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K-theory for real C^*-algebras via unitary elements with symmetries

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Abstract. We prove that all eight KO groups for a real C^*-algebra can be constructed from homotopy classes of unitary matrices that respect a variety of symmetries. In this manifestation of the KO groups, all eight boudary maps in the 24-term exact sequences associated to an ideal in a real C^*-algebra can be computed as exponential or index maps with formulas that are nearly identical to the complex case.

Contents

1. Introduction

In the common picture of K-theory for C^*-algebras, the abelian groups $K_0(A)$ and $K_1(A)$ arise from projections and unitaries in $M_n(A)$, respectively. Because of Bott periodicity, we do not worry about independent descriptions of $K_i(A)$ for other integer values of i. In the case of real C^*-algebras, the same pictures carry over to give us concrete descriptions of

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TABLE 1. Unitary Picture of K-theory

K-group	unitary symmetries
$KO_{-1}(A,\tau)$	$u^\tau = u$
$KO_0(A,\tau)$	$u = u^*$, $u^\tau = u^*$
$KO_1(A,\tau)$	$u^\tau = u^*$
$KO_2(A,\tau)$	$u = u^*$, $u^\tau = -u$
$KO_3(A,\tau)$	$u^{\tau\otimes\sharp} = u$
$KO_4(A,\tau)$	$u = u^*$, $u^{\tau\otimes\sharp} = u^*$
$KO_5(A,\tau)$	$u^{\tau\otimes\sharp} = u^*$
$KO_6(A,\tau)$	$u = u^*$, $u^{\tau\otimes\sharp} = -u$

The classes in $KO_j(A,\tau)$, for a unital C^*-algebra with real structure are, in our picture, given by unitary elements of $M_n(\mathbb{C}) \otimes A$ with the symmetries as indicated. See Theorem 7.1 and Table 3 for details.

$KO_0(A)$ and $KO_1(A)$ in terms of projections and unitaries. The higher K-theory groups (for $i \neq 0, 1$) can be defined using suspensions or using Clifford algebras. While this reliance on suspensions allows the theoretical development of K-theory to proceed nicely, it leaves much to be desired in terms of being able to represent specific K-theory classes for purposes of computation.

We rectify this situation by putting forward a unified description of all ten K-theory groups (eight KO-groups and two KU-groups) of a real C^*-algebra A using unitaries in $M_n(\widetilde{A}_{\mathbb{C}})$ satisfying appropriate symmetries, completing the project that we began in [7]. This unified description is summarized in condensed form in Table 1. A complete description of our picture of K-theory can be found in Theorem 7.1 and Table 3, which summarize the results developed in detail through Sections 5 and 6. A salient feature of our picture is that all of the groups are obtained without using the Grothendieck construction, so any KO-element can be represented exactly by a single unitary.

The boundary maps associated to $I \to A \to A/I$ can be critical when calculating K-theory groups. In the complex case, both boundary maps have explicit formulas in terms of lifting problems associated to projections and unitaries. Any picture of real K-theory should have computable boundary maps in the 24-term exact sequence of abelian groups associated to a short exact sequence of real C^*-algebras.

For real C^*-algebras, we have had explicit pictures for KO_j for all j except $j = 3$ and $j = 7$ [19]. There were some details missing for $j = 2$ and $j = 6$ to adapt to the C^*-algebra setting, but essentially these cases were dealt with already in [41]. The boundary map has been less developed. Given $I \to A \to A/I$ in the real case, it is a folk-theorem that the usual formulas in the complex case work to determine both $\partial_1 : KO_1(A/I) \to KO_0(I)$ and $\partial_5 : KO_5(A/I) \to KO_4(I)$. For this form of ∂_5 it is essential to work with the isomorphism $KO_{j+4}(D) \cong KO_j(D \otimes \mathbb{H})$ where $\mathbb{H}$ is the algebra of the quaternions.

We seek a consistent picture of the KO and KU groups that will allow us to have essentially only two formulas for the boundary maps, one for the even-to-odd cases and one for the odd-to-even cases. It will also tie real K-theory more closely to classical mathematics. For example, the isomorphism $KO_2(\mathbb{R}) \cong \mathbb{Z}_2$ can be given simply as sign of the Pfaffian of a self-adjoint unitary that is purely imaginary.

We work with the complexified form of a real C^*-algebra with the real structure determined by a generalized involution. That is, our objects are typically pairs (A, τ) where A is a complex C^*-algebra and $\tau : A \to A$ an involution that is antimultiplicative and written a^τ. In the case where A has a unit, the unitaries we consider live in $M_n(\mathbb{R}) \otimes A$ and the symmetries are in terms of the usual involution $*$ and one of two extensions of τ to matrix algebras over A. These extensions are $\tau = \mathrm{Tr} \otimes \tau$ and $\sharp \otimes \tau$ where Tr is the familiar transpose and $\sharp$ is the dual operation, discussed in detail later, that is based on the derived involution on the complexification of $\mathbb{H}$.

Recently there has been much interest in physics regarding real K-theory. This has been true in string theory, to classifying D-branes, and in condensed matter physics, to classify topological insulators. There are mathematical reasons to study our constructions in real K-theory, but lets us briefly review some of the physics.

In string theory, the utility of real K-theory in classifying D-branes was discovered by Witten [40]. A more recent work more closely related to this paper is [3]. More recent developments coming from this connection have involved twisted KR-theory, as in [14]. In condensed matter physics, real K-theory is used to classify topological insulators [21, 36]. Many of the invariants, for example the computable invariant used to detected 3D topological insulators [16], do not seem at first to be part of an KO group. Recently detailed studies of KR-theory of low-dimensional spaces [12, 13] explain the place in K-theory for such invariants, but only in the case of no disorder. For methods that handle disorder, see [15, 28, 32].

The ten-fold way in physics [36] was a key motivation for this work. The Altland–Zirnbauer [1] classification of the essential antiunitary symmetries on a quantum system has ten symmetry classes, named according to associated Cartan labels. These ten classes correspond to the two complex and eight real K-theory groups, as in Table 3. It is hoped that the consistent

and simple formulas presented here for all ten boundary maps will be of utility in understanding the indices being developed in physics.

A typical problem involving topological insulators and K-theory involves a collection of maps

$$\varphi_t : C(\mathbb{T}^2) \to \widetilde{\mathbb{K}}$$

that are asymptotically multiplicative, while exactly preserving addition, adjoint and the given real structure. That is, we have an element of real E-theory. To identify that element, we need only pair it with each of the two generators of $KO_{-2}(C(\mathbb{T}^2), \mathrm{id})$. Other spaces and involutions arise in a similar fashion, as in [28, 30]. A typical real structure on $C(X)$ is $f^\tau = f$ on the domain and a typical real structure on the compact operators is the dual operation. Thus the initial problem is how to calculate an explicit generator of $KO_{-2}(C(\mathbb{T}^2), \mathrm{id})$. Let us revisit how the calculation would look in the familiar complex case, where we need a generator of the reduced KU_0 group.

Consider the short exact sequence

$$0 \to C_0((0,1)^2) \to C(\mathbb{T}^2) \to C(S^1 \vee S^1) \to 0$$

coming from the closed copy of $S^1 \times S^1$ consisting of points (z, w) that have either $z = 1$ or $w = 1$. We need to compute the boundary map

$$\partial_1 : KU_1(C(S^1 \vee S^1)) \to KU_0(C_0((0,1)^2)).$$

This is easy. One generator of $KU_1(C(S^1 \vee S^1))$ is u_1 defined by $(z, w) \mapsto z$. This lifts as a unitary v_1 to $C(\mathbb{T}^2)$. The same is true of the other generator so $\partial_1 = 0$. Therefore

$$\iota_* : KU_0(C_0((0,1)^2)) \to KU_0(C(\mathbb{T}^2))$$

is an inclusion, and the element we need comes from the generator of $KU_0(C_0((0,1)^2))$. To find that, one can look at the exact sequence

$$0 \to C_0(U) \to C(\mathbb{D}) \to C(S^1) \to 0$$

and compute ∂_1 on the unitary $u(z) = z$. Here U is the open disk.

With a few modifications, the standard method to compute $\partial_1([u])$ for a unitary in B is as follows, assuming

$$0 \to I \to A \to B \to 0$$

is exact with A unital. The first step is to lift u to an element a in A with $\|a\| \leq 1$ and then form the projection

$$p = \begin{pmatrix} aa^* & a\sqrt{1-a^*a} \\ a^*\sqrt{1-aa^*} & 1-a^*a \end{pmatrix}. \tag{1}$$

To see how this arises from the more usual formulas [34, §9.1], notice

$$v = \begin{pmatrix} a & -\sqrt{1-aa^*} \\ \sqrt{1-a^*a} & a^* \end{pmatrix}$$

is a unitary in A (cf. [34, Lemma 9.2.1]) that is a lift of $\mathrm{diag}(u, u^*)$. Then $p = v\mathrm{diag}(1,0)v^*$ and so, up to identifying $M_2(\widetilde{I})$ with a subalgebra of $M_2(B)$, we have $\partial_1([u]) = [p] - [1]$.

Applying (1) in the case $u(z) = z$ on the circle we lift (extend) to a function $a(z) = z$ on the disk. Then

$$p(z) = \begin{pmatrix} |z|^2 & z\sqrt{1-|z|^2} \\ \overline{z}\sqrt{1-|z|^2} & 1-|z|^2 \end{pmatrix}.$$

Taken as a map on the sphere, this is a degree-one mapping of S^2 onto the set of projections in $M_2(\mathbb{C})$ of trace one. In terms of the real coordinates (x, y, z) restricted to the unit sphere, we find the desired element of $KU(C_0(U))$ is

$$\left[\begin{pmatrix} \frac{1}{2}z & \frac{1}{2}x - \frac{i}{2}y \\ \frac{1}{2}x + \frac{i}{2}y & \frac{1}{2} - \frac{1}{2}z \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Pushing this forward to $C(\mathbb{T}^2)$ is a little tricky. One solution is the element

$$\left[\begin{pmatrix} f(z) & g(z) + h(z)\overline{w} \\ g(z) + h(z)w & 1 - f(z) \end{pmatrix}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \tag{2}$$

where f, g and h are certain real-valued functions on the circle satisfying $gh = 0$ and $f^2 + g^2 + h^2 = 1$, as discussed in [27].

Our immediate goal is to allow the calculation of generators of KO_* groups to proceed in essentially the same manner as in the preceeding calcuation. In particular, the generator of $KO_{-1}(C(S^1), \mathrm{id})$ will be $[u]$ where $u(z) = z$. What will be new is having to check that this matrix is symmetric.

Given

$$0 \to I \to A \to B \to 0$$

exact, and unital, but now with real structures, given u a unitary in B with $u^\tau = u$, we have a representative of a KO_{-1} class. To calculate the boundary, we lift to a with $\|a\| \leq 1$ and $a^\tau = a$ and form

$$w = \begin{pmatrix} 2aa^* - 1 & 2a\sqrt{1-a^*a} \\ 2a^*\sqrt{1-aa^*} & 1 - 2a^*a \end{pmatrix}.$$

Then w is unitary, self-adjoint, and with the more subtle symmetry that is component-wise given as

$$w_{11}^\tau = -w_{22}, \quad w_{12}^\tau = w_{12}, \quad w_{21}^\tau = w_{21}.$$

We will see this is valid to specify an element of $KO_{-2}(I)$. Thus the boundary map $\partial_{-1} : KO_{-1}(B) \to KO_{-2}(I)$ looks very much like the odd boundary map in the complex case. We will see that the generator of $KO_{-2}(C_0(U), \mathrm{id})$ is

$$\left[\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}\right].$$

The generator of $KO_{-2}(C(\mathbb{T}^2), \mathrm{id})$ will be the same as in (2) with just a small modification of the three functions.

The even boundary maps will also be given as a lifting problem. A unitary u with $u^* = u$ and other symmetries gets lifted to x with $-1 \leq x \leq 1$ and other symmetries. The unitary needed is then

$$-\exp(\pi i x)$$

which is again very close to the complex case. Indeed, by reformulating the complex case in terms of self-adjoint unitaries for KU_0 this will be the formula for the even boundary map. It should be noted that we are losing track of the order structure on KO_0 and KU_0. In principle we can recover this, but have no present need.

As preliminary work to developing this picture of real K-theory, but of independent interest, we also present a collection of classifying algebras A_i for $i \in \{0, 1, \ldots, 8\}$. These are real semiprojective homotopy symmetric C^*-algebras that classify K-theory in the sense that

$$KO_i(D) \cong [A_i, \mathbb{K}^{\mathbb{R}} \otimes D] \cong \lim_{n\to\infty} [A_i, M_n(\mathbb{R}) \otimes D]$$

for all i, as we show in Theorem 4.13. The algebras A_i are thus real analogs of the complex C^*-algebras $q\mathbb{C}$ and $C_0(\mathbb{R}, \mathbb{C})$ which are classifying algebras for K-theory in the category of complex C^*-algebras in the same sense. Not unexpectedly, the algebras A_i will all be real forms of matrix algebras over $q\mathbb{C}$ and $C_0(\mathbb{R}, \mathbb{C})$.

In Section 3, we introduce the real C^*-algebras A_i for $0 \leq i < 8$ and we calculate their united K-theory, finding that $KO_*(A_i) \cong \Sigma^{-i} K_*(\mathbb{R})$. It follows from this (or rather from the stronger statement $K^{CRT}(A_i) \cong \Sigma^{-i} K^{CRT}(\mathbb{R})$) and the universal coefficient theorem that there is a real KK-equivalence between A_i and $S^{-i}\mathbb{R}$ and that $KO_i(B) \cong KKO_0(A_i, B)$ for any real separable C^*-algebra B in the UCT bootstrap category. Also in Section 3, we will show that each A_i is semiprojective, following a short detour to prove a key semiprojectivity closure theorem. Then in Section 4 we will prove that each A_i is homotopy symmetric. We validate the real version of unsuspended E-theory, and it then follows that these algebras represent K-theory in the strong sense that $KO_i(B) \cong \lim_{n\to\infty} [A_i, B \otimes M_n(\mathbb{R})]$ for any separable real C^*-algebra B.

In Sections 5 and 6 we will develop the unitary picture of K-theory, first in the even degrees and then in the odd degrees. We note that we are not attempting to accomplish a complete development of K-theory from scratch using the unitary picture — although that would be an interesting project. Instead, we take it for granted that K-theory is an established entity with known properties. We will define a sequence of groups $KO_i^u(A)$ in terms of unitaries and will then develop its properties mainly to get to the point of being able to prove that there is a natural isomorphism $KO_i(A) \cong KO_i^u(A)$ in each case.

Section 7 explores some examples where the generators of the KO groups can be found easily by comparing with the complex case. Section 8 finds

and describes formulas for the eight boundary maps, and these are applied in Section 9 in finding more explicit generators of real K-theory groups, for several examples.

2. Preliminaries

The category of interest in this paper is the category $\mathcal{R}^*$ of real C^*-algebras (also known as R^*-algebras), with real *-algebra homomorphisms. A real C^*-algebra (as in Section 1 of [37]) is a real Banach *-algebra satisfying the norm condition $\|a^*a\| = \|a\|^2$ and the condition that $\mathbb{1} + a^*a$ is invertible (in $\widetilde{A}$) for all $a \in A$.

The category $\mathcal{R}^*$ is equivalent to the category $\mathcal{R}^{*,\tau}$ of $C^{*,\tau}$-algebras with $C^{*,\tau}$-algebra homomorphisms (see [31]). A $C^{*,\tau}$-algebra is a pair (A, τ) where A is a (complex) C^*-algebra and τ is an involutive antiautomorphism on A. Given a $C^{*,\tau}$-algebra (A, τ), the corresponding real C^*-algebra is

$$A^\tau = \{a \in A \mid a^\tau = a^*\}.$$

Conversely, given a real C^*-algebra A there is a unique complexification $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$, which as an algebra is isomorphic to $A + iA$. The formula $(a + ib) \mapsto (a^* + ib^*)$ is an antimultiplicative involution on $A_{\mathbb{C}}$. This construction gives a functor from $\mathcal{R}^*$ to $\mathcal{R}^{*,\tau}$, which is inverse (up to isomorphism) to the functor described in the previous paragraph.

We will slide back and forth easily between these two categories, as is appropriate for the situation. In particular, whereas our unitary description of $KO_0^u(-)$ and $KO_1^u(-)$ can be made in terms of a real C^*-algebra A, our description of $KO_i^u(-)$ for other values of i requires the context of a $C^{*,\tau}$-algebra. Hence we present our unified picture of $KO_i(-)$ for all i in the setting of a $C^{*,\tau}$-algebra (see Section 7). This approach is analogous to the development of K-theory for topological spaces with involution in [2].

If (A, τ) is a $C^{*,\tau}$-algebra, then so is $(M_n(\mathbb{C}) \otimes A, \tau_n)$ where $\tau_n = \mathrm{Tr}_n \otimes \tau$ and Tr_n is the transpose operation on $M_n(\mathbb{C})$. We will frequently neglect the subscripts on τ and Tr when we can do so without sacrificing clarity. Similarly, we will let τ also denote the involution on $\mathbb{K} \otimes A$ induced by τ_n through a choice of isomorphism $\lim_{n\to\infty}(M_n(\mathbb{C})) \cong \mathbb{K}$. These constructions correspond to the real C^*-algebra constructions of tensoring by $M_n(\mathbb{R})$ or by $\mathbb{K}^{\mathbb{R}}$, the real C^*-algebra of compact operators on a separable real Hilbert space.

There is a related antiautomorphism $\widetilde{\mathrm{Tr}}$ on $M_2(\mathbb{C})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\widetilde{\mathrm{Tr}}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

This involution is equivalent to Tr in the sense that there is an isomorphism of $C^{*,\tau}$-algebras, $(M_2(\mathbb{C}), \mathrm{Tr}) \cong (M_2(\mathbb{C}), \widetilde{\mathrm{Tr}})$. Indeed, the reader can check

that $(WxW^*)^{\mathrm{Tr}} = W x^{\widetilde{\mathrm{Tr}}} W^*$ where

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

More generally, we have $(M_2(\mathbb{C}) \otimes A, \tau) \cong (M_2(\mathbb{C}) \otimes A, \widetilde{\tau})$ where the automorphism $\widetilde{\tau}$ is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\widetilde{\tau}} = \begin{pmatrix} d^\tau & b^\tau \\ c^\tau & a^\tau \end{pmatrix}.$$

There is yet another real structure on $M_{2n}(A)$ and on $\mathbb{K} \otimes A$, which is genuinely distinct from Tr. Define $\sharp\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then $(M_2(\mathbb{C}), \sharp)$ corresponds to the real C^*-algebra $\mathbb{H}$ of quaternions, and $(M_{2n}(\mathbb{C}), \sharp \otimes \mathrm{Tr}_n)$ corresponds to the real C^*-algebra $M_n(\mathbb{H})$. More generally, if (A, τ) is a $C^{*,\tau}$-algebra, then $(M_{2n}(\mathbb{C}) \otimes A, \sharp \otimes \mathrm{Tr}_n \otimes \tau)$ is a $C^{*,\tau}$-algebra that corresponds to the real C^*-algebra $M_n(\mathbb{H}) \otimes A^\tau$.

We will be dealing with these matrix algebras frequently in the subsequent work, and the technicalities require that we clarify the conventions for the action of $\sharp \otimes \tau$ on a matrix in $M_{2n}(A)$, since this action requires a particular choice of isomorphism $M_2(A) \otimes M_n(A) \cong M_{2n}(A)$. The two obvious choices of such an isomorphism lead to two conventions for $\sharp \otimes \tau$ that we will make use of regularly. The first is shown by organizing the matrix $a \in M_{2n}(A)$ as an $n \times n$ matrix whose entries are 2×2 blocks, denoted by $b_{ij} \in M_2(A)$. Then

$$a^{\sharp\otimes\tau} = \begin{pmatrix} b_{1\,1} & b_{1\,2} & \dots & b_{1\,n} \\ b_{2\,1} & b_{2\,2} & \dots & b_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n\,1} & b_{n\,2} & \dots & b_{n\,n} \end{pmatrix}^{\sharp\otimes\tau} = \begin{pmatrix} b_{1\,1}^{\sharp\otimes\tau} & b_{2\,1}^{\sharp\otimes\tau} & \dots & b_{n\,1}^{\sharp\otimes\tau} \\ b_{1\,2}^{\sharp\otimes\tau} & b_{2\,2}^{\sharp\otimes\tau} & \dots & b_{n\,2}^{\sharp\otimes\tau} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1\,n}^{\sharp\otimes\tau} & b_{2\,n}^{\sharp\otimes\tau} & \dots & b_{n\,n}^{\sharp\otimes\tau} \end{pmatrix}.$$

The second convention for an involution on $M_{2n}(A)$ will be denoted by $\widetilde{\sharp} \otimes \tau$ and is shown by organizing the matrix $a \in M_{2n}(A)$ as a 2×2 matrix whose entries are $n \times n$ blocks, denoted by $c_{ij} \in M_n(A)$. Then

$$a^{\widetilde{\sharp}\otimes\tau} = \begin{pmatrix} c_{1\,1} & c_{1\,2} \\ c_{2\,1} & c_{2\,2} \end{pmatrix}^{\widetilde{\sharp}\otimes\tau} = \begin{pmatrix} c_{2\,2}^{\tau_n} & -c_{1\,2}^{\tau_n} \\ -c_{2\,1}^{\tau_n} & c_{1\,1}^{\tau_n} \end{pmatrix}.$$

The first convention for $\sharp \otimes \tau$ will be our preferred convention.

As mentioned, we will take for granted the full development and known properties of both K-theory and KK-theory for real C^*-algebras. The development of KK-theory for real C^*-algebras goes back to [20] while much what is known about both K-theory and KK-theory can be found in [37].

For a real C^*-algebra A, we will also occasionally make reference to the united K-theory $K^{CRT}(A)$, as developed in [5]. Briefly, $K^{CRT}(A)$ consists of

the eight real K-theory groups $KO_i(A)$, the two complex K-theory groups $KU_i(A)$ (coinciding with the K-theory of the complexification of A), and the four self-conjugate K-theory groups $KT_i(A)$; as well as the several natural transformations among them. The main result about united K-theory that we will make use of is the Universal Coefficient Theorem proven in [6], which implies that united K-theory classifies KK-equivalence for real C^*-algebras that are nuclear, separable, and in the bootstrap class for the UCT.

For a final note regarding conventions, we will use $\mathbb{1}$ to denote the adjoined unit in $\widetilde{A}$ for any C^*-algebra A (unital or not). Similarly $\mathbb{1}_n$ will denote the diagonal identity matrix in $M_n(\widetilde{A})$.

3. Semiprojective suspension C^*-algebras

Let $q\mathbb{C} = \{f \in C_0((0,1], M_2(\mathbb{C})) \mid f(1) \in \mathbb{C}^2\}$ where we are identifying $\mathbb{C}^2$ with the subalgebra of diagonal elements of $M_2(\mathbb{C})$. The algebras A_i for i even are defined as follows. Three are real structures of $q\mathbb{C}$ and one is a real structure of $M_2(q\mathbb{C})$.

$$\begin{aligned} A_0 &= \{f \in C_0((0,1], M_2(\mathbb{R})) \mid f(1) \in \mathbb{R}^2\} \\ A_2 &= \{f \in C_0((0,1], \mathbb{H}) \mid f(1) \in \mathbb{C}\} \\ A_4 &= \{f \in C_0((0,1], M_2(\mathbb{H})) \mid f(1) \in \mathbb{H}^2\} \\ A_6 &= \{f \in C_0((0,1], M_2(\mathbb{R})) \mid f(1) \in \mathbb{C}\}. \end{aligned}$$

For i odd, the algebras A_i are defined as follows. Each is a real structure of either $C_0(S^1 \setminus \{1\}, \mathbb{C})$ or $C_0(S^1 \setminus \{1\}, M_2(\mathbb{C}))$.

$$\begin{aligned} A_{-1} &= S\mathbb{R} = \{f \in C(S^1, \mathbb{R}) \mid f(1) = 0\} \\ A_1 &= S^{-1}\mathbb{R} = \{f \in C(S^1, \mathbb{C}) \mid f(1) = 0 \text{ and } f(\overline{z}) = \overline{f(z)}\} \\ A_3 &= S\mathbb{H} = \{f \in C(S^1, \mathbb{H}) \mid f(1) = 0\} \\ A_5 &= S^{-1}\mathbb{H} = \{f \in C(S^1, M_2(\mathbb{C})) \mid f(1) = 0 \text{ and } f(\overline{z})^\sharp = \overline{f(z)}\}. \end{aligned}$$

These real C^*-algebras have corresponding objects in the cateogry of $C^{*,\tau}$-algebras as shown in Table 2. In this table, the involution ζ denotes the involution on $C_0(S^1 \setminus \{1\}, \mathbb{C})$ induced by the involution $z \mapsto \overline{z}$ on S^1.

Proposition 3.1. *$K^{CRT}(A_i) \cong \Sigma^{-i} K^{CRT}(\mathbb{R})$ for all $i \in \{0, 2, 4, 6\}$.*

Proof. In each case, $A_i \otimes \mathbb{C} \cong q\mathbb{C}$ or $A_i \otimes \mathbb{C} \cong M_2(q\mathbb{C})$. So $K_*(A_i \otimes \mathbb{C}) \cong K_*(q\mathbb{C}) \cong K_*(\mathbb{C})$. Thus by Theorem 3.2 of [9], $K^{CRT}(A_i)$ is a free *CRT*-module. Furthermore, from Section 2.4 of [9], the only free *CRT*-module that has the complex part isomorphic to $K_*(\mathbb{C})$ is $K^{CRT}(\mathbb{R})$ up to an even suspension. Therefore there are only four possibilities for $K^{CRT}(A_i)$ up to isomorphism. A full description of the *CRT*-module $K^{CRT}(\mathbb{R})$ is in Table 1 of [5], but in particular recall that the real part of it is given by $KO_*(\mathbb{R})$ as

TABLE 2. The Classifying Algebras

	R^*-algebra	$C^{*,\tau}$-algebra
even cases	A_0	$(q\mathbb{C}, \mathrm{Tr})$
	A_2	$(q\mathbb{C}, \sharp)$
	A_4	$(M_2(\mathbb{C}) \otimes q\mathbb{C}, \sharp \otimes \mathrm{Tr})$
	A_6	$(q\mathbb{C}, \widetilde{\mathrm{Tr}})$
odd cases	A_{-1}	$(C_0(S^1 \setminus \{1\}, \mathbb{C}), \mathrm{id})$
	A_1	$(C_0(S^1 \setminus \{1\}, \mathbb{C})), \zeta)$
	A_3	$(M_2(\mathbb{C}) \otimes C_0(S^1 \setminus \{1\}, \mathbb{C}), \sharp \otimes \mathrm{id})$
	A_5	$(M_2(\mathbb{C}) \otimes C_0(S^1 \setminus \{1\}, \mathbb{C}), \sharp \otimes \zeta)$

This table shows the real C^*-algebras A_i and the corresponding objects in the category of $C^{*,\tau}$-algebras. They classify real K-theory in the sense of Theorem 4.13.

shown below. In each case, this will be enough to determine which of the four possible suspensions is isomorphic to $K^{CRT}(A_i)$.

i	0	1	2	3	4	5	6	7
$KO_i(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

We first consider A_0. Use the extension of real C^*-algebras

$$0 \to SM_2(\mathbb{R}) \xrightarrow{\iota} A_0 \xrightarrow{\mathrm{ev}_1} \mathbb{R}^2 \to 0 \tag{3}$$

where ev_1 is the evaluation map at $t = 1$. Then we have the long exact sequence

$$\cdots \to K^{CRT}(\mathbb{R}^2) \xrightarrow{\partial} K^{CRT}(\mathbb{R}) \xrightarrow{\iota_*} K^{CRT}(A_0) \xrightarrow{(\mathrm{ev}_1)_*} K^{CRT}(\mathbb{R}^2) \xrightarrow{\partial} \cdots .$$

The map ∂ as written has degree 0 and can be determined by its action on the generators of the two $K^{CRT}(\mathbb{R})$ summands, which are elements in $KO_0(\mathbb{R}) \cong \mathbb{Z}$. The complex part of this long exact sequence arises from the complexification of Sequence (3), which is

$$0 \to SM_2(\mathbb{C}) \xrightarrow{\iota} q\mathbb{C} \xrightarrow{\mathrm{ev}_1} \mathbb{C}^2 \to 0$$

and for which the boundary map $\partial\colon K_0(\mathbb{C}^2) \to K_0(M_2(\mathbb{C}))$ is known to be $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ up to isomorphism. In the commutative diagram below, we know that the complexification maps c are both isomorphisms, so it follows that the boundary map $\partial\colon K_0(\mathbb{R}^2) \to K_0(M_2(\mathbb{R}))$ is also isomorphic

to $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$.

$$\begin{array}{ccc} KO_0(\mathbb{R}^2) & \xrightarrow{\partial} & KO_0(M_2(\mathbb{R})) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ K_0(\mathbb{C}^2) & \xrightarrow{\partial} & K_0(M_2(\mathbb{C})) \end{array}$$

It follows that $\partial\colon K^{CRT}(\mathbb{R}^2) \to K^{CRT}(\mathbb{R})$ is surjective and has kernel isomorphic to $K^{CRT}(\mathbb{R})$. Thus $K^{CRT}(A_0) \cong K^{CRT}(\mathbb{R})$.

For A_2, we have the short exact sequence

$$0 \to S\mathbb{H} \to A_2 \xrightarrow{\mathrm{ev}_1} \mathbb{C} \to 0 \tag{4}$$

and the corresponding long exact sequence

$$\cdots \to K^{CRT}(\mathbb{C}) \xrightarrow{\partial} K^{CRT}(\mathbb{H}) \to K^{CRT}(A_2) \to K^{CRT}(\mathbb{C}) \xrightarrow{\partial} K^{CRT}(\mathbb{H}) \to \cdots .$$

The complexification of Sequence (4) is the same as that of Sequence (3) so again we can use the complexification map to calculate the boundary map. The commutative diagram we obtain is as follows which shows that $\partial\colon KO_0(\mathbb{C}) \to KO_0(\mathbb{H})$ is an isomorphism from $\mathbb{Z}$ to $\mathbb{Z}$.

$$\begin{array}{ccc} KO_0(\mathbb{C}) & \xrightarrow{\partial} & KO_0(\mathbb{H}) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ K_0(\mathbb{C}\oplus\mathbb{C}) & \xrightarrow{\partial} & K_0(M_2(\mathbb{C})) \end{array} \quad \text{isomorphic to} \quad \begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ {\scriptstyle \binom{1}{1}}\downarrow & & \downarrow{\scriptstyle 2} \\ \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow{(1\ 1)} & \mathbb{Z}. \end{array}$$

Then the long exact sequence shows that $KO_0(A_2) \cong 0$. Of the four possibilities for $K^{CRT}(A_2)$, there is only one that is consistent with this fact. Thus we conclude that $K^{CRT}(A_2) \cong \Sigma^{-2}K^{CRT}(\mathbb{R})$.

