}_k)$ and $h(\tilde{\mathbf{p}}_k) + n$. Then

$$h(\tilde{\mathbf{p}}_{k+\tilde{f}_k}) < h(\tilde{\mathbf{p}}_k) + n \leq h(\tilde{\mathbf{p}}_{k+\tilde{f}_k+1}).$$

Also, it follows from (35) and the equality $n - 1 = f_k + \tilde{f}_k$ that

$$h(\tilde{\mathbf{p}}_{k+\tilde{f}_k+1}) \leq h(\tilde{\mathbf{p}}_{k+n-j-1}) < h(\tilde{\mathbf{p}}_{k+n-j}).$$

Thus $h(\tilde{\mathbf{p}}_k) + n < h(\tilde{\mathbf{p}}_{k+n-j})$. This implies that $\langle \tilde{\mathbf{p}}_{k+n-1}, z\tilde{\mathbf{p}}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = 0$, which yields that $d_k^{(n-j)} = 0$ whenever (33) holds. \square

Corollary 4.1. *The matrix representation of the operator of multiplication by the independent variable in $L_2(\mathbb{R}, \tilde{\sigma})$ with respect to the basis $\{\tilde{\mathbf{p}}_k\}_{k=1}^N$ is a matrix in $\mathcal{M}(n, N)$.*

Proof. Taking into account (31), it follows from Lemma 4.1 and 4.2 that the matrix $\{c_{kl}\}_{k,l=1}^N$ whose entries are given by (30) is in the class $\mathcal{M}(n, N)$. \square

Remark 4.4. Since the matrix $\{c_{kl}\}_{k,l=1}^N$ is in $\mathcal{M}(n, N)$, there are numbers $\{m_i\}_{i=1}^n$ associated with the matrix (see Introduction). This numbers can be found from Lemma 4.2 which tells us that a degeneration occurs when

there exists $k \in \{1, \dots, N\}$ such that $h(z\tilde{\mathbf{p}}_k) = h(\tilde{\mathbf{q}}_{j+1})$ (this happens for each $j \in \{0, 1, \dots, n-1\}$). Thus,

$$(36) \quad h(z\tilde{\mathbf{p}}_{m_i}) = h(\tilde{\mathbf{q}}_i), \quad \forall i \in \{1, \dots, n\}.$$

It is straightforward to verify that (11) is equivalent to the fact that $\mathbf{e}_i(z)$ is not in the equivalence class of zero in $L(\mathbb{R}, \tilde{\sigma})$ for $i \in \{1, \dots, n\}$. Therefore, the first n elements of $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N$ are obtained by applying Gram-Schmidt to the set $\{\mathbf{e}_i(z)\}_{i=1}^n$. Thus, if one defines

$$(37) \quad t_{ij} := \langle \delta_i, \tilde{\mathbf{p}}_j \rangle_{\mathbb{C}^n}, \quad \forall i, j \in \{1, \dots, n\},$$

the matrix $\mathcal{T} = \{t_{ij}\}_{i,j=1}^n$ turns out to be upper triangular real and $t_{jj} \neq 0$ for all $j \in \{1, \dots, n\}$. Now, for this matrix \mathcal{T} and \mathcal{A} construct the solutions $\varphi^{(j)}(z)$ satisfying (4). Hence, the vector polynomials $\{\mathbf{p}_1(z), \dots, \mathbf{p}_n(z)\}$ defined by (16) satisfy (37). In other words

$$\mathbf{p}_j(z) = \tilde{\mathbf{p}}_j(z), \quad \forall j \in \{1, \dots, n\}.$$

Consider the recurrence equation, which is obtained from (28), but only for the case (iii) of the Remark 4.2 taking into account (31) and Lemma 4.2. That is,

