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Equivariant bundles and adapted connections

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Abstract. Given a complex manifold M equipped with a holomorphic action of a connected complex Lie group G, and a holomorphic principal H–bundle E_H over X equipped with a G–connection h, we investigate the connections on the principal H–bundle E_H that are (strongly) adapted to h. Examples are provided by holomorphic principal H–bundles equipped with a flat partial connection over a foliated manifold.

Contents

1. Introduction

Let X be a complex manifold, G a connected complex Lie group and $\rho : G \times X \longrightarrow X$ a holomorphic action of G on X. The Lie algebra of G is denoted by $\mathfrak{g}$. Let $p : E_H \longrightarrow X$ be a holomorphic principal H–bundle, where H is a complex Lie group. A G–connection on E_H is a $\mathbb{C}$–linear map $h : \mathfrak{g} \longrightarrow H^0(E_H, TE_H)^H$ such that for every $v \in \mathfrak{g}$, the vector field $dp \circ h(v)$ on X coincides with the one defined by v using the above action ρ

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(see Section 2.2). In [BP17], G–connections were investigated, in particular, a criterion was given for the existence of a G–connection.

Here we continue the investigations of G–connections. More precisely, we study the interactions of G–connections on E_H with the holomorphic connections on the principal H–bundle E_H. There are two possible compatibility conditions between them which are called "adapted" and "strongly adapted" (see Section 3.1). To explain these conditions, if h is given by a holomorphic action ρ_E of G on E_H, then a holomorphic connection η on the principal H–bundle E_H is adapted to h if and only if η is preserved by ρ_E; such an adapted connection η is called strongly adapted if the image of the homomorphism h is contained in the horizontal subbundle of TE_H for the connection η.

The property of a holomorphic connection η on a holomorphic principal H–bundle E_H that it is strongly adapted to a G–connection h on E_H can also be formulated in the context of foliated manifolds and principal H–bundles on them equipped with a flat partial connection; the details are in Section 5.

2. Preliminaries

2.1. Atiyah bundle. Let H be a complex Lie group. Its Lie algebra will be denoted by $\mathfrak{h}$. Let X be a connected complex manifold and

$$p\,:\, E_H \longrightarrow X \tag{2.1}$$

a holomorphic principal H–bundle over X. This means that E_H is a complex manifold equipped with a holomorphic right action of H

$$a\,:\, E_H \times H \longrightarrow E_H$$

such that

- $p \circ a = p \circ p_{E_H}$, where p_{E_H} is the projection of $E_H \times H$ to E_H, and
- the map $(p_{E_H},\, a)\,:\, E_H \times H \longrightarrow E_H \times_X E_H$ is an isomorphism.

Note that the first condition means that the action of H takes a fiber of p to itself, so the image of the map $(p_{E_H},\, a)$ is contained in the fiber product $E_H \times_X E_H$. The second condition above means that the action of H on a fiber of p is free and transitive.

The adjoint bundle for E_H

$$\mathrm{ad}(E_H) \,:=\, E_H \times^H \mathfrak{h} \longrightarrow X$$

is the holomorphic vector bundle over X associated to E_H for the adjoint action of H on the Lie algebra $\mathfrak{h}$.

The holomorphic tangent (respectively, cotangent) bundle of a complex manifold Y will be denoted by TY (respectively, T^*Y). The tangent bundle of a real manifold Y will be denoted by $T^{\mathbb{R}}Y$.

The *Atiyah bundle* for E_H

$$\mathrm{At}(E_H) \,:=\, (TE_H)/H \longrightarrow E_H/H \,=\, X$$

is a holomorphic vector bundle over X whose rank is $\dim X + \dim \mathfrak{h}$; see [At57]. Let

$$T_{E_H/X} \subset TE_H$$

be the relative tangent bundle for the projection p in (2.1). The subbundle

$$(T_{E_H/X})/H \subset (TE_H)/H = \mathrm{At}(E_H)$$

is identified with the adjoint vector bundle $\mathrm{ad}(E_H)$. This identification is a consequence of the isomorphism of $T_{E_H/X}$ with the trivial vector bundle $E_H \times \mathfrak{h} \longrightarrow E_H$ given by the action of H on E_H. Therefore, the short exact sequence

$$0 \longrightarrow T_{E_H/X} \longrightarrow TE_H \xrightarrow{dp} p^*TX \longrightarrow 0\,,$$

where dp is the differential of p, produces a short exact sequence on X

$$0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}(E_H) \xrightarrow{dp} TX \longrightarrow 0\,, \tag{2.2}$$

which is known as the *Atiyah exact sequence* for E_H. For simplicity, we have used the same notation dp for the differential $TE_H \longrightarrow p^*TX$ over E_H as well as its descent $\mathrm{At}(E_H) \longrightarrow TX$ to X. A holomorphic connection on E_H is a holomorphic homomorphism

$$\eta\,:\, TX \longrightarrow \mathrm{At}(E_H) \tag{2.3}$$

such that $(dp)\circ\eta = \mathrm{Id}_{TX}$, where dp is the homomorphism in (2.2). For a holomorphic connection η on E_H, the homomorphism

$$\bigwedge\nolimits^2 TX \longrightarrow \mathrm{ad}(E_H)\,, \quad v\otimes w - w\otimes v \longmapsto 2([\eta(v),\, \eta(w)] - \eta([v,\, w]))\,,$$

where v and w are locally defined holomorphic sections of TX, produces a holomorphic section of $(\bigwedge^2 T^*X)\otimes \mathrm{ad}(E_H)$. This holomorphic section of $(\bigwedge^2 T^*X)\otimes \mathrm{ad}(E_H)$ is called the *curvature* of the connection η.