From the Künneth Formula we know that $K^{CRT}(B) \cong \Sigma^4 K^{CRT}(\mathbb{H}\otimes B)$ for any real C^*-algebra. Hence, $K^{CRT}(A_6)$ and $K^{CRT}(A_4)$ are determined by the isomorphisms $M_2(\mathbb{R})\otimes A_2 \cong \mathbb{H}\otimes A_6$ and $A_4 \cong \mathbb{H}\otimes A_0$. □

Proposition 3.2. *$K^{CRT}(A_i) \cong \Sigma^{-i}K^{CRT}(\mathbb{R})$ for all $i \in \{-1, 1, 3, 5\}$.*

Proof. For $i = \pm 1$ this follows from Proposition 1.20 of [5]. For $i = 3, 5$, this follows from the Künneth Formula and the isomorphisms $A_i \cong \mathbb{H}\otimes A_{i-4}$. □

As in Section 1 of [25], consider the following relations for elements h, x, k in a C^*-algebra A:

$$\begin{aligned} h^*h + x^*x &= h, \\ k^*k + xx^* &= k, \\ kx &= xh, \\ hk &= 0. \end{aligned} \tag{5}$$

The same relations can be formally encoded by

$$\begin{aligned} hk &= 0, \\ T(h,x,k)^2 = T(h,x,k)^* &= T(h,x,k). \end{aligned} \tag{6}$$

where

$$T(h,x,k) = \begin{pmatrix} \mathbb{1}-h & x^* \\ x & k \end{pmatrix} \in M_2(\widetilde{A}).$$

Of particular interest, we have the elements

$$h_0 = t \otimes e_{11}, \qquad k_0 = t \otimes e_{22}, \qquad x_0 = \sqrt{t-t^2} \otimes e_{21}$$

that satisfy (6) in $q\mathbb{C}$. Recall from Lemma 2 of [25] that $q\mathbb{C}$ is the universal C^*-algebra generated by h, x, k subject to the relations (6). The following theorem gives a version of this result for A_0, A_2, and A_6; characterizing $C^{*,\tau}$-algebra-homomorphisms from $(q\mathbb{C}, \mathrm{Tr})$, $(q\mathbb{C}, \sharp)$, and $(q\mathbb{C}, \widetilde{\mathrm{Tr}})$.

Proposition 3.3. *Let (A,τ) be a $C^{*,\tau}$-algebra.*

1. *Given elements h,k,x in A satisfying $h^\tau = h$, $k^\tau = k$, $x^\tau = x^*$ and Equations (5), then there exists a unique homomorphism*
$$\alpha\colon (q\mathbb{C}, \mathrm{Tr}) \to (A,\tau)$$
such that $\alpha(h_0) = h$, $\alpha(k_0) = k$, and $\alpha(x_0) = x$.
2. *Given elements h,k,x in A satisfying $h^\tau = k$, $k^\tau = h$, $x^\tau = -x$ and Equations (5), then there exists a unique homomorphism*
$$\alpha\colon (q\mathbb{C}, \sharp) \to (A,\tau)$$
such that $\alpha(h_0) = h$, $\alpha(k_0) = k$, and $\alpha(x_0) = x$.
3. *Given elements h,k,x in A satisfying $h^\tau = k$, $k^\tau = h$, $x^\tau = x$ and Equations (5), then there exists a unique homomorphism*
$$\alpha\colon (q\mathbb{C}, \widetilde{\mathrm{Tr}}) \to (A,\tau)$$
such that $\alpha(h_0) = h$, $\alpha(k_0) = k$, and $\alpha(x_0) = x$.

Proof. Under the hypotheses of Part (1), Lemma 1 of [25] gives a unique C^*-algebra homomorphism $\alpha\colon q\mathbb{C} \to A$ satisfying $\alpha(h_0) = k_0$, $\alpha(k_0) = k$, and $\alpha(x_0) = x$. It is only required here to verify that α respects the real structures; that is to verify that

$$\alpha(a^{\mathrm{Tr}}) = \alpha(a)^\tau \tag{7}$$

holds for all $a \in q\mathbb{C}$. In $q\mathbb{C}$ we have

$$h_0^{\mathrm{Tr}} = h_0, \quad k_0^{\mathrm{Tr}} = k_0, \quad \text{and} \quad x_0^{\mathrm{Tr}} = x_0^*$$

from which it follows that (7) holds for $a = h_0, k_0, x_0$. But since these elements generate $q\mathbb{C}$ and since the set of elements that satisfy (7) is a subalgebra of $q\mathbb{C}$, it follows that (7) holds for on $q\mathbb{C}$.

The proofs in the second and third cases are the same, noting that in $q\mathbb{C}$ we have

$$h_0^\sharp = k_0, \quad k_0^\sharp = h_0, \quad \text{and} \quad x_0^\sharp = -x_0$$

and

$$h_0^{\widetilde{\mathrm{Tr}}} = k_0, \quad k_0^{\widetilde{\mathrm{Tr}}} = h_0, \quad \text{and} \quad x_0^{\widetilde{\mathrm{Tr}}} = x_0. \qquad \square$$

The concepts of projectivity and semiprojectivity were first introduced and developed in the context of real C^*-algebras (and $C^{*,\tau}$-algebras) in Section 3 of [31]. In what follows, we extend that work by proving a significant closure theroem, namely that $A \otimes \mathbb{H}$ and $A \otimes M_n(\mathbb{R})$ are semiprojective if A is semiprojective (Theorem 3.10). This result will subsequently be applied to show that each of the real C^*-algebras A_i is semiprojective.

The cone $CM_2(\mathbb{C}) = C_0((0,1], M_2(\mathbb{C}))$ has two real structures, corresponding to the antiautomorphisms Tr and $\sharp$ defined pointwise on $CM_2(\mathbb{C})$. The corresponding real C^*-algebras are $CM_2(\mathbb{R}) = C_0((0,1], M_2(\mathbb{R}))$ and $C\mathbb{H} = C_0((0,1], \mathbb{H})$. More generally $CM_n(\mathbb{C})$ has one real structure for n odd (corresponding to Tr) and two real structures for n even (corresponding to Tr and to $\sharp \otimes \mathrm{Tr}$).

Lemma 3.4. *Let (A,τ) and (B,τ) be $C^{*,\tau}$-algebras and let*

$$\pi\colon (A,\tau) \to (B,\tau)$$

be a surjective $C^{*,\tau}$-algebra homomorphism. Let h and k be positive orthogonal elements in B.

1. *If $h^\tau = h$ and $k^\tau = k$, then there are positive orthogonal elements h' and k' in A that satisfy $(h')^\tau = h'$ and $(k')^\tau = k'$.*
2. *If $h^\tau = k$ and $k^\tau = h$, then there are positive orthogonal elements h' and k' in A that satisfy $(h')^\tau = k'$ and $(k')^\tau = h'$.*

Furthermore, h' and k' can be taken to satisfy $\|h'\| \leq \|h\|$ and $\|k'\| \leq \|k\|$.

Proof. Let $a = h - k$. Let $a' \in A$ be a self-adjoint lift of a. Furthermore, in case (1) we have $a^\tau = a$ and we can take a' to satisfy the same by replacing a' with $\frac{1}{2}\left(a' + (a')^\tau\right)$. Let $f_+, f_-\colon \mathbb{R} \to \mathbb{R}$ be defined by $f_+(t) = \max\{0,t\}$ and $f_-(t) = -\min\{t,0\}$ so that $(f_+ - f_-)(t) = t$. Let $h' = f_+(a')$ and $k' = f_-(a')$. Then h' and k' are positive and orthogonal. Also,

$$\pi(h') = \pi(f_+(a')) = f_+(\pi(a')) = f_+(a) = h$$

and similarly, $\pi(k') = k$. Finally,

$$(h')^\tau = (f_+(a'))^\tau = f_+((a')^\tau) = f_+(a') = h'$$

and similarly, $(k')^\tau = k'$.

In case (2) we have $a^\tau = -a$ and we can take a' to satisfy the same by replacing a' with $\frac{1}{2}\left(a' - (a')^\tau\right)$. Using h' and k' as above, we obtain

$$(h')^\tau = (f_+(a'))^\tau = f_+((a')^\tau) = f_+(-a') = k'$$

and similarly $(k')^\tau = h'$.

In either case, the norm condition can be obtained by truncating the elements h' and k' using the functions $g_K(t) = \min\{t, K\}$ where $K = \|h\|$ and $K = \|k\|$ respectively. $\square$

Proposition 3.5. *The $C^{*,\tau}$-algebras $(CM_n(\mathbb{C}), \mathrm{Tr})$ and $(CM_2(\mathbb{C}), \sharp)$ are projective.*

Proof. A proof that $(CM_n(\mathbb{C}), \mathrm{Tr})$ is projective can be obtained by examining a proof that $CM_n(\mathbb{C})$ is projective in the context of complex C^*-algebras. For example, see Theorem 10.2.1 in [24] or Theorem 3.5 of [29].

To show that $(CM_2(\mathbb{C}), \sharp)$ is projective let $\phi\colon (CM_2(\mathbb{C}), \sharp) \to (B, \tau)$ be a $C^{*,\tau}$-algebra homomorphism and let $\pi\colon (A, \tau) \to (B, \tau)$ be a surjective $C^{*,\tau}$-algebra homomorphism. Let $x = \phi(t \otimes e_{12})$. Then x satisfies $\|x\| \leq 1$, $x^2 = 0$, and $x^\tau = -x$. In fact $(CM_2(\mathbb{C}), \sharp)$ is universal for these relations so it suffices to show that x can be lifted to an element in A satisfying the same.

Using Lemma 3.4, lift $\phi(t^{1/3} \otimes e_{11})$ and $\phi(t^{1/3} \otimes e_{22})$ to elements $h, k \in A$ satisfying $0 \leq h, k \leq 1$, $hk = 0$, and $h^\tau = k$. Lift $\phi(t^{1/3} \otimes e_{12}) = x^{1/3}$ to an element $y \in A$ satisfying $y^\tau = -y$. Let $z = kyh$ so that $\pi(z) = x$, $z^2 = 0$, and $z^\tau = -z$.

To finish, let

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ t^{-1/2} & 1 \leq t \end{cases} \qquad \text{and} \qquad w = zf(z^*z).$$

Then $\|w\| \leq 1$ since $w^*w = f(z^*z)z^*zf(z^*z) = g(z^*z)$ where

$$g(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 \leq t. \end{cases}$$

Since $\pi(w) = xf(x^*x) = x$, we have that w is still a lift of x. We also have $w^2 = f(zz^*)zzf(z^*z) = 0$. Finally, we show that $w^\tau = -w$. Check that $(z^*z)^\tau = z^\tau z^{*\tau} = -z(-z^*) = zz^*$ so that

$$w^\tau = (zf(z^*z))^\tau = f(z^*z)^\tau z^\tau = -f(zz^*)z = -w. \qquad \square$$

We remark the Proposition 3.5 can be strengthened to state that the cone $(CM_{2n}(\mathbb{C}), \sharp \otimes \mathrm{Tr}_n)$ is projective using a similar proof to the above. This however will be a direct consequence of Proposition 3.9 below and the stronger result is not required for us before that point.

Lemma 3.6. *Suppose $\varphi : CM_n(\mathbb{C}) \to B$ is a $*$-homomorphism of C^*-algebras. Denote by B_0 and B_n the hereditary subalgebras of B generated by $\varphi(C_0(0,1] \otimes e_{11})$ and $\varphi(CM_n(\mathbb{C}))$, respectively. Then there is a natural isomorphism*

$$\Phi : B_0 \otimes M_n(\mathbb{C}) \to B_n.$$

defined by

$$\Phi\left(\varphi(f \otimes e_{1r})b\varphi(g \otimes e_{s1}) \otimes e_{jk}\right) = \varphi(f \otimes e_{jr})b\varphi(g \otimes e_{sk})$$

for $f, g \in C_0(0,1]$ and $b \in B$.

Furthermore, suppose $\varphi\colon (CM_n(\mathbb{C}), \mathrm{Tr}) \to (B, \tau)$ *is a* $C^{*,\tau}$*-algebra homomorphism of* $C^{*,\tau}$*-algebras. Then there is a natural isomorphism of* $C^{*,\tau}$*-algebras*

$$\Phi\colon (B_0 \otimes M_n(\mathbb{C}), \tau \otimes \mathrm{Tr}) \to (B_n, \tau)$$

given by the same formula as above.

Proof. We start with some nice descriptions of B_0 and B_n. Since $C(0,1] \otimes e_{11}$ is generated by $t \otimes e_{11}$, we have

$$B_0 = \overline{\varphi(t \otimes e_{11}) B \varphi(t \otimes e_{11})} \qquad \text{and} \qquad B_n = \overline{\varphi(t \otimes 1_n) B \varphi(t \otimes 1_n)}.$$

On the other hand the nice factorization result, Corollary 4.6 of [33], implies that

$$\begin{aligned} B_0 &= \varphi(C_0(0,1] \otimes e_{11}) B \varphi(C_0(0,1] \otimes e_{11}), \\ B_n &= \varphi(C_0(0,1] \otimes 1_n) B \varphi(C_0(0,1] \otimes 1_n), \end{aligned}$$

which shows that it is enough to define Φ on the elements of the form

$$x = \varphi(f \otimes e_{11}) b \varphi(g \otimes e_{11}).$$

We first establish that Φ is well-defined as a map restricted to $B_0 \otimes e_{jk}$. Suppose

$$\varphi(f \otimes e_{1r}) b \varphi(g \otimes e_{s1}) = \varphi(h \otimes e_{1p}) b' \varphi(k \otimes e_{q1}).$$

Select any μ_n that is an approximate identity in $C_0(0,1]$ and calculate:

$$\begin{aligned} &\varphi(f \otimes e_{jr}) b \varphi(g \otimes e_{sk}) \\ &\quad = \lim_m \lim_n \varphi(\mu_n \otimes e_{j1}) \varphi(f \otimes e_{1r}) b \varphi(g \otimes e_{s1}) \varphi(\mu_m \otimes e_{1k}) \\ &\quad = \lim_m \lim_n \varphi(\mu_n \otimes e_{j1}) \varphi(h \otimes e_{1p}) b' \varphi(k \otimes e_{q1}) \varphi(\mu_m \otimes e_{1k}) \\ &\quad = \varphi(h \otimes e_{jp}) b' \varphi(k \otimes e_{qk}). \end{aligned}$$

To see this is additive, consider two elements in B_0,

$$x = \varphi(f \otimes e_{11}) b \varphi(g \otimes e_{11}), \quad y = \varphi(h \otimes e_{11}) b' \varphi(k \otimes e_{11}).$$

We claim that we can rewrite these elements so that $f = h$ and $g = k$. Indeed, we can factor the functions as $f = \mu f_1$ and $h = \mu h_1$ where

$$\mu(x) = \sqrt{|f(x)| + |h(x)|}$$

to get $f \otimes e_{11} = (\mu \otimes e_{11})(f_1 \otimes e_{11})$ and $h \otimes e_{11} = (\mu \otimes e_{11})(f_1 \otimes e_{11})$,

$$x = \varphi(\eta \otimes e_{11}) b''' \varphi(g \otimes e_{11}), \quad y = \varphi(\eta \otimes e_{11}) b'' \varphi(k \otimes e_{11}).$$

Perform a similar procedure using the functions g and k. Therefore, we can assume that we have

$$x = \varphi(f \otimes e_{11}) b \varphi(g \otimes e_{11}), \quad y = \varphi(f \otimes e_{11}) b' \varphi(g \otimes e_{11}).$$

Then we prove additivity as follows,

$$\begin{aligned}
&\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk}+\varphi(f\otimes e_{11})b'\varphi(g\otimes e_{11})\otimes e_{jk}\right)\\
&\quad=\Phi\left(\varphi(f\otimes e_{11})(b+b')\varphi(g\otimes e_{11})\otimes e_{jk}\right)\\
&\quad=\varphi(f\otimes e_{j1})(b+b')\varphi(g\otimes e_{1k})\\
&\quad=\varphi(f\otimes e_{j1})b\varphi(g\otimes e_{1k})+\varphi(f\otimes e_{j1})b'\varphi(g\otimes e_{1k})\\
&\quad=\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk}\right)+\Phi\left(\varphi(f\otimes e_{11})b'\varphi(g\otimes e_{11})\otimes e_{jk}\right).
\end{aligned}$$

Now we easily conclude that Φ is a well-defined linear map on all of $B\otimes M_n(\mathbb{C})$.

As to the product, we observe

$$\begin{aligned}
&\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk}\right)\Phi\left(\varphi(g\otimes e_{11})b'\varphi(h\otimes e_{11})\otimes e_{kl}\right)\\
&\quad=\varphi(f\otimes e_{j1})b\varphi(g\otimes e_{1k})\varphi(h\otimes e_{k1})b\varphi(k\otimes e_{1l})\\
&\quad=\varphi(f\otimes e_{j1})b\varphi(gh\otimes e_{11})b\varphi(k\otimes e_{1l})\\
&\quad=\Phi\left(\varphi(f\otimes e_{11})b\varphi(gh\otimes e_{11})b\varphi(k\otimes e_{11})\otimes e_{jl}\right).
\end{aligned}$$

Proving that Φ is a $*$-homomorphism is easier:

$$\begin{aligned}
\Phi\left((\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk})^*\right)&=\Phi\left(\varphi(\bar g\otimes e_{11})b^*\varphi(\bar f\otimes e_{11})\otimes e_{kj}\right)\\
&=\varphi(\bar g\otimes e_{k1})b^*\varphi(\bar f\otimes e_{1j})\\
&=\left(\varphi(f\otimes e_{j1})b\varphi(g\otimes e_{1k})\right)^*\\
&=\left(\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk}\right)\right)^*.
\end{aligned}$$

To prove Φ is onto, we start with an element

$$\varphi(f\otimes 1_n)b\varphi(g\otimes 1_n)$$

which we expand as

$$\sum\varphi(f\otimes e_{jj})b\varphi(g\otimes e_{kk})=\sum\Phi(\varphi(f\otimes e_{1j})b\varphi(g\otimes e_{k1})\otimes e_{jk}).$$

Injectivity is easy since Φ will be injective if and only if its restriction to $B_0\otimes e_{11}$ is injective, and

$$\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{11}\right)=\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}).$$

For naturality, suppose that $\gamma:B\to C$ is a homomorphism of C^*-algebras or $C^{*,\tau}$-algebras. Then define $\psi=\gamma\circ\varphi$ and subsequently define

$$\Psi\colon C_0\otimes M_n(\mathbb{C})\to C_n$$

as above. Check that $\gamma(B_0)\subseteq C_0$ and $\gamma(B_n)\subseteq C_n$. Then we show

$$\Psi(\gamma(x)\otimes e_{jk})=\gamma(\Psi(x\otimes e_{jk}))$$

as follows:

$$\begin{aligned}\Psi(\gamma(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))\otimes e_{jk}) &= \Psi(\psi(f\otimes e_{11})\gamma(b)\psi(g\otimes e_{11}))\otimes e_{jk})\\ &= \psi(f\otimes e_{j1})\gamma(b)\psi(g\otimes e_{1k}))\\ &= \gamma(\varphi(f\otimes e_{j1}))\gamma(b)\gamma(\varphi(g\otimes e_{1k}))\\ &= \gamma(\varphi(f\otimes e_{j1})b\varphi(g\otimes e_{1k}))\\ &= \gamma\Phi(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk}).\end{aligned}$$

In the case that there is an involution τ on B with $\varphi(x^{\mathrm{Tr}}) = \varphi(x)^\tau$ for $x \in CM_2(\mathbb{C})$, we show that $\Phi((x\otimes e_{jk})^{\tau\otimes\mathrm{Tr}}) = \Phi(x\otimes e_{jk})^\tau$ for $x\otimes e_{jk}$ in $B_0\otimes M_n(\mathbb{C})$:

$$\begin{aligned}\Phi\left((\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^\tau\otimes e_{jk}^{\mathrm{Tr}}\right) &= \Phi\left(\varphi(g\otimes e_{11})b^\tau\varphi(f\otimes e_{11})\otimes e_{kj}\right)\\ &= \varphi(g\otimes e_{k1})b^\tau\varphi(f\otimes e_{1j})\\ &= \varphi((g\otimes e_{1k})^{\mathrm{Tr}})b^\tau\varphi((f\otimes e_{j1})^{\mathrm{Tr}})\\ &= (\varphi(f\otimes e_{j1})b\varphi(g\otimes e_{1k}))^\tau\\ &= \Phi(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{jk})^\tau.\ \square\end{aligned}$$

Corollary 3.7. *Let h be a strictly positive element in a C^*-algebra B. There is an embedding $C\mathbb{C}\hookrightarrow B$ sending the canonical generator to h. Similarly, there is an embedding $C\mathbb{C}\to CM_n(\mathbb{C})$ by $f\mapsto f\otimes e_{11}$. Then there is an isomorphism*

$$B *_{C\mathbb{C}} CM_n(\mathbb{C}) \cong B\otimes M_n(\mathbb{C})$$

given by

$$b\mapsto b\otimes e_{11} \qquad \text{and} \qquad f\otimes e_{jk}\mapsto f(h)\otimes e_{jk}\ .$$

If there is a real structure τ on B and if h satisfies $h^\tau = h$, then the isomorphism is τ-preserving.

Lemma 3.8. *Suppose $\varphi\colon (CM_2(\mathbb{C}),\sharp)\to(B,\tau)$ is a $C^{*,\tau}$-algebra homomorphism of $C^{*,\tau}$-algebras. Then there is a natural isomorphism of $C^{*,\tau}$-algebras*

$$\Phi\colon (B_0\otimes M_2(\mathbb{C}),\sigma\otimes\sharp)\to(B_2,\tau)$$

where B_0, B_2, and Φ are as in Lemma 3.6 *and where σ is an antimultiplicative involution on B_0 defined by*

$$(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^\sigma = \varphi(g\otimes e_{12})b^\tau\varphi(f\otimes e_{21}).$$

Furthermore, the construction $\varphi\mapsto(B_0,\sigma)$ is natural.

Proof. We already know from Lemma 3.6 that Φ is a well defined isomorphism. Suppose now that $\varphi\colon CM_2(\mathbb{C})\to B$ satisfies $\varphi(x^\sharp) = \varphi(x)^\tau$. We first

show that σ is well-defined and is a real structure on B_0. Since

$$\begin{aligned}(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^{\sigma}\otimes e_{11} &= \varphi(g\otimes e_{12})b^{\tau}\varphi(f\otimes e_{21})\otimes e_{11}\\ &= \Phi^{-1}\left(\varphi(g\otimes e_{12})b^{\tau}\varphi(f\otimes e_{21})\right)\\ &= \Phi^{-1}\left(\varphi(g\otimes e_{12}^{\sharp})b^{\tau}\varphi(f\otimes e_{21}^{\sharp})\right)\\ &= \Phi^{-1}\left((\varphi(f\otimes e_{21})b\varphi(g\otimes e_{12}))^{\tau}\right)\\ &= \Phi^{-1}\left((\Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{22}\right))^{\tau}\right)\end{aligned}$$

we see that

$$x^{\sigma}\otimes e_{11} = \Phi^{-1}\left((\Phi(x\otimes e_{22}))^{\tau}\right)$$

and so σ is an anti-$*$-homomorphism, being a composition of four homomorphisms and one anti-homomorphism. That σ is an involution on B_0 is shown by:

$$\begin{aligned}&(\varphi(fg\otimes e_{11})b\varphi(hk\otimes e_{11}))^{\sigma^2}\\ &= (\varphi(kh\otimes e_{12})b^{\tau}\varphi(gf\otimes e_{21}))^{\sigma}\\ &= (\varphi(k\otimes e_{11})\varphi(h\otimes e_{12})b^{\tau}\varphi(g\otimes e_{21})\varphi(f\otimes e_{11}))^{\sigma}\\ &= \varphi(f\otimes e_{12})\left(\varphi(h\otimes e_{12})b^{\tau}\varphi(g\otimes e_{21})\right)^{\tau}\varphi(k\otimes e_{21})\\ &= \varphi(f\otimes e_{12})\varphi(g\otimes e_{21}^{\sharp})b\varphi(h\otimes e_{12}^{\sharp})\varphi(k\otimes e_{21})\\ &= \varphi(f\otimes e_{12})\varphi(g\otimes e_{21})b\varphi(h\otimes e_{12})\varphi(k\otimes e_{21})\\ &= \varphi(fg\otimes e_{11})b\varphi(hk\otimes e_{11}).\end{aligned}$$

Now we show that Φ commutes with the appropriate real structures; that is we prove that $\Phi((x\otimes e_{jk})^{\sigma\otimes\sharp}) = \Phi(x\otimes e_{jk})^{\tau}$ for all $x\otimes e_{jk}\in B_0\otimes M_2(\mathbb{C})$. Of the four cases to consider, we will show the calculations for the cases $x\otimes e_{11}$ and $x\otimes e_{12}$ since the cases for $x\otimes e_{22}$ and $x\otimes e_{21}$ are similar.

$$\begin{aligned}\Phi\left((\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^{\sigma}\otimes e_{11}^{\sharp}\right) &= \Phi\left(\varphi(g\otimes e_{12})b^{\tau}\varphi(f\otimes e_{21})\otimes e_{22}\right)\\ &= \varphi(g\otimes e_{22})b^{\tau}\varphi(f\otimes e_{22})\\ &= (\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^{\tau}\\ &= \Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{11}\right)^{\tau}\end{aligned}$$

and

$$\begin{aligned}\Phi\left((\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11}))^{\sigma}\otimes e_{12}^{\sharp}\right) &= \Phi\left(\varphi(g\otimes e_{12})b^{\tau}\varphi(f\otimes e_{21})\otimes -e_{12}\right)\\ &= -\varphi(g\otimes e_{12})b^{\tau}\varphi(f\otimes e_{22})\\ &= (\varphi(f\otimes e_{11})b\varphi(g\otimes e_{12}))^{\tau}\\ &= \Phi\left(\varphi(f\otimes e_{11})b\varphi(g\otimes e_{11})\otimes e_{12}\right)^{\tau}.\end{aligned}$$

Finally, we consider the question of naturality. For a $C^{*,\tau}$-algebra homomorphism $\gamma:(B,\tau)\to(C,\tau)$ we define $\psi=\gamma\circ\varphi$. We obtain a real stucture

σ on C_0,

$$(\psi(f \otimes e_{11})c\psi(g \otimes e_{11}))^\sigma = \psi(g \otimes e_{12})c^\tau\psi(f \otimes e_{21}).$$

The claim of naturality is the claim that the restriction of γ to a map $B_0 \to C_0$ is a $C^{*,\tau}$-algebra homomorphism $\gamma_* : (B_0, \sigma) \to (C_0, \sigma)$. Indeed,

$$\begin{aligned}\gamma(\varphi(f \otimes e_{11})b\varphi(g \otimes e_{11}))^\sigma &= (\psi(f \otimes e_{11})\gamma(b)\psi(g \otimes e_{11}))^\sigma \\ &= \psi(g \otimes e_{12})\gamma(b^\tau)\psi(f \otimes e_{21}) \\ &= \gamma(\varphi(g \otimes e_{12})b^\tau\varphi(f \otimes e_{21})) \\ &= \gamma((\varphi(f \otimes e_{11})b\varphi(g \otimes e_{11}))^\sigma). \qquad \square\end{aligned}$$

An important special case of Lemma 3.8 occurs when $B = C \otimes M_2(\mathbb{C})$ with involution $\tau \otimes \sharp$ and when the map $\varphi : (CM_2(\mathbb{C}), \sharp) \to (C \otimes M_2(\mathbb{C}), \tau \otimes \sharp)$ sends $f \otimes e_{jk}$ to $f(h) \otimes e_{jk}$ for some strictly positive self-τ element h in C. In that case, $B_2 = C \otimes M_2(\mathbb{C})$ and $B_0 = C \otimes e_{11}$. Then the induced real structure σ on B_0, defined by

$$(\varphi(f \otimes e_{11})b\varphi(g \otimes e_{11}))^\sigma = \varphi(g \otimes e_{12})b^{\tau\otimes\sharp}\varphi(f \otimes e_{21})$$

satisfies

$$\begin{aligned}(hbh \otimes e_{11})^\sigma &= ((h \otimes e_{11})(b \otimes e_{11})(h \otimes e_{11}))^\sigma \\ &= (h \otimes e_{12})(b^\tau \otimes e_{22})(h \otimes e_{21}) \\ &= hb^\tau h \otimes e_{11}.\end{aligned}$$

Thus we find that σ is just $\tau \otimes \mathrm{id}$, restricted to $B_0 = C \otimes e_{11}$.

Proposition 3.9. *Let A be a real C^*-algebra . If A is projective then $A \otimes \mathbb{H}$ is projective. If A is semiprojective then $A \otimes \mathbb{H}$ is semiprojective.*

Proof. We work in the category of $C^{*,\tau}$-algebras. Suppose that (A, τ) is projective, that $(B, \tau), (C, \tau)$ are $C^{*,\tau}$-algebras, and that we have $C^{*,\tau}$-algebra homomorphisms φ and π as in the diagram

$$\begin{array}{ccc} & & B \\ & & \big\downarrow{\scriptstyle \pi} \\ A \otimes M_2(\mathbb{C}) & \xrightarrow{\varphi} & C \end{array}$$

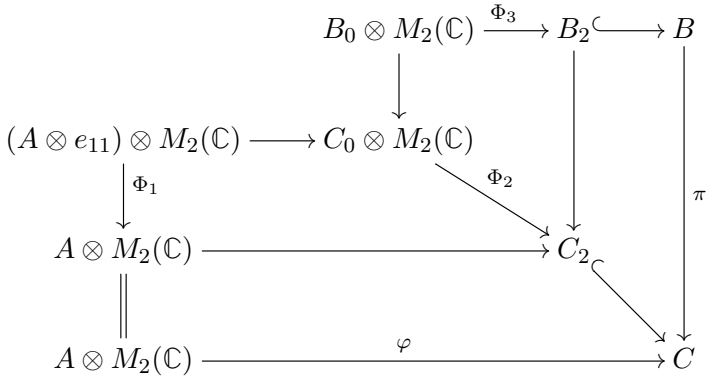
where π is surjective and the involution on $A \otimes M_2(\mathbb{C})$ is $\tau \otimes \sharp$. We select a strictly positive element $h \in A$ satisfying $h^\tau = h$ and define

$$\gamma : (CM_2(\mathbb{C}), \sharp) \to (A \otimes M_2(\mathbb{C}), \tau \otimes \sharp)$$

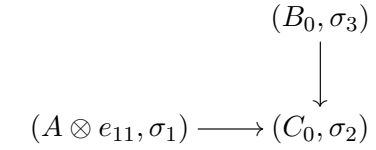
by $\gamma(f \otimes e_{jk}) = f(h) \otimes e_{jk}$. By Proposition 3.5 there is a homomorphism

$$\varphi_1 : (CM_2(\mathbb{C}), \sharp) \to (B, \tau)$$

with $\pi \circ \varphi_1 = \varphi \circ \gamma$. We apply Lemma 3.8 to get two commutative diagrams of real C^*-algebras. The first diagram is



where each Φ_j is an isomorphism, and the real structures on the algebras closest to the upper left of the diagram are all $\sigma_j \otimes \sharp$ where the σ_j are in the second diagram:



By the remark following Lemma 3.8 we know that $(A \otimes e_{11}, \sigma_1)$ is isomorphic to (A, τ) and so we get a lift in the second diagram by the hypothesis on A. Tensoring by the identity on $M_2(\mathbb{C})$ now gives a lift in the upper-left portion of the first diagram, which then provides the desired lift of φ.