$$(38) \quad \begin{aligned} d_1^{(0)}\tilde{\mathbf{p}}_1 + \dots + d_1^{(n)}\tilde{\mathbf{p}}_{n+1} &= z\tilde{\mathbf{p}}_1 \\ d_1^{(1)}\tilde{\mathbf{p}}_1 + d_2^{(0)}\tilde{\mathbf{p}}_2 + \dots + d_2^{(n)}\tilde{\mathbf{p}}_{n+2} &= z\tilde{\mathbf{p}}_2 \\ &\vdots \\ d_{m_1-1-n}^{(n)}\tilde{\mathbf{p}}_{m_1-1-n} + \dots + d_{m_1-1}^{(0)}\tilde{\mathbf{p}}_{m_1-1} + d_{m_1-1}^{(1)}\tilde{\mathbf{p}}_{m_1} + d_{m_1-1}^{(n)}\tilde{\mathbf{p}}_{m_1-1+n} &= z\tilde{\mathbf{p}}_{m_1-1} \\ \dots + d_{m_1+1}^{(0)}\tilde{\mathbf{p}}_{m_1+1} + d_{m_1+1}^{(1)}\tilde{\mathbf{p}}_{m_1+2} + \dots + d_{m_1+1}^{(n-1)}\tilde{\mathbf{p}}_{m_1+n} + S_{m_1+1,1}\tilde{\mathbf{q}}_1 &= z\tilde{\mathbf{p}}_{m_1+1} \\ &\vdots \\ \dots + d_{m_2-1}^{(0)}\tilde{\mathbf{p}}_{m_2-1} + d_{m_2-1}^{(1)}\tilde{\mathbf{p}}_{m_2} + d_{m_2-1}^{(n-1)}\tilde{\mathbf{p}}_{m_2-2+n} + S_{m_2-1,1}\tilde{\mathbf{q}}_1 &= z\tilde{\mathbf{p}}_{m_2-1} \\ \dots + d_{m_2+1}^{(0)}\tilde{\mathbf{p}}_{m_2+1} + d_{m_2+1}^{(1)}\tilde{\mathbf{p}}_{m_2+2} + \dots + d_{m_2+1}^{(n-2)}\tilde{\mathbf{p}}_{m_2-1+n} + \sum_{i=1}^2 S_{m_2+1,i}\tilde{\mathbf{q}}_i &= z\tilde{\mathbf{p}}_{m_2+1} \\ &\vdots \end{aligned}$$

Since $\mathbf{p}_k(z)$ and $\tilde{\mathbf{p}}_k(z)$ satisfy the same recurrence equation for any k in $\{1, \dots, m_1 - 1 + n\}$, one has

$$\mathbf{p}_k(z) = \tilde{\mathbf{p}}_k(z), \quad \forall k \in \{1, \dots, m_1 - 1 + n\}.$$

From the system of equations (38), consider the equation containing the vector polynomial $z\tilde{\mathbf{p}}_{m_1+1}(z)$. By comparing this equation with the corresponding one from (17), one concludes

$$\mathbf{p}_{m_1+n}(z) = \tilde{\mathbf{p}}_{m_1+n}(z) + S(z)\tilde{\mathbf{q}}_1(z),$$

where $S(z)$ is a scalar polynomial, so $S(z)\tilde{\mathbf{q}}_1(z)$ is in the equivalence class of zero of $L_2(\mathbb{R}, \tilde{\sigma})$. Note that $h(\tilde{\mathbf{p}}_{m_1+n}) > h(S\tilde{\mathbf{q}}_1)$ since $h(\tilde{\mathbf{p}}_{m_1+n}) = h(z\tilde{\mathbf{p}}_{m_1+1})$

in the equation containing $z\tilde{\mathbf{p}}_{m_1+1}(z)$ and the height of $z\tilde{\mathbf{p}}_{m_1+1}(z)$ does not coincide with the height of $S(z)\tilde{\mathbf{q}}_1(z)$. Recursively, for $k > m_1 + n$, one obtains the following lemma.

Lemma 4.3. *The vector polynomials $\{\mathbf{p}_k(z)\}_{k=1}^N$ and $\{\tilde{\mathbf{p}}_k(z)\}_{k=1}^N$ defined above (see the text below (37) and above Remark 4.1, respectively) satisfy that*

$$(39) \quad \mathbf{p}_k(z) = \tilde{\mathbf{p}}_k(z) + \tilde{\mathbf{r}}_k(z)$$

for all $k \in \{1, \dots, N\}$, where $\tilde{\mathbf{r}}_k(z)$ is in the equivalence class of zero of $L_2(\mathbb{R}, \tilde{\sigma})$ and $h(\tilde{\mathbf{r}}_k) < h(\mathbf{p}_k)$. Therefore,

$$h(\mathbf{p}_k) = h(\tilde{\mathbf{p}}_k), \quad \forall k \in \{1, \dots, N\}.$$