The vector bundle $TE_H\otimes p^*(TX)^*$ on E_H has a natural action of H given by the action of H on TE_H and the tautological action of H on $p^*(TX)^*$. We note that a holomorphic connection on E_H is an H–invariant holomorphic section of $TE_H\otimes p^*(TX)^*$.

2.2. *G*–connections on E_H. Let G be a connected complex Lie group; its Lie algebra will be denoted by $\mathfrak{g}$. The identity element of G will be denoted by e. Let

$$\rho\,:\, G\times X \longrightarrow X \tag{2.4}$$

be a holomorphic action of G on X. Consider the holomorphic homomorphism

$$\rho'\,:\, \mathrm{At}(E_H)\oplus(X\times\mathfrak{g}) \longrightarrow TX\,, \quad (v\,,w) \longmapsto dp(v) - d'\rho(w)\,,$$

where dp is the homomorphism in (2.2), and

$$d'\rho\,:\, X\times\mathfrak{g} \longrightarrow TX\,, \quad (x,\, v) \longmapsto (d\rho)(e,x)(v,0)\,, \tag{2.5}$$

with $(d\rho)(e,x) : \mathfrak{g} \oplus T_xX \longrightarrow T_xX$ being the differential of ρ at $(e,\, x) \in G \times X$. Define the subsheaf

$$\mathrm{At}_\rho(E_H) := (\rho')^{-1}(0) \subset \mathrm{At}(E_H) \oplus (X \times \mathfrak{g})\,. \tag{2.6}$$

Since the differential dp is surjective, it follows that ρ' is surjective. This implies that $\mathrm{At}_\rho(E_H)$ is a holomorphic subbundle of $\mathrm{At}(E_H) \oplus (X \times \mathfrak{g})$. The vector bundle $\mathrm{At}_\rho(E_H)$ fits in a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{ad}(E_H) & \longrightarrow & \mathrm{At}_\rho(E_H) & \stackrel{q}{\longrightarrow} & X \times \mathfrak{g} & \longrightarrow & 0 \\ & & \| & & \Big\downarrow J & & \Big\downarrow d'\rho & & \\ 0 & \longrightarrow & \mathrm{ad}(E_H) & \longrightarrow & \mathrm{At}(E_H) & \stackrel{dp}{\longrightarrow} & TX & \longrightarrow & 0 \end{array} \tag{2.7}$$

where J (respectively, q) is given by the projection of $\mathrm{At}(E_H) \oplus (X \times \mathfrak{g})$ to $\mathrm{At}(E_H)$ (respectively, $X \times \mathfrak{g}$). (See [BP17].)

A *holomorphic G–connection* on E_H is a holomorphic homomorphism of vector bundles

$$h : X \times \mathfrak{g} \longrightarrow \mathrm{At}_\rho(E_H) \tag{2.8}$$

such that $q \circ h = \mathrm{Id}_{X\times\mathfrak{g}}$, where q is the homomorphism in (2.7). The curvature of a G–connection h

$$(s,\, t) \longmapsto [h(s),\, h(t)] - h([s,\, t])$$

is a holomorphic section

$$\mathcal{K}(h) \in H^0(X,\, \mathrm{ad}(E_H) \otimes \bigwedge^2 (X \times \mathfrak{g})^*) = H^0(X,\, \mathrm{ad}(E_H)) \otimes \bigwedge^2 \mathfrak{g}^*\,. \tag{2.9}$$

We will give examples of G–connection.

Let $a : E_H \times H \longrightarrow E_H$ be the action of H on the principal H–bundle E_H.

A G–action on the principal bundle E_H is a holomorphic action of G on the total space of E_H

$$\rho_E : G \times E_H \longrightarrow E_H \tag{2.10}$$

such that

1. $p \circ \rho_E = \rho \circ (\mathrm{Id}_G \times p)$, where p and ρ are the maps in (2.1) and (2.4) respectively, and
2. $\rho_E \circ (\mathrm{Id}_G \times a) = a \circ (\rho_E \times \mathrm{Id}_H)$ as maps from $G \times E_H \times H$ to E_H (this condition means that the actions of G and H on E_H commute).

An equivariant principal H–bundle is a holomorphic principal H–bundle with a G–action.