Adjusting the given proof to the semiprojectivity case proceeds exactly as in Section 14.2 of [24]. □

Theorem 3.10. *If a real C^*-algebra A is projective then $A \otimes M_n(\mathbb{R})$ and $A \otimes M_n(\mathbb{R}) \otimes \mathbb{H}$ are projective for all n. If A is semiprojective then $A \otimes M_n(\mathbb{R})$ and $A \otimes M_n(\mathbb{R}) \otimes \mathbb{H}$ are semiprojective for all n.*

Proof. Suppose that A is projective. The statement that $A \otimes M_n(\mathbb{R})$ is projective is proven exactly as in the complex case, Theorem 3.3 of [23]. Similarly, if A is semiprojective, the proof of Theorem 4.3 of [23] applies to the case of real C^*-algebras to show that $A \otimes M_n(\mathbb{R})$ is semiprojective. Proposition 3.9 completes the proof. □

Proposition 3.11. *A_i is semiprojective for i even.*

Proof. First we consider A_0. Suppose that $J_1 \subseteq J_2 \subseteq \ldots$ be an increasing sequence of τ-invariant ideals in a $C^{*,\tau}$-algebra (B, τ) and let $J = \overline{\cup_n J_n}$. We will use the same notation τ for the involution τ passing to each quotient algebra B/J_n and B/J. Establish the following notation for the natural

quotient maps, which all commute with τ:

$$\begin{aligned}\pi_n &\colon B \to B/J_n\\ \pi_\infty &\colon B \to B/J\\ \pi_{n,m} &\colon B/J_n \to B/J_m\\ \pi_{n,\infty} &\colon B/J_n \to B/J.\end{aligned}$$

Let $\phi\colon (q\mathbb{C}, \mathrm{Tr}) \to (B/J, \tau)$ be a $C^{*,\tau}$-algebra homomorphism. We will produce a $C^{*,\tau}$-algebra homomorphism $\psi\colon (q\mathbb{C}, \mathrm{Tr}) \to (B/J_n, \tau)$ for some n such that $\pi_{n,\infty} \circ \psi = \phi$.

Let $h_\infty = \phi(h_0)$, $k_\infty = \phi(k_0)$, $x_\infty = \phi(x_0)$ in B/J. The elements h_∞ and k_∞ are positive, contractions, orthogonal, and fixed by τ. Thus by Lemma 3.4, there are elements $h, k \in B$ with the same properties such that $\pi_\infty(h) = h_\infty$ and $\pi_\infty(k) = k_\infty$.

We will take $x \in B$ to be a lift of x_∞. As in the proof of Theorem 6 of [25], this can be arranged so that $x \in k^{1/8} B h^{1/8}$. Furthermore, replacing x by $\frac{1}{2}(x + x^{\tau *})$ we can assume that $x^\tau = x^*$ holds. Then $T = T(h, x, k)$ is an element in the subalgebra

$$\widehat{B} = \begin{pmatrix} \mathbb{C}\cdot\mathbb{1} \oplus \overline{hBh} & \overline{hBk} \\ \overline{kBh} & \mathbb{C}\cdot\mathbb{1} \oplus \overline{kBk} \end{pmatrix} \subseteq M_2(\widetilde{B}).$$

Furthermore, T satisfies $T^{\tau\otimes \mathrm{Tr}} = T^* = T$ and is a lift of

$$T_\infty = T(h_\infty, x_\infty, k_\infty) \in M_2(\widetilde{B}/J).$$

Since $\pi_\infty(T)$ is a projection, there is an n large enough so that the spectrum of $T_n := \pi_n(T) \in M_2(\widetilde{B}/J_n)$ does not contain $1/2$. Then $T_n' = f(T_n)$ is a projection in $M_2(\widetilde{B}/J_n)$ where

$$f_{1/2}(t) = \begin{cases} 0 & \text{if } t < 1/2 \\ 1 & \text{if } t > 1/2. \end{cases}$$

Furthermore, T_n' is a lift of T_∞ and the relation $T_n' = (T_n')^{\tau\otimes \mathrm{Tr}}$ holds. Write

$$T_n' = \begin{pmatrix} \mathbb{1} - h_n' & (x_n')^* \\ x_n' & k_n' \end{pmatrix}$$

where h_n', k_n', x_n' are elements of $\widetilde{B}/J_n$ and are necessary lifts of h_∞, k_∞, and x_∞ respectively. Since we have $T_n' = (T_n')^\tau$, it follows that $h_n' = (h_n')^\tau$, $k_n' = (k_n')^\tau$, and $(x_n')^* = (x_n')^\tau$. We claim that h_n' and k_n' are orthogonal. Indeed, we know that $h_n = \pi_n(h)$ and $k_n = \pi_n(k)$ are orthogonal and that T_n (and hence $T_n' = f(T_n)$) lies in the subalgebra

$$\widehat{B_n} = \begin{pmatrix} \mathbb{C}\cdot\mathbb{1} \oplus \overline{h_n B h_n} & \overline{h_n B k_n} \\ \overline{k_n B h_n} & \overline{k_n B k_n} \end{pmatrix} \subseteq M_2(\widetilde{B}/J_n),$$

proving our claim.

Therefore, the elements h'_n, x'_n, and k'_n are elements in B/J_n which satisfy the universal relations for $q\mathbb{C}$ as in Proposition 3.3, so there exists a homomorphism $\psi\colon (q\mathbb{C}, \mathrm{Tr}) \to (B/J_n, \tau)$ that maps h_0, k_0, and x_0 to h'_n, k'_n, and x'_n respectively. Since $\pi_\infty(h'_n) = h_\infty$, $\pi_\infty(k'_n) = k_\infty$, and $\pi_\infty(x'_n) = x_\infty$; it follows that ψ is a lift of ϕ.

For A_2 the proof is quite similar to that for A_0. The initial difference is that we are using the involution $\sharp$ on $q\mathbb{C}$. So ϕ is assumed to satisfy $\phi(a^\sharp) = \phi(a)^\tau$ and we must find a lift ψ which satisfies the same.

If we let h_∞, k_∞, and x_∞ be as in the proof above, then we have $h_\infty^\tau = k_\infty$, $k_\infty^\tau = h_\infty$, and $x_\infty^\tau = -x_\infty$. We use Lemma 3.4 to find elements h and k in B that satisfy $h^\tau = k$ and $k^\tau = h$. Lift x_∞ to an element $x \in k^{1/8}Bh^{1/8}$ that satisfies $x^\tau = -x$ (using the adjustment $\frac{1}{2}(x - x^\tau)$). Then $T = T(h, x, k)$ is in $\widehat{B}$ as before and satisfies $T = T^*$. Now we have

$$T = \begin{pmatrix} \mathbb{1} - h & x^* \\ x & k \end{pmatrix} \quad \text{and} \quad T^{\tau\otimes\widetilde{\mathrm{Tr}}} = \begin{pmatrix} h & -x^* \\ -x & \mathbb{1} - k \end{pmatrix}$$

so we have $T^{\tau\otimes\widetilde{\mathrm{Tr}}} = \mathbb{1}_2 - T$.

Then as in the proof for A_0, find n large enough so that $1/2$ is in the spectral gap for T_n and let $T'_n = f_{1/2}(T_n)$. Then T'_n is a projection and satisfies $(T'_n)^{\tau\otimes\widetilde{\mathrm{Tr}}} = \mathbb{1}_2 - T'_n$ (since $f'(\mathbb{1}_2) = \mathbb{1}_2$). So we can write

$$T'_n = \begin{pmatrix} \mathbb{1} - h'_n & (x'_n)^* \\ x'_n & k'_n \end{pmatrix}$$

where $(h'_n)^\tau = k'_n, (k'_n)^\tau = h'_n$, and $(x'_n)^\tau = -x'_n$. Then by Proposition 3.3, there exists a homomorphism ψ which is the desired lift of ϕ.

Now we consider the case of A_6. In this case the we have elements in B/J that satisfy $h_\infty^\tau = k_\infty$, $k_\infty^\tau = h_\infty$, and $x_\infty^\tau = x_\infty$ in B/J which are lifted to elements in B that satisfy $h^\tau = k$, $k^\tau = h$, and $x^\tau = x$. So T satisfies $T^{\tau\otimes\sharp} = \mathbb{1}_2 - T$. Then for n large enough we obtain

$$T'_n = \begin{pmatrix} \mathbb{1} - h'_n & (x'_n)^* \\ x'_n & k'_n \end{pmatrix}$$

where $(h'_n)^\tau = k'_n, (k'_n)^\tau = h'_n$, and $(x'_n)^\tau = x'_n$ and we apply Proposition 3.3 as before.

Finally, to show that A_4 is semiprojective we make use of the isomorphism $A_4 \cong A_0 \otimes \mathbb{H}$. Since A_0 is semiprojective, Proposition 3.9 implies that A_4 is semiprojective. □

Proposition 3.12. *A_i is semiprojective for i odd.*

Proof. For $n = 1$ and $n = -1$, this is Example 3.10 and Corollary 3.12 of [31]. Then the cases $n = 3$ and $n = 5$ follow by Proposition 3.9. □

4. Unsuspended E-theory for real C^*-algebras

In this section, we develop the theory of homotopy symmetric real C^*-algebras, along the lines of [11] in the complex case. Our main theoretical result states (as in the complex case) that a real C^*-algebra A is homotopy symmetric if and only if the usual natural homomorphism $[[A \otimes \mathbb{K}^{\mathbb{R}}, B \otimes \mathbb{K}^{\mathbb{R}}]]_{\mathcal{R}} \to E(A,B)$ is an isomorphism for all real C^*-algebras B. Furthermore, we prove that homotopy symmetry has permanence with respect to complexification: a real C^*-algebra A is homotopy symmetric if and only if $A_{\mathbb{C}}$ is homotopy symmetric (in the category of C^*-algebras). It will follow that all of the algebras A_i introduced in the previous section are homotopy symmetric. We introduce a standing assumption in this section that all real C^*-algebras are separable. This will apply to all of our discussion of E-theory and of homotopy symmetry. However our main result Theorem 4.13 will be proven in full generality for all real C^*-algebras.

We refer the reader to Section 4 of [7] and Section 8 of [8] for the development of asymptotic morphisms for real C^*-algebras. In what follows we will use the notation $[[A,B]]_{\mathcal{R}}$ to denote the homotopy classes of asymptotic morphisms in the category of real C^*-algebras and $[[A,B]]_{\mathcal{C}}$ to denote the same in the category of complex C^*-algebras, unless the meaning is clear from context. In both cases, this set has the structure of a semigroup if B is stable. And in both cases, as we shall see, the property of homotopy symmetry is connected to the question of whether or not this semigroup has inverses.

Let e be a rank 1 projection in $\mathbb{K}^{\mathbb{R}} \subset \mathbb{K}$. Then $\mathrm{id}_A(a) = a \otimes e$ defines a homomorphism, either $A \to A \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}$ in the category of real C^*-algebras or $A \to A \otimes \mathbb{K}$ in the category of complex C^*-algebras. If A and B are real C^*-algebras, then complexification induces a natural semigroup homomorphism

$$\theta_{A,B} \colon [[A, B \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}]]_{\mathcal{R}} \to [[A_{\mathbb{C}}, B_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{K}]]_{\mathcal{C}}.$$

In particular, we have $\theta_{A,A}(\mathrm{id}_A) = \mathrm{id}_{A_{\mathbb{C}}}$.

Definition 4.1 (See Section 5 of [11])**.** A C^*-algebra A is *homotopy symmetric* if the class $[[\mathrm{id}_A]]$ is invertible in $[[A, A \otimes_{\mathbb{C}} \mathbb{K}]]_{\mathcal{C}}$. A real C^*-algebra A is *homotopy symmetric* if the class $[[\mathrm{id}_A]]$ is invertible in $[[A, A \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}]]_{\mathcal{R}}$.

Lemma 4.2. *Suppose that A and B are real stable C^*-algebra with A homotopy symmetric. Then $[[A,B]]_{\mathcal{R}}$ is a group. In particular the asymptotic morphism η_A that is inverse to id_A is unique up to homotopy.*

Proof. Suppose that η_A is an asymptotic morphism such that $[\eta_A]$ is inverse to $[\mathrm{id}_A]$. Then $\mathrm{id}_A \oplus \eta_A$ is null-homotopic in $[[A,A]]_{\mathcal{R}}$, and it follows that $[\psi \circ \eta_A]$ is an inverse to $[\psi]$ in $[[A,B]]_{\mathcal{R}}$. □

Lemma 4.3. *Let A, B, and D be real C^*-algebras and let $\alpha \colon A \to B$ be a homomorphism. Then*

$$[[D, SA]]_{\mathcal{R}} \xrightarrow{(S\alpha)_*} [[D, SB]]_{\mathcal{R}} \xrightarrow{\partial} [[D, C_\alpha]]_{\mathcal{R}} \xrightarrow{\kappa_*} [[D, A]]_{\mathcal{R}} \xrightarrow{\alpha *} [[D, B]]_{\mathcal{R}}$$

is an exact sequence where C_α is the mapping cone of α and $\kappa\colon C_\alpha \to A$ is defined by $\kappa(a, f) = a$.

Proof. As in Proposition 6 of [10]. □

Lemma 4.4. *For any split exact sequence*

$$0 \to J \to A \xrightarrow{\pi} B \to 0$$

of real C^*-algebras and any real C^*-algebra D, the exact sequence

$$0 \to [[D, J]] \to [[D, A]] \xrightarrow{\pi} [[D, B]] \to 0$$

is split.

Proof. As in Proposition 3.2 of [11]. □

Recall that if A and B are stable C^*-algebras, then we have natural homomorphisms $\Sigma\colon [[A, B]] \to [[SA, SB]]$ and $\Sigma^{-1}\colon [[A, B]] \to [[S^{-1}A, S^{-1}B]]$.

Lemma 4.5. *If A and B are real C^*-algebras and B is stable, then*

$$\Sigma\colon [[SA, SB]] \to [[S^2A, S^2B]] \quad \textit{and} \quad \Sigma^{-1}\colon [[SA, SB]] \to [[S^{-1}SA, S^{-1}SB]]$$

are isomorphisms.

Proof. The first statement is Lemma 4.5 of [7] and the second statement can be proven in a similar way. Instead of using the elements in $E(\mathbb{R}, S^8\mathbb{R})$ and $E(S^8\mathbb{R}, \mathbb{R})$ associated with the Bott isomorphism, we use elements in $E(\mathbb{R}, S^{-1}S\mathbb{R})$ and $E(S^{-1}S\mathbb{R}, \mathbb{R})$ that are inverses to each other arising from the KK-equivalence between $\mathbb{R}$ and $S^{-1}S\mathbb{R}$. □

The following definition is from Section 4 of [7].

Definition 4.6. Let A and B be real separable C^*-algebras. Then we define

$$E(A, B) = [[SA, SB \otimes \mathbb{K}^{\mathbb{R}}]].$$

Lemma 4.7. *There exists an asymptotic morphism $\alpha_t\colon SS^{-1}\mathbb{R} \to \mathbb{K}^{\mathbb{R}}$ such that $\alpha_*\colon KO_0(SS^{-1}\mathbb{R}) \to KO_0(\mathbb{K}^{\mathbb{R}})$ is an isomorphism. Thus α_* is an isomorphism on $K^{CRT}(-)$.*

Proof. Note that $KO_0(SS^{-1}\mathbb{R}) \cong KO_0(\mathbb{K}^{\mathbb{R}}) \cong \mathbb{Z}$. In fact, $K^{CRT}(SS^{-1}\mathbb{R}) \cong K^{CRT}(\mathbb{K}^{\mathbb{R}})$ is isomorphic to the free CRT-module with a generator in the real part in degree 0. So the Universal Coefficient Theorem for real C^*-algebras implies that

$$\begin{aligned} KKO(SS^{-1}\mathbb{R}, \mathbb{K}^{\mathbb{R}}) \cong KKO(\mathbb{R}, \mathbb{R}) &\cong \mathrm{Hom}_{CRT}(K^{CRT}(\mathbb{R}), K^{CRT}(\mathbb{R})) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(KO_0(\mathbb{R}), KO_0(\mathbb{R})) \cong \mathbb{Z}. \end{aligned}$$

As in the remarks preceding Theorem 5.2 of [7], the isomorphism

$$KKO(SS^{-1}\mathbb{R}, \mathbb{K}^{\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{Z}}(KO_0(\mathbb{R}), KO_0(\mathbb{R}))$$

factors through $E(S^{-1}\mathbb{R}, \mathbb{K}^{\mathbb{R}}) \cong [[SS^{-1}\mathbb{R}, \mathbb{K}^{\mathbb{R}}]]$, giving the existence of α as desired. □

We now pause to establish some notation and to define several more homomorphisms that we will make use of for the rest of this section. We will use ζ to denote an involution on either $\mathbb{R}^n$ or on a sphere S^{n-1}, given by multiplication by -1 in exactly one coordinate (let us take it to be the y-coordinate). Then for example,

$$C_0(\mathbb{R}^2;\zeta) = \{f \in C_0(\mathbb{R}^2,\mathbb{C}) \mid f(x,y) = \overline{f(x,-y)}\} \cong SS^{-1}\mathbb{R}.$$

More generally, $C_0(\mathbb{R}^n;\zeta) \cong S^{n-1}S^{-1}\mathbb{R}$. There is a split exact sequence

$$0 \to C_0(\mathbb{R}^n;\zeta) \xrightarrow{i} C_0(S^n;\zeta) \xrightarrow{\varepsilon} \mathbb{R} \to 0$$

where i is the standard inclusion via stereographic projection and ε is evaluation at any point fixed by ζ.

Now consider the projection

$$p_0(x,y,z) = \tfrac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix},$$

in $C(S^2,\mathbb{C})$. We know that p_0 satisfies $[p_0] = (1,1) \in KO_0(C(S^2,\mathbb{C})) \cong \mathbb{Z}\oplus\mathbb{Z}$ (see Example 6.2.3 of [35]). Since $p_0^{\zeta\otimes \mathrm{Tr}} = p_0$, it follows that $[p_0]$ is an element in $KO_0(C(S^2;\zeta)) \cong \mathbb{Z}\oplus\mathbb{Z}$. Since the complexification functor

$$c\colon KO_0(A) \to K_0(A_{\mathbb{C}})$$

is known to be an isomorphism in this case where $A = C(S^2;\zeta)$, we conclude that

$$[p_0] = (1,1) \in KO_0(C(S^2;\zeta))$$

in the usual identification of $KO_0(C(S^2;\zeta)) \cong \mathbb{Z}\oplus\mathbb{Z}$. More precisely, this means that $\varepsilon_*([p_0])$ is a generator of $KO_0(\mathbb{R}) \cong \mathbb{Z}$; and that $[p_0] - [\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)]$ is a generator of $KO_0(C(\mathbb{R}^2;\zeta)) \cong \ker(\varepsilon_*) \cong \mathbb{Z}$. For future reference, we can take ε to be evaluation at the point $(0,0,-1)$ and we obtain the exact formula

$$\varepsilon(p_0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We define a *-homomorphism γ_1 and an asymptotic morphism γ_2 by

$$\gamma_1\colon A \to A \otimes C(S^2;\zeta) \otimes M_2(\mathbb{R}) \qquad \text{by } \gamma_1(a) = a \otimes p_0$$
$$\gamma_2\colon A \to A \otimes C(S^2;\zeta) \qquad \text{by } \gamma_2(a) = \eta_t(a) \otimes 1$$

For later reference we note that we have

$$((\mathrm{id}_A \otimes \varepsilon \otimes \mathrm{id}_{M_2(\mathbb{R})}) \circ \gamma_1)(a) = a \otimes \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\0&a\end{smallmatrix}\right) \in A \otimes M_2(\mathbb{R}).$$

Proposition 4.8. *Suppose that A is stable and homotopy symmetric. There exists an asymptotic morphism*

$$\beta_t^A\colon A \to A \otimes C(S^2;\zeta) \otimes M_3(\mathbb{R})$$

unique up to homotopy, so that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta^A} & A \otimes C_0(\mathbb{R}^2;\zeta) \otimes M_3(\mathbb{R}) \\ & \searrow^{\gamma_1 \oplus \gamma_2} & \downarrow i \\ & & A \otimes C(S^2;\zeta) \otimes M_3(\mathbb{R}) \end{array} \tag{8}$$

commutes up to homotopy. Furthermore, $\Sigma\beta^A$ *is homotopic to* $\mathrm{id}_A \otimes \beta^{S\mathbb{R}}$ *as asymptotic morphisms from* SA *to* $SA \otimes C_0(\mathbb{R}^2;\zeta) \otimes M_3(\mathbb{R})$*, and* $\beta^{S\mathbb{R}}$ *is an isomorphism on* K*-theory.*

Proof. Composing ε and $\gamma_1 \oplus \gamma_2$ we have

$$(\mathrm{id}_A \otimes \varepsilon \otimes \mathrm{id}_{M_2(\mathbb{R})})(\gamma_1 \oplus \gamma_2)(a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \eta_t(a) \end{pmatrix}$$

where η_t is the asymptotic inverse to id_A. Thus this composition is null-homotopic. So from the split exact sequence

$$0 \to C_0(\mathbb{R}^2;\zeta) \xrightarrow{i} C(S^2;\zeta) \xrightarrow{\varepsilon} \mathbb{R} \to 0$$

(or rather from the split exact sequence obtained by tensoring the above with $A \otimes M_3(\mathbb{R})$), Lemma 4.4 implies that there is a unique asymptotic morphism β^A making Diagram (8) commute.

Taking the special case $A = S\mathbb{R}$, we obtain the diagram

$$\begin{array}{ccc} S\mathbb{R} & \xrightarrow{\beta^{S\mathbb{R}}} & S\mathbb{R} \otimes C_0(\mathbb{R}^2;\zeta) \otimes M_3(\mathbb{R}) \\ & \searrow^{\gamma_1 \oplus \gamma_2} & \downarrow i \\ & & S\mathbb{R} \otimes C(S^2;\zeta) \otimes M_3(\mathbb{R}). \end{array} \tag{9}$$

Now, we construct two diagrams that both look like

$$\begin{array}{ccc} SA & \xrightarrow{\beta^{SA}} & SA \otimes C_0(\mathbb{R}^2;\zeta) \otimes M_3(\mathbb{R}) \\ & \searrow^{\gamma_1 \oplus \gamma_2} & \downarrow i \\ & & SA \otimes C(S^2;\zeta) \otimes M_3(\mathbb{R}) \end{array}$$

by either suspending Diagram (8) or by tensoring Diagram (9) by A. In these two diagrams, the homomorphisms i and γ_1 are exactly the same and the homomorphism γ_2 is the same up to homotopy in since

$$[[\eta_{SA}]] = [[\eta_{S\mathbb{R}} \otimes \mathrm{id}_A]] = [[\mathrm{id}_{S\mathbb{R}} \otimes \eta_A]]$$

(using Lemma 4.2). Therefore, by uniqueness of β^{SA} we have

$$[[\beta^{SA}]] = [[\mathrm{id}_A \otimes \beta^{S\mathbb{R}}]] = [[\mathrm{id}_{S\mathbb{R}} \otimes \beta^A]].$$

To prove the statement about K-theory, we can calculate the action of γ_1 and γ_2 on $KO_{-1}(S\mathbb{R})$ as in Diagram (9). We can write

$$\gamma_1^{S\mathbb{R}} = \mathrm{id}_{S\mathbb{R}} \otimes \gamma_1^{\mathbb{R}} \colon S\mathbb{R} \otimes \mathbb{R} \to S\mathbb{R} \otimes C(S^2;\zeta) \otimes M_2(\mathbb{R})$$

and see that $(\gamma_1^{S\mathbb{R}})_*$ maps the generator of $KO_{-1}(S\mathbb{R}) \cong \mathbb{Z}$ to the class corresponding to $[p_0]$ in $KO_{-1}(S\mathbb{R} \otimes C(S^2;\zeta)) \cong KO_0(C(S^2;\zeta)) \cong \mathbb{Z} \oplus \mathbb{Z}$. At the same time, $(\gamma_2^{S\mathbb{R}})_*$ maps the generator of $KO_{-1}(S\mathbb{R})$ to the additive inverse of the class representing the unit in the same group. Thus, we see that $(\gamma_1 \oplus \gamma_2)_*$ maps the generator of $KO_{-1}(S\mathbb{R})$ to the kernel of ε_* (which we already knew) and to the generator of

$$\mathrm{image}\,(i_*) \cong KO_{-1}(S\mathbb{R} \otimes C_0(\mathbb{R}^2;\zeta)) \cong KO_0(C_0(\mathbb{R}^2;\zeta)) \cong \mathbb{Z}.$$

This proves $\beta^{S\mathbb{R}}$ is an isomorphism on $KO_{-1}(-)$ and hence on $K^{CRT}(-)$. $\square$

Theorem 4.9. *Let A be a stable homotopy symmetric real C^*-algebra. Then*

$$\Sigma \colon [[A,B]] \to E(A,B)$$

is an isomorphism for all real stable C^*-algebras B.

Proof. From Lemma 4.5 use the isomorphism $E(A,B) \cong [[SS^{-1}A, SS^{-1}B]]$ to show that

$$\Sigma\,\Sigma^{-1} \colon [[A,B]] \to [[SS^{-1}A, SS^{-1}B]]$$

is an isomorphism with inverse

$$\Theta \colon [[SS^{-1}A, SS^{-1}B]] \to [[A,B]]$$

defined by

$$\Theta([[\varphi]]) = [[\mathrm{id}_B \otimes \alpha \otimes \mathrm{id}_{M_2(\mathbb{R})}]] \circ [[\varphi \otimes M_3(\mathbb{R})]] \circ [[\beta^A]].$$

By the Yoneda Lemma, it suffices to consider the case $A = B$ and to then show that id_A maps to id_A under the homomorphism

$$\Theta \circ \Sigma\,\Sigma^{-1} \colon [[A,A]] \to [[A,A]].$$

We have $\Sigma\,\Sigma^{-1}(\mathrm{id}_A) = \mathrm{id}_{SS^{-1}\mathbb{R}}$ and we have

$$\Theta(\mathrm{id}_{SS^{-1}\mathbb{R}}) = (\mathrm{id}_A \otimes \alpha \otimes \mathrm{id}_{M_3(\mathbb{R})}) \circ \beta^A.$$

So we need to show that

$$(\mathrm{id}_A \otimes \alpha \otimes \mathrm{id}_{M_3(\mathbb{R})}) \circ \beta^A \colon A \to A \otimes \mathbb{K}^{\mathbb{R}} \otimes M_3(\mathbb{R})$$

is homotopic to id_A as an asymptotic morphism. For this we use the commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\beta^A} & A \otimes C_0(\mathbb{R}^2;\zeta) \otimes M_3(\mathbb{R}) & \xrightarrow{\mathrm{id}\otimes\alpha\otimes\mathrm{id}} & A \otimes \mathbb{K}^{\mathbb{R}} \otimes M_3(\mathbb{R}) \\
& \searrow^{\gamma_1\oplus\gamma_2} & \downarrow i & & \downarrow i \\
& & A \otimes C(S^2;\zeta) \otimes M_3(\mathbb{R}) & \xrightarrow{\mathrm{id}\otimes\widetilde{\alpha}\otimes\mathrm{id}} & A \otimes \widetilde{\mathbb{K}}^{\mathbb{R}} \otimes M_3(\mathbb{R}).
\end{array}$$

By Lemma 4.4, the homomorphism

$$[[A, A \otimes \mathbb{K}^{\mathbb{R}} \otimes M_3(\mathbb{R})]] \xrightarrow{i_*} [[A, A \otimes \widetilde{\mathbb{K}}^{\mathbb{R}} \otimes M_3(\mathbb{R})]]$$

is injective, so (in yet another reduction) it suffices to show that

$$(\mathrm{id}_A \otimes \widetilde{\alpha} \otimes \mathrm{id}_{M_3(\mathbb{R})}) \circ (\gamma_1 \oplus \gamma_2) \colon A \to A \otimes \widetilde{\mathbb{K}}^{\mathbb{R}} \otimes M_3(\mathbb{R})$$

is homotopic to $i \circ \mathrm{id}_A$.

For any projection p in a real C^*-algebra B, let $j_p \colon \mathbb{R} \to B$ be the homomorphism given by $j_p(t) = tp$. Let $q_t = \widetilde{\alpha}_t(p_0) \in \widetilde{\mathbb{K}}^{\mathbb{R}} \otimes M_2(\mathbb{R})$. Since q_t is asymptotically a projection, there exists an actual projection $q_0 \in \widetilde{\mathbb{K}}^{\mathbb{R}} \otimes M_2(\mathbb{R})$ such that $(\alpha_t) \circ j_{p_0}$ is homotopic to j_{q_0}. Furthermore, by calculating the class $\alpha_*([p_0]) = [q_0] \in KO_0(\mathbb{K}^{\mathbb{R}})$, we know that q_0 is homotopic to the projection $q_0' = \left(\begin{smallmatrix} e & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where e is a rank one projection in $\mathbb{K}^{\mathbb{R}}$. So we can and do assume that $q_0 = q_0'$. Then up to a homotopy of asymptotic morphisms we have

$$(\mathrm{id}_A \otimes \widetilde{\alpha} \otimes \mathrm{id}_{M_3(\mathbb{R})}) \circ (\gamma_1 \oplus \gamma_2)(a) = \begin{pmatrix} a \otimes e & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \eta_A(a) \end{pmatrix}$$

and thus $(\mathrm{id}_A \otimes \widetilde{\alpha} \otimes \mathrm{id}_{M_3(\mathbb{R})}) \circ (\gamma_1 \oplus \gamma_2)$ is homotopic to id_A.

For the other direction, again by the Yoneda Lemma it suffices to compute $\Sigma\Sigma^{-1} \circ \Theta$ applied to $\mathrm{id}_A \otimes \mathrm{id}_{SS^{-1}\mathbb{R}}$. But we have just seen that $\Theta(\mathrm{id}_{SS^{-1}\mathbb{R}}) = \mathrm{id}_A$ and that $\Sigma\Sigma^{-1}(\mathrm{id}_A) = \mathrm{id}_{SS^{-1}\mathbb{R}}$, which completes the proof. □

Proposition 4.10. *A real C^*-algebra A is homotopy symmetric if and only if the complexification $A_{\mathbb{C}}$ is homotopy symmetric.*

Proof. We assume that $A_{\mathbb{C}}$ is homotopy symmetric (in the category of C^*-algebras), so there is an asymptotic morphism $\eta_A \in [[A_{\mathbb{C}}, A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{K}]]_{\mathcal{C}}$ such that $\mathrm{id}_{A_{\mathbb{C}}} \oplus \eta_A$ is null-homotopic through asymptotic morphisms of complex C^*-algebras. Let $c \colon A \to A_{\mathbb{C}}$ be the standard inclusion and let r be the homomorphism

$$r \colon A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{K} \to A \otimes_{\mathbb{R}} M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}.$$

Notice that $r \circ \mathrm{id}_{A_{\mathbb{C}}} \circ c = \mathrm{id}_A \oplus \mathrm{id}_A$, since for any $a \in A$ we have

$$(r \circ \mathrm{id}_{A_{\mathbb{C}}} \circ c)(a) = \begin{pmatrix} a \otimes e & 0 \\ 0 & a \otimes e \end{pmatrix}.$$

By hypothesis, then, the composition $r \circ (\mathrm{id}_{A_{\mathbb{C}}} \oplus \eta_A) \circ c$ is null-homotopic. On the other hand, we have

$$\begin{aligned} r \circ (\mathrm{id}_{A_{\mathbb{C}}} \oplus \eta_A) \circ c &= (r \circ \mathrm{id}_{A_{\mathbb{C}}} \circ c) \oplus (r \circ \eta_A \circ c) \\ &= \mathrm{id}_A \oplus \mathrm{id}_A \oplus (r \circ \eta_A \circ c) \end{aligned}$$

which shows that $[[\mathrm{id}_A \oplus (r \circ \eta_A \circ c)]]$ is an inverse for $[[\mathrm{id}_A]]$ in $[[A, A \otimes \mathbb{K}^{\mathbb{R}}]]_{\mathcal{R}}$.