On the other hand, for the particular case $k = m_1$, (28) and (36) imply that

$$z\tilde{\mathbf{p}}_{m_1} = d_{m_1-n}^{(n)}\tilde{\mathbf{p}}_{m_1-n} + \dots + d_{m_1}^{(0)}\tilde{\mathbf{p}}_{m_1} + d_{m_1}^{(1)}\tilde{\mathbf{p}}_{m_1+1} + \dots + d_{m_1}^{(n-1)}\tilde{\mathbf{p}}_{m_1+n-1} + \gamma_1\tilde{\mathbf{q}}_1,$$

where $\gamma_1 \neq 0$.

In general, one verifies that for all $j \in \{1, \dots, n\}$

$$\begin{aligned} z\tilde{\mathbf{p}}_{m_j} &= d_{m_j-n}^{(n)}\tilde{\mathbf{p}}_{m_j-n} + \dots + d_{m_j}^{(0)}\tilde{\mathbf{p}}_{m_j} \\ &\quad + d_{m_j}^{(1)}\tilde{\mathbf{p}}_{m_j+1} + \dots + d_{m_j}^{(n-j)}\tilde{\mathbf{p}}_{m_j+n-j} + \sum_{i < j} S_{m_j,i}\tilde{\mathbf{q}}_i + \gamma_j\tilde{\mathbf{q}}_j, \end{aligned}$$

where $\gamma_j \neq 0$ and $S_i(z)$ is a scalar polynomial. Hence,

$$(40) \quad \begin{aligned} \gamma_j\tilde{\mathbf{q}}_j &= \left(z - d_{m_j}^{(0)} \right) \tilde{\mathbf{p}}_{m_j} - \left(d_{m_j-n}^{(n)}\tilde{\mathbf{p}}_{m_j-n} + \dots + d_{m_j-1}^{(1)}\tilde{\mathbf{p}}_{m_j-1} \right. \\ &\quad \left. + d_{m_j}^{(1)}\tilde{\mathbf{p}}_{m_j+1} + \dots + d_{m_1}^{(n-j)}\tilde{\mathbf{p}}_{m_j+n-j} + \sum_{i < j} S_{m_j,i}\tilde{\mathbf{q}}_i \right) \end{aligned}$$

for all $j \in \{1, \dots, n\}$.

Let us define the set of vector polynomials $\{\mathbf{q}_1(z), \dots, \mathbf{q}_n(z)\}$ by means of (18) using $\{\mathbf{p}_1(z), \dots, \mathbf{p}_N(z)\}$, as was done in Section 2.

Lemma 4.4. *Let $\tilde{\mathbf{q}}_j(z)$ be j -generator of $\mathbb{S}(\{\tilde{\sigma}_k\}_{k=1}^N, \{\tilde{x}_k\}_{k=1}^N)$, and $\mathbf{q}_j(z)$ be defined as above. Then $h(\mathbf{q}_j) = h(\tilde{\mathbf{q}}_j)$ for all $j \in \{1, \dots, n\}$ and*

$$(41) \quad \mathbf{q}_j(z) = \sum_{i \leq j} S_i(z)\tilde{\mathbf{q}}_i(z), \quad S_j \neq 0,$$

where $S_i(z)$ are scalar polynomials.

Proof. It follows from (18), (39) and (40) that

$$(42) \quad \mathbf{q}_j(z) = \gamma_j\tilde{\mathbf{q}}_j(z) + \tilde{\mathbf{s}}_j(z), \quad \text{for all } j \in \{1, \dots, n\},$$

where $\tilde{\mathbf{s}}_j(z)$ is in the equivalence class of the zero of $L_2(\mathbb{R}, \tilde{\sigma})$ and its height is strictly less than the height of $\tilde{\mathbf{q}}_j(z)$ since, due to (36), the height of $\tilde{\mathbf{q}}_j(z)$

is strictly greater than the height of any other term in the equation with $k = m_j$ in the system (28). Thus, $h(\mathbf{q}_j) = h(\tilde{\mathbf{q}}_j)$ for all $j \in \{1, \dots, n\}$.