Let $\rho_E : G \times E_H \longrightarrow E_H$ be a G–action on E_H. Consider the homomorphism

$$\widetilde{h} : E_H \times \mathfrak{g} \longrightarrow TE_H$$

given by the differential $d\rho_E$ of the action ρ_E; more precisely,

$$\widetilde{h}(z, v) = d\rho_E(e, z)(v, 0),$$

so $\widetilde{h}$ is the homomorphism in (2.5) when X is substituted by E_H. Since the actions of G and H on E_H commute, this homomorphism $\widetilde{h}$ produces a G–connection

$$h_0 : X \times \mathfrak{g} \longrightarrow \mathrm{At}_\rho(E_H) \tag{2.11}$$

on E_H; the curvature of this G–connection h_0 vanishes identically [BP17, p. 355, Lemma 4.1].

Let Y be a connected compact complex manifold such that TY is holomorphically trivial. Then Y is holomorphically isomorphic to G/Γ, where G is a connected complex Lie group and $\Gamma \subset G$ is a cocompact lattice [Wa54]; in fact, G is the connected component, containing the identity element, of the group of all holomorphic automorphisms of Y. Consider the left–translation action of G on $G/\Gamma = Y$. A G–connection on a holomorphic principal H–bundle E_H on Y is an usual holomorphic connection on the principal H–bundle.

2.3. Distributions under a flow. Let Y be a connected C^∞ manifold and

$$\mathcal{D} \subset T^{\mathbb{R}}Y$$

a C^∞ subbundle. In other words, $\mathcal{D}$ is a distribution on Y. The fiber of $\mathcal{D}$ over any point $z \in Y$ will be denoted by $\mathcal{D}_z$.

Let ξ be a C^∞ vector field on Y. Given any point $x \in Y$, there is an open neighborhood $x \in U_x \subset Y$ and an open interval $0 \in I_x \subset \mathbb{R}$, such that ξ integrates to a flow

$$\Phi_x : U_x \times I_x \longrightarrow Y\,.$$

For any $t \in I_x$, define

$$\Phi_{x,t} : U_x \longrightarrow Y\,, \quad z \longmapsto \Phi_x(z,t)\,.$$

Lemma 2.1. *The following two are equivalent:*

(1) *For every $x \in Y$ and $z \in U_x$ as above,*

$$(d\Phi_{x,t})(z)(\mathcal{D}_z) = \mathcal{D}_{\Phi_{x,t}(z)}\,,$$

where $d\Phi_{x,t}(z) : T^{\mathbb{R}}_z Y \longrightarrow T^{\mathbb{R}}_{\Phi_{x,t}(z)}Y$ is the differential of the map $\Phi_{x,t}$ at z.

(2) $[\xi, \mathcal{D}] \subset \mathcal{D}$.

Proof. Let $\mathcal{W}$ denote the space of all C^∞ 1–forms on Y that vanish on $\mathcal{D}$. The first statement is equivalent to the statement that

$$L_\xi(w) \in \mathcal{W} \quad \forall \; w \in \mathcal{W}\,, \tag{2.12}$$

where L_ξ denotes the Lie derivative with respect to the vector field ξ.

First assume that

$$[\xi, \mathcal{D}] \subset \mathcal{D}\,. \tag{2.13}$$

To prove that (2.12) holds, take any $w \in \mathcal{W}$ and any C^∞ section θ of $\mathcal{D}$. We have

$$(L_\xi(w))(\theta) = \xi(w(\theta)) - w(L_\xi\theta) = \xi(w(\theta)) - w([\xi,\, \theta])\,.$$

Now, $w(\theta) = 0$, and $[\xi,\, \theta]$ is section of $\mathcal{D}$ by (2.13). Hence $(L_\xi(w))(\theta) = 0$, which implies that (2.12) holds.

Now assume that (2.12) holds. To prove (2.13), let θ be any C^∞ section of $\mathcal{D}$. Take any $w \in \mathcal{W}$. We have

$$w([\xi,\, \theta]) = w(L_\xi\theta) = \xi(w(\theta)) - (L_\xi w)(\theta)\,.$$

Now, $w(\theta) = 0$, and also $(L_\xi w)(\theta) = 0$ because $L_\xi w \in \mathcal{W}$ by (2.12). Hence (2.13) holds. □

3. Connections and (strongly) adapted connections

3.1. Definitions. Let E_H be a holomorphic principal bundle over X such that E_H is equipped with a holomorphic connection

$$\eta \,:\, TX \longrightarrow \mathrm{At}(E_H)$$

(see (2.3)). Since $\mathrm{At}(E_H) = (TE_H)/H$, the image of η is a holomorphic distribution on E_H; it is known as the *horizontal distribution* for the connection η.

As before, a connected complex Lie group G acts holomorphically on X.

Given a holomorphic G–connection $h \,:\, X \times \mathfrak{g} \longrightarrow \mathrm{At}_\rho(E_H)$ on E_H (see (2.8)), the connection η is said to be *adapted* to h if

$$[J \circ h(X \times \{v\}),\, \eta(TX)] \subset \eta(TX) \quad \forall\, v \in \mathfrak{g}\,, \tag{3.1}$$

where J is the homomorphism in (2.7). Note that a C^∞ section of $\mathrm{At}(E_H)$ defines a H–invariant vector field on E_H of type (1, 0).