For the other direction, we have a semigroup homomorphism

$$\theta_{A,A} \colon [[A, A \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}]]_{\mathcal{R}} \to [[A_{\mathbb{C}}, A_{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{K}]]_{\mathcal{C}}.$$

So if $[[\mathrm{id}_A]]$ is invertible in the former, then it immediately follows that $\theta_{A,A}([[\mathrm{id}_A]]) = [[\mathrm{id}_{A_{\mathbb{C}}}]]$ is invertible in the latter. □

Corollary 4.11. *The real C^*-algebras A_i are homotopy symmetric for all $0 \leq i < 8$.*

Proof. For all i, we have that $(A_i)_{\mathbb{C}}$ is isomorphic to one of the following C^*-algebras: $q\mathbb{C}, q\mathbb{C} \otimes M_2(\mathbb{C}), S\mathbb{C}$, and $S\mathbb{C} \otimes M_2(\mathbb{C})$. From the comments at the beginning of Section 5 of [11] we know that $q\mathbb{C}$ and $S\mathbb{C}$ are homotopy symmetric; and from Lemma 5.1 of [11] we know that $q\mathbb{C} \otimes M_n(\mathbb{C})$ and $S\mathbb{C} \otimes M_n(\mathbb{C})$ are homotopy symmetric. Therefore Proposition 4.10 implies that A_i is homotopy symmetric for all i. □

Lemma 4.12. *Let D and B be real C^*-algebras, with D semiprojective. Then:*

(1) $[D, B] \cong [[D, B]]$.
(2) *If $B = \lim_{n\to\infty} B_n$, then $[D, B] \cong \lim_{n\to\infty} [D, B_n]$.*

Proof. Both of these results have proofs that carry over directly to the real case from the complex case. The proofs in the complex case are found at the beginning of Section 6 of [11] and as the proof to Corollary 15.1.3 of [24], respectively. □

Theorem 4.13. *For each integer i in the range $0 \leq i < 8$ and for any real C^*-algebra B (not necessarily separable), there is a natural isomorphism*

$$KO_i(B) \cong [A_i, \mathbb{K}^{\mathbb{R}} \otimes B] \cong \lim_{n\to\infty} [A_i, M_n(B)].$$

If B is stable, then

$$KO_i(B) \cong [A_i, B].$$

Proof. First consider the case that B is separable. From Propositions 3.1 and 3.2 we have $K^{CRT}(A_i) \cong \Sigma^{-i} K^{CRT}(\mathbb{R})$ for all i, so the Universal Coefficient Theorem (Corollary 4.11 of [6]) implies that A_i is KK-equivalent to $S^{-i}\mathbb{R}$. We note that the condition for the Universal Coefficient Theorem to apply is that the complexification of A_i is in the bootstrap category of separable nuclear C^*-algebras. This is easy to check since the complexifications of these algebras are all stably isomorphic to a commutative C^*-algebra or to $q\mathbb{C}$. Furthermore, each A_i is semiprojective by Propositions 3.11 and 3.12. Therefore,

$$\begin{aligned}
KO_i(B) &\cong KKO(S^{-i}\mathbb{R}, B) \\
&\cong KKO(A_i, B) \\
&\cong E(A_i, B) && \text{by Theorem 4.6 of [7]} \\
&\cong [[A_i, \mathbb{K}^{\mathbb{R}} \otimes B]] && \text{by Theorem 4.9} \\
&\cong [A_i, \mathbb{K}^{\mathbb{R}} \otimes B] && \text{by Lemma 4.12(1)} \\
&\cong \lim_{n\to\infty} [A_i, M_n(B)] && \text{by Lemma 4.12(2).}
\end{aligned}$$

To address the general case, let $F_i(B) = \lim_{n\to\infty}[A_i, M_n(B)]$ and consider the natural homomorphism

$$\alpha_B \colon F_i(B) \to KO_i(B)$$

defined by $\alpha_B([\phi]) = \phi_*(\xi_i)$ where ξ_i is a generator of $KO_i(A_i) \cong \mathbb{Z}$. This homomorphism exists for all real C^*-algebras and is an isomorphism when B is separable. In general, write B as the inductive limit $B = \lim_{\lambda} B_\lambda$ where $\{B_\lambda\}$ is the net of all separable subalgebras of B. We leave it to the reader to verify that F_i is continuous with respect to inductive limits, using the fact that A_i is semiprojective. Then since both functors F_i and KO_i are continuous with respect to inductive limits and since α_{B_λ} is an isomorphism for all λ, it follows that α_B is an isomorphism. □

5. *K*-theory via unitaries — the even cases

In the next two sections, we develop pictures of all eight fundamental KO-groups in terms of unitaries. We use the notation KO_i^u for these functors defined on the category of real C^*-algebras or (equivalently) the category of $C^{*,\tau}$-algebras. For $i = 0, 1$, we will write down the definition both in terms of a real C^*-algebra A and in terms of a $C^{*,\tau}$-algebra (A, τ). However for $i \neq 0, 1$, we will only consider $KO_i^u(A, \tau)$ in the context of a $C^{*,\tau}$-algebra(A, τ), since that picture gives the most direct and consistent definitions for varying values of i. For a real C^*-algebra A, one should consider the associated $C^{*,\tau}$-algebra $(A_{\mathbb{C}}, \tau)$. Thus $KO_i^u(A) = KO_i^u(A_{\mathbb{C}}, \tau)$.

In each case, we have a picture in terms of unitaries in matrix algebras over A satisfying certain relations. In each case, we will prove that our picture is a well-defined group and then prove that it is naturally isomorphic to the standard version of K-theory. A reader who wishes to skip our detailed development can see the final pictures summarized in Section 7, where we also include a description of complex K-theory, $KU_i(A, \tau)$.

5.1. KO_0 via unitaries.

Definition 5.1. Let A be a unital real C^*-algebra. Let $U_\infty^{(0)}(A)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{2n}(A)$ satisfying $u^2 = 1$ (equivalently, unitaries u that satisfy $u = u^*$). Let $\sim_0$ be the equivalence relation on $U_\infty^{(0)}(A)$, generated by

(1) $u_0 \sim_0 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of self-adjoint unitaries on $[0, 1]$; and
(2) $u \sim_0 \iota_n^{(0)}(u)$ for $u \in M_{2n}(A)$ where $\iota_n^{(0)} \colon M_{2n}(A) \to M_{2n+2}(A)$ is given by

$$\iota_n^{(0)}(a) = \operatorname{diag}\left(a, I^{(0)}\right) \quad \text{where } I^{(0)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Then we define $KO_0^u(A) = U_\infty^{(0)}(A)/\sim_0$, with a binary operation given by $[u] + [v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(0)}(A)$.

In addition to the notation for $I^{(0)}$ given above, we will use

$$I_n^{(0)} = \operatorname{diag}(I^{(0)}, I^{(0)}, \dots, I^{(0)}) \in M_{2n}(\mathbb{C}).$$

Proposition 5.2. *$KO_0^u(A)$ is a homotopy invariant functor from the category of all unital C^*-algebras to the category of abelian groups. The inverse of an element $[u] \in KO_0^u(A)$ is $[-u]$.*

Proof. The binary operation on $KO_0^u(A)$ is clearly associative and by definition the element $I^{(0)}$ represents the identity.

Consider self-adjoint unitaries $u \in M_{2n}(A)$ and $v \in M_{2m}(A)$. Let w be a unitary in $M_{2n+2m}(\mathbb{R})$ such that $w \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) w^* = \left(\begin{smallmatrix} v & 0 \\ 0 & u \end{smallmatrix}\right)$ and $\det w = 1$. This can be done by taking w to be a change of basis matrix corresponding to a particular even permutation of the basis elements. There is then a path of unitaries $w_t \in M_{2n+2m}(\mathbb{R})$ such that $w_0 = 1_{2n+2m}$ and $w_1 = w$. Then $w_t \cdot \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) \cdot w_t^*$ is a self-adjoint unitary for all t showing that $[u]+[v] = [v]+[u]$.

Let $u \in M_{2n}(A)$ be a self-adjoint unitary. Let

$$r_t = \begin{pmatrix} \cos((\pi/2)t) \cdot 1_{2n} & -\sin((\pi/2)t) \cdot 1_{2n} \\ \sin((\pi/2)t) \cdot 1_{2n} & \cos((\pi/2)t) \cdot 1_{2n} \end{pmatrix}$$

and let $u_t = r_t \cdot \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot r_t^*$ be the path from $\left(\begin{smallmatrix} u & 0 \\ 0 & 1_{2n} \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1_{2n} & 0 \\ 0 & u \end{smallmatrix}\right)$. Then using the relation $u^2 = 1_{2n}$, one can show that u_t commutes with $\left(\begin{smallmatrix} 1_{2n} & 0 \\ 0 & -u \end{smallmatrix}\right)$. Hence $u_t \cdot \left(\begin{smallmatrix} 1_{2n} & 0 \\ 0 & -u \end{smallmatrix}\right)$ gives a path in $U_\infty^{(0)}(A)$ from $\left(\begin{smallmatrix} u & 0 \\ 0 & -u \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1_{2n} & 0 \\ 0 & -uu \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1_{2n} & 0 \\ 0 & -1_{2n} \end{smallmatrix}\right)$. This last matrix in $M_{4n}(A)$ is equivalent to $I_{2n}^{(0)}$ representing the identity element in $KO_0^u(A)$, as shown in the proof of Proposition 5.3 below. □

Proposition 5.3. *$KO_0^u(\mathbb{R}) \cong \mathbb{Z}$.*

Proof. Consider the map $\phi\colon U_\infty^{(0)}(\Bbbk) \to \mathbb{Z}$ given by $\phi(u) = \frac{1}{2}\operatorname{trace}(u)$. If u is a self-adjoint unitary, then u is unitarily equivalent to a diagonal matrix with eigenvalues in $\{1, -1\}$. It follows that the range of ϕ is exactly $\mathbb{Z}$. Furthermore, since ϕ is continuous, is invariant under unitary equivalence, and satisfies $\phi(u) = \phi(\iota_n^{(0)}(u))$; it follows that ϕ is well-defined on $KO_0^u(\mathbb{R})$.

Suppose now that u and v are self-adjoint unitaries with entries in $\mathbb{R}$ having the same trace. We may assume that u and v have the same dimension by perhaps replacing u with $\iota_n^{(0)}(u)$ or replacing v with $\iota_n^{(0)}(v)$. Let $u = xvx^*$ where x is a unitary in $M_{2n}(\mathbb{R})$. We can assume that x is in the same component as the identity among unitaries in $M_{2n}(\mathbb{R})$. For if $\det x = -1$, we can replace u, v, x by $\iota_n^{(0)}(u), \iota_n^{(0)}(v), \iota_n^{(0)}(x)$ in $M_{2n+2}(\mathbb{R})$ and note that $\det \iota_n^{(0)}(x) = -\det x$.

Now let x_t be a path of unitaries from 1_{2n} to x. Then $u_t = x_t v x_t^*$ is a path of self-adjoint unitaries from v to u showing that $[u] = [v]$. The result follows. □

Definition 5.4. Let A be any unital C^*-algebra. Then we define $KO_0^u(A) = \ker(\lambda_*)$ where $\lambda\colon \widetilde{A} \to \mathbb{R}$ is the natural projection from the unitization of A with kernel isomorphic to A.

We note that the formula in this definition is valid also in case A is unital. Therefore we have a picture in which any element of $KO_0^u(A)$ is represented by a self-adjoint unitary u in $M_{2n}(\widetilde{A})$ such that $\operatorname{trace}(\lambda_{2n}(u)) = 0$. The following proposition makes clear the picture of $KO_0^u(A)$ that we are presenting.

Proposition 5.5. *Let A be a real C^*-algebra. Any element of $KO_0^u(A)$ can be represented as $[u]$ where $u \in M_{2n}(\widetilde{A})$ is a self-adjoint unitary satisfying $\lambda(u) = I_n^{(0)}$.*

Proof. Suppose u is a self-adjoint unitary in $M_{2n}(\widetilde{A})$ and $\lambda_*([u]) = 0$ in $KO_0^u(\mathbb{R})$. Then $\operatorname{trace}(\lambda(u)) = 0$. So there is a unitary $v \in M_{2n}(\mathbb{R})$ such that $v\lambda(u)v^* = I_n^{(0)}$. Let $u' = vuv^*$, so that $\lambda(u') = I_n^{(0)}$. Furthermore, as in the proof of Proposition 5.3 we can choose v so that $\det v = 1$ (possibly by increasing n). Then there is a path of unitaries v_t in $M_{2n}(\mathbb{R})$ from v to $\mathbb{1}_{2n}$; so $u_t = v_t u v_t^*$ is a path of self-adjoint unitaries from u' to u showing that $[u'] = [u]$. □

Theorem 5.6. *Let A be a real C^*-algebra. Then there is a natural isomorphism $\theta\colon KO_0^u(A) \to KO_0(A)$. The isomorphism θ is given by*

$$\theta([u]) = \left[\tfrac{1}{2}(u + \mathbb{1}_{2n})\right] - [\mathbb{1}_n]$$

for any self-adjoint unitary $u \in M_{2n}(\widetilde{A})$.

Proof. It suffices to consider the case where A is unital and $u \in M_{2n}(A)$. The reader can check that if u is a self-adjoint unitary, then $\frac{1}{2}(u + 1_{2n})$ is a projection in $M_{2n}(A)$, and that if u_t is a path of self-adjoint unitaries in $M_{2n}(A)$, then $\frac{1}{2}(u_t + 1_{2n})$ is a path of projections in $M_{2n}(A)$. For θ to be well-defined, we also have

$$\begin{aligned}
\theta\left(\left[\begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right]\right) &= \left[\tfrac{1}{2}\begin{pmatrix} u+1_{2n} & 0 & 0 \\ 0 & 2\cdot 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] - [1_{n+1}] \\
&= \left[\begin{pmatrix} \frac{1}{2}(u+1_{2n}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] - [1_{n+1}] \\
&= \left[\tfrac{1}{2}(u + 1_{2n})\right] - [1_n] \\
&= \theta([u]).
\end{aligned}$$

To show that θ is a group homomorphism, we check that for $u \in M_{2m}(A)$ and $v \in M_{2n}(A)$ we have

$$\begin{aligned}
\theta\left(\left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right]\right) &= \left[\tfrac{1}{2}\left(\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} + 1_{2m+2n}\right)\right] - [1_{m+n}] \\
&= \left[\tfrac{1}{2}(u + 1_{2m})\right] - [1_m] + \left[\tfrac{1}{2}(v + 1_{2n})\right] - [1_n] \\
&= \theta([u]) + \theta([v]).
\end{aligned}$$

It remains to show that θ is a bijection. To show that θ is onto it suffices to show that for any projection $p \in M_n(A)$, the element $[p] \in KO_0(A)$ is in

the range of θ. In fact, taking $u = \left(\begin{smallmatrix} 2p-1_n & 0 \\ 0 & 1_n \end{smallmatrix}\right) \in M_{2n}(A)$, we have

$$\theta([u]) = \left[\left(\begin{smallmatrix} p & 0 \\ 0 & 1_n \end{smallmatrix}\right)\right] - [1_n] = [p].$$

To show that θ is one-to-one, suppose that u and v are unitaries such that $\theta([u]) = \theta([v])$. We can assume that $u, v \in M_{2n}(A)$ for some n. Then we have $[\frac{1}{2}(u+1_{2n})] = [\frac{1}{2}(v+1_{2n})]$ in $KO_0(A)$. It follows that there is an integer m such that the projections $p = \frac{1}{2}(u+1_{2n}) \oplus 1_m$ and $q = \frac{1}{2}(v+1_{2n}) \oplus 1_m$ are homotopic in $M_{2n+2m}(A)$. Up to a homotopy of projections in $M_{2n+2m}(A)$, we can now write p and q in the form

$$\begin{aligned} p &= \tfrac{1}{2}\left(\operatorname{diag}(u, 1_m, -1_m) + 1_{2(n+m)}\right) \\ &\sim \tfrac{1}{2}\left(\operatorname{diag}\left(u, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) + 1_{2(n+m)}\right) \\ q &= \tfrac{1}{2}\left(\operatorname{diag}(v, 1_m, -1_m) + 1_{2(n+m)}\right) \\ &\sim \tfrac{1}{2}\left(\operatorname{diag}\left(v, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) + 1_{2(n+m)}\right). \end{aligned}$$

Then $2p - 1_{2(n+m)}$ and $2q - 1_{2(n+m)}$ are homotopic through unitaries that satisfy $u^2 = u$, so it follows that

$$\operatorname{diag}\left(u, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) \sim \operatorname{diag}\left(v, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \ldots, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$$

where there are m copies of the block $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ along the diagonal. Therefore $[u] = [v]$ in $KO_0^u(A)$. □

For elements h, k, x in a C^*-algebra A, recall from Section 3 that

$$\begin{aligned} T(h, x, k) &= \begin{pmatrix} \mathbb{1} - h & x^* \\ x & k \end{pmatrix}, \\ U(h, x, k) &= 2T - \mathbb{1}_2 = \begin{pmatrix} \mathbb{1} - 2h & 2x^* \\ 2x & 2k - \mathbb{1} \end{pmatrix}, \end{aligned}$$

which are both elements in $M_2(\widetilde{A})$. In particular, let

$$u_0 = U(h_0, x_0, k_0) = \begin{pmatrix} \mathbb{1} - 2t & 0 & 0 & 2\sqrt{t - t^2} \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 2\sqrt{t - t^2} & 0 & 0 & 2t - \mathbb{1} \end{pmatrix} \in M_2(\widetilde{A_0})$$

where A_0 is defined as in Section 3.

Proposition 5.7. *The class $[u_0]$ is a generator of $KO_0^u(A_i) \cong \mathbb{Z}$.*

Proof. Evidently, u_0 is a self-adjoint unitary. We write

$$\widetilde{A_0} = \{f\colon [0,1] \to M_2(\mathbb{R}) \mid f(0) = \left(\begin{smallmatrix} t & 0 \\ 0 & t \end{smallmatrix}\right), f(1) = \left(\begin{smallmatrix} r & 0 \\ 0 & s \end{smallmatrix}\right), r, s, t \in \mathbb{R}\}$$

and the map $\lambda\colon \widetilde{A_0} \to \mathbb{R}$ coincides with evaluation at 0. Since we have $\lambda_2(u_0) = I^{(0)}$ (where $\lambda_2\colon M_2(\widetilde{A_0}) \to M_2(\mathbb{R})$), we know $[u_0]$ is a class in $KO_0^u(A_0)$.

Now, we have the map $\operatorname{ev}_1\colon \widetilde{A_0} \to \mathbb{R}^2$ which is evaluation at $t = 1$ and we have $\pi_1\colon \mathbb{R}^2 \to \mathbb{R}$ which is projection onto the first coordinate. It follows from

the calculation of $KO_*(A_0)$ in Section 3 that the map $\phi = \pi_1 \circ \mathrm{ev}_1 \colon \widetilde{A_0} \to \mathbb{R}$ is an isomorphism on standard K-theory and hence on KO_0^u. We calculate that $\phi(u_0) = -1_2$, which is a generator of $KO_0^u(\mathbb{R}) = \mathbb{Z}$. □

Proposition 5.8. *For any real C^*-algebra A, the map $[\phi] \mapsto \phi_*[u_0]$ defines a natural isomorphism*

$$\Theta \colon [A_0, A \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}] \to KO_0^u(A).$$

To be precise, the formula for Θ is

$$\Theta([\phi]) = (\phi_2)_*([u_0]) = [\phi_2(u_0)]$$

where $\phi_2(u_0) \in M_2(\widetilde{A})$.

We note that it already follows from Theorems 4.13 and 5.6 that the groups in question are isomorphic. However, we give a direct proof here since it establishes the concrete formula for the isomorphism and since it will serve as a model for the proof that $KO_2^u(A) \cong [A_2, A \otimes_{\mathbb{R}} \mathbb{K}^{\mathbb{R}}]$ in the next section.

Proof. Since $[A_0, \mathbb{K}^{\mathbb{R}} \otimes_{\mathbb{C}} A] \cong \lim_{n\to\infty} [A_0, M_n(A)]$ it suffices to define Θ for $\phi \colon A_0 \to M_n(A)$. To show that the formula above gives a well-defined function Θ, we first mention that $\phi_*([u_0])$ does not depend on the homotopy class of ϕ. Here let us show more carefully that the homomorphisms

$$\phi \colon A_0 \to M_n(A) \quad \text{and} \quad \phi' = \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right) : A_0 \to M_{n+1}(A)$$

will give the same element of $KO_0^u(A)$. We have

$$\phi_2(u_0) = U(\phi(h_0), \phi(x_0), \phi(k_0)) = \begin{pmatrix} \mathbb{1}_n - 2\phi(h_0) & 2\phi(x_0)^* \\ 2\phi(x_0) & 2\phi(k_0) - \mathbb{1}_n \end{pmatrix}$$

and

$$\begin{aligned} \phi_2'(u_0) &= U(\phi'(h_0), \phi'(x_0), \phi'(k_0)) \\ &= \begin{pmatrix} \mathbb{1}_n - 2\phi(h_0) & 0 & 2\phi(x_0)^* & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 2\phi(x_0) & 0 & 2\phi(k_0) - \mathbb{1}_n & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix} \end{aligned}$$

showing that $[\phi_2(u_0)] = [\phi_2'(u_0)]$ in $KO_0^u(A)$, hence $\Theta([\phi]) = \Theta([\phi'])$. Therefore Θ is well-defined.

Now suppose that we have an element $[u] \in KO_0^u(A)$ where $u \in M_{2n}(\widetilde{A})$ is a unitary that satisfies $u^* = u$ and $\lambda_*([u]) = 0$ in $KO_0^u(\mathbb{R})$. After conjugating by a unitary in $M_{2n}(\mathbb{R})$ we can assume that $\lambda(u) = \left(\begin{smallmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{smallmatrix}\right)$ and there exists $h, k, x \in M_n(A)$ such that

$$u = U(h, x, k) = \begin{pmatrix} \mathbb{1}_n - 2h & 2x^* \\ 2x & 2k - \mathbb{1}_n \end{pmatrix}.$$

Since it is not guaranteed that h and k are orthogonal, we define the elements

$$h' = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}, \qquad k' = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, \qquad \text{and} \qquad x' = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \tag{10}$$

in $M_{2n}(A)$. Then u represents the same element of $KO_0^u(A)$ as the self-adjoint unitary

$$u' := U(h', x', k') = \begin{pmatrix} \mathbb{1}_n - 2h & 0 & 0 & 2x^* \\ 0 & \mathbb{1}_n & 0 & 0 \\ 0 & 0 & -\mathbb{1}_n & 0 \\ 2x & 0 & 0 & 2k - \mathbb{1}_n \end{pmatrix} \in M_{4n}(\widetilde{A}).$$

Since h' and k' are orthogonal and $U(h', x', k')$ is a unitary, by Proposition 3.3 there is a homomorphism $\phi_u \colon A_0 \to M_{2n}(A)$ such that h_0, k_0, and x_0 map to h', k', and x' respectively. Then

$$\Theta([\phi_u]) = [\phi_u(U(h_0, x_0, k_0))] = [u'] = [u]$$

showing that Θ is surjective.

In fact, we show that the construction of ϕ_u in the previous paragraph defines a homomorphism Φ from $KO_0^u(A)$ to $[A_0, \lim_{n\to\infty} M_n(A)]$; and that Φ is inverse to Θ. First of all, if u and v are self-adjoint unitaries in $M_{2n}(\widetilde{A})$ that are homotopic through self-adjoint unitaries satisfying $\lambda(u_t) = \left(\begin{smallmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{smallmatrix}\right)$, then the construction in the previous paragraph results in a homotopy between ϕ_u and ϕ_v. Now let $u \in M_{2n}(\widetilde{A})$ and let $v = \left(\begin{smallmatrix} u & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & -\mathbb{1} \end{smallmatrix}\right) \in M_{2n+2}(\widetilde{A})$. We show that ϕ_u and ϕ_v are equivalent elements of $[A_0, \mathbb{K}^{\mathbb{R}} \otimes_{\mathbb{R}} A]$. If we write $u = \left(\begin{smallmatrix} \mathbb{1}_n - 2h & 2x^* \\ 2x & 2k - \mathbb{1}_n \end{smallmatrix}\right)$ then we can write

$$v = \begin{pmatrix} \mathbb{1}_n - 2h & 2x^* & 0 & 0 \\ 2x & 2k - \mathbb{1}_n & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}.$$

In order to have $\lambda(v)$ of the right form, we conjugate v by a unitary in $M_{2n+2}(\mathbb{R})$ and we write instead

$$\begin{aligned} v &= \begin{pmatrix} \mathbb{1}_n - 2h & 0 & 0 & 2x^* \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 2x & 0 & 0 & 2k - \mathbb{1}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{1}_{n+1} - 2\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} & 2\begin{pmatrix} 0 & x^* \\ 0 & 0 \end{pmatrix} \\ 2\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} & 2\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} - \mathbb{1}_{n+1} \end{pmatrix}. \end{aligned}$$

Hence, ϕ_v will map the elements h_0, k_0, and x_0, respectively, to the elements

$$h'_v = \begin{pmatrix} h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad k'_v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \quad \text{and} \quad x'_v = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \end{pmatrix}$$

in $M_{2n+2}(\widetilde{A})$ and thus we see that ϕ_v is unitary equivalent to $\left(\begin{smallmatrix} \phi_u & 0 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that Φ is well-defined.

We have already seen above that $\Theta \circ \Phi$ is the identity on $KO_0^u(A)$. To see that $\Phi \circ \Theta$ is the identity on $[A_0, \mathbb{K}^{\mathbb{R}} \otimes_{\mathbb{R}} A]$, let $\phi\colon A_0 \to M_n(A)$ be a given homomorphism. Let $h = \phi(h_0)$, $x = \phi(x_0)$, and $k = \phi(k_0)$. Then $\Theta(\phi) = [\phi_2(u_0)] = [U(h, x, k)]$. Then the reader can verify that $(\Phi \circ \Theta)(\phi)$ carries h_0, x_0, and k_0 to h', x', and k' in $M_{2n}(A)$ as given by Equations (10). We will show that $\left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $(\Phi \circ \Theta)(\phi)$ are homotopic by producing a homotopy of triples $\{h_t, x_t, k_t\}$ from $\{h, x, k\}$ to $\{h', x', k'\}$ in $M_{2n}(A)$ that satisfy (for each t) the conditions that $h_t k_t = 0$ and that $U(h_t, x_t, k_t)$ is a unitary in $M_{4n}(\widetilde{A})$. For $t \in [0, 1]$, let

$$r_t = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Then let $h_t = h', k_t = r_t k' r_t^*$, and $x_t = r_t x'$. The reader can verify directly that $h_t k_t = 0$ and that $U(h_t, x_t, k_t)$ is a unitary for all t since

$$U(h_t, x_t, k_t) = \begin{pmatrix} 1_{2n} & 0 \\ 0 & r_t \end{pmatrix} \cdot U(h', x', k') \cdot \begin{pmatrix} 1_{2n} & 0 \\ 0 & r_t^* \end{pmatrix}. \qquad \square$$

We end this section by giving a rephrasing of the definition of KO_0^u in the context of $C^{*,\tau}$-algebras. This gives a description of $KO_0^u(A, \tau)$ that is parallel to the forthcoming descriptions of $KO_j^u(A, \tau)$ for all values of j.

Definition 5.9. Let (A, τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(0)}(A, \tau)$ be the set of all unitaries u in $\cup_{n \in \mathbb{N}} M_{2n}(A)$ satisfying $u^2 = 1$ and $u^\tau = u$. Let $\sim_0$ be the equivalence relation on $U_\infty^{(0)}(A, \tau)$, generated by

(1) $u_0 \sim_0 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of self-adjoint unitaries satisfying $u_t^\tau = u_t$; and
(2) $u \sim_0 \iota_n^{(0)}(u)$ for $u \in M_{2n}(A)$ where $\iota_n^{(0)}\colon M_{2n}(A) \to M_{2n+2}(A)$ is given by

$$\iota_n^{(0)}(a) = \operatorname{diag}\left(a, I^{(0)}\right) \quad \text{where } I^{(0)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Then we define $KO_0^u(A, \tau) = U_\infty^{(0)}(A, \tau)/\sim_0$, with a binary operation given by $[u] + [v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(0)}(A, \tau)$.

Since the sets $U_\infty^{(0)}(A, \tau)$ and $U_\infty^{(0)}(A^\tau)$ are identical and have the same equivalence relation, the following is immediate.

Proposition 5.10. *Let (A,τ) be a $C^{*,\tau}$-algebra and let*

$$A^\tau = \{a \in A \mid a^\tau = a^*\}$$

be the associated real C^*-algebra. Then there is an isomorphism

$$KO_0^u(A,\tau) \cong KO_0^u(A^\tau).$$

5.2. KO_2 via unitaries. In this section, we will produce the definition of $KO_2^u(A,\tau)$ and we will prove that

$$KO_2^u(A,\tau) \cong [(q\mathbb{C},\sharp),(\mathbb{K}\otimes A,\tau)] \cong [A_2, \mathbb{K}^{\mathbb{R}}\otimes A^\tau].$$

From Theorem 4.13, we know that $[A_2, \mathbb{K}^{\mathbb{R}}\otimes A^\tau] \cong KO_2(A)$. This will then imply that $KO_2^u(A) \cong KO_2(A)$ (where the latter is defined in terms of projections in the double suspension).

Definition 5.11. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(2)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{2n}(A)$ satisfying $u^2 = 1$ and $u^\tau = -u$. Let $\sim_2$ be the equivalence relation on $U_\infty^{(2)}(A,\tau)$, generated by

(1) $u_0 \sim_2 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of self-adjoint unitaries satisfying $u_t^\tau = -u_t$; and
(2) $u \sim_2 \iota_n^{(2)}(u)$ for $u \in M_{2n}(A)$ where $\iota_n^{(2)}\colon M_{2n}(A,\tau) \to M_{2n+2}(A)$ is given by

$$\iota_n^{(2)}(a) = \operatorname{diag}(a, I^{(2)}) \quad \text{where } I^{(2)} = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right).$$

Then we define $KO_2^u(A,\tau) = U_\infty^{(2)}(A,\tau)/\sim_2$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(2)}(A,\tau)$.