Equation (42) also shows that $\mathbf{q}_i(z) \in \mathbb{S}(\{\tilde{\sigma}_k\}_{k=1}^N, \{\tilde{x}_k\}_{k=1}^N)$ and, due to Proposition 3.3, (41) is satisfied. \square

Lemma 4.5. *Let $\mathbf{r}(z)$ and $\mathbf{s}(z)$ be any two n -dimensional vector polynomials. Then,*

$$\langle \mathbf{r}, \mathbf{s} \rangle_{L_2(\mathbb{R}, \sigma)} = \langle \mathbf{r}, \mathbf{s} \rangle_{L_2(\mathbb{R}, \tilde{\sigma})}.$$

Proof. Any vector polynomial $\mathbf{r}(z)$ can be written as

$$\mathbf{r}(z) = \sum_{k=1}^N c_k \mathbf{p}_k(z) + \sum_{j=1}^n S_j(z) \mathbf{q}_j(z),$$

where $c_k = \langle \mathbf{r}, \mathbf{p}_k \rangle_{L_2(\mathbb{R}, \sigma)}$ and $S_j(z)$ are scalar polynomials. Thus,

$$\begin{aligned} \langle \mathbf{r}, \tilde{\mathbf{p}}_k \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} &= \left\langle \sum_{l=1}^N c_l \mathbf{p}_l + \sum_{j=1}^n S_j \mathbf{q}_j, \tilde{\mathbf{p}}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \\ &= \left\langle \sum_{l=1}^N c_l (\tilde{\mathbf{p}}_l + \tilde{\mathbf{r}}_l) + \sum_{j=1}^n S_j \left(\sum_{i \leq j} S_i \tilde{\mathbf{q}}_i \right), \tilde{\mathbf{p}}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} \\ &= \left\langle \sum_{l=1}^N c_l \tilde{\mathbf{p}}_l, \tilde{\mathbf{p}}_k \right\rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = c_k. \end{aligned} \quad \square$$

For the functions $\sigma(t)$ and $\tilde{\sigma}(t)$ in $\mathfrak{M}(n, N)$ consider the points x_k and \tilde{x}_k , where, respectively, $\sigma(t)$ and $\tilde{\sigma}(t)$ have jumps σ_k and $\tilde{\sigma}_k$. By definition, k takes all the values of the set $\{1, \dots, N\}$.

Lemma 4.6. *The points where the jumps of the matrix valued functions $\sigma(t)$ and $\tilde{\sigma}(t)$ take place coincide, i.e.,*

$$x_k = \tilde{x}_k, \quad \text{for all } k \in \{1, \dots, N\}.$$

Proof. Define the n -dimensional vector polynomial

$$\mathbf{r}(z) := \prod_{l=1}^N (z - x_l) \mathbf{e}_1(z)$$

(see (25)). Therefore,

$$\langle \mathbf{r}, \mathbf{r} \rangle_{L_2(\mathbb{R}, \sigma)} = \sum_{k=1}^N \langle \mathbf{r}(x_k), \sigma_k \mathbf{r}(x_k) \rangle_{\mathbb{C}^n} = 0.$$

Now, if one assumes that $\{\tilde{x}_k\}_{k=1}^N \setminus \{x_k\}_{k=1}^N \neq \emptyset$, then

$$\langle \mathbf{r}, \mathbf{r} \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \sum_{k=1}^N \langle \mathbf{r}(\tilde{x}_k), \tilde{\sigma}_k \mathbf{r}(\tilde{x}_k) \rangle_{\mathbb{C}^n} > 0$$

due to (11). In view of Lemma 4.5 our assumption has lead to a contradiction, so $\{\tilde{x}_k\}_{k=1}^N \subset \{x_k\}_{k=1}^N$. Analogously, one proves that $\{x_k\}_{k=1}^N \subset \{\tilde{x}_k\}_{k=1}^N$. \square

Lemma 4.7. *The jumps of the matrix valued functions $\sigma(t)$ and $\tilde{\sigma}(t)$ coincide, namely, for all $k \in \{1, \dots, N\}$,*

$$\sigma_k = \tilde{\sigma}_k.$$

Proof. Define, for each $i \in \{1, \dots, n\}$, the n -dimensional vector polynomial by

$$\mathbf{r}_{ki}(z) := \prod_{\substack{l=1 \\ l \neq k}}^N (z - x_l) \mathbf{e}_i(z).$$