The connection η is said to be *strongly adapted* to h if it is adapted to h, and furthermore

$$\mathrm{image}(J \circ h) \subset \mathrm{image}(\eta)\,. \tag{3.2}$$

We will now give examples to show that the conditions in (3.1) and (3.2) are independent.

Consider the trivial action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on X. Let E be a holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle on X admitting a holomorphic connection, for example E can be the trivial holomorphic principal $\mathrm{GL}(r, \mathbb{C})$–bundle $X \times \mathrm{GL}(r, \mathbb{C})$ on X. The center of $\mathrm{GL}(r, \mathbb{C})$ is identified with $\mathbb{C}^*$ by sending any $c \in \mathbb{C}^*$ to $c \cdot \mathrm{Id}_{\mathbb{C}^r} \in \mathrm{GL}(r, \mathbb{C})$. Using this identification, the action of the center of $\mathrm{GL}(r, \mathbb{C})$ on E produces an action of $\mathbb{C}^*$ on E. Since $\mathbb{C}^*$ is in the center of $\mathrm{GL}(r, \mathbb{C})$, the actions of $\mathbb{C}^*$ and $\mathrm{GL}(r, \mathbb{C})$ on E commute. If E' is the vector bundle of rank r associated to E by the standard representation of $\mathrm{GL}(r, \mathbb{C})$, then this action of $\mathbb{C}^*$ on E corresponds to the action of $\mathbb{C}^*$ on E' as scalar multiplications. Let h be the holomorphic $\mathbb{C}^*$–connection on E given by this action of $\mathbb{C}^*$ on E (see

(2.11)). Any holomorphic connection on the principal $\mathrm{GL}(r,\mathbb{C})$–bundle E is adapted to h. But (3.2) fails for every holomorphic connection on E.

Now take $X = \mathbb{C}^2$ and $G = \mathbb{C} = H$. Let E_H be the trivial principal $\mathbb{C}$–bundle $\mathbb{C}^2 \times \mathbb{C} \longrightarrow \mathbb{C}^2$. Take ρ to be the action of $\mathbb{C}$ on $\mathbb{C}^2$ defined by

$$(z,\, (x,\, y)) \longmapsto (x+z,\, y)\,, \quad z \in \mathbb{C}\,, \quad (x,\, y) \in \mathbb{C}^2\,.$$

This action of $\mathbb{C}$ on X and the trivial action of $\mathbb{C}$ on $\mathbb{C}$ together define an action of $\mathbb{C}$ on $E_H = X \times \mathbb{C}$. Let h be the holomorphic $\mathbb{C}$–connection on E_H associated to this action of $\mathbb{C}$ on E_H (see (2.11)). Let D be the holomorphic connection on the principal H–bundle E_H defined by $f \longmapsto df + xf\cdot dy$, where f is any holomorphic function on $\mathbb{C}^2$ (holomorphic sections of E_H are holomorphic functions) and d denotes the standard de Rham differential. Then (3.2) holds while (3.1) fails.

3.2. Equivariant bundles and adaptable connections. As in (2.10), take a G–action ρ_E on E_H. As mentioned earlier, there is a natural G–connection on E_H

$$h_0 \,:\, X \times \mathfrak{g} \longrightarrow \mathrm{At}_\rho(E_H) \tag{3.3}$$

corresponding to ρ_E.

Let $p_X \,:\, G\times X \longrightarrow X$ be the natural projection. The action ρ_E produces a holomorphic isomorphism of principal H–bundles

$$\beta \,:\, p_X^* E_H \longrightarrow \rho^* E_H\,, \quad \beta(g,x)(z) \,=\, \rho_E(g,z) \tag{3.4}$$

for all $g \in G$, $x \in X$ and $z \in (E_H)_x$, where ρ is the map in (2.4).

For any $g \in G$, let

$$j_g \,:\, X \longrightarrow G \times X\,, \quad x \longmapsto (g,\, x)$$

be the embedding. For all $g \in G$, the isomorphism β in (3.4) produces a holomorphic isomorphism of principal H–bundles

$$\beta^g \,:\, E_H \longrightarrow (\rho\circ j_g)^* E_H\,, \quad z \longmapsto \beta(g,\, x)(z) \,=\, \rho_E(g,z) \tag{3.5}$$

for all $x \in X$ and $z \in (E_H)_x$. The map from the holomorphic connections on E_H to the holomorphic connections on $(\rho\circ j_g)^* E_H$ induced by the above isomorphism β^g will be denoted by β^g_*; note that β^g_* is a bijection.

Proposition 3.1. *A holomorphic connection η on E_H is adapted to the G–connection h_0 in* (3.3) *associated to ρ_E if and only if for all $g \in G$,*

$$(\rho\circ j_g)^*\eta \,=\, \beta^g_*(\eta) \tag{3.6}$$

(both are connections on the principal H–bundle $(\rho\circ j_g)^* E_H$).