Proposition 5.12. *$KO_2^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups.*

Proof. We note that the set $U_\infty^{(2)}(A,\tau)$ is closed under conjugation by elements in $O(2n)$. Furthermore, since $SO(2n)$ is connected, conjugation by any element in $SO(2n)$ induces the identity automorphism on $KO_2^u(A,\tau)$. For any $u, v \in U_\infty^{(2)}(A,\tau)$, with $u \in M_{2m}(A)$ and $v \in M_{2n}(A)$ we have

$$\begin{pmatrix} 0 & 1_{2n} \\ 1_{2m} & 0 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} 0 & 1_{2m} \\ 1_{2n} & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} 0 & 1_{2n} \\ 1_{2m} & 0 \end{smallmatrix}\right) \in SO(2n+2m)$, it follows that $[u]+[v] = [v]+[u]$.

By definition the element $[I^{(2)}]$ is an identity for the semigroup. We defer the proof that there are inverses until Proposition 5.14 below. □

Proposition 5.13. $KO_2^u(\mathbb{R}) = KO_2^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}_2$.

Proof. Let $u \in M_{2n}(\mathbb{C})$ be a self-adjoint, skew-symmetric unitary. As in Section 4 of [27], there is a factorization $u = x\cdot\operatorname{diag}(I^{(2)},\dots,I^{(2)})\cdot x^{\mathrm{Tr}}$ where $x \in O(2n)$. Then depending on whether $x \in SO(2n)$ or not, we can find a path from x either to 1_{2n} or to $\operatorname{diag}(1_{2n-1}, -1)$. Therefore, we either have $u \sim_2 \operatorname{diag}(I^{(2)},\dots,I^{(2)},I^{(2)})$ or $u \sim_2 \operatorname{diag}(I^{(2)},\dots,I^{(2)},-I^{(2)})$.

Thus there are at most two equivalence classes of $U_\infty^{(2)}(\mathbb{C}, \mathrm{id})$. Furthermore, $I^{(2)} \nsim_{(2)} -I^{(2)}$ as the two elements are distinguished by the value of the Pfaffian (see Theorem 4.1 of [27]). Since

$$\mathrm{diag}(I^{(2)}, I^{(2)}) \sim_2 \mathrm{diag}(-I^{(2)}, -I^{(2)})$$

it follows that $KO_2^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}_2$. □

The proof above indicates that two self-adjoint skew-symmetric unitaries in $M_{2n}(\mathbb{C})$ represent the same element of $KO_2^u(\mathbb{C}, \mathrm{id})$ if and only if they have the same Pfaffian (although there are different conventions regarding how the Pfaffian is actually defined, especially when n is odd). We establish the notation $I_n^{(2)} = \mathrm{diag}(I^{(2)}, \ldots, I^{(2)}) \in M_{2n}(\mathbb{C})$ for the standard representative of the identity in $KO_2^u(\mathbb{C}, \mathrm{id})$.

Proposition 5.14. *Every element of $KO_2^u(A, \tau)$ has an inverse. If $u \in M_{2n}(A)$ and n is even, then the inverse of $[u]$ is $[-u]$.*

Proof. Let A be unital and let u be a self-adjoint, skew-symmetric unitary in $M_{2n}(A)$. Then the matrix

$$u_t = \begin{pmatrix} \cos(\pi t/2) \cdot u & i \sin(\pi t/2) \cdot 1_{2n} \\ i \sin(\pi t/2) \cdot 1_{2n} & -\cos(\pi t/2) \cdot u \end{pmatrix}$$

for $t \in [0, 1]$, gives a continuous path of self-adjoint skew-symmetric unitaries from $\left(\begin{smallmatrix} u & 0 \\ 0 & -u \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & i \cdot 1_{2n} \\ -i \cdot 1_{2n} & 0 \end{smallmatrix}\right)$. Depending on the parity of n, the latter matrix is similar via conjugation by a special orthogonal matrix either to $I_{2n}^{(2)}$ or to $\mathrm{diag}(I_{2n-1}^{(2)}, -I^{(2)})$. So in $KO_2^u(A, \tau)$, we either have $[u] + [-u] = 0$ or $[u] + [-u] + [-I^{(2)}] = 0$. □

Definition 5.15. For a $C^{*,\tau}$-algebra (A, τ), we define $KO_2^u(A, \tau) = \ker(\lambda_*)$ where $\lambda \colon \widetilde{A} \to \mathbb{R}$ is the natural projection on the unitization of A.

Proposition 5.16. *Let A be a real C^*-algebra. Any element of $KO_2^u(A)$ can be represented as $[u]$ where $u \in M_{2n}(\widetilde{A})$ is a unitary satisfying $u^\tau = u$ and $\lambda(u) = I_n^{(2)}$.*

Proof. Let u be a skew-symmetric self-adjoint unitary in $M_{2n}(\widetilde{A})$ such that $[\lambda(u)] = 0$ in $KO_2^u(\mathbb{C}, \mathrm{id})$. Then $\lambda(u) = x I_n^{(2)} x^{\mathrm{Tr}}$ for some $x \in SO(2n)$. Let $v = x^{\mathrm{Tr}} u x \in M_{2n}(\widetilde{A})$. Then $u \sim_2 v$ and $\lambda(v) = I_n^{(2)}$ as desired. □

Recall from Section 2 that there is an isomophism

$$(M_2(A), \tau) \cong (M_2(A), \widetilde{\tau}),$$

where $\widetilde{\tau}$ is an alternate form of the transpose operator on matrices. Thus one can make the following alternative definition of $KO_2^u(A)$ using $\widetilde{\tau}$ in place of τ.

Definition 5.17. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $\widetilde{U}^{(2)}_\infty(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}}M_{2n}(A)$ satisfying $u^2=1$ and $u^{\widetilde{\tau}}=-u$. Let $\sim_2$ be the equivalence relation on $\widetilde{U}^{(2)}_\infty(A,\tau)$, generated by

(1) $u_0\sim_2 u_1$ if $u_t\in M_{2n}(A)$ is a continuous path of self-adjoint unitaries satisfying $u_t^{\widetilde{\tau}}=-u_t$; and
(2) $u\sim_2 \iota^{(2)}_n(u)$ for $u\in M_{2n}(A)$ where $\iota^{(2)}_n\colon M_{2n}(A)\to M_{2n+2}(A)$ is given by
$$\iota^{(2)}_n(a)=\operatorname{diag}(a,1,-1).$$

Then we define $\widetilde{KO}^u_2(A,\tau)=\widetilde{U}^{(2)}_\infty(A,\tau)/\sim_2$, with a binary operation given by $[u]+[v]=[\left(\begin{smallmatrix}u&0\\0&v\end{smallmatrix}\right)]$ for $u,v\in\widetilde{U}^{(2)}_\infty(A,\tau)$.

There is a natural isomorphism $KO^u_2(A,\tau)\cong\widetilde{KO}^u_2(A,\tau)$ given by conjugation by the matrix W introduced in Section 2. This model of $KO^u_2(A,\tau)$ will be used only in this section to obtain our results.

We set a different convention for the action of the involution $\widetilde{\tau}_{2n}=\widetilde{\operatorname{Tr}}\otimes\tau_n$ on $M_{2n}(A)$ (similar to our discussion of the action of $\sharp$ and $\widetilde{\sharp}$ in Section 2). We use the formula
$$\begin{pmatrix}x&y\\z&w\end{pmatrix}^{\widetilde{\tau}}=\begin{pmatrix}w^{\tau_n}&y^{\tau_n}\\z^{\tau_n}&x^{\tau_n}\end{pmatrix}$$
where $x,y,z,w\in M_n(A)$. This convention changes the formula for
$$\iota^{(2)}_n\colon M_{2n}(A)\to M_{2n+2}(A)$$
given in Definition 5.17. Instead of the formula there we have
$$\iota^{(2)}_n\begin{pmatrix}x&y\\z&w\end{pmatrix}=\begin{pmatrix}x&0&y&0\\0&1&0&0\\z&0&w&0\\0&0&0&-1\end{pmatrix}$$
for $x,y,z,w\in M_n(A)$. With this notation, if $x\in M_{2n}(A)$ satisfies $x^{\widetilde{\tau}}=-x$, then the element $y=\iota^{(2)}_n(x)$ also satisfies $y^{\widetilde{\tau}}=-y$. With this notation, the trivial element of $\widetilde{KO}^u_2(\mathbb{R})\cong\mathbb{Z}_2$ is represented by $\left(\begin{smallmatrix}1_n&0\\0&-1_n\end{smallmatrix}\right)$ for any n.

Theorem 5.18. *For any $C^{*,\tau}$-algebra (A,τ), there is a natural isomorphism*
$$[(q\mathbb{C},\sharp),(\mathbb{K}\otimes A,\tau)]\cong KO^u_2(A).$$

Proof. Assume A is unital. We will prove that there is an isomorphism between $[(q\mathbb{C},\sharp),(\mathbb{K}^{\mathbb{R}}\otimes A,\tau)]$ and $\widetilde{KO}^u_2(A,\tau)$ using a similar proof to that of Proposition 5.8. Let
$$u_0=U(h_0,x_0,k_0)=\begin{pmatrix}1-2t&0&0&2\sqrt{t-t^2}\\0&1&0&0\\0&0&-1&0\\2\sqrt{t-t^2}&0&0&2t-1\end{pmatrix}\in M_2(\widetilde{q\mathbb{C}})$$

and note that we have $u_0^{\sharp\otimes\widetilde{\mathrm{Tr}}} = -u_0$; so $[u_0] \in KO_2^u(q\mathbb{C}, \sharp)$. If $\phi\colon (q\mathbb{C}, \sharp) \to (M_n(A), \tau)$ is a $C^{*,\tau}$-algebra homomorphism, then $\phi(u_0) \in \widetilde{U}_\infty^{(2)}(A, \tau)$. We define $\Theta(\phi) = [\phi(u_0)] \in \widetilde{KO}_2^u(A, \tau)$. To show this is well-defined, we need to consider $\phi' = \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

We can now check that as in the proof of Proposition 5.8,

$$\phi_2(u_0) = \begin{pmatrix} 1_n - 2\phi(h_0) & 2\phi(x_0)^* \\ 2\phi(x_0) & 2\phi(k_0) - 1_n \end{pmatrix},$$

$$\phi_2'(u_0) = \begin{pmatrix} 1_n - 2\phi(h_0) & 0 & 2\phi(x_0)^* & 0 \\ 0 & 1 & 0 & 0 \\ 2\phi(x_0) & 0 & 2\phi(k_0) - 1_n & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus we have $\Theta(\phi'(u_0)) = \iota_n^{(2)}(\phi(u_0))$, showing that Θ is well defined.

To construct an inverse to Θ (as in the proof of Proposition 5.8), suppose $u \in M_{2n}(\widetilde{A})$ is a unitary satisfying $u^2 = 1$, $u^{\widetilde{\tau}} = -u$, and $\lambda(u) = \left(\begin{smallmatrix} 1_n & 0 \\ 0 & -1_n \end{smallmatrix}\right)$. Then there exist $h, x, k \in M_n(A)$ such that

$$u = U(h, x, k) = \begin{pmatrix} 1_n - 2h & 2x^* \\ 2x & 2k - 1_n \end{pmatrix}.$$

The conditions that $u = u^*$ and $u^{\widetilde{\tau}} = -u$ are equivalent to the conditions that h and k are self-adjoint, they are interchanged by τ, and $x^\tau = -x$.

We now construct modified elements h', k', x' that satisfy the same conditions as well as the condition $h'k' = 0$. Let

$$W_{2n} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i \cdot 1_n & 1_n \\ 1_n & i \cdot 1_n \end{pmatrix} \in M_{2n}(\mathbb{C})$$

(generalizing the definition of W from Section 2) and define

$$h' = W_{2n} \left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n}^* = \tfrac{1}{2} \left(\begin{smallmatrix} h & ih \\ -ih & h \end{smallmatrix}\right),$$
$$k' = W_{2n}^* \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n} = \tfrac{1}{2} \left(\begin{smallmatrix} k & -ik \\ ik & k \end{smallmatrix}\right),$$
$$x' = W_{2n}^* \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n}^* = \tfrac{1}{2} \left(\begin{smallmatrix} -x & -ix \\ -ix & x \end{smallmatrix}\right),$$

in $M_{2n}(A)$. We leave it to the reader to check that h' and k' are self-adjoint, that $h'k' = 0$, that $(h')^\tau = k'$, and that $(x')^\tau = -x'$. The following calculation shows that $u' = U(h', x', k')$ is a unitary:

$$\begin{aligned} u' &= \left(\begin{smallmatrix} 1_{2n} - 2h' & 2(x')^* \\ 2x' & 2k' - 1_{2n} \end{smallmatrix}\right) \\ &= \begin{pmatrix} 1_{2n} - 2W_{2n}\left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n}^* & 2W_{2n}\left(\begin{smallmatrix} x^* & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n} \\ 2W_{2n}^*\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n}^* & 2W_{2n}^*\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) W_{2n} - 1_{2n} \end{pmatrix} \\ &= \left(\begin{smallmatrix} W_{2n} & 0 \\ 0 & W_{2n}^* \end{smallmatrix}\right) \begin{pmatrix} 1_{2n} - 2\left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right) & 2\left(\begin{smallmatrix} x^* & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ 2\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) & 2\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) - 1_{2n} \end{pmatrix} \left(\begin{smallmatrix} W_{2n}^* & 0 \\ 0 & W_{2n} \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} W_{2n} & 0 \\ 0 & W_{2n}^* \end{smallmatrix}\right) \cdot \iota^{(2)}(u) \cdot \left(\begin{smallmatrix} W_{2n}^* & 0 \\ 0 & W_{2n} \end{smallmatrix}\right). \end{aligned}$$

where $\iota^{(2)}\colon M_{2n}(A) \to M_{4n}(A)$ is the composition $\iota^{(2)}_{2n-1} \dots \iota^{(2)}_{n+1}\iota^{(2)}_n$ and

$$\iota^{(2)}(u) = \begin{pmatrix} 1_n - 2h & 0 & 2x^* & 0 \\ 0 & 1_n & 0 & 0 \\ 2x & 0 & 2k - 1_n & 0 \\ 0 & 0 & 0 & -1_n \end{pmatrix}.$$

Thus $u' = U(h', x', k')$ is a unitary that satisfies

$$u' = (u')^* \quad \text{and} \quad (u')^{\widetilde{\tau}} = -u'.$$

By Proposition 3.3, there is a homomorphism

$$\phi_u\colon (q\mathbb{C}, \sharp) \to (M_{2n}(A), \tau)$$

such that $\phi_u(h_0) = h'$, $\phi_u(k_0) = k'$, and $\phi_u(x_0) = x'$. Thus $(\phi_u)_2(u_0) = u'$.

We claim that $u \sim_2 u'$, which will imply that $\Theta(\phi_u) = [u]$ as desired. Specifically, we will show that u' is homotopic to $\iota^{(2)}(u)$ in $M_{4n}(A)$. The previous calculation shows that $u' = v\iota^{(2)}(u)v^*$ where $v = \left(\begin{smallmatrix} w & 0 \\ 0 & w^* \end{smallmatrix}\right)$. So it suffices to show that $vxv^* \sim_2 x$ for any self-adjoint unital $x \in M_{4n}(A)$ satisfying $x^{\widetilde{\tau}} = -x$. Let

$$W_{4n} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i \cdot \mathbb{1}_{2n} & \mathbb{1}_{2n} \\ \mathbb{1}_{2n} & i \cdot \mathbb{1}_{2n} \end{pmatrix}.$$

For any $x \in M_{4n}(A)$ we have $x^{\widetilde{\tau}} = -x$ if and only if

$$(W_{4n} x W_{4n}^*)^\tau = -(W_{4n} x W_{4n}^*).$$

Then as $U^{(2)}(A, \tau)$ is closed under conjugation by $SO(4n)$, so $\widetilde{U}^{(2)}(A, \tau)$ is closed under conjugation by elements in

$$\widetilde{SO}(4n) = W_{4n}^* SO(4n) W_{4n}.$$

Now $v \in \widetilde{SO}(4n)$ since

$$W_{4n} v W_{4n}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \in SO(4n).$$

Since $SO(4n)$ is connected, so is $\widetilde{SO}(4n)$. Therefore there is path in $\widetilde{SO}(4n)$ from v to the identity which proves that $vxv^* \sim_2 x$.

This completes the proof that $\Theta(\phi_u) = [u]$. The rest of the proof consists in showing that the construction described is actually an inverse to Θ. This proceeds as in the proof of Proposition 5.8. $\square$

5.3. KO_4 via unitaries.

Definition 5.19. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(4)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{4n}(A)$ satisfying $u^2 = 1$ and $u^{\sharp\otimes\tau} = u$. Let $\sim_4$ be the equivalence relation on $U_\infty^{(4)}(A,\tau)$, generated by

(1) $u_0 \sim_4 u_1$ if $u_t \in M_{4n}(A)$ is a continuous path of self-adjoint unitaries satisfying $u_t^{\sharp\otimes\tau} = u_t$; and
(2) $u \sim_4 \iota_n^{(4)}(u)$ for $u \in M_{4n}(A)$ where $\iota_n^{(4)}\colon M_{4n}(A) \to M_{4n+4}(A)$ is given by

$$\iota_n^{(4)}(a) = \operatorname{diag}(a, I^{(4)}) \quad \text{where } I^{(4)} = \operatorname{diag}(1,1,-1,-1).$$

Then we define $KO_4^u(A,\tau) = U_\infty^{(4)}(A,\tau)/\sim_4$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(0)}(A,\tau)$.

In the above definition, the formulas for $\iota_n^{(4)}$, for $I^{(4)}$, and for addition implicitly assume the particular convention for the action of the involution $\sharp\otimes\tau$ on $M_{4n}(A)$ as discussed in Section 2. Under this convention, the addition formula and the formula for $\iota_n^{(4)}$ in Definition 5.19 preserve membership in $U_\infty^{(4)}(A)$.

Proposition 5.20. *If (A,τ) is a unital $C^{*,\tau}$-algebra, then*

$$KO_4^u(A,\tau) \cong KO_4(A,\tau).$$

In particular, $KO_4^u(\mathbb{C},\mathrm{id}) = \mathbb{Z}$.

Proof. An element of $U_\infty^{(4)}(A,\tau)$ is given by a self-adjoint unitary $u \in M_2(\mathbb{C}) \otimes M_{2n}(A)$ that satisfies $u^{\sharp\otimes\tau} = u$. This is the same as an element of $U_\infty^{(0)}(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau)$. Therefore $U_\infty^{(4)}(A,\tau) \cong U_\infty^{(0)}(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau)$ and hence $KO_4^u(A,\tau) \cong KO_0^u(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau)$.

As a special case of the Künneth formula (the Main Theorem of [5]), for a real C^*-algebra A we know that $KO_n(A) \cong KO_{n+4}(\mathbb{H}\otimes A)$. The same statement in terms of $C^{*,\tau}$-algebras is that

$$KO_n(A,\tau) \cong KO_{n+4}(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau).$$

Combining this with Theorem 5.6,

$$\begin{aligned} KO_4^u(A,\tau) &\cong KO_0^u(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau) \\ &\cong KO_0(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau) \\ &\cong KO_4(A,\tau). \end{aligned}$$

□

The identity element of $KO_4^u(\mathbb{C},\mathrm{id}) \cong \mathbb{Z}$ is represented by $I^{(4)} \in M_4(\mathbb{C})$ or, more generally, by $I_n^{(4)} = \operatorname{diag}(I^{(4)},\dots,I^{(4)}) \in M_{4n}(\mathbb{C})$. The isomorphism $KO_4^u(\mathbb{C},\mathrm{id}) \to \mathbb{Z}$ can be written as $[u] \mapsto \frac{1}{4}\mathrm{trace}(u)$.

Theorem 5.21. *$KO_4^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_4^u(A,\tau)$ is given by $[-u]$.*

Proof. The first statement follows immediately from Proposition 5.20. The statement about inverses follows from Proposition 5.2 and the statement in the proof of Proposition 5.20 that $U_\infty^{(4)}(A,\tau) \cong U_\infty^{(0)}(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau)$. □

Definition 5.22. Let (A,τ) be any $C^{*,\tau}$-algebra. Then we define

$$KO_4^u(A,\tau) = \ker(\lambda_*)$$

where $\lambda_*\colon KO_4^u(\widetilde{A},\tau) \to KO_4^u(\mathbb{C},\mathrm{id})$.

Combining Theorem 5.21 with this definition gives the following.

Theorem 5.23. *If (A,τ) is any $C^{*,\tau}$-algebra, then $KO_4^u(A,\tau) \cong KO_4(A,\tau)$.*

Theorem 5.24. *Let (A,τ) be a $C^{*,\tau}$-algebra. Any element of $K_4^u(A,\tau)$ can be represented as $[u]$ where $u \in M_n(\widetilde{A})$ satisfies $u^2 = 1$, $u^{\sharp\otimes\tau} = u$, and $\lambda_*(u) = I_n^{(4)}$.*

Proof. Consider the following commutative diagram in which each row is a short exact sequence and g is the isomorphism described in the proof of Proposition 5.20.

$$\begin{array}{ccccc}
KO_0^u(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau) & \longrightarrow & KO_0^u((M_2(\mathbb{C})\otimes A)^\sim, \sharp\otimes\tau) & \xrightarrow{\lambda_*} & KO_0^u(\mathbb{C},\mathrm{id}) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle k_*} & & \downarrow{\scriptstyle \iota_*} \\
KO_0^u(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau) & \longrightarrow & KO_0^u(M_2(\mathbb{C})\otimes \widetilde{A}, \sharp\otimes\tau) & \xrightarrow{\lambda_*} & KO_0^u(M_2(\mathbb{C}),\sharp) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\
KO_4^u(A,\tau) & \longrightarrow & KO_4^u(\widetilde{A},\tau) & \xrightarrow{\lambda_*} & KO_4^u(\mathbb{C},\mathrm{id}).
\end{array}$$

It follows from the diagram that any element of $KO_4^u(A,\tau)$ can be written as $gk_*([u])$ where $[u] \in KO_0^u((M_2(\mathbb{C})\otimes A)^\sim, \sharp\otimes\tau)$ and $\lambda_*([u]) = 0$. Using Proposition 5.5, we can take $u \in M_{2n}((M_2(\mathbb{C})\otimes A)^\sim)$ such that $u^2 = 1$, $u^{\sharp\otimes\tau} = u$, and $\lambda(u) = I_{2n}^{(0)}$. Then $v = gk(u) \in M_{4n}(\widetilde{A})$ satisfies $v^2 = 1$, $v^{\sharp\otimes\tau} = v$, and $\lambda(v) = I_n^{(4)}$ as desired. □

5.4. KO_6 via unitaries.

Definition 5.25. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(6)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{2n}(A)$ satisfying $u^2 = 1$ and $u^{\sharp\otimes\tau} = -u$. Let $\sim_6$ be the equivalence relation on $U_\infty^{(6)}(A,\tau)$, generated by

(1) $u_0 \sim_6 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of self-adjoint unitaries satisfying $u_t^{\sharp\otimes\tau} = u_t$; and
(2) $u \sim_6 \iota_n^{(4)}(u)$ for $u \in M_{2n}(A)$ where $\iota_n^{(6)}\colon M_{2n}(A) \to M_{2n+2}(A)$ is given by

$$\iota_n^{(6)}(a) = \mathrm{diag}(a, I^{(6)}) \quad \text{where } I^{(6)} = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right).$$

Then we define $KO_6^u(A,\tau) = U_\infty^{(6)}(A,\tau)/\sim_6$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u,v \in U_\infty^{(0)}(A,\tau)$.

If, in the definition above, we only used unitaries in $M_{4n}(A)$, then it would be clear that $KO_6^u(A,\tau)$ is isomorphic to $KO_2^u(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau)$. However, the allowed inclusion of unitaries in $M_{2n}(A)$ does not change the group in the limit. Therefore, the following results follow a development that is similar to that for $KO_4^u(A,\tau)$.

Proposition 5.26. *If (A,τ) is a unital $C^{*,\tau}$-algebra, then*

$$KO_6^u(A,\tau) \cong KO_6(A,\tau).$$

In particular, $KO_6^u(\mathbb{C},\mathrm{id}) = 0$.

Proposition 5.27. *$KO_6^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_6^u(A,\tau)$ is given by $[-u]$ if $u \in M_{2n}(A)$.*

Definition 5.28. Let (A,τ) be any $C^{*,\tau}$-algebra. Then we define

$$KO_6^u(A,\tau) = \ker(\lambda_*)$$

where $\lambda_*\colon KO_6^u(\widetilde{A},\tau) \to KO_6^u(\mathbb{C},\mathrm{id})$.

Proposition 5.29. *If (A,τ) is any $C^{*,\tau}$-algebra, then*

$$KO_6^u(A,\tau) \cong KO_6(A,\tau).$$

Proposition 5.30. *Let (A,τ) be a $C^{*,\tau}$-algebra. Any element of $K_6^u(A,\tau)$ can be represented as $[u]$ where $u \in M_{4n}(\widetilde{A})$ satisfies $u^2 = 1$, $u^{\sharp\otimes\tau} = u$, and $\lambda_*(u) = I_n^{(6)}$.*

6. *K*-theory via unitaries — the odd cases

6.1. KO_1 via unitaries. The following definitions and theorems represent the standard development of $KO_1(A)$ as in Chapter 8 of [34] for the complex case. They are included here for reference and terminology; and the proofs will be omitted as appropriate.

Definition 6.1. Let A be a real unital C^*-algebra. Let $U_\infty^{(1)}(A)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_n(A)$. Let $\sim_1$ the equivalence relation on $U_\infty^{(1)}(A)$, generated by

(1) $u_0 \sim_1 u_1$ if $u_t \in M_n(A)$ is a continuous path of unitaries for $t \in [0,1]$; and
(2) $u \sim_1 \iota_n^{(1)}(u)$ for $u \in M_n(A)$ where $\iota_n^{(1)}\colon M_n(A) \to M_{n+1}(A)$ is given by

$$\iota_n^{(1)}(a) = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) = \mathrm{diag}(a,1).$$

Then we define $KO_1^u(A) = U_\infty^{(1)}(A)/\sim_1$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u,v \in U_\infty^{(1)}(A)$.

Proposition 6.2. *$KO_1^u(A)$ is a homotopy invariant functor from the category of unital C^*-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_1^u(A)$ is given by $[u^*]$.*

Proposition 6.3. *$KO_1^u(\mathbb{R}) \cong \mathbb{Z}_2$. The isomorphism $KO_1^u(\mathbb{R}) \to \mathbb{Z}_2$ is given by $[u] \mapsto \det(u)$.*

Definition 6.4. Let A be any unital C^*-algebra. Then we define $KO_1^u(A) = \ker(\lambda_*)$ where $\lambda\colon \widetilde{A} \to \mathbb{R}$ is the natural projection from the unitization of A with kernel isomorphic to A.

Proposition 6.5. *Let A be a real C^*-algebra. Any element of $KO_1^u(A)$ can be represented as $[u]$ where $u \in M_n(\widetilde{A})$ is a unitary satisfying $\lambda(u) = 1_n$.*

Proof. Let $u \in M_n(\widetilde{A})$ be a unitary element satisfying $\lambda_*([u]) = 0 \in KO_1^u(\mathbb{R})$. This implies that $\det(\lambda(u)) = 1$ so there is a path u_t of unitaries in $M_n(\mathbb{R})$ such that $u_0 = 1_n$ and $u_1 = \lambda(u)$. Then $v_t = u(u_t)^*$ is a path of unitaries in $M_n(\widetilde{A})$ such that $v_0 = u$ and $\lambda(v_1) = \lambda(u)u_1^* = 1_n$. □

Proposition 6.6. *For any real C^*-algebra there is an isomorphism*

$$\Gamma\colon KO_1^u(A) \to KO_0^u(SA)$$

given as follows. Let $u \in M_n(\widetilde{A})$ be a unitary, satisfying $\lambda(u) = 1_n$. Let $v_t \in M_{2n}(\widetilde{A})$ be a continuous path of unitaries such that $v_0 = \mathbb{1}_{2n}$, $v_1 = \operatorname{diag}(u, u^*)$, and $\lambda(v_t) = 1_{2n}$ for all $t \in [0,1]$. Then

$$\Gamma([u]) = \left[2v_t \left(\begin{smallmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{smallmatrix}\right) v_t^* - \mathbb{1}_{2n}\right].$$

Proof. There is an isomorphism $\gamma\colon KO_1^u(A) \to KO_0(SA)$ given by

$$\gamma([u]) = \left[v \left(\begin{smallmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{smallmatrix}\right) v^*\right] - [\mathbb{1}_n]$$

where v is as in the statement of the theorem above. This is well-known in the complex case (see for example Theorem 10.1.3 of [34]) and works the same in the real case. Then the isomorphism Γ is given by $\Gamma = \Theta \circ \gamma$ where Θ is from Theorem 5.6. □

Corollary 6.7. *For any real C^*-algebra A, there is a natural isomorphism $KO_1^u(A) \cong KO_1(A)$.*

We end this section by giving a rephrase of the definition of KO_1^u in the context of a $C^{*,\tau}$-algebra. This gives a description of $KO_1^u(A,\tau)$ that is parallel to the forthcoming descriptions of $KO_j^u(A,\tau)$ for all values of j.

Definition 6.8. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(1)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_n(A)$ satisfying $u^\tau = u^*$. Let $\sim_1$ be the equivalence relation on $U_\infty^{(1)}(A,\tau)$, generated by

(1) $u_0 \sim_1 u_1$ if $u_t \in M_n(A)$ is a continuous path of unitaries satisfying $u_t^\tau = u_t^*$; and
(2) $u \sim_1 \iota_n^{(1)}(u)$ for $u \in M_n(A)$ where $\iota_n^{(1)}\colon M_n(A) \to M_{n+1}(A)$ is given by

$$\iota_n^{(1)}(a) = \operatorname{diag}(a, 1).$$

Then we define $KO_1^u(A,\tau) = U_\infty^{(1)}(A,\tau)/\sim_1$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(1)}(A,\tau)$.

As in Proposition 5.10, the sets $U_\infty^{(1)}(A,\tau)$ and $U_\infty^{(1)}(A^\tau)$ are easily seen to be identical.

Proposition 6.9. *Let (A,τ) be a $C^{*,\tau}$-algebra and let*

$$A^\tau = \{a \in A \mid a^\tau = a^*\}$$

be the associated real C^*-algebra. Then there is an isomorphism

$$KO_1^u(A,\tau) \cong KO_1^u(A^\tau).$$

6.2. KO_{-1} via unitaries.

Definition 6.10. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(-1)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_n(A)$ that satisfy $u^\tau = u$. Let $\sim_{(-1)}$ be the equivalence relation on $U_\infty^{(-1)}(A,\tau)$, generated by

(1) $u_0 \sim_{(-1)} u_1$ if $u_t \in M_n(A)$ is a continuous path of unitaries satisfying $u_t^\tau = u_t$; and
(2) $u \sim_{(-1)} \iota_n^{(-1)}(u)$ for $u \in M_n(A)$ where $\iota_n^{(-1)} \colon M_n(A) \to M_{n+1}(A)$ is given by

$$\iota_n^{(-1)}(a) = \operatorname{diag}(a, 1).$$

Then we define $KO_{-1}^u(A,\tau) = U_\infty^{(-1)}(A,\tau)/\sim_{(-1)}$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U_\infty^{(-1)}(A,\tau)$.