Thus, for all $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} \langle \mathbf{r}_{ki}, \mathbf{r}_{kj} \rangle_{L_2(\mathbb{R}, \sigma)} &= \sum_{s=1}^N \langle \mathbf{r}_{ki}(x_s), \sigma_s \mathbf{r}_{kj}(x_s) \rangle_{\mathbb{C}^n} \\ &= \sum_{s=1}^N \prod_{\substack{l=1 \\ l \neq k}}^N |(x_s - x_l)|^2 \alpha_i(x_s) \overline{\alpha_j(x_s)} \\ &= \prod_{\substack{l=1 \\ l \neq k}}^N |x_k - x_l|^2 \alpha_i(x_k) \overline{\alpha_j(x_k)}. \end{aligned}$$

Analogously,

$$\langle \mathbf{r}_{ki}, \mathbf{r}_{kj} \rangle_{L_2(\mathbb{R}, \tilde{\sigma})} = \prod_{\substack{l=1 \\ l \neq k}}^N |x_k - x_l|^2 \tilde{\alpha}_i(x_k) \overline{\tilde{\alpha}_j(x_k)},$$

where Lemma 4.6 was used together with the fact that the numbers $\tilde{\alpha}_i(x_k)$ define the entries of the matrix $\tilde{\sigma}_k$ (see (14)). Therefore, by Lemma 4.5

$$\sigma_k = \tilde{\sigma}_k, \quad \text{for all } k \in \{1, \dots, N\}. \quad \square$$

The above results can be summarized in the following assertion.

Theorem 4.1. *Let $\tilde{\sigma}(t)$ be an element of $\mathfrak{M}(n, N)$ and $\{c_{kl}\}_{k,l=1}^N \in \mathcal{M}(n, N)$ be the corresponding matrix that results from applying the method of reconstruction to the matrix valued function $\tilde{\sigma}(t)$ (see (28) and (30)). If A is the operator whose matrix representation with respect to the basis $\{\delta_1, \dots, \delta_N\}$ in \mathcal{H} , is $\{c_{kl}\}_{k,l=1}^N$, then there is an upper triangular real matrix \mathcal{T} with no zeros in the main diagonal such that the corresponding spectral function $\sigma(t)$ for the operator A coincides with $\tilde{\sigma}(t)$.*

Remark 4.5. Theorem 4.1, can be paraphrased as follows: A matrix-valued function $\tilde{\sigma}$ in $\mathfrak{M}(n, N)$ determines uniquely a matrix in $\mathcal{M}(n, N)$ such that $\tilde{\sigma}$ is a spectral measure of it.

Let \mathcal{A} be in $\mathcal{M}(n, N)$ and A be the corresponding operator. Denote by V_θ the unitary operator whose matrix representation with respect to the canonical basis is $\text{diag}\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$ with $\theta_k \in [0, 2\pi)$ for any $k \in \{1, \dots, N\}$. Define

$$(43) \quad B = V_\theta A V_\theta^*.$$

The matrix representation of B is in $\mathcal{M}(n, N)$ if and only if $\theta_k = \theta_{k+ln}$ for all $k \in \{1, \dots, n\}$ such that $k + ln \leq N$ with l being a positive integer. Thus, there are various elements of the family of unitarily equivalent matrices corresponding to (43) in $\mathcal{M}(n, N)$.

5. Alternative inverse spectral methods

For Jacobi matrices there are two ways of recovering the matrix from the spectral function ρ . The first one is based on the fact that the sequence of orthonormal polynomials, constructed via the application of Gram-Schmidt procedure to the sequence of functions $\{t^{k-1}\}_{k=1}^\infty$ in $L_2(\mathbb{R}, \rho)$, determines the entries of the matrix (see [Akh65, Chap.1, Sec.1 and Chap.4 Sec.2] and [Sim98, Sec.1]. The second method uses the fact that the asymptotic expansion of the m -Weyl function corresponding to ρ yields the matrix entries [GeS97, Sec.3]. In the case of tridiagonal block matrices, these two methods also work with some restrictions. Indeed, consider a finite tridiagonal block matrix