Proof. First assume that η is adapted to h_0. Take any $v \in \mathfrak{g}$. The flow on E_H generated by v sends any $t \in \mathbb{R}$ to the biholomorphism

$$F_t \,:\, E_H \longrightarrow E_H\,, \quad z \longmapsto \rho_E(\exp(tv),\, z)\,.$$

Note that F_t coincides with $\beta^{\exp(tv)}$ constructed in (3.5). Consider the H-invariant distribution

$$D^\eta := \text{image}(\eta) \subset TE_H .$$

Its fiber over any point $z \in E_H$ will be denoted by D^η_z. Since η is adapted to h_0, from Lemma 2.1 it follows that

$$(dF_t)(z)(D^\eta_z) = D^\eta_{F_t(z)} \tag{3.7}$$

for all $z \in E_H$ and $t \in \mathbb{R}$, where $(dF_t)(z) : T_zE_H \longrightarrow T_{F_t(z)}E_H$ is the differential of the map F_t. Since the subset $\{\exp(tv)\}_{v\in\mathfrak{g},t\in\mathbb{R}} \subset G$ is dense in the analytic topology (recall that G is connected), and also $F_t = \beta^{\exp(tv)}$, from (3.7) we conclude that (3.6) holds for all $g \in G$.

Now assume that (3.6) holds for all $g \in G$. This implies that (3.7) holds for all $z \in E_H$ and $t \in \mathbb{R}$. Consequently, from Lemma 2.1 we conclude that η is adapted to h_0. □

Take any point $x \in X$. Define

$$\rho_x : G \longrightarrow X , \quad g \longmapsto \rho \circ j_g(x) = \rho(g, x) .$$

Consider the map

$$\rho_{E,x} : G \times (E_H)_x \longrightarrow \rho^*_x E_H , \quad (g, z) \longmapsto \rho_E(g, z) .$$

Since this $\rho_{E,x}$ is H–equivariant (recall that the actions of G and H on E_H commute), it identifies the pulled back principal H–bundle $\rho^*_x E_H$ with the trivial principal H–bundle $G \times (E_H)_x \longrightarrow G$. Let D^0_x be the holomorphic connection on the principal H–bundle $\rho^*_x E_H$ induced by the trivial connection on $G \times (E_H)_x$ using the above isomorphism $\rho_{E,x}$. Note that $\rho^*_x E_H$ is identified with the restriction of $\rho^* E_H$ to $G \times \{x\}$, because ρ_x is the restriction of ρ to $G \times \{x\}$. Therefore, $\rho^*\eta|_{G\times\{x\}}$ is also a connection on $\rho^*_x E_H$.

Proposition 3.2. *A holomorphic connection η on E_H is strongly adapted to the G–connection h_0 in* (3.3) *if and only if the following two hold:*

(1) *For all $g \in G$,*

$$(\rho \circ j_g)^*\eta = \beta^g_*(\eta) .$$

(2) *For every $x \in X$, the connection D^0_x on $\rho^*_x E_H$ coincides with the connection $\rho^*\eta|_{G\times\{x\}}$.*

Proof. First assume that η is strongly adapted to h_0. Since η is adapted to h_0, Proposition 3.1 says that $(\rho \circ j_g)^*\eta = \beta^g_*(\eta)$ for all $g \in G$. The given condition (3.2) implies that the connection D^0_x coincides with $\rho^*\eta|_{G\times\{x\}}$.

The converse is similarly proved. Assume that the two statements in the proposition hold. From Proposition 3.1 we know that η is adapted to h_0. The second condition in the proposition implies that (3.2) holds. □

4. Criterion for adapted connection

Let $\eta : TX \longrightarrow \mathrm{At}(E_H)$ be a holomorphic connection on E_H. Let

$$\widetilde{\eta} : X \times \mathfrak{g} \longrightarrow \mathrm{At}(E_H) \oplus (X \times \mathfrak{g}) \tag{4.1}$$

be the $\mathcal{O}_X$–linear homomorphism defined by

$$(x, v) \longmapsto (\eta(d'\rho(x,v)), (x, v)),$$

where $d'\rho$ is the homomorphism in (2.5). Since we have $(dp) \circ \eta = \mathrm{Id}_{TX}$, where dp is the homomorphism in (2.2), it follows immediately that the image of $\widetilde{\eta}$ is contained in $\mathrm{At}_\rho(E_H) := (\rho')^{-1}(0)$ (see (2.6)). The homomorphism $\widetilde{\eta}$ evidently is a G–connection on E_H.

Let $\mathcal{K}(\eta) \in H^0(X, \Omega^2_X \otimes \mathrm{ad}(E_H))$ be the curvature of the connection η, where $\Omega^2_X = \bigwedge^2 T^*X$. For any $w \in T_xX$, let

$$i_w(\mathcal{K}(\eta)(x)) \in (T^*X)_x \otimes \mathrm{ad}(E_H)_x = (T^*X \otimes \mathrm{ad}(E_H))_x \tag{4.2}$$

be the contraction of $\mathcal{K}(\eta)(x) \in (\Omega^2_X \otimes \mathrm{ad}(E_H))_x$ by the tangent vector $w \in T_xX$.