Proposition 6.11. $KO_{-1}^u(\mathbb{C}, \mathrm{id}) = 0$.

Proof. Let $u \in M_n(\mathbb{C})$ be a unitary element such that $u^{\mathrm{Tr}} = u$. By Corollary 4.4.4 of [22], there exists a unitary v such that $u = v^{\mathrm{Tr}} v$. Since the group of unitaries in $M_n(\mathbb{C})$ in path connected, we can find a path v_t from v to 1_n and let $u_t = v_t^{\mathrm{Tr}} v_t$ be the path of unitaries from u to $\mathbb{1}_n$ satisfying $u_t^{\mathrm{Tr}} = u_t$. □

Proposition 6.12. *$KO_{-1}^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_{-1}^u(A,\tau)$ is given by $[u^*]$.*

Proof. We leave the question of functoriality and homotopy invariance to the reader, and show that the binary operation is commutative.

Let $u \in M_n(A)$ and $v \in M_m(A)$ be unitaries satisfying $u^\tau = u$ and $v^\tau = v$. First we claim that there exists a unitary w in $M_{n+m}(\mathbb{R})$ such that $w \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) w^* = \left(\begin{smallmatrix} v & 0 \\ 0 & u \end{smallmatrix}\right)$ and $\det w = 1$. If either m or n is even, then w can be taken to be the obvious change of basis matrix corresponding to an even permutation of the basis elements. On the other hand if m and n are both odd then let w' be the odd permutation matrix that satisfies $w' \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) (w')^* = \left(\begin{smallmatrix} v & 0 \\ 0 & u \end{smallmatrix}\right)$. Then let $w = w' \left(\begin{smallmatrix} 1_n & 0 \\ 0 & -1_m \end{smallmatrix}\right)$. This proves the claim. Then let w_t be a path of special orthogonal matrices from 1_{n+m} to w and

consider the path $x_t = w_t \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right) w_t^*$. Verifying that this satisfies $x_t^\tau = x_t$ completes the proof.

For the statement about inverses, we claim that if u is a unitary in $M_n(A)$ that satisfies $u^\tau = u$, then $\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right) \sim_{(-1)} \left(\begin{smallmatrix} 1_n & 0 \\ 0 & 1_n \end{smallmatrix}\right)$. Indeed,

$$v_t = \begin{pmatrix} \cos((\pi/2)t) \cdot u & \sin((\pi/2)t) \cdot 1_n \\ \sin((\pi/2)t) \cdot 1_n & -\cos((\pi/2)t) \cdot u^* \end{pmatrix}$$

is a homotopy from $\left(\begin{smallmatrix} u & 0 \\ 0 & -u^* \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right)$ satisfying $v_t^\tau = v_t$. Since any matrix of the form $w = \left(\begin{smallmatrix} u & 0 \\ 0 & \lambda u^* \end{smallmatrix}\right)$ for $|\lambda| = 1$ satisfies $w^\tau = w$, we have

$$\left(\begin{smallmatrix} u & 0 \\ 0 & u^* \end{smallmatrix}\right) \sim_{(-1)} \left(\begin{smallmatrix} u & 0 \\ 0 & -u^* \end{smallmatrix}\right).$$

The argument given in the proof of Proposition 6.11 above implies that $\left(\begin{smallmatrix} 0 & 1_n \\ 1_n & 0 \end{smallmatrix}\right) \sim_{(-1)} \left(\begin{smallmatrix} 1_n & 0 \\ 0 & 1_n \end{smallmatrix}\right)$. □

Definition 6.13. Let (A, τ) be any $C^{*,\tau}$-algebra. Then we define

$$KO^u_{-1}(A, \tau) = \ker(\lambda_*).$$

Since $KO^u_{-1}(\mathbb{C}, \mathrm{id}) = 0$ it follows that $KO^u_{-1}(A, \tau) = KO^u_{-1}(\widetilde{A}, \tau)$.

Proposition 6.14. *Let (A, τ) be a $C^{*,\tau}$-algebra and let $\widetilde{A}$ be the unitization. Any element of $KO^u_{-1}(A, \tau)$ can be represented as $[u]$ where $u \in M_n(\widetilde{A})$ is a unitary satisfying $u^\tau = u$ and $\lambda(u) = 1_n$.*

Proof. Let u be a unitary in $M_n(\widetilde{A})$ such that $u^\tau = u$. Let $x = \lambda(u) \in M_n(\mathbb{C})$. As in the proof of Proposition 6.11, write $x = y^{\mathrm{Tr}} y$ where y is a unitary in $M_n(\mathbb{C})$ and let y_t be a path of unitaries from 1_n to y. Then $z_t = (y_t^*)^{\mathrm{Tr}} u y_t^*$ is a path from u to a unitary z_1 in $M_n(\widetilde{A})$ that satisfies $\lambda(z_1) = 1_n$. Also, note that we have $z_t^\tau = z_t$ for all t. □

Proposition 6.15. *For any $C^{*,\tau}$-algebra (A, τ), there is a natural isomorphism*

$$KO^u_{-1}(A, \tau) \cong [(C_0(S^1), \mathrm{id}), (\mathbb{K}^{\mathbb{R}} \otimes_{\mathbb{R}} A, \tau)].$$

Proof. Let u_0 be the identity function in $C(S^1)$. That is, $u_0(z) = z$ for all $z \in S^1$. Then $[u_0] \in KO^u_{-1}(C(S^1), \tau) \cong \mathbb{Z}$. For any $\phi\colon (C_0(S^1), \mathrm{id}) \to (M_n(A), \tau)$, we can extend to the unitization to obtain $\phi\colon (C(S^1), \mathrm{id}) \to (M_n(\widetilde{A}), \tau)$ and write $\theta([\phi]) = [\phi(u_0)] \in KO^u_{-1}(A)$. For $\phi' = \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)$ we have $\phi'(u_0) = \left(\begin{smallmatrix} \phi(u_0) & 0 \\ 0 & \mathbb{1} \end{smallmatrix}\right)$, so $\theta([\phi]) = \theta([\phi'])$. This gives us a well defined natural transformation

$$\Theta\colon [(C_0(S^1), \mathrm{id}), (\mathbb{K}^{\mathbb{R}} \otimes A, \tau)] \to KO^u_{-1}(A, \tau).$$

Conversely suppose $u \in M_n(\widetilde{A})$ is a unitary satisfying $u^\tau = u$. There is a unique unital homomorphism $\phi\colon C(S^1, \mathbb{C}) \to M_n(\widetilde{A})$ such that $\phi(u_0) = u$, and it is easily seen that ϕ satisfies $\phi(x^{\mathrm{id}}) = \phi(x)^\tau$ for all x. Thus ϕ is a $C^{*,\tau}$-algebra homomorphism $\phi\colon (C(S^1), \mathrm{id}) \to (M_n(\widetilde{A}), \tau)$. The restriction

yields a homomorphism $\phi\colon (C_0(S^1), \mathrm{id}) \to (M_n(A), \tau) \subset (\mathbb{K}^{\mathbb{R}} \otimes_{\mathbb{R}} A, \tau)$. This construction gives an inverse to Θ. □

Corollary 6.16. *For any $C^{*,\tau}$-algebra (A, τ), there is a natural isomorphism*

$$KO^u_{-1}(A, \tau) \cong KO_{-1}(A, \tau).$$

Proof. This follows from Proposition 6.14 above and Theorem 4.13 (remembering that $A_{-1} = (C_0(S^1 \setminus \{1\}), \mathrm{id})$). □

6.3. KO_3 via unitaries.

Definition 6.17. Let (A, τ) be a unital $C^{*,\tau}$-algebra. Let $U^{(3)}_\infty(A, \tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{2n}(A)$ satisfying $u^{\sharp\otimes\tau} = u$. Let $\sim_3$ be the equivalence relation on $U^{(3)}_\infty(A, \tau)$, generated by

(1) $u_0 \sim_3 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of unitaries satisfying $u_t^{\sharp\otimes\tau} = u_t$; and
(2) $u \sim_3 \iota^{(3)}_n(u)$ for $u \in M_{2n}(A)$ where $\iota^{(3)}_n\colon M_{2n}(A) \to M_{2n+2}(A)$ is given by

$$\iota^{(3)}_n = \mathrm{diag}\,(a, 1_2)\,.$$

Then we define $KO^u_3(A, \tau) = U^{(3)}_\infty(A, \tau)/\sim_3$, with a binary operation given by $[u] + [v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)]$ for $u, v \in U^{(3)}_\infty(A, \tau)$.

Here we are using the same convention on the involution $\sharp \otimes \tau$ as we did in Definition 5.19 for $KO^u_4(A, \tau)$.

Proposition 6.18. *If (A, τ) is a unital $C^{*,\tau}$-algebra, then*

$$KO^u_3(A, \tau) \cong KO_3(A, \tau).$$

In particular, $KO^u_3(\mathbb{C}, \mathrm{id}) = 0$.

Proof. An element of $U^{(3)}_\infty(A, \tau)$ is given by a unitary $u \in M_2(\mathbb{C}) \otimes M_n(A)$ that satisfies $u^{\sharp\otimes\tau} = u$. This is the same as an element of

$$U^{(-1)}_\infty(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau).$$

Therefore $U^{(3)}_\infty(A, \tau) \cong U^{(-1)}_\infty(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau)$; and hence

$$KO^u_3(A, \tau) = KO^u_{-1}(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau).$$

As a special case of the Künneth formula (the Main Theorem of [5]), for a real C^*-algebra A we know that $KO_n(A) \cong KO_{n+4}(\mathbb{H}\otimes A)$. In terms of $C^{*,\tau}$-algebras, this is the statement that $KO_n(A, \tau) \cong KO_{n+4}(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau)$. Therefore

$$\begin{aligned} KO^u_3(A, \tau) &\cong KO^u_{-1}(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau) \\ &\cong KO_{-1}(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau) \\ &\cong KO_3(A, \tau). \end{aligned}$$

□

Proposition 6.19. *$KO_3^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_3^u(A,\tau)$ is given by $[u^*]$.*

Proof. This follows immediately from Proposition 6.18. The statement about inverses follows from Proposition 6.12 and the statement in the proof of Proposition 6.18 that $U_\infty^{(3)}(A,\tau) \cong U_\infty^{(-1)}(M_2(\mathbb{C})\otimes A, \sharp\otimes\tau)$. □

Definition 6.20. Let (A,τ) be any $C^{*,\tau}$-algebra. Then we define

$$KO_3^u(A,\tau) = \ker(\lambda_*)$$

where $\lambda_*\colon KO_3^u(\widetilde{A},\tau) \to KO_3^u(\mathbb{C},\mathrm{id})$.

Combining Proposition 6.19 with this definition gives the following.

Proposition 6.21. *If (A,τ) is any $C^{*,\tau}$-algebra, then*

$$KO_3^u(A,\tau) \cong KO_3(A,\tau).$$

Proposition 6.22. *Let (A,τ) be a $C^{*,\tau}$-algebra. Any element of $KO_3^u(A,\tau)$ can be represented as $[u]$ where $u \in M_n(\widetilde{A})$ satisfies $u^{\sharp\otimes\tau} = u$ and $\lambda_*(u) = 1_n$.*

Proof. Let $u \in U_3(\widetilde{A},\tau) = U_{-1}(M_2(\mathbb{C})\otimes\widetilde{A}, \sharp\otimes\tau)$. The unital homomorphism $(M_2(\mathbb{C})\otimes A)^\sim \hookrightarrow M_2(\mathbb{C})\otimes\widetilde{A}$ induces an isomorphism on $KO_{-1}^u(-)$. Therefore, using Proposition 6.14, we can replace u by an equivalent unitary v in $M_n(M_2(\mathbb{C})\otimes\widetilde{A}) \subset U_{-1}(M_2(\mathbb{C})\otimes\widetilde{A}, \sharp\otimes\tau)$ that satisfies $\lambda_n(v) = \mathbb{1}_{2n}$ where $\lambda\colon (M_2(\mathbb{C})\otimes A)^\sim \to \mathbb{C}$ and $\lambda_n\colon M_n((M_2(\mathbb{C})\otimes A)^\sim) \to M_n(\mathbb{C})$.

Now, we consider the same unitary v as an element in $U_3(\widetilde{A},\tau)$. In that context it is a unitary in $M_{2n}(\widetilde{A})$ and it satisfies $\lambda_{2n}(v) = \mathbb{1}_{2n}$ where $\lambda\colon \widetilde{A} \to \mathbb{C}$ and $\lambda_{2n}\colon M_{2n}(\widetilde{A}) \to M_{2n}(\mathbb{C})$. □

6.4. KO_5 via unitaries.

Definition 6.23. Let (A,τ) be a unital $C^{*,\tau}$-algebra. Let $U_\infty^{(5)}(A,\tau)$ be the set of all unitaries u in $\cup_{n\in\mathbb{N}} M_{2n}(A)$ satisfying $u^{\sharp\otimes\tau} = u^*$. Let $\sim_5$ be the equivalence relation on $U_\infty^{(5)}(A,\tau)$, generated by

(1) $u_0 \sim_5 u_1$ if $u_t \in M_{2n}(A)$ is a continuous path of unitaries satisfying $u_t^{\sharp\otimes\tau} = u_t^*$; and
(2) $u \sim_5 \iota_n^{(5)}(u)$ for $u \in M_{2n}(A)$ where $\iota_n^{(5)}\colon M_{2n}(A) \to M_{2n+2}(A)$ is given by
$$\iota_n^{(5)}(a) = \mathrm{diag}\,(a, 1_2)\,.$$

Then we define $KO_5^u(A,\tau) = U_\infty^{(5)}(A,\tau)/\sim_5$, with a binary operation given by $[u]+[v] = [\left(\begin{smallmatrix} u & 0 \\ 0 & v\end{smallmatrix}\right)]$ for $u,v \in U_\infty^{(5)}(A,\tau)$.

In this definition we use the same convention for $\sharp\otimes\tau$ as discussed for Definition 6.17.

Proposition 6.24. *If (A,τ) is a unital $C^{*,\tau}$-algebra, then*

$$KO_5^u(A,\tau) \cong KO_5(A,\tau).$$

In particular, $KO_5^u(\mathbb{C},\mathrm{id}) = 0$.

Proof. The proof is the same as the proof of Proposition 6.18, using

$$KO_5^u(A,\tau) = KO_1^u(M_2(\mathbb{C}) \otimes A, \sharp \otimes \tau). \qquad \square$$

Proposition 6.25. *$KO_5^u(A,\tau)$ is a homotopy invariant functor from the category of unital $C^{*,\tau}$-algebras to the category of abelian groups. The inverse of an element $[u]$ in $KO_5^u(A,\tau)$ is given by $[u^*]$.*

Proof. This follows from Proposition 6.24. $\square$

Definition 6.26. Let (A,τ) be any $C^{*,\tau}$-algebra. Then we define

$$KO_5^u(A,\tau) = \ker(\lambda_*)$$

where $\lambda_*\colon KO_5^u(\widetilde{A},\tau) \to KO_5^u(\mathbb{C},\mathrm{id})$.

The following result is immediate from our development so far.

Proposition 6.27. *If (A,τ) is any $C^{*,\tau}$-algebra, then*

$$KO_5^u(A,\tau) \cong KO_5(A,\tau).$$

Proposition 6.28. *Let (A,τ) be a $C^{*,\tau}$-algebra. Any element of $KO_5^u(A,\tau)$ can be represented as $[u]$ where $u \in M_n(\widetilde{A})$ satisfies $u^{\sharp\otimes\tau} = u^*$ and $\lambda_*(u) = \mathbb{1}_n$.*

Proof. There is a proof similar to that of Proposition 6.22, but instead we give the following slightly more constructive proof.

Let $u \in M_{2n}(\widetilde{A},\tau)$ be a unitary satisfying $u^{\sharp\otimes\tau} = u^*$. Then $\lambda(u)$ is a unitary in $M_{2n}(\mathbb{C})$ satisfying $u^\sharp = u^*$, which is to say that u is a unitary in $M_n(\mathbb{H})$. Since the unitary group of $M_n(\mathbb{H})$ is connected (this follows for example from Theorem 1 of [39]), there exists a path v_t from $\mathbb{1}_{2n}$ to $\lambda(u)$ in $M_{2n}(\mathbb{C})$ satisfying $v_t^\sharp = v_t^*$. Then uv_t^* is the desired path from u to a unitary $w = uv_1^*$ satisfying $\lambda(w) = \mathbb{1}_n$. $\square$

7. Summary and examples

The following theorem and Table 3 summarize the unitary description of KO-theory from the previous two sections. The statements about KU will be clarified later in this section.

Theorem 7.1. *Let (A,τ) be a $C^{*,\tau}$-algebra, not necessarily unital. Let n_i be the positive integer, $\mathscr{S}_i$ be the symmetry relation, and $I^{(i)} \in M_{n_i}(\mathbb{C})$ be the neutral element as specified in Table 3, for $i \in \{-1,0,\ldots,6\}$.*

Then there exist natural isomorphisms $KO_i^u(A,\tau) \cong KO_i(A^\tau)$ for all i, where $KO_i^u(A,\tau)$ is defined to be group of equivalence classes of unitaries u in $\cup_{n\in\mathbb{N}} M_{n_i\cdot n}(\widetilde{A})$ that satisfy $\mathscr{S}_i$ and satisfy $\lambda(u) = \mathrm{diag}(I^{(i)},\ldots,I^{(i)})$.

TABLE 3. Unitary Picture of K-theory — The Ten-Fold Way

	K-group	n_i	$\mathscr{S}_i$	$I^{(i)}$
complex	$KU_0^u(A,\tau)$	2	$u = u^*$	$\left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$
	$KU_1^u(A,\tau)$	1	–	$\mathbb{1}$
real	$KO_{-1}^u(A,\tau)$	1	$u^\tau = u$	$\mathbb{1}$
	$KO_0^u(A,\tau)$	2	$u = u^*$, $u^\tau = u^*$	$\left(\begin{smallmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{smallmatrix}\right)$
	$KO_1^u(A,\tau)$	1	$u^\tau = u^*$	$\mathbb{1}$
	$KO_2^u(A,\tau)$	2	$u = u^*$, $u^\tau = -u$	$\left(\begin{smallmatrix} 0 & i\cdot\mathbb{1} \\ -i\cdot\mathbb{1} & 0 \end{smallmatrix}\right)$
	$KO_3^u(A,\tau)$	2	$u^{\sharp\otimes\tau} = u$	$\mathbb{1}_2$
	$KO_4^u(A,\tau)$	4	$u = u^*$, $u^{\sharp\otimes\tau} = u^*$	$\mathrm{diag}(\mathbb{1}_2, -\mathbb{1}_2)$
	$KO_5^u(A,\tau)$	2	$u^{\sharp\otimes\tau} = u^*$	$\mathbb{1}_2$
	$KO_6^u(A,\tau)$	2	$u = u^*$, $u^{\sharp\otimes\tau} = -u$	$\left(\begin{smallmatrix} 0 & i\cdot\mathbb{1} \\ -i\cdot\mathbb{1} & 0 \end{smallmatrix}\right)$

In the unitary picture, the K-theory of a $C^{*,\tau}$-algebra (A,τ) consists of unitaries in matrix algebras over A satisfying the symmetry $\mathscr{S}_i$. See Theorem 7.1.

The equivalence relation is generated by path homotopy (within unitaries satisfying $\mathscr{S}_i$) and by the relation $u \sim \mathrm{diag}(u, I^{(i)})$. The binary operation is defined by $[u] + [v] = [\mathrm{diag}(u,v)]$.

Similar statements are made for $KU_i^u(A,\tau)$.

Remark 7.2. The inverse of an element $[u] \in KO_i^u(A,\tau)$ is given by $[u^*]$ if i is odd. In the even case, the inverse of $[u] \in KO_i^u(A,\tau)$ is $[-u]$ when $i = 0, 4$; or when $i = 2, 6$ and $u \in M_{n_i\cdot n}(\widetilde{A})$ with n even.

Remark 7.3. The restriction that $\lambda(u) = I_n^{(i)} = \mathrm{diag}(I^{(i)}, \ldots, I^{(i)})$ could be replaced by the weaker condition that $[\lambda(u)] = 0 \in KO_I^u(\mathbb{C}, \mathrm{id}) = KO_i^u(\mathbb{R})$. We have shown in each case that a representative of $KO_i^u(A,\tau)$ can always be found that satisfies the stronger λ condition. We have not proven, but we believe to be true in each of the ten cases, that the equivalence relation can be taken to be path homotopy not only within unitaries satisfying the appropriate symmetry, but also within unitaries satisfying the stronger condition $\lambda(u) = \mathrm{diag}(I^{(i)}, \ldots, I^{(i)})$.

Remark 7.4. If (A,τ) is already a unital $C^{*,\tau}$-algebra, it is not necessary to work in the unitization $\widetilde{A}$. We can realize $KO_i^u(A,\tau)$ using unitaries in $M_{n_i}(A)$ satisfying the correct symmetries (and without any λ restriction). The isomorphism between the picture of $KO_i^u(A)$ given by

unitaries in $M_{n_i}(A)$ and that given by unitaries in $M_{n_i}(\widetilde{A})$ is given by $[u] \mapsto [u - I_n^{(i)} \cdot (1_A)_n + I_n^{(i)} \cdot \mathbb{1}_n]$.

Remark 7.5. In the cases where the matrices are required to be of dimensions that are multiples of 2 or 4, it is possible to write down a picture of K-theory so that any unitary (satisfying the symmetry) in any dimension of square matrices represents a K-class. However, this would require a carefully specified and consistent choice of each of the embeddings $M_n(A) \hookrightarrow M_{n+1}(A)$. These choices would not be canonical and the designated "neutral element" would look different for different values of n.

Remark 7.6. Let u be a unitary in $M_n(\widetilde{A})$ and let $x \in O(n) \subset M_n(\mathbb{R})$. If u satisfies any symmetry $\mathscr{S}_i$ for $-1 \leq i \leq 2$, then so does xux^*. Furthermore, $[u] = [xux^*] \in KO_i^u(A)$ if $x \in SO(n)$ (since $SO(n)$ is connected). The equality $[u] = [xux^*]$ also holds if $x \in O(n)$ and $-1 \leq i \leq 1$. Indeed, if $\det x = -1$, then $\operatorname{diag}(x, \mathbb{1}, -\mathbb{1}) \in SO(n+2)$ and $\operatorname{diag}(x, -\mathbb{1}) \in SO(n+1)$ so we have

$$\begin{aligned}
[u] = [\operatorname{diag}(u, \mathbb{1}, -\mathbb{1})] &= [\operatorname{diag}(x, \mathbb{1}, -\mathbb{1}) \cdot \operatorname{diag}(u, \mathbb{1}, -\mathbb{1}) \cdot \operatorname{diag}(x^*, \mathbb{1}, -\mathbb{1})] \\
&= [\operatorname{diag}(xux^*, \mathbb{1}, -\mathbb{1})] = [xux^*] \qquad (\text{for } i = 0). \\
[u] = [\operatorname{diag}(u, \mathbb{1})] &= [\operatorname{diag}(x, -\mathbb{1}) \cdot \operatorname{diag}(u, \mathbb{1}) \cdot \operatorname{diag}(x^*, -\mathbb{1})] \\
&= [\operatorname{diag}(xux^*, \mathbb{1})] = [xux^*] \qquad (\text{for } i = \pm 1).
\end{aligned}$$

However, for $i = 2$, we may have $[u] \neq [xux^*] \in KO_2^u(A)$ if $x \in O(n)$.

Similar comments hold for $3 \leq i \leq 6$, with respect to conjugation by elements in the image of the injective homomorphism $O(n) \hookrightarrow O(2n)$ or $SO(n) \hookrightarrow SO(2n)$ given by

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}\mathbb{1}_2 & x_{12}\mathbb{1}_2 & \dots & x_{1n}\mathbb{1}_2 \\ x_{21}\mathbb{1}_2 & x_{22}\mathbb{1}_2 & \dots & x_{2n}\mathbb{1}_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}\mathbb{1}_2 & x_{n2}\mathbb{1}_2 & \dots & x_{nn}\mathbb{1}_2 \end{pmatrix}.$$

7.1. $KU_i^u(A)$ for real C^*-algebras. First note that Definitions 5.1 and 6.1 carry over to the complex setting and give pictures of the K-theory groups $K_0^u(A)$ and $K_1^u(A)$ for any complex C^*-algebra A. Specifically, $K_0^u(A)$ is given in terms of self-adjoint unitaries in $M_{2n}(A)$ and $K_1^u(A)$ is given in terms of unitaries in $M_n(A)$. The same proofs carry over to show that $K_0^u(A) \cong K_0(A)$ and $K_1^u(A) \cong K_1(A)$ for any complex C^*-algebra A.

Following the convention in [5], [6], and later papers; we define $KU_i(A)$ as a functor on the category of real C^*-algebra via complexification as follows:

Definition 7.7. For a real C^*-algebra A, define

$$KU_i(A) = K_i(A_{\mathbb{C}}) \cong K_i^u(A_{\mathbb{C}})$$

for $i = 0, 1$.

Alternatively, if (A,τ) is a $C^{*,\tau}$-algebra, then we have $KU_i(A,\tau) \cong K_i(A)$ since A is exactly the complexification of the real C^*-algebra A^τ corresponding to (A,τ). Thus we end up with unitary pictures $KU_0^u(A,\tau)$ and $KU_1^u(A,\tau)$ in terms of self-adjoint unitaries and unitaries in matrix algebras over A, exactly as described in Theorem 7.1 referring to the first two lines of Table 3. The fact that the symmetry relations for $KU_i^u(A,\tau)$ do not actually involve τ reflects the fact that these groups depend only on the underlying C^*-algebra A and not on the real structure imposed on A.

In fact, for each i there is a natural transformation

$$c_i^u \colon KO_i^u(A,\tau) \to \begin{cases} KU_0^u(A,\tau) & \text{for } i \text{ even} \\ KU_1^u(A,\tau) & \text{for } i \text{ odd.} \end{cases}$$

defined simply by $[u] \mapsto [u]$. These are clearly well-defined and natural, since in each case we are forgetting the extra symmetry requirement involving τ. To simplify notation, we define $KU_i^u(A,\tau)$ for all i by

$$KU_i^u(A,\tau) = \begin{cases} KU_0^u(A,\tau) & \text{for } i \text{ even} \\ KU_1^u(A,\tau) & \text{for } i \text{ odd} \end{cases}$$

so that we can simply write $c_i^u \colon KO_i^u(A,\tau) \to KU_i^u(A,\tau)$ in all cases.

We will verify in Proposition 7.10 below that, for any real C^*-algebra B, the homomorphism c_i^u coincides with the frequently used homomorphism $c_i \colon KO_i(B) \to KU_i(B)$ induced by the injective *-algebra homomorphism $c \colon B \to B_{\mathbb{C}}$, This natural transformation appears for example in Section 1.4 of [37] and Section 1.2 of [5] and forms one of the maps of the crucial long exact sequence relating real and complex K-theory.

First, note that the complexification functor $B \rightsquigarrow B_{\mathbb{C}}$, rephrased in terms of $C^{*,\tau}$-algebras, is equivalent to the functor $(A,\tau) \rightsquigarrow (A \oplus A, \sigma)$ where $(a_1,a_2)^\sigma = (a_2^\tau, a_1^\tau)$. The *-homomorphism $c \colon B \to B_{\mathbb{C}}$, rephrased in terms of $C^{*,\tau}$-algebras, is the injective $(*,\tau)$-homomorphism $\widetilde{c} \colon (A,\tau) \to (A \oplus A, \sigma)$ given by $\widetilde{c}(a) = (a,a)$. To verify these claims, one can verify that the restricted map $\widetilde{c} \colon A^\tau \to (A \oplus A)^\sigma$ is the same, up to isomorphism, as the canonical inclusion of A^τ into its complexification A. Indeed,

$$A^\tau = \{a \in A \mid a^* = a^\tau\}$$

and

$$\begin{aligned} (A \oplus A)^\sigma &= \{(a_1,a_2) \in A \oplus A \mid (a_1,a_2)^\sigma = (a_1,a_2)^*\} \\ &= \{(a_1, a_1^{*\tau}) \mid d \in A\} \cong A. \end{aligned}$$

Lemma 7.8. *Let (A,τ) be a $C^{*,\tau}$-algebra. Consider the $C^{*,\tau}$-algebra*

$$(A \oplus A, \sigma).$$

Then there is an isomorphism

$$\Gamma \colon KU_i^u(A,\tau) \to KO_i^u(A \oplus A, \sigma).$$

Proof. For $i = 0, 1$, define a homomorphism by $[x] \mapsto [(x, x^{*\tau})]$, and check that it is well-defined and is a bijection on the appropriate symmetry classes of unitaries. For $i = -1$ use $[x] \mapsto [(x, x^{\tau})]$, and for $i = 2$ use $[x] \mapsto [(x, -x^{\tau})]$. For $i = 3, 4, 5, 6$, use the same formulas replacing τ with $\sharp\otimes\tau$. □

Lemma 7.9. *The diagram below commutes.*

$$\begin{array}{ccc} KO_i^u(A,\tau) & \xrightarrow{c_i^u} & KU_i^u(A,\tau) \\ & \searrow^{\widetilde{c}_*} & \downarrow \Gamma \\ & & KO_i^u(A \oplus A, \sigma). \end{array}$$

Proof. Let $[x] \in KO_i^u(A,\tau)$. For each i, we use the formulas $\widetilde{c}_i([x]) = [(x,x)]$ and $c_i([x]) = [x]$, and the formula for $\theta([x])$ given in the proof of Lemma 7.8. Combining these formulas with the symmetries that x is assumed to satisfy, it follows that the diagram commutes for each i. □

Proposition 7.10. *Let (A,τ) be a $C^{*,\tau}$-algebra and let A^τ be the corresponding real C^*-algebra. Then $c_i^u \colon KO_i^u(A,\tau) \to KU_i^u(A,\tau)$ corresponds to the natural transformation $c_i \colon KO_i(A^\tau) \to KU_i(A^\tau)$ via the identifications $KO_i^u(A,\tau) \cong KO_i(A^\tau)$ and $KU_i^u(A,\tau) \cong KU_i(A^\tau)$.*

Proof. The claim is that for any $C^{*,\tau}$-algebra (A,τ) the diagram

$$\begin{array}{ccc} KO_i^u(A,\tau) & \xrightarrow{c_i^u} & KU_i^u(A,\tau) \\ \downarrow & & \downarrow \\ KO_i(A,\tau) & \xrightarrow{c_i} & KU_i(A,\tau) \\ \| & & \| \\ KO_i(A^\tau) & \xrightarrow{c_i} & KU_i(A^\tau) \end{array}$$

commutes, where the vertical arrows represent the appropriate natural isomorphisms from the previous sections.