$$(44) \quad \begin{pmatrix} Q_1 & B_1^* & 0 & \cdots & 0 \\ B_1 & Q_2 & B_2^* & \ddots & \vdots \\ 0 & B_2 & Q_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B_{K-1}^* \\ 0 & \dots & 0 & B_{K-1} & Q_K \end{pmatrix},$$

where B_k is invertible for all $k = 1, \dots, K - 1$. According to [Ber68, Chap.7 Sec.2.8], one recovers the matrix entries Q_1, \dots, Q_K and B_1, \dots, B_{K-1} from a matrix valued function obtained from the spectral function. This corresponds to the first method outlined above. There is also an analogue of the second method which is based on the function $M(z)$ given by [Ber68, Chap.7 Eq.2.63] which satisfy

$$(45) \quad M(z)^{-1} = Q_1 - zI - B_1 \widetilde{M}(z) B_1^*,$$

where $\widetilde{M}(z)$ is the function given by [Ber68, Chap.7 Eq.2.63] for the tridiagonal block matrix obtained from (44) by deleting the first block row and block column. Equation (45) is the block analogue of [GeS97, Eq.2.15]. On the basis of the asymptotic behavior of \widetilde{M} , one finds Q_1 and $B_1 B_1^*$ from

(45). Since, in our setting, the matrix B_1 is upper triangular with positive main diagonal, one can actually obtain the entries of B_1 from $B_1 B_1^*$. It is possible to obtain the next matrix entries by considering (45) for the next truncated matrix.

Any matrix of the class $\mathcal{M}(n, N)$ can be written as (44) whenever $N/n = K$. Note that if a matrix in $\mathcal{M}(n, N)$ undergoes degeneration, then there is k_0 such that B_k is not invertible for all $k = k_0, \dots, K - 1$. Thus, the methods cited above can be used for the inverse spectral analysis of the elements of $\mathcal{M}(n, N)$ which, do not undergo degenerations and for which $N/n \in \mathbb{N}$.

The procedure developed in Section 4 is applicable to the whole class $\mathcal{M}(n, N)$, which shows that it is more general than the methods described above. In the reconstruction technique of Section 4, degenerations can be treated on the basis of the solution of the linear interpolation problem for n -dimensional vector polynomials.

Appendix A. Mass-spring systems

This appendix briefly describes how Newton’s laws of motion and the Hooke law yield a finite difference equation which can be written by a finite band symmetric matrix.

Consider the finite mass-spring system given by Figure 5, where we have assumed that N is even.

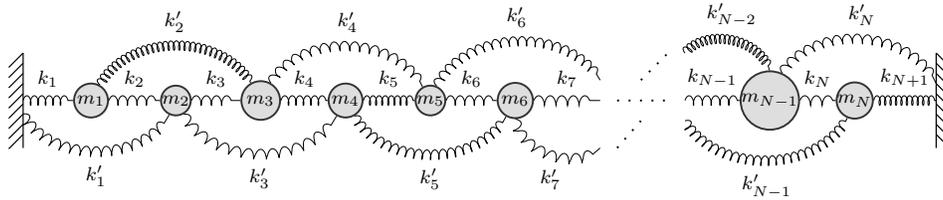


FIGURE 5. Mass-spring system of a matrix in $\mathcal{M}(2, N)$: non-degenerated case

In Figure 5, m_j and k_j, k'_j stand, respectively, for the j -th mass, the j -th spring constant connecting immediate neighbors, and the j -th spring constant connecting mediated neighbors.

Due to the Hooke law, the forces F_i acting on the masses m_i are given by

$$F_i = k'_{i+1}x_{i+2} + k_{i+1}x_{i+1} - (k_{i+1} + k'_{i+1} + k_i + k'_{i-1})x_i + k_i x_{i-1} + k'_{i-1}x_{i-2}.$$