Lemma 4.1. *The connection η on E_H is strongly adapted to the above constructed G–connection $\widetilde{\eta}$ if and only if for all $v \in \mathfrak{g}$ and $x \in X$,*

$$i_{d'\rho(x,v)}(\mathcal{K}(\eta)(x)) = 0, \tag{4.3}$$

where $d'\rho$ is defined in (2.5) *(see* (4.2) *for the contraction).*

Proof. From the construction of $\widetilde{\eta}$ in (4.1) it follows immediately that the condition in (3.2) holds. We need to show that (3.1) holds if and only if (4.3) holds.

To prove this, we recall a construction of the curvature $\mathcal{K}(\eta)$. Given a point $x \in X$ and holomorphic tangent vectors $v, w \in T_xX$, extend v, w to vector fields $\widetilde{v}, \widetilde{w}$ of type (1, 0) on some open neighborhood of the point x. Let $\widehat{v} = \eta(\widetilde{v})$ and $\widehat{w} = \eta(\widetilde{w})$ be the horizontal lifts of $\widetilde{v}$ and $\widetilde{w}$ respectively, for the connection η. Then

$$\mathcal{K}(\eta)(x)(v,w) = ([\widehat{v}, \widehat{w}]_{\mathrm{Vert}})|_{p^{-1}(x)},$$

where $[\widehat{v}, \widehat{w}]_{\mathrm{Vert}}$ is the component of the Lie bracket $[\widehat{v}, \widehat{w}]$ in the vertical direction (for the projection p). We note that the section $([\widehat{v}, \widehat{w}]_{\mathrm{Vert}})|_{p^{-1}(x)}$ of $T_{E_H/X}$ over $p^{-1}(x)$ is H–invariant and hence it defines an element of the fiber $\mathrm{ad}(E_H)_x$ over x; recall that $\mathrm{ad}(E_H)$ is identified with $(T_{E_H/X})/H$. The element $([\widehat{v}, \widehat{w}]_{\mathrm{Vert}})|_{p^{-1}(x)} \in \mathrm{ad}(E_H)_x$ does not depend on the choice of the extensions $\widetilde{v}$ and $\widetilde{w}$ of v and w respectively. From this description of $\mathcal{K}(\eta)$ it follows immediately that (3.1) holds if and only if (4.3) holds. □

From the proof of Lemma 4.1 we have the following:

Corollary 4.2. *The connection η on E_H is adapted to the above constructed G–connection $\widetilde{\eta}$ if and only if the condition in* (4.3) *holds. In other words, the connection η on E_H is strongly adapted to $\widetilde{\eta}$ if η is adapted to $\widetilde{\eta}$.*

Take a $\mathbb{C}$–linear map

$$\varphi_0 : \mathfrak{g} \longrightarrow H^0(X, \operatorname{ad}(E_H)). \tag{4.4}$$

For any $v \in \mathfrak{g}$, the section $\varphi_0(v) \in H^0(X, \operatorname{ad}(E_H))$ defines a holomorphic vertical tangent vector field on E_H for the projection p. This vertical tangent vector field on E_H will be denoted by $\varphi(v)$. Let $U \subset X$ be an open subset and V a C^∞ vector field on U of type $(1, 0)$. Let $V' = \eta(V)$ be the horizontal lift of V on $p^{-1}(U)$ for the holomorphic connection η on E_H. Let f_0 be any C^∞ function on U. Then $V'(f_0 \circ p)$ is a H–invariant function on $p^{-1}(U)$, and hence

$$\varphi(v)(V'(f_0 \circ p)) = 0. \tag{4.5}$$

On the other hand,

$$\varphi(v)(f_0 \circ p) = 0 \tag{4.6}$$

because $\varphi(v)$ is a vertical vector field. From (4.5) and (4.6) we conclude that

$$[\varphi(v), V'](f_0 \circ p) = 0.$$

In other words,

$$[\varphi(v), V'] = [\varphi(v), V']_{\mathrm{Vert}}, \tag{4.7}$$

where $[\varphi(v), V']_{\mathrm{Vert}}$ is the vertical component of $[\varphi(v), V']$. The vector field $[\varphi(v), V']$ is H–invariant because both $\varphi(v)$ and V' are H–invariant. If f_1 is a C^∞ function on U, then note that

$$[\varphi(v), (f_1 \circ p) \cdot V'] = (f_1 \circ p) \cdot [\varphi(v), V']$$

because $\varphi(v)(f_1 \circ p) = 0$. Clearly, the vector field $(f_1 \circ p) \cdot V'$ is the horizontal lift of the vector field $f_1 \cdot V$ on U for the connection η. From these observations we conclude that there is a homomorphism

$$\widetilde{\varphi} : \mathfrak{g} \otimes_{\mathbb{C}} TX \longrightarrow \operatorname{ad}(E_H) \tag{4.8}$$

that sends $v \otimes w \in \mathfrak{g} \otimes T_xX$ to $[\varphi(v), V'](x)$, where $V' = \eta(V)$ is the horizontal lift, with respect to the connection η, of a vector field V defined on a neighborhood of the point $x \in X$ with $V(x) = w$. Note that $[\varphi(v), V'](x)$ does not depend on the choice of the extension V of w.