By Lemma 7.9, it follows that c_i^u is equivalent to the natural homomorphism induced by the homomorphism $A \hookrightarrow A_{\mathbb{C}}$. Since c_i is induced by the same homomorphism, and since the isomorphisms $KO_i^u(A,\tau) \cong KO_i(A^\tau)$ and $KU_i^u(A,\tau) \cong KU_i(A^\tau)$ are natural with respect to real *-algebra homomorphisms, the result follows. □

7.2. Examples. The following theorem will identify for each i a specific unitary that generates the K-theory for each of the corresponding classifying

algebra A_i. Let

$$Q = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix} \in M_4(\mathbb{C})$$

be the unitary matrix (from Lemma 1.3 of [19]) which facilitates an equivalence between the involutions $\widetilde{\sharp\otimes\sharp}$ and Tr_4. That is, the equations $Qx^{\mathrm{Tr}}Q^* = (QxQ^*)^{\widetilde{\sharp\otimes\sharp}}$ and $Q^*x^{\widetilde{\sharp\otimes\sharp}}Q = (Q^*xQ)^{\mathrm{Tr}}$ hold for all $x \in M_4(A)$ for any C^*-algebra A.

Example 7.11. For each i, the class of a generator of $KO_i^u(A_i) \cong \mathbb{Z}$ is given by a unitary element x_i as described below.

- $[x_{-1}] \in KO_{-1}^u(A_{-1})$. $(C_0(S^1 \setminus \{1\}, \mathbb{C}), \mathrm{id})$ is the associated $C^{*,\tau}$-algebra. The unitary is
$$x_{-1} = z \in C(S^1, \mathbb{C})$$
which satsifies $(x_{-1})^{\mathrm{id}} = x_{-1}$.
- $[x_0] \in KO_0^u(A_0)$. The associated $C^{*,\tau}$-algebra is, $(q\mathbb{C}, \mathrm{Tr})$. The unitary is
$$x_0 = \begin{pmatrix} 1-2t & 0 & 0 & 2\sqrt{t-t^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2\sqrt{t-t^2} & 0 & 0 & 2t-1 \end{pmatrix} \in M_2(\widetilde{q\mathbb{C}})$$
which satisfies $x_0 = x_0^*$ and $x_0^{\mathrm{Tr}} = x_0^*$.
- $[x_1] \in KO_1^u(A_1)$. The associated $C^{*,\tau}$-algebra is $(C_0(S^1 \setminus \{1\}, \mathbb{C}), \zeta)$. The unitary is
$$x_1 = z \in C(S^1, \mathbb{C})$$
which satsifies $(x_1)^{\zeta} = x_1^*$.
- $[x_2] \in KO_2^u(A_2)$. The associated $C^{*,\tau}$-algebra is, $(q\mathbb{C}, \sharp)$. The unitary is
$$x_2 = W x_0 W^* \in M_2(\widetilde{q\mathbb{C}})$$
which satisfies $x_2 = x_2^*$ and $x_2^{\sharp} = -x_2$ (where W is as in Section 2).
- $[x_3] \in KO_3^u(A_3)$. The associated $C^{*,\tau}$-algebra is
$$(M_2(\mathbb{C}) \otimes C_0(S^1 \setminus \{1\}, \mathbb{C}), \sharp \otimes \mathrm{id}).$$
The unitary is
$$x_3 = Q\mathrm{diag}(z, 1, 1, 1)Q^* \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes C_0(S^1, \mathbb{C})$$
which satisfies $(x_3)^{\widetilde{\sharp\otimes\sharp\otimes\mathrm{id}}} = x_3$.
- $[x_4] \in KO_4^u(A_4)$. The associated $C^{*,\tau}$-algebra is, $(M_2(\mathbb{C}) \otimes q\mathbb{C}, \sharp \otimes \mathrm{Tr})$. The unitary is
$$x_4 = Q\mathrm{diag}(x_0, 1, 1, -1, -1)Q^* \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \widetilde{q\mathbb{C}}$$
which satisfies $(x_4)^{\widetilde{\sharp\otimes\sharp\otimes\mathrm{Tr}}} = x_4^*$.

- $[x_5] \in KO_5^u(A_5)$. The associated $C^{*,\tau}$-algebra is
$$(M_2(\mathbb{C}) \otimes C_0(S^1 \setminus \{1\}, \mathbb{C}), \sharp \otimes \zeta).$$
The unitary is
$$x_5 = Q\mathrm{diag}(z, 1, 1, 1)Q^* \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes C_0(S^1, \mathbb{C})$$
which satisfies $(x_5)^{\widetilde{\sharp\otimes\sharp\otimes\zeta}} = x_5^*$.
- $[x_6] \in KO_6^u(A_6)$. The associated $C^{*,\tau}$-algebra is, $(q\mathbb{C}, \widetilde{\mathrm{Tr}})$. The unitary is
$$x_6 = x_0 \in M_2(\widetilde{q\mathbb{C}})$$
which satisfies $x_6 = x_6^*$ and $(x_6)^{\sharp\otimes\widetilde{\mathrm{Tr}}} = -x_6$.

Proof. For each i, we know from Propositions 3.1 and 3.2 that $KO_i^u(A_i) \cong \mathbb{Z}$ and that $c_i \colon KO_i^u(A_i) \to KU_i^u(A_i)$ is an isomorphism.

The statement that $[x_0]$ generates $KO_0^u(A_0) \cong \mathbb{Z}$ is Proposition 5.7. It follows that $c_0^u[x_0] = [x_0]$ generates $KU_0^u(A_0) \cong \mathbb{Z}$. (This fact can also be derived from the fact that $[\frac{1}{2}(x_0 + \mathbb{1})] - [\mathbb{1}]$ is the generator of $KU_0(q\mathbb{C})$ as in Section 3 of [25].) Now for $i = 2, 4, 6$, we also have $c_i^u([x_i]) = [x_0] \in KU_i^u(A_i) \cong KU_0^u(q\mathbb{C})$. Since c_i^u is an isomorphism on $KO_i^u(A_i)$ it follows that $[x_i]$ must be generator of $KO_i^u(A_i)$.

For i odd, let z be the identity function on S^1. It is known that $[z]$ is a generator of $KU_1^u(C_0(S^1 \setminus \{1\}))$. Again in each case, we have $c_i^u([x_i]) = [z]$ so it follows that $[x_i]$ is a generator of $KO_i(A_i) \cong \mathbb{Z}$. □

Example 7.12. Recall that the groups of $KO_*(\mathbb{R}) \cong KO_*(\mathbb{C}, \mathrm{id})$, are given by
$$KO_*(\mathbb{R}) = \begin{cases} \mathbb{Z}, & i = 0, 4, \\ \mathbb{Z}_2, & i = 1, 2, \\ 0, & i = 3, 5, 6, 7. \end{cases}$$
Summarizing from discussions in the previous sections, we identify explicit generators for the non-zero groups, with our unitary picture.

(1) The generator of $KO_0^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}$ is $[1_2]$.
(2) The generator of $KO_1^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}_2$ is $[-1]$.
(3) The generator of $KO_2^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}_2$ is $[\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)]$.
(4) The generator of $KO_4^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}$ is $[1_4]$.

Notice also that $KU_0^u(\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}$ is generated by $[1_2]$. In terms of these generators, it is easy to verify that the homomorphisms $c_i \colon KO_i^u(\mathbb{C}, \mathrm{id}) \to KU_i^u(\mathbb{C}, \mathrm{id})$ agree with their known behavior. For example $c_0 \colon \mathbb{Z} \to \mathbb{Z}$ is an isomorphism. The class of $[-1]$ is non-trivial in $KO_2^u(\mathbb{C}, \mathrm{id})$ but is trivial in $KU_2^u(\mathbb{C}, \mathrm{id})$; this corresponds to the fact that $c_2 \colon \mathbb{Z}_2 \to \mathbb{Z}$ is trivial (as it must be of course). Also, we have that $c_4 \colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2, since $[1_4]$ is twice the generator of $KU_0^u(\mathbb{C}, \mathrm{id})$.

Example 7.13. Let ζ be the reflection on $C(S^1,\mathbb{C})$ or $C(S^2,\mathbb{C})$ corresponding to negation of the y coordinate. $(x,y,z)\mapsto(x,-y,z)$. Let 1 denote the point $(1,0,\dots,0)$ in S^{n-1} (for $n=2,3,4$).

(1) The generator of $KO_1^u\left(C_0(S^1\setminus\{1\}),\zeta\right)\cong\mathbb{Z}$ is the class of the unitary u, where
$$u(x+iy)=x+iy.$$
(2) The generator of $KO_{-1}^u\left(C_0(S^1\setminus\{1\}),\mathrm{id}\right)\cong\mathbb{Z}$ is the class of the unitary u, where
$$u(x,y)=x+iy.$$
(3) The generator of $KO_0^u\left(C_0(S^2\setminus\{1\},\zeta\right)\cong\mathbb{Z}$ is the class of the unitary u, where
$$u(x,y,z)=\begin{pmatrix} z & x-iy\\ x+iy & -z\end{pmatrix}.$$
(4) The generator of $KO_{-2}^u\left(C_0(S^2\setminus\{1\}),\mathrm{id}\right)\cong\mathbb{Z}$ is the class of the unitary u, where
$$u(x,y,z)=\begin{pmatrix} z & x-iy\\ x+iy & -z\end{pmatrix}.$$
(5) The generator of $KO_{-3}^{\mathrm{u}}\left(C_0(S^3\setminus\{1\}),\mathrm{id}\right)\cong\mathbb{Z}$ is the class of the unitary u, where
$$u(x,y,z,w)=\begin{pmatrix} iz-w & ix+y\\ ix-y & -iz-w\end{pmatrix}.$$

Proof. Results (1) and (2) are restatements from Example 7.11.

For (3), first check that $u=u^{\zeta}=u^*$. The equation $\lambda(u)=I^{(0)}$ does not hold exactly, but it does hold on the level of $KO_0^u(\mathbb{C})$. Using the isomorphism θ from Theorem 5.6, we have $\theta([u])=[\frac{1}{2}(u+1_2)]-[1]=[p_0]-[1]$ where $p_0=\frac{1}{2}\begin{pmatrix}1+z & x-iy\\ x+iy & 1-z\end{pmatrix}$. But we know from the discussion preceding Proposition 4.8 that $KO_0(C_0(S^2\setminus\{1\}),\zeta)\cong\mathbb{Z}$ is generated by $[p_0]-[1]$. This proves (3).

Since $c_0^u\colon KO_0(C_0(S^2\setminus\{1\}),\zeta)\to KU_0(C_0(S^2\setminus\{1\}),\zeta)$ is an isomorphism, it follows that $[u]$ also generates
$$KU_0(C_0(S^2\setminus\{1\}),\zeta)=KU_0(C_0(S^2\setminus\{1\}),\mathrm{id}).$$

For (4), check that $u^{\sharp\otimes\mathrm{id}}=-u$. Now we also know that
$$c_{-2}\colon KO_{-2}^u\left(C_0(S^2\setminus\{1\};\mathrm{id}\right)\to KU_{-2}^u\left(C_0(S^2\setminus\{1\};\mathrm{id}\right)$$
is an isomorphism and $[u]$ is the same generator of
$$KU_{-2}^u\left(C_0(S^2\setminus\{1\};\mathrm{id}\right)=KU_0(C_0(S^2\setminus\{1\});\mathrm{id})$$
identified in the previous paragraph. So $[u]$ is also a generator of
$$KO_{-2}(C_0(S^2\setminus\{1\}),\mathrm{id}).$$

For (5), check that u is a unitary and that $u^{\sharp\otimes\mathrm{id}} = u^*$. The transformation

$$c_{-3} : KO^u_{-3}\left(C_0(S^3 \setminus \{1\}), \mathrm{id}\right) \to KU^u_1\left(C_0(S^3 \setminus \{1\}), \mathrm{id}\right)$$

is known to be an isomorphism and sends u to the known generator of $\pi_3(SO(2))$ so also to a generator of $KU_1\left(C_0(S^3 \setminus \{1\}), \mathrm{id}\right)$. □

We will return to specific computations of $KO(-)$ groups and elements of $KO(-)$ represented by unitaries in Section 9.

8. The boundary map

There exist known natural boundary maps $\partial_i \colon KO_i(B,\tau) \to KO_{i-1}(I,\tau)$ for a short exact sequence

$$0 \to (I,\tau) \to (A,\tau) \to (B,\tau) \to 0. \tag{11}$$

In this section, we will derive concrete formulas for these boundary maps described in term of the unitary pictures of K-theory. The approach we will take is to first write down specific formulas for maps

$$\clubsuit_i : KO^u_i(B,\tau) \to KO^u_{i-1}(I,\tau),$$

then prove that those formulas give well defined and natural homomorphisms, and finally prove that the homomorphisms coincide with ∂_i via the natural isomorphisms $KO^u_i(-) \cong KO_i(-)$. We start with the odd cases $i = -1, 1, 3, 5$. For each of these the basic formula will be the same, but we will have to conjugate by a different unitary $Y^{(i)}$ in each case in order to obtain a unitary v that is in the correct symmetry class and satisfies $\lambda(u) = I_n^{(i-1)}$. Easier formulas would be possible if we relaxed the λ condition (see Remark 7.3).

We must introduce notation for several classes of unitaries that will be used for conjugation in these definitions. First let us recall from Section 5 that we have the matrix

$$W_{2n} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} i \cdot 1_n & 1_n \\ 1_n & i \cdot 1_n \end{pmatrix} \in M_{2n}(\mathbb{C}),$$

which satisfies

$$W_{2n} \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} W_{2n}^* = \begin{pmatrix} 0 & i \cdot 1_n \\ -i \cdot 1_n & 0 \end{pmatrix}.$$

Generalizing the matrix Q used in Section 7, we define

$$Q_{4n} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1_{2n} & -I_n^{(2)} \\ I_n^{(2)} & 1_{2n} \end{pmatrix} \in M_{4n}(\mathbb{C}).$$

The key property, as in Lemma 1.3 of [19], is that for all $x \in M_{4n}(A)$ we have $Q_{4n} x^\tau Q_{4n}^* = (Q_{4n} x Q_{4n}^*)^{\widetilde{\sharp}\otimes\sharp\otimes\tau}$ where $\widetilde{\sharp}$ and $\sharp$ are as defined in Section 2.

Let $V_{2n} \in M_{2n}(\mathbb{R})$ be the unique permutation matrix such that (for diagonal matrices) we have

$$V_{2n}\mathrm{diag}(\lambda_1, \ldots, \lambda_{2n}) V_{2n}^* = \mathrm{diag}(\lambda_1, \lambda_{n+1}, \lambda_2, \lambda_{n+2}, \ldots, \lambda_n, \lambda_{2n}).$$

Then for all matrices $x \in M_{2n}(A)$ we have

$$V_{2n} x^{\widetilde{\sharp}\otimes\tau} V_{2n}^* = (V_{2n} x V_{2n}^*)^{\sharp\otimes\tau}$$

where $\sharp$ and $\widetilde{\sharp}$ are defined as in Section 2. Similarly, let $X_{4n} \in M_{4n}(\mathbb{R})$ be the unique permutation matrix such that (for diagonal matrices) we have

$$\begin{aligned} &X_{4n}\mathrm{diag}(\lambda_1, \ldots, \lambda_{4n}) X_{4n}^* \\ &\quad = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_{2n+1}, \lambda_{2n+2}, \lambda_3, \lambda_4, \lambda_{2n+3}, \lambda_{2n+4}, \ldots, , \lambda_{4n}). \end{aligned}$$

Definition 8.1. Suppose we have an exact sequence as in Sequence (11). Let π denote both the quotient map $\pi : A \to B$ and its extension to $M_n(\widetilde{A}) \to M_n(\widetilde{B})$ for every n. Furthermore, we assume $I = \ker(\pi)$ and we identify the unit in $\widetilde{I}$ with that of $\widetilde{A}$.

(1) Suppose $[u] \in KO_1^u(B, \tau)$ where $u \in M_n(\widetilde{B})$ is a unitary with $u^\tau = u^*$ and $\lambda(u) = I_n^{(1)}$. Then define

$$\clubsuit_1([u]) = \left[Y_{2n}^{(1)} \begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix} Y_{2n}^{(1)*} \right] \in KO_0^u(I, \tau)$$

where a in $M_n(\widetilde{A})$ is any lift of u with $\|a\| \leq 1$ and $a^\tau = a^*$; and $Y_{2n}^{(1)} = V_{2n}$.

(2) Suppose $[u] \in KO_{-1}^u(B, \tau)$ where $u \in M_n(\widetilde{B})$ is a unitary with $u^\tau = u$ and $\lambda_n(u) = I_n^{(-1)}$. Then define

$$\clubsuit_{-1}([u]) = \left[Y_{2n}^{(-1)} \begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix} Y_{2n}^{(-1)*} \right] \in KO_6^u(I, \tau)$$

where a in $M_n(\widetilde{A})$ is any lift of u with $\|a\| \leq 1$ and $a^\tau = a$; and $Y_{2n}^{(-1)} = V_{2n} W_{2n}$.

(3) Suppose that $[u] \in KO_5^u(B, \tau)$ where $u \in M_{2n}(\widetilde{B})$, is a unitary with $u^{\sharp\otimes\tau} = u^*$ and $\lambda_{2n}(u) = I_n^{(5)}$. Then define

$$\clubsuit_5([u]) = \left[Y_{4n}^{(5)} \begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix} Y_{4n}^{(5)*} \right] \in KO_4^u(I, \tau)$$

where a in $M_{2n}(\widetilde{A})$ is any lift of u with $\|a\| \leq 1$ and $a^{\sharp\otimes\tau} = a^*$; and $Y_{4n}^{(5)} = X_{4n}$.

(4) Suppose that $[u] \in KO_3^u(B, \tau)$ where $u \in M_{2n}(\widetilde{B})$ is a unitary with $u^{\sharp\otimes\tau} = u$ and $\lambda_{2n}(u) = I_n^{(3)}$. Then define

$$\clubsuit_3([u]) = \left[Y_{4n}^{(3)} \begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix} Y_{4n}^{(3)*} \right]$$

where a in $M_{2n}(\widetilde{A})$ is any lift of u with $\|a\| \leq 1$ and $a^{\sharp\otimes\tau} = a$; and $Y_{4n}^{(3)} = V_{4n} Q_{4n} W_{4n}$.

Lemma 8.2. *The maps $\clubsuit_i$ are well-defined group homomorphisms, for i odd.*

Proof. We need to show that the unitaries constructed all satisfy the correct relations, that the choice of lift is not important, that some lift is always available, that homotopy is respected, that embedding into larger matrices via $\iota_n^{(i)}$ does not effect the outcome, and that the addition is respected at the level of K-theory.

For convenience, we define

$$B(a) = \begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix}.$$

Making repeated use of the equality $2a\sqrt{\mathbb{1} - a^*a} = 2\sqrt{\mathbb{1} - aa^*}a$, we find that $B(a)^* = B(a)$ and that

$$B(a)^2 = \begin{pmatrix} (2aa^* - \mathbb{1}_n)^2 + 4a\,(\mathbb{1}_n - a^*a)\,a^* & 4aa^*a\sqrt{\mathbb{1}_n - a^*a} - 4a\sqrt{\mathbb{1}_n - a^*a}a^*a \\ 4a^*\sqrt{\mathbb{1}_n - aa^*}aa^* - 4a^*aa^*\sqrt{\mathbb{1}_n - aa^*} & (\mathbb{1}_n - 2a^*a)^2 + 4a^*\,(\mathbb{1}_n - aa^*)\,a \end{pmatrix} = \mathbb{1}_{2n}$$

so that $B(a)$ is always self-adjoint unitary. In each case, we will check that $B(a)$ satisfies the appropriate symmetry based on the symmetry that a satisfies.

(1) First we consider the map for $i = 1$,

$$\clubsuit_1 : KO_1^u(B, \tau) \to KO_0^u(I, \tau).$$

We start with u in $M_n(\widetilde{B})$, a unitary with $u^\tau = u^*$ and $\lambda_n(u) = \mathbb{1}_n$. There exists a lift x in $M_n(\widetilde{A})$ such that $\pi(x) = u$ and then we necessarily have $\lambda_n(x) = \mathbb{1}_n$. Set $y = \frac{1}{2}(x + x^{\tau *})$ to obtain the relation $y^\tau = y^*$. We utilize the usual function $f(\lambda) = \min(\sqrt{1/\lambda}, 1)$ and set $a = yf(y^*y)$. Then a satisfies $\|a\| \leq 1$, $a^\tau = a^*$, and $\pi(a) = u$. The condition $\lambda(a) = \mathbb{1}_n$ follows automatically since a is a lift of u. This shows that an appropriate lift exists. Now, suppose that a is any suitable lift of u and let $B'(a) = Y_{2n}^{(1)} B(a) Y_{2n}^{(1)*}$. Then

$$\lambda(B'(a)) = Y_{2n}^{(1)} \lambda(B(a)) Y_{2n}^{(1)*} = Y_{2n}^{(1)} \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} Y_{2n}^{(1)*} = I_n^{(0)}.$$

This shows that $B'(a) \in M_{2n}(\widetilde{I})$.

Using $(a^*a)^\tau = a^*a$ and $(aa^*)^\tau = aa^*$, we have

$$\begin{aligned} B(a)^\tau &= \begin{pmatrix} (2aa^* - \mathbb{1})^\tau & \left(2a^*\sqrt{\mathbb{1} - aa^*}\right)^\tau \\ \left(2a\sqrt{\mathbb{1} - a^*a}\right)^\tau & (\mathbb{1} - 2a^*a)^\tau \end{pmatrix} \\ &= \begin{pmatrix} 2aa^* - \mathbb{1} & 2a\sqrt{\mathbb{1} - a^*a} \\ 2a^*\sqrt{\mathbb{1} - aa^*} & \mathbb{1} - 2a^*a \end{pmatrix} = B(a). \end{aligned}$$

Thus $B(a)^\tau = B(a)$. Since $Y_{2n}^{(1)}$ is a real orthogonal matrix, $B'(a) = Y_{2n}^{(1)} B(a) Y_{2n}^{(1)*}$ satisfies the same relation, which means that $B'(a)$ is the right sort of unitary to define an element of $KO_0^u(I, \tau)$. (In fact, in this

case we have $[B(a)] = [B'(a)] \in KO_0^u(I,\tau)$. We used $B'(a)$ in our definition because it satisfies $\lambda(B'(a)) = I_n^{(0)}$.)

If we have two lifts a_0 and a_1 with the required relations, then the straight line

$$a_t = (1-t)a_0 + ta_1$$

satisfies the relations at every point, and so $B(a_t)$ provides the needed homotopy showing that $\big[B'(a_1)\big] = \big[B'(a_2)\big]$.

Suppose now that we have a homotopy u_t of unitaries in $M_n(\widetilde{B})$ satisfying $u_t^\tau = u^*$. By working with the surjection

$$M_{2n}((C[0,1],\widetilde{A})) \to M_{2n}((C[0,1],\widetilde{B}))$$

induced by π, the techniques of the first paragraph of this proof show that there is a lift a_t of the homotopy u_t, which then is a homotopy between a lift of u_0 and one for u_1.

To complete the proof that $\clubsuit_1$ is well defined, we need to show that the results of this construction for u and for

$$v = \iota_n^{(1)}(u) = \begin{pmatrix} u & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

are the same element of $KO_0^u(I)$. In fact, this result follows from a special case (taking $v = \mathbb{1}$) of the argument below that $\clubsuit_1$ is additive.

Suppose that $u \in M_m(\widetilde{B})$ and $v \in M_n(\widetilde{B})$ are unitaries representing elements in $KO_1^u(B,\tau)$; and let a and b be self-adjoint unitary lifts of u and v, respectively, such that $a^\tau = a^*$ and $b^\tau = b^*$. Then $\operatorname{diag}(a,b)$ is a lift of $\operatorname{diag}(u,v)$. Consider the two matrices

$$B\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 2aa^* - \mathbb{1}_m & 0 & 2a\sqrt{\mathbb{1}_m - a^*a} & 0 \\ 0 & 2bb^* - \mathbb{1}_n & 0 & 2b\sqrt{\mathbb{1}_n - b^*b} \\ 2a^*\sqrt{\mathbb{1}_m - aa^*} & 0 & \mathbb{1}_m - 2a^*a & 0 \\ 0 & 2b^*\sqrt{\mathbb{1}_n - bb^*} & 0 & \mathbb{1}_n - 2b^*b \end{pmatrix}$$

and

$$\begin{pmatrix} B(a) & 0 \\ 0 & B(b) \end{pmatrix} = \begin{pmatrix} 2aa^* - \mathbb{1}_m & 2a\sqrt{\mathbb{1}_m - a^*a} & 0 & 0 \\ 2a^*\sqrt{\mathbb{1}_m - aa^*} & \mathbb{1}_m - 2a^*a & 0 & 0 \\ 0 & 0 & 2bb^* - \mathbb{1}_n & 2b\sqrt{\mathbb{1}_n - b^*b} \\ 0 & 0 & 2b^*\sqrt{\mathbb{1}_n - bb^*} & \mathbb{1}_n - 2b^*b \end{pmatrix}$$

in $M_{2m+2n}(\widetilde{I})$ and observe that

$$V_{2m+2n} B\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) V_{2m+2n}^* = \begin{pmatrix} V_{2m}B(a)V_{2m}^* & 0 \\ 0 & V_{2n}B(b)V_{2n}^* \end{pmatrix}$$

showing that $\clubsuit_1([u]+[v]) = \clubsuit_1([u]) + \clubsuit_1([v])$.

(2) Now we consider

$$\clubsuit_{-1} : KO_{-1}^u(B,\tau) \to KO_6^u(I,\tau).$$

This time, we start with a unitary $u \in M_n(\widetilde{B})$ that satisfies $u^\tau = u$ and $\lambda(u) = \mathbb{1}_n$. Using a similar construction as in the previous case, we find

an element $a \in M_n(\widetilde{A})$ that is a lift of u, has norm at most 1, and satisfies $a^\tau = a$. For any such a, define $B'(a) = Y_{2n}^{(-1)} B(a) Y_{2n}^{(-1)*}$. Then we have

$$\lambda(B'(a)) = V_{2n} W_{2n} \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} W_{2n}^* V_{2n}^* = V_{2n} \begin{pmatrix} 0 & i\mathbb{1}_n \\ -i\mathbb{1}_n & 0 \end{pmatrix} V_{2n}^* = I_n^{(6)}.$$

We have $B(a)^2 = \mathbb{1}_{2n}$ as before, so $B'(a)^2 = \mathbb{1}_{2n}$. We show that

$$(W_{2n} B(a) W_{2n}^*)^{\widetilde{\sharp \otimes \tau}} = -W_{2n} B(a) W_{2n}^*$$

from which it follows that $(B')^{\sharp \otimes \tau} = -B'$. Indeed,

$$\begin{aligned} (W_{2n} B(a) W_{2n}^*)^{\widetilde{\sharp \otimes \tau}} &= W_{2n} B(a)^{\widetilde{\sharp \otimes \tau}} W_{2n}^* && (\text{since } W_{2n}^{\widetilde{\sharp}} = -W_{2n}^*) \\ &= W_{2n}(-B(a)) W_{2n}^* && (\text{using } a^\tau = a) \\ &= -W_{2n} B(a) W_{2n}^*. \end{aligned}$$

Then the formua $V_{2n} x^{\widetilde{\sharp \otimes \tau}} V_{2n}^* = (V_{2n} x V_{2n}^*)^{\sharp \otimes \tau}$ implies that $B'(a)^{\sharp \otimes \tau} = -B'(a)$. So $[B'(a)]$ is an element of $KO_6(I, \tau)$ as desired.

The proof that $\clubsuit_{-1}$ is independent of the choice of lift and of the homotopy class of $[u]$ is similar to that in the previous case. To show that $\clubsuit_{-1}([u]) = \clubsuit_{-1}([\mathrm{diag}(u, \mathbb{1})])$, we again appeal to a special case of the additivity argument in the next paragraph.

Let $a \in M_m(\widetilde{A})$ and $b \in M_n(\widetilde{B})$ be lifts of unitaries u and v, satisfying $a^\tau = a$ and $b^\tau = b$. Then check that

$$\begin{aligned} B' \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &= V_{2m+2n} W_{2m+2n} B \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} W_{2m+2n}^* V_{2m+2n}^* \\ &= \begin{pmatrix} V_{2m} W_{2m} B(a) W_{2m}^* V_{2m}^* & 0 \\ 0 & V_{2n} W_{2n} B(b) W_{2n}^* V_{2n}^* \end{pmatrix} \\ &= \begin{pmatrix} B'(a) & 0 \\ 0 & B'(b) \end{pmatrix}. \end{aligned}$$

(3) For

$$\clubsuit_5 \colon KO_5^u(B, \tau) \to KO_4^u(I, \tau),$$

we will focus on the two crucial aspects: that the proposed element satisfies the symmetries required to be an element of $KO_4^u(I, \tau)$ and that it is respects addition. The other aspects are similar to the previous cases.

Start with a unitary $u \in M_{2n}(\widetilde{B})$ satisfying $u^{\sharp \otimes \tau} = u^*$ and $\lambda(u) = I_n^{(5)} = \mathbb{1}_{2n}$ and suppose that $a \in M_{2n}(\widetilde{A})$ satisfies $a^{\sharp \otimes \tau} = a^*$, $\|a\| \leq 1$, and $\lambda(a) = \mathbb{1}_{2n}$. Let $B'(a) = Y_{4n}^{(5)} B(a) Y_{4n}^{(5)*}$. Then we have

$$\lambda(B'(a)) = Y_{4n}^{(5)} \begin{pmatrix} \mathbb{1}_{2n} & 0 \\ 0 & -\mathbb{1}_{2n} \end{pmatrix} Y_{4n}^{(5)*} = \mathrm{diag}(\mathbb{1}_2, -\mathbb{1}_2, \ldots \mathbb{1}_2, -\mathbb{1}_2) = I_n^{(4)}.$$

Using the fact that $a^{\sharp \otimes \tau} = a^*$, we can show that $B(a)^{\sharp \otimes \tau} = B(a)^*$ just as in the proof of (1) (with the involution $\sharp \otimes \tau$ in place of τ). Conjugation by

$Y_{4n}^{(5)}$ rearranges the 2×2 blocks of the matrix and the action of $\sharp$ is contained within each such block, so we have

$$Y_{4n}^{(5)} x^{\sharp\otimes\tau} Y_{4n}^{(5)*} = \left(Y_{4n}^{(5)} x Y_{4n}^{(5)*}\right)^{\sharp\otimes\tau}.$$

Thus $B'(a)^{\sharp\otimes\tau} = B'(a)^*$.

Let $a \in M_{2m}(\widetilde{A})$ and $b \in M_{2n}(\widetilde{B})$ be lifts of unitaries u and v, satisfying $a^{\sharp\otimes\tau} = a^*$ and $b^{\sharp\otimes\tau} = b^*$. Then

$$\begin{aligned} B'\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &= X_{4m+4n} B \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} X_{4m+4n}^* \\ &= \begin{pmatrix} X_{4m} B(a) X_{4m}^* & 0 \\ 0 & X_{4n} B(b) X_{4n}^* \end{pmatrix} \\ &= \begin{pmatrix} B'(a) & 0 \\ 0 & B'(b) \end{pmatrix}. \end{aligned}$$

(4) Show that $\clubsuit_3 \colon KO_3^u(B,\tau) \to KO_2^u(I,\tau)$ is well-defined. We will prove that the proposed element satisfies the symmetries required to be an element of $KO_2^u(I,\tau)$ and that it is additive.