This system of equations, due to Newton’s second law, can be written as

$$(46) \quad M\ddot{x} = Kx,$$

where

$$M = \begin{pmatrix} m_1 & & & & & \\ & m_2 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & m_N & \end{pmatrix} \quad K = \begin{pmatrix} \alpha_1 & k_2 & k'_2 & 0 & \dots & 0 \\ k_2 & \alpha_2 & k_3 & k'_3 & \dots & 0 \\ k'_2 & k_3 & \alpha_3 & k_4 & \ddots & 0 \\ 0 & k'_3 & k_4 & \alpha_4 & & k'_{N-1} \\ \vdots & & \ddots & & \ddots & k_N \\ 0 & \dots & 0 & k'_{N-1} & k_N & \alpha_N \end{pmatrix}$$

with $\alpha_i = -(k_{i+1} + k'_{i+1} + k_i + k'_{i-1})$. The system (46) is equivalent to $\ddot{U} = LU$, where $U = M^{1/2}X$ and

$$L = M^{-1/2}KM^{-1/2} = \begin{pmatrix} \frac{\alpha_1}{m_1} & \frac{k_2}{\sqrt{m_1 m_2}} & \frac{k'_2}{\sqrt{m_1 m_3}} & 0 & \dots & 0 \\ \frac{k_2}{\sqrt{m_1 m_2}} & \frac{\alpha_2}{m_2} & \frac{k_3}{\sqrt{m_2 m_3}} & \frac{k'_3}{\sqrt{m_2 m_4}} & \dots & 0 \\ \frac{k'_2}{\sqrt{m_1 m_3}} & \frac{k_3}{\sqrt{m_2 m_3}} & \frac{\alpha_3}{m_3} & \frac{k_4}{\sqrt{m_3 m_4}} & \ddots & 0 \\ 0 & \frac{k'_3}{\sqrt{m_2 m_4}} & \frac{k_4}{\sqrt{m_3 m_4}} & \frac{\alpha_4}{m_4} & & \frac{k'_{N-1}}{\sqrt{m_{N-2} m_N}} \\ \vdots & & \ddots & & \ddots & \frac{k_N}{\sqrt{m_{N-1} m_N}} \\ 0 & \dots & 0 & \frac{k'_{N-1}}{\sqrt{m_{N-2} m_N}} & \frac{k_N}{\sqrt{m_{N-1} m_N}} & \frac{\alpha_N}{m_N} \end{pmatrix}.$$

Thus, according to our notation, the diagonals are given by

$$(47) \quad d_j^{(0)} = -\frac{k_{j+1} + k'_{j+1} + k_j + k'_{j-1}}{m_j}$$

$$(48) \quad d_j^{(1)} = \frac{k_{j+1}}{\sqrt{m_{j+1} m_j}}$$

$$(49) \quad d_j^{(2)} = \frac{k'_{j+1}}{\sqrt{m_{j+2} m_j}}.$$

The eigenvalues of this matrix determine the frequencies of the harmonic oscillations whose superposition yields the movement of the mechanical system.

For Jacobi matrices, viz. when the masses are connected only with their immediate neighbor, it is possible to give a finite continued fraction which yields the quotients k_j/m_j for any j from the quotient k_1/m_1 [dRKS13, Rem. 11] (see also [Mar05, pag. 76]). This reconstruction is physically meaningful. In the general case, one can construct the following continued fractions from (47), (48), and (49). Note that the first equation reduces to the continued fraction of [dRKS13, Rem. 11] when $k'_j = 0$.

$$\begin{aligned}
(50) \quad & \frac{k_{j+1} + k'_j}{m_{j+1}} \\
&= \frac{\left(d_j^{(1)}\right)^2 + \sqrt{\frac{m_{j+2}}{m_{j+1}}} d_j^{(1)} d_j^{(2)} + \sqrt{\frac{m_{j-1}}{m_j}} d_{j-1}^{(2)} d_j^{(1)} + \sqrt{\frac{m_{j-1} m_{j+2}}{m_{j+1} m_j}} d_j^{(2)} d_{j-1}^{(2)}}{d_j^{(0)} + \frac{k_j + k'_{j-1}}{m_j}} \\
&= \frac{\left(d_j^{(1)}\right)^2 + \frac{k'_{j+1}}{k_{j+1}} \left(d_j^{(1)}\right)^2 + \frac{k_{j-1}}{k'_{j-1}} \frac{d_j^{(1)} d_{j-1}^{(2)} d_{j-2}^{(2)}}{d_{j-2}^{(1)}} + \frac{k'_{j+1} k_{j-1}}{k_{j+1} k'_{j-1}} \frac{d_j^{(1)} d_{j-1}^{(2)} d_{j-2}^{(2)}}{d_{j-2}^{(1)}}}{d_j^{(0)} + \frac{k_j + k'_{j-1}}{m_j}}.
\end{aligned}$$

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