The contraction in (4.2) produces a homomorphism

$$\Pi : \mathfrak{g} \otimes_{\mathbb{C}} TX \longrightarrow \operatorname{ad}(E_H) \tag{4.9}$$

that sends $v \otimes w \in \mathfrak{g} \otimes T_xX$ to

$$i_w i_{d'\rho(x,v)}(\mathcal{K}(\eta)(x)) \in \operatorname{ad}(E_H)_x,$$

which is the contraction of $i_{d'\rho(x,v)}(\mathcal{K}(\eta)(x)) \in (T^*X)_x \otimes \operatorname{ad}(E_H)_x$ (see (2.5), (4.2)) by the tangent vector $w \in T_xX$.

Theorem 4.3. *Let X be a complex manifold equipped with a holomorphic action of G and E_H a holomorphic principal H–bundle on X equipped with a holomorphic connection η. Then there is a G–connection h on E_H such that η is adapted to h if and only if there is a homomorphism φ_0 as in* (4.4) *such that the homomorphism $\widetilde{\varphi}$ in* (4.8) *coincides with the homomorphism $-\Pi$, where Π is constructed in* (4.9)*.*

Proof. Let $h : \mathfrak{g} \longrightarrow H^0(X, \mathrm{At}_\rho(E_H))$ be a G–connection on E_H such that η is adapted to h. For any $v \in \mathfrak{g}$, consider

$$J \circ h(v) - \eta(v') \in H^0(X, \mathrm{At}(E_H)),$$

where J is the homomorphism in (2.7) and v' is the holomorphic vector field on X defined by $x \longmapsto d'\rho(x, v)$ (see (2.5)). Note that $dp \circ J \circ h(v) = v'$, where dp is the homomorphism in (2.2). Therefore, we have

$$J \circ h(v) - \eta(v') \in H^0(X, \mathrm{ad}(E_H)) \subset H^0(X, \mathrm{At}(E_H))$$

(see (2.7)). Now define

$$\varphi_0 : \mathfrak{g} \longrightarrow H^0(X, \mathrm{ad}(E_H)), \quad v \longmapsto J \circ h(v) - \eta(v').$$

We will show that the homomorphism $\widetilde{\varphi}$ in (4.8) for this φ_0 coincides with the homomorphism $-\Pi$.

Take any $v \in \mathfrak{g}$. Given any $x \in X$ and any $w \in T_xX$, let V be any C^∞ vector field of type (1, 0), defined on an open neighborhood of $x \in X$, such that

$$[v', V] = 0.$$

Since η is adapted to h, the Lie bracket $[J \circ h(v), \eta(V)]$ lies in the horizontal subbundle $\eta(TX) \subset TE_H$. In other words, the vertical component of $[J \circ h(v), \eta(V)]$ vanishes identically.

The Lie bracket $[\eta(v'), \eta(V)]$ is vertical because

$$dp([\eta(v'), \eta(V)]) = [v', V] = 0.$$

From (4.7) we know that the Lie bracket $[\varphi(v), \eta(V)]$ is vertical, where $\varphi(v)$ is the vertical vector field corresponding to

$$\varphi_0(v) \in H^0(X, \mathrm{ad}(E_H)).$$

This and the fact that $[\eta(v'), \eta(V)]$ is vertical together imply that

$$[\varphi(v) + \eta(v'), \eta(V)] = [J \circ h(v), \eta(V)] \tag{4.10}$$

is vertical. But it was shown above that the vertical component of $[J \circ h(v), \eta(V)]$ vanishes identically. Hence we conclude that

$$[J \circ h(v), \eta(V)] = 0.$$

Consequently, we have

$$[\varphi(v), \eta(V)] = -[\eta(v'), \eta(V)] \tag{4.11}$$

for all $v \in \mathfrak{g}$. Since $[\varphi(v), \eta(V)] = \widetilde{\varphi}(v \otimes V)$ and $[\eta(v'), \eta(V)] = \Pi(v \otimes V)$, from (4.11) it follows that

$$\widetilde{\varphi} = -\Pi .$$

To prove the converse, take any homomorphism φ_0 as in (4.4) such that

$$\widetilde{\varphi} = -\Pi . \tag{4.12}$$

Now define a G–connection

$$h : \mathfrak{g} \longrightarrow H^0(X, \mathrm{At}_\rho(E_H)), v \longmapsto (\varphi_0(v) + \eta(v'), X \times \{v\}) .$$

We will show that η is adapted to h.

Let V be a C^∞ vector field of type $(1, 0)$ defined on an open subset $U \subset X$. Take any $v \in \mathfrak{g}$. The Lie bracket $[\varphi(v), \eta(V)]$ is vertical (see (4.7)), where $\varphi(v)$, as before, is the vertical vector field for the projection p corresponding to the section $\varphi_0(v)$ of $\mathrm{ad}(E_H)$. We have

$$\widetilde{\varphi}(v \otimes V) = [\varphi(v), \eta(V)] ,$$

and $\Pi(v \otimes V)$ is the vertical component of $[\eta(v'), \eta(V)]$. Consequently, from (4.12) and the definition of h it follows that the vertical component of $[J \circ h(v), \eta(V)]$ vanishes. This implies that η is adapted to h. □

Let $h : \mathfrak{g} \longrightarrow H^0(X, \mathrm{At}_\rho(E_H))$ be a G–connection on E_H. Take any section

$$\theta \in C^\infty(X, \mathrm{At}(E_H)^{\otimes a} \otimes (\mathrm{At}(E_H)^*)^{\otimes b}) ,$$

where a and b are nonnegative integers. Note that θ defines a H–invariant section of the vector bundle $(TE_H)^{\otimes a} \otimes (T^*E_H)^{\otimes b}$ on E_H; this section of $(TE_H)^{\otimes a} \otimes (T^*E_H)^{\otimes b}$ will be denoted by Θ. We say that θ is preserved by h if

$$L_{J \circ h(v)}\Theta = 0 \quad \forall \ v \in \mathfrak{g} ,$$

where $L_{J \circ h(v)}$ is the Lie derivative with respect to the vector field $J \circ h(v)$ on E_H (the homomorphism J is constructed in (2.7)).

If h is the G–connection associated to a G–action ρ_E on E_H, then it is straight-forward to check that θ is preserved by h if and only if the section Θ is preserved by the action ρ_E on E_H.

5. Holomorphic foliations and strongly adapted connections

As before, X is a complex manifold. Let

$$\mathcal{F} \subset TX$$

be a holomorphic foliation on X, which means that $\mathcal{F}$ is a holomorphic subbundle of TX such that for any two sections s and t of $\mathcal{F}$ defined over some open subset of X, the Lie bracket $[s, t]$ is also a section of $\mathcal{F}$ [La77]. Let E_H be a holomorphic principal H–bundle on X.

Consider the Atiyah exact sequence for E_H in (2.2). Define

$$\mathrm{At}_{\mathcal{F}}(E_H) := (dp)^{-1}(\mathcal{F}) \subset \mathrm{At}(E_H) .$$

So, from (2.2) we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathrm{ad}(E_H) \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H) \xrightarrow{\widetilde{dp}} \mathcal{F} \longrightarrow 0\,, \tag{5.1}$$

where $\widetilde{dp}$ is the restriction of dp to $\mathrm{At}_{\mathcal{F}}(E_H)$. A *holomorphic partial connection* on E_H is a homomorphism

$$D\,:\,\mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$$

such that $\widetilde{dp}\circ D = \mathrm{Id}_{\mathcal{F}}$ [La77].

Given such a holomorphic partial connection D, the homomorphism

$$\bigwedge^2 \mathcal{F} \longrightarrow \mathrm{ad}(E_H)\,,\quad v\otimes w - w\otimes v \longmapsto 2([D(v),\, D(w)] - D([v,\, w]))\,,$$

where v and w are locally defined holomorphic sections of $\mathcal{F}$, produces a holomorphic section of $(\bigwedge^2 \mathcal{F}^*)\otimes \mathrm{ad}(E_H)$. This holomorphic section of $(\bigwedge^2 \mathcal{F}^*)\otimes \mathrm{ad}(E_H)$ is called the *curvature* of the partial connection D. A holomorphic partial connection is called *flat* if its curvature vanishes identically.

Let $\eta\,:\,TX \longrightarrow \mathrm{At}(E_H)$ be a holomorphic connection on the principal H–bundle E_H. As before, the curvature of η will be denoted by $\mathcal{K}(\eta)$. Let $D\,:\,\mathcal{F} \longrightarrow \mathrm{At}_{\mathcal{F}}(E_H)$ be a flat holomorphic partial connection on E_H.

The connection η is said to be *strongly adapted* to D if

- the restriction $\eta|_{\mathcal{F}}\,:\,\mathcal{F} \longrightarrow \mathrm{At}(E_H)$ coincides with D, and
- for any $x \in X$ and $w \in \mathcal{F}_x$, the contraction
$$i_w\mathcal{K}(\eta)(x) \,\in\, T^*_xX\otimes \mathrm{ad}(E_H)_x$$
vanishes.

Corollary 5.1. *Suppose that $\mathcal{F}$ is given by a holomorphic action ρ of a connected complex Lie group G on X (so the leaves of $\mathcal{F}$ are the orbits of G), and also assume that D is given by a G–action ρ_E on E_H (so the tangent spaces to the leaves in E_H are the horizontal subspaces). Then η is strongly adapted to D if and only if η is strongly adapted to the G–connection on E_H given by ρ_E.*

Proof. The above condition that $\eta|_{\mathcal{F}} = D$ is equivalent to the condition that the G–connection $\widetilde{\eta}$ constructed in (4.1) from η coincides with the G–connection on E_H given by the above G–action ρ_E. Therefore, the result follows from Lemma 4.1. □

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