Suppose that u is a unitary in $M_{2n}(\widetilde{B})$ satisfying $u^{\sharp\otimes\tau} = u$ and $\lambda(u) = I_n^{(3)} = \mathbb{1}_{2n}$, and that $a \in M_{2n}(\widetilde{A})$ is lift of norm not more than 1 satisfying the same symmetry. Let $B(a)$ be as before and let $B'(a) = Y_{4n}^{(3)} B(a) Y_{4n}^{(3)*}$ where $Y_{4n}^{(3)} = V_{4n} Q_{4n} W_{4n}$. Then we have

$$\begin{aligned} \lambda(B'(a)) &= Y_{4n}^{(3)} \lambda(B(a)) Y_{4n}^{(3)*} \\ &= V_{4n} Q_{4n} W_{4n} \operatorname{diag}(\mathbb{1}_{2n}, -\mathbb{1}_{2n}) W_{4n}^* Q_{4n}^* V_{4n}^* \\ &= \tfrac{1}{2} V_{4n} \begin{pmatrix} \mathbb{1}_{2n} & -I_n^{(2)} \\ I_n^{(2)} & \mathbb{1}_{2n} \end{pmatrix} \begin{pmatrix} 0 & i\cdot\mathbb{1}_{2n} \\ -i\cdot\mathbb{1}_{2n} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2n} & I_n^{(2)} \\ -I_n^{(2)} & \mathbb{1}_{2n} \end{pmatrix} V_{4n}^* \\ &= V_{4n} \begin{pmatrix} 0 & i\cdot\mathbb{1}_{2n} \\ -i\cdot\mathbb{1}_{2n} & 0 \end{pmatrix} V_{4n}^* \\ &= \operatorname{diag}(I^{(2)}, \dots, I^{(2)}) = I_{2n}^{(2)}. \end{aligned}$$

Now $a^{\sharp\otimes\tau} = a^*$ implies that $B(a)^{\widetilde{\sharp}\otimes\sharp\otimes\tau} = -B(a)$ similar to case (2). Also $W_{4n}^{\widetilde{\sharp}\otimes\sharp} = -W_{4n}^*$ so

$$(W_{4n} B(a) W_{4n}^*)^{\widetilde{\sharp}\otimes\sharp\otimes\tau} = W_{4n} B(a)^{\widetilde{\sharp}\otimes\sharp\otimes\tau} W_{4n}^* = -W_{4n} B(a) W_{4n}^*.$$

Now we have $Q_{4n} x^\tau Q_{4n}^* = (Q_{4n} x Q_{4n}^*)^{\widetilde{\sharp}\otimes\sharp\otimes\tau}$ for all x, from it which it follows that $Q_{4n}^* x^{\widetilde{\sharp}\otimes\sharp\otimes\tau} Q_{4n} = (Q_{4n} x Q_{4n}^*)^\tau$. Thus

$$(Q_{4n} W_{4n} B(a) W_{4n}^* Q_{4n}^*)^\tau = -Q_{4n} W_{4n} B(a) W_{4n}^* Q_{4n}^*.$$

Since V_{4n} is special orthogonal, the same formula holds for

$$B'(a) = V_{4n} Q_{4n} W_{4n} B(a) W_{4n}^* Q_{4n}^* V_{4n}^*.$$

showing that $B'(a)$ represents an element in $KO_2^u(I,\tau)$.

For additivity, let $a \in M_{2m}(\widetilde{A})$ and $b \in M_{2n}(\widetilde{B})$ be lifts of unitaries u and v, satisfying $a^{\sharp\otimes\tau} = a$ and $b^{\sharp\otimes\tau} = b$. Then

$$B\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 2aa^* - \mathbb{1}_{2m} & 0 & 2a\sqrt{\mathbb{1}_{2m} - a^*a} & 0 \\ 0 & 2bb^* - \mathbb{1}_{2n} & 0 & 2b\sqrt{\mathbb{1}_{2n} - b^*b} \\ 2a^*\sqrt{\mathbb{1}_{2m} - aa^*} & 0 & \mathbb{1}_{2m} - 2a^*a & 0 \\ 0 & 2b^*\sqrt{\mathbb{1}_{2n} - bb^*} & 0 & \mathbb{1}_{2n} - 2b^*b \end{pmatrix}$$

as before. Taking advantage of the block matrix structure of Q_{4n} and W_{4n} and $B\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we have

$$Q_{4m+4n}W_{4m+4n}B\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}W_{4m+4n}^*Q_{4m+4n}^* = \begin{pmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & B_{11} & 0 & B_{12} \\ A_{21} & 0 & A_{22} & 0 \\ 0 & B_{21} & 0 & B_{22} \end{pmatrix}$$

where

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = Q_{4m}W_{4m}B(a)W_{4m}^*Q_{4m}^*,$$
$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = Q_{4n}W_{4n}B(b)W_{4n}^*Q_{4n}^*.$$

Hence,

$$V_{4m+4n}Q_{4m+4n}W_{4m+4n}B\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}W_{4m+4n}^*Q_{4m+4n}^*V_{4m+4n}^* = \begin{pmatrix} Q_{4m}W_{4m}B(a)W_{4m}^*Q_{4m}^* & 0 \\ 0 & Q_{4n}W_{4n}B(b)W_{4n}^*Q_{4n}^* \end{pmatrix},$$

which shows that $B'\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} B'(a) & 0 \\ 0 & B'(b) \end{pmatrix}$. □

Definition 8.3. Suppose we have an exact sequence in Sequence (11). Let π denote both the quotient map $\pi : A \to B$ and its extension to $M_n(\widetilde{A}) \to M_n(\widetilde{B})$ for every n. Furthermore, we assume $I = \ker(\pi)$ and we identify the unit in $\widetilde{I}$ with that of $\widetilde{A}$.

(1) Suppose $[u] \in KO_2^u(B,\tau)$ where $u \in M_{2n}(\widetilde{B})$ is a unitary with $u^\tau = -u$, $u^* = u$, and $\lambda(u) = I_n^{(2)}$. Then define
$$\clubsuit_2([u]) = [-\exp(\pi i a)] \in KO_1^u(I,\tau)$$
where a in $M_{2n}(\widetilde{A})$ is any lift of u with $-1 \le a \le 1$ and $a^\tau = -a$.

(2) Suppose $[u] \in KO_0^u(B,\tau)$ where $u \in M_{2n}(\widetilde{B})$ is a unitary with $u^\tau = u^* = u$, and $\lambda(u) = I_n^{(0)}$. Then define
$$\clubsuit_0([u]) = [-\exp(\pi i a)] \in KO_{-1}^u(I,\tau)$$
where a in $M_{2n}(\widetilde{A})$ is any lift of u with $-1 \le a \le 1$ and $a^\tau = a$.

(3) Suppose $[u] \in KO_6^u(B,\tau)$ where $u \in M_{2n}(\widetilde{B})$ is a unitary with $u^{\sharp\otimes\tau} = -u$, $u^* = u$, and $\lambda(u) = I_n^{(6)}$. Then define

$$\clubsuit_6([u]) = [-\exp(\pi i a)] \in KO_5^u(I,\tau)$$

where a in $M_{2n}(\widetilde{A})$ is any lift of u with $-1 \le a \le 1$ and $a^{\sharp\otimes\tau} = -a$.

(4) Suppose $[u] \in KO_4^u(B,\tau)$ where $u \in M_{4n}(\widetilde{B})$ is a unitary with $u^{\sharp\otimes\tau} = u^* = u$ and $\lambda(u) = I_n^{(4)}$. Then define

$$\clubsuit_4([u]) = [-\exp(\pi i a)] \in KO_3^u(I,\tau)$$

where a in $M_{4n}(\widetilde{A})$ is any lift of u with $-1 \le a \le 1$ and $a^{\sharp\otimes\tau} = a$.

Lemma 8.4. *The maps $\clubsuit_i$ are well-defined group homomorphisms, for i even.*

Proof. This involves proving all the same assertions as in the proof of Lemma 8.2 above.

(1) First we consider $\clubsuit_2 \colon KO_2^u(B,\tau) \to KO_1^u(I,\tau)$. Suppose u is a unitary in $M_{2n}(\widetilde{B})$ with $u^\tau = -u$ and $u^* = u$ and $\lambda(u) = I_n^{(2)}$. We can lift by standard methods to an element $a \in M_n(\widetilde{A})$ with $-1 \le a \le 1$ and are guaranteed $\lambda(a) = I_n^{(2)}$ automatically. Replacing with the element $\frac{1}{2}(a - a^\tau)$, we can assume the relation $a^\tau = -a$. This shows that an appropriate lift exists.

Now consider any lift a of u satisfying $-1 \le a \le 1$, $a^\tau = -a$, and $\lambda(a) = I_n^{(2)}$. Let $E(a) = -\exp(\pi i a)$, which is of course a unitary when a is self-adjoint; and satisfies $E(a^\tau) = E(a)^\tau$ and $E(-a) = E(a)^*$. Since $I^{(2)}$ has eigenvalues ± 1 we find $\lambda(E(a)) = \mathbb{1}_{2n}$ and we have also the familiar fact that $E(a)$ is a unitary when a is self-adjoint. As to the real structure, we check

$$E(a)^\tau = E(a^\tau) = E(-a) = E(a)^*,$$

so we have indeed obtained a representative of an element in $KO_1^u(I,\tau)$.

If we have two lifts a_0 and a_1 with the required relations, then the straight line

$$a_t = (1-t)a_0 + t a_1$$

satisfies the relations at every point, and so $E(a_t)$ provides the needed homotopy showing $[E(a_1)] = [E(a_2)]$. To deal with a homotopy from u_t in $M_n(\widetilde{B})$, we again use the surjection

$$M_{2n}((C[0,1],\widetilde{A})) \to M_{2n}((C[0,1],\widetilde{B}))$$

induced by π to get a_t, a homotopy of lifts covering the u_t. Then $E(a_t)$ is the needed homotopy.

Next we compare the results of this construction for u and that for

$$v = \operatorname{diag}(u, I^{(2)}) = \begin{pmatrix} u & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}.$$

Take a to be a lift of u with $-1 \leq a \leq 1$ and $a^\tau = -a$, and take $b = \operatorname{diag}(a, I^{(2)})$ as an appropriate lift of v. Then $E(b) = \operatorname{diag}(E(a), E(I^{(2)})) = \operatorname{diag}(E(a), \mathbb{1}_2)$ showing $[E(a)] = [E(b)]$.

Finally, showing that $\clubsuit_i$ is a group homomorphism is straightforward in each even case, since we have $E(\operatorname{diag}(a, b)) = \operatorname{diag}(E(a), E(b))$ exactly (rather than just up to homotopy as in the odd cases).

(2) Next we show that $\clubsuit_0 \colon KO^u_0(B, \tau) \to KO^u_{-1}(I, \tau)$ is well defined. Suppose u in $M_{2n}(\widetilde{B})$ is unitary with $u^\tau = u = u^*$ and $\lambda_{2n}(u) = I^{(0)}_n$. Lift to an element $a \in M_n(\widetilde{A})$ with $-1 \leq a \leq 1$ and the condition $\lambda_{2n}(a) = I^{(0)}_n$ is immediate. Make the appropriate replacement to obtain $a^\tau = a$.

Now for any lift a of u with $-1 \leq a \leq 1$, $a^\tau = a$, and $\lambda(a) = I^{(0)}_n$, we see even more easily this time that $\lambda(E(a)) = I_{2n}$ and that $E(a)$ is unitary. This time the real structure calculation is

$$E(a)^\tau = E(a^\tau) = E(a),$$

so we have $[E(a)] \in KO^u_{-1}(I, \tau)$ as desired.

Dealing with different lifts and dealing with a homotopy of u_t, is accomplished just as in (1).

Finally, we need to compare the results of this construction for u and for $v = \operatorname{diag}(u, I^{(0)})$. Let a be a lift of u and $b = \operatorname{diag}(a, I^{(0)})$ be a lift of v. Then $E(b) = \operatorname{diag}(E(a), E(I^{(0)})) = \operatorname{diag}(E(a), \mathbb{1}_2)$, showing that $\clubsuit_0([u]) = \clubsuit_0([v])$.

(3), (4) The proofs that $\clubsuit_4$ and $\clubsuit_6$ are well defined are the same as the proofs that $\clubsuit_0$ and $\clubsuit_2$ are well defined, using $\sharp \otimes \tau$ instead of τ everywhere. □

Lemma 8.5. *Each $\clubsuit_i$ is natural with respect to morphisms of short exact sequences of real C^*-algebras.*

Proof. Suppose we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (I_1, \tau) & \longrightarrow & (A_1, \tau) & \xrightarrow{\pi_1} & (B_1, \tau) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & (I_2, \tau) & \longrightarrow & (A_2, \tau) & \xrightarrow{\pi_2} & (B_2, \tau) & \longrightarrow & 0 \end{array}$$

of real $C^{*,\tau}$-algebras, with exact rows. We show that $\iota_* \circ \clubsuit_i = \clubsuit_i \circ \beta_*$ for all i.

Suppose that $[u_1] \in KO^u_i(B_1, \tau)$ is given by a unitary $u_1 \in M_n(\widetilde{B_1})$ satisfying the specific symmetry relations and λ requirement. Let $u_2 = \beta(u_1) \in M_n(\widetilde{B_2})$. Then select an element $a_1 \in M_n(\widetilde{A_1})$ such that $\pi_1(a_1) = u$ and a_1 satisfies the requirements for the lift described in the definition of $\clubsuit_i$. Then $a_2 = \alpha(a_1)$ satisfies $\pi_2(a_2) = u_2$ and a_2 satisfies the requirements to be an appropriate lift of u_2. In the case that i is odd, since $B(\alpha(a_1)) =$

$\alpha(B(a_1))$, we have

$$\begin{aligned}\alpha_*\clubsuit_i([u_1]) &= \alpha_*([Y^{(i)}B(a_1)Y^{(i)*}]) \\ &= [Y^{(i)}\alpha(B(a_1))Y^{(i)*}] \\ &= [Y^{(i)}B(a_2)Y^{(i)*}] \\ &= \clubsuit_i([u_2]) \\ &= \clubsuit_i\alpha_*([u_1]).\end{aligned}$$

A similar calculation using $E(a_1)$ instead of $B(a_1)$ addresses the even cases. □

For reference, we include a parallel definition of the index maps in the complex case.

Definition 8.6. Suppose we have a exact sequence of real C^*-algebras. Let π denote both the quotient map $\pi : A \to B$ and its extension to $M_n(\widetilde{A}) \to M_n(\widetilde{B})$ for every n. Furthermore, we assume $I = \ker(\pi)$ and we identify the unit in $\widetilde{I}$ with that of $\widetilde{A}$.

(1) Suppose $[u] \in KU_1^u(B)$ where $u \in M_n(\widetilde{B})$ is a unitary with $\lambda(u) = I_n^{(1)}$. Then define

$$\clubsuit_1([u]) = \left[\begin{pmatrix} 2aa^* - \mathbb{1}_n & 2a\sqrt{\mathbb{1}_n - a^*a} \\ 2a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - 2a^*a \end{pmatrix}\right] \in KU_0^u(I)$$

where a in $M_n(\widetilde{A})$ is any lift of u with $\|a\| \leq 1$.

(2) Suppose $[u] \in KU_0^u(B)$ where $u \in M_n(\widetilde{B})$ is a unitary with $u = u^*$ and $\lambda(u) = I_n^{(0)}$. Then define

$$\clubsuit_0([u]) = \left[-e^{\pi i a}\right] \in KU_1^u(I)$$

where a in $M_n(\widetilde{A})$ is any lift of u with $-1 \leq a \leq 1$.

These homomorphisms are well-defined and natural, as can be shown by proofs similar to those we have just performed in the real case. In a sense made precise by the following lemma, these definitions of the index map are equivalent to the standard definitions found in the literature.

Lemma 8.7. *Let $0 \to I \to A \to B \to 0$ be a short exact sequence of real C^*-algebras.*

(1) *For all $0 \leq i < 8$, the following diagram commutes*

$$\begin{array}{ccc} KO_i^u(B) & \xrightarrow{\clubsuit_i} & KO_{i-1}^u(I) \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_{i-1}} \\ KU_i^u(B) & \xrightarrow{\clubsuit_i} & KU_{i-1}^u(I). \end{array}$$

(2) *For all* $0 \leq i < 2$*, the following diagrams commute up to sign*

$$\begin{array}{ccc} KU_1^u(B) & \xrightarrow{\clubsuit_1} & KU_0^u(I) \\ & \searrow^{\partial_1} & \downarrow \Theta \\ & & KU_0(I) \end{array} \qquad \begin{array}{ccc} KU_0^u(B) & \xrightarrow{\clubsuit_0} & KU_1^u(I) \\ \Theta \downarrow & \nearrow_{\partial_0} & \\ KU_0(B) & & \end{array}$$

where Θ *is the isomorphism of Theorem* 5.6 *and* ∂_i *is the boundary map of the literature in the complex setting.*

Proof. Statement (1) is immediate from the definitions of $\clubsuit_i$ for each i, noting that $KU_i^u(-)$ classes are unchanged by conjugation by any unitary in $M_n(\mathbb{C})$.

For statement (2), first let $[u] \in KU_1^u(B)$ where $u \in M_n(\widetilde{B})$ is a unitary with $\lambda(u) = \mathbb{1}_n$. Find a lift $a \in M_n(\widetilde{A})$ of norm at most 1. Then

$$\begin{aligned} (\Theta \circ \clubsuit_1)([u]) &= \Theta([B(a)]) \\ &= \left[\tfrac{1}{2}(B(a) + \mathbb{1}_{2n})\right] - [\mathbb{1}_n] \\ &= \left[\begin{pmatrix} aa^* + \mathbb{1}_n & a\sqrt{\mathbb{1}_n - a^*a} \\ a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - a^*a \end{pmatrix}\right] - [\mathbb{1}_n]\,. \end{aligned}$$

On the other hand, using the description of ∂_1 from Proposition 9.2.2 of [34], it is defined in terms of the same lift a and works out to

$$\partial_1([u]) = [\mathbb{1}_n] - \left[\begin{pmatrix} aa^* + \mathbb{1}_n & a\sqrt{\mathbb{1}_n - a^*a} \\ a^*\sqrt{\mathbb{1}_n - aa^*} & \mathbb{1}_n - a^*a \end{pmatrix}\right].$$

Hence the diagram commutes after adjusting by a factor of -1.

Now let $[u] \in KU_0^u(B)$ where $u \in M_{2n}(\widetilde{B})$ is a unitary with $u = u^*$ and $\lambda(u) = I_n^{(0)}$. Then $\Theta([u]) = \left[\frac{1}{2}(u + \mathbb{1}_{2n})\right] - [\mathbb{1}_n]$. Let $a \in M_{2n}(\widetilde{A})$ be a lift of u satisfying $-1 \leq a \leq 1$. Then $a' = \frac{1}{2}(a + \mathbb{1}_{2n})$ is a self-adjoint lift of the projection $p = \frac{1}{2}(u + \mathbb{1}_{2n})$ so using the formulas for ∂_0 from Proposition 12.2.2 of [34] or 9.3.2 of [4],

$$\begin{aligned} (\partial_0 \circ \Theta)([u]) &= \partial_0([p] - [\mathbb{1}_n]) \\ &= [\exp(2\pi i a')] \\ &= [\exp(\pi i(a + \mathbb{1}_{2n}))] \\ &= [-\exp(\pi i a)] \\ &= \clubsuit_0([u]). \end{aligned}$$

□

Our goal for the rest of this section is to prove that in the real case the homomorphisms $\clubsuit_i$ and ∂_i are the same, up to the same sign adjustment necessary in Lemma 8.7 above. Since any convention of the index map can be adjusted by a sign, we will henceforth assume that the diagrams in Lemma 8.7 commute exactly and prove that $\clubsuit_i = \partial_i$ exactly for all i.

For each $i \in \{0, 1, \ldots, 7\}$, the algebra A_i is generated by a finite number of elements. By Theorem 5.1.5 of [38], which is the real C^*-algebra counterpart of Theorem 2.10 of [26], there exist universal real C^*-algebras on a given set of generators subject to the relation that these generators are bounded in norm by 1. Thus, for each i, there is such a real C^*-algebra P_i and a surjective homomorphism

$$\rho_i \colon P_i \to A_i.$$

Furthermore, since the relation is liftable, the algebras P_i are projective. From this we obtain the short exact sequences

$$0 \to J_i \to P_i \xrightarrow{\rho_i} A_i \to 0$$

which are universal for the boundary map in a sense that we will take advantage of in the proof of Theorem 8.9 below.

Lemma 8.8. *For all i we have $\partial_i = \clubsuit_i \colon KO_i^u(A_i) \to KO_{i-1}^u(J_i)$.*

Proof. Since P_i is projective, we have $K^{CRT}(P_i) \cong 0$, so $\partial_i \colon KO_i^u(A_i) \to K_{i-1}^u(J_i)$ is an isomorphism of degree -1. By Theorems 3.1 and 3.2, both of these groups are isomorphic to $\mathbb{Z}$. In fact, since $K^{CRT}(A_i) \cong \Sigma^{-i} K^{CRT}(\mathbb{R})$, the structure of this *CRT*-module also implies that all four groups in the diagram below are isomorphic to $\mathbb{Z}$, and that the vertical maps c_i and c_{i-1} are isomorphisms:

$$\begin{array}{ccc} KO_i^u(A_i) & \xrightarrow{\partial_i} & KO_{i-1}^u(J_i) \\ \downarrow{\scriptstyle c_i} & & \downarrow{\scriptstyle c_{i-1}} \\ KU_i^u(A_i) & \xrightarrow{\partial_i} & KU_{i-1}^u(J_i). \end{array}$$

This diagram commutes by the naturality of the index map and the naturality of the complexification map. By Lemma 8.7, the diagram also commutes if we replace ∂_i in the upper horizontal arrow with $\clubsuit_i$. It follows that these two homomorphisms must coincide. □

Theorem 8.9. *For all i, $\partial_i = \clubsuit_i$.*

Proof. Let

$$0 \to I \to A \to B \to 0$$

be a short exact sequence of $C^{*,\tau}$-algebras. Let $\xi \in KO_i^u(B)$. Then by Theorem 4.13 for some integer n there exists a homomorphism $\phi \colon A_i \to M_n(B)$, such that $\phi_*([x_i]) = \xi$. Since P_i is projective, there exists a homomorphism ψ and we obtain a homomorphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_i & \longrightarrow & P_i & \xrightarrow{\rho_i} & A_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & M_n(I) & \longrightarrow & M_n(A) & \longrightarrow & M_n(B) & \longrightarrow & 0 \end{array}$$

which then induces a commutative diagram on K-theory

$$\begin{array}{ccc} KO_i^u(A_i) & \xrightarrow{\partial_i \text{ or } \clubsuit_i} & KO_{i-1}^u(J_i) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow \\ KO_i^u(M_n(B)) & \xrightarrow{\partial_i \text{ or } \clubsuit_i} & KO_{i-1}^u(M_n(I_i)). \end{array}$$

This diagram commutes if we take the horizontal homomorphisms to be both ∂_i or both $\clubsuit_i$. Since these two choices coincide for the upper arrow, they must coincide for the lower arrow on ξ.

Finally, consider the commutative diagrams below arising from the morphism of the original short exact sequence into the same one tensored by M_n:

$$\begin{array}{ccc} KO_i^u(B) & \xrightarrow{\partial_i \text{ or } \clubsuit_i} & KO_{i-1}^u(I_i) \\ \downarrow & & \downarrow \\ KO_i^u(M_n(B)) & \xrightarrow{\partial_i \text{ or } \clubsuit_i} & KO_{i-1}^u(M_n(I_i)). \end{array}$$

Since the vertical arrows are isomorphisms, the result of the previous paragraph shows that $\partial_i = \clubsuit_i \colon KO_i^u(B) \to KO_{i-1}^u(I)$. □

9. Boundary map examples: spheres and Calkin algebras

Example 9.1. Let σ be the involution on $C(S^1)$ given by $f^\sigma(z) = f(-z)$. The corresponding real C^*-algebra is $\{f \in C(S^1, \mathbb{C}) \mid f(-z) = \overline{f(z)}\}$ which is isomorphic to the real C^*-algebra T associated with self-conjugate K-theory discussed in [5]. The groups $KO_i(T)$ are calculated in Corollary 1.6 of [5], but we will present a self-contained calculation of $KO_*(C(S^1), \sigma)$ and also find unitary elements representing generators of the non-trivial KO-classes.

Let σ also denote the involution on $\mathbb{C} \oplus \mathbb{C}$ given by $(z, w)^\sigma = (w, z)$. Then there is a short exact sequence

$$0 \to (C_0(S^1 \setminus \{\pm 1\}), \sigma) \to (C(S^1), \sigma) \xrightarrow{\pi} (\mathbb{C} \oplus \mathbb{C}, \sigma) \to 0$$

where $\pi = (\mathrm{ev}_1, \mathrm{ev}_{-1})$ and we will describe the boundary maps

$$\partial_i \colon KO_i^u(\mathbb{C} \oplus \mathbb{C}, \sigma) \to KO_{i-1}^u(C_0(S^1 \setminus \{\pm 1\}), \sigma).$$

By Lemma 7.8, we have

$$KO_i^u(\mathbb{C} \oplus \mathbb{C}, \sigma) \cong KU_i(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Similarly, since there is an isomorphism

$$(C_0(S^1 \setminus \{\pm 1\}), \sigma) \cong (C_0(0,1) \oplus C_0(0,1), \sigma),$$

we have

$$KO_i^u(C_0(S^1 \setminus \{\pm 1\}), \sigma) \cong \begin{cases} 0 & i \text{ even} \\ \mathbb{Z} & i \text{ odd.} \end{cases}$$

Thus for i even we have $\partial_i \colon \mathbb{Z} \to \mathbb{Z}$ and we claim that

$$\partial_i = \begin{cases} 0 & i = 0, 4 \\ 2 & i = 2, 6 \end{cases}$$

(up to sign determined by the choices of isomorphism).

From Example 7.12 and Lemma 7.8, the generator of $KO_0^u(\mathbb{C} \oplus \mathbb{C}, \sigma)$ is $[w]$ where $w = (1_2, 1_2)$. This lifts to $a = 1_2 \in M_2(C(S^1, \mathbb{C}))$, which still satisfies $a^* = a$ and $a^\sigma = a^*$. Then $\partial_0([w]) = [-\exp(\pi i 1_2)] = [1_2]$, so $\partial_0 = 0$.

The generator of $KO_2^u(\mathbb{C} \oplus \mathbb{C}, \sigma)$ is $[w]$ where $w = (1_2, -1_2)$. One lift of w is $a \in M_2(C(S^1, \mathbb{C}))$ defined by

$$a(e^{2\pi i t}) = \begin{pmatrix} f(t) & 0 \\ 0 & f(t) \end{pmatrix} \text{ where } f(t) = \begin{cases} 1 - 4t & 0 \le t \le 1/2 \\ -3 + 4t & 1/2 \le t \le 1. \end{cases}$$

Check that $a^* = a$ and $a^\sigma = -a$. Then $\partial_2([w]) = [E(a)] = [-\exp(\pi i a)]$. We have

$$E(a)(e^{2\pi i t}) = \begin{pmatrix} -\exp(\pi i f(t)) & 0 \\ 0 & -\exp(\pi i f(t)) \end{pmatrix}$$

so

$$E(a) = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \text{ where } v(z) = \begin{cases} \overline{z}^2 & \operatorname{Im}(z) \ge 0 \\ z^2 & \operatorname{Im}(z) < 0. \end{cases}$$

Using the natural isomorphism

$$(C_0(S^1 \setminus \{\pm 1\}), \sigma) \cong (C_0(0,1) \oplus C_0(0,1), \sigma)$$

and combining Example 7.12 and Lemma 7.8, we see that $[v]$ is a generator of $KO_1^u(C_0(S^1 \setminus \{\pm 1\}), \sigma)$. Thus $[E(a)]$ is two times a generator.

For $i = 4$, the generator of $KO_4^u(\mathbb{C} \oplus \mathbb{C}, \sigma)$ is $[w]$ where

$$w = (\operatorname{diag}(1,1,1,-1), \operatorname{diag}(1,1,-1,1)) \in M_4(\mathbb{C}) \oplus M_4(\mathbb{C}).$$

To verify this, note that $w^{\sharp \otimes \sigma} = w$ and that $[\operatorname{diag}(1,1,1,-1)]$ is a generator of $KU_0(\mathbb{C}, \mathrm{id})$. Then a lift of w is a where

$$a(x,y) = \operatorname{diag}\left(1_2, \left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right)\right) \in M_4(C(S^1)).$$

which satisfies $a^{\sharp \otimes \sigma} = a$. Since a in fact is a self-adjoint unitary, then $[E(a)]$ is the trivial class.

Finally, for $i = 6$, the generator of $KO_6^u(\mathbb{C} \oplus \mathbb{C}, \sigma)$ is $[w]$ where

$$w = (1_2, -1_2)\,.$$

This satisfies $w^{\sharp \otimes \sigma} = -w$ and a lift a that satisfies $a^{\sharp \otimes \sigma} = -a$ is

$$a(e^{2\pi i t}) = \operatorname{diag}(f(t), f(t)).$$

Then $E(a) = \operatorname{diag}(v, v)$. Again this implies that $[E(a)]$ is two times a generator.

Therefore, we have

i	0	1	2	3	4	5	6	7
$KO_i(C(S^1), \sigma)$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

We will show furthermore that the non-trivial KO-groups have generators represented by the following unitaries.

- $KO^u_{-1}(C(S^1), \sigma) \cong \mathbb{Z}$ generated by $[w_{-1}]$ where $w_{-1}(z) = z^2$.
- $KO^u_0(C(S^1), \sigma) \cong \mathbb{Z}$ generated by $[w_0]$ where $w_0 = 1_2$.
- $KO^u_1(C(S^1), \sigma) \cong \mathbb{Z}_2$ generated by $[w_1]$ where $w_1(z) = -1$
- $KO^u_3(C(S^1), \sigma) \cong \mathbb{Z}$ generated by $[w_3]$ where $w_3(z) = \operatorname{diag}(z, -z)$.
- $KO^u_4(C(S^1), \sigma) \cong \mathbb{Z}$ generated by $[w_4]$ where

$$w_4(x, y) = \operatorname{diag}\left(1_2, \left(\begin{smallmatrix} x & y \\ y & -x \end{smallmatrix}\right)\right).$$

- $KO^u_5(C(S^1), \sigma) \cong \mathbb{Z}_2$ generated by $[w_5]$ where $w_5(z) = \operatorname{diag}(z, \overline{z})$.

For $i = 0, 4$, we know that π_* is surjective, so it is just a matter of checking that the induced class $[\pi(w_i)]$ is a known generator of $KO^u_i(\mathbb{C} \oplus \mathbb{C}, \sigma)$. For i odd, in each case we start with a known generator $[x_i]$ of

$$KO^u_i(C_0(S^1 \setminus \{\pm 1\}), \sigma)$$

and find the induced element in $KO^u_i(C_0(S^1), \sigma)$.

For example, for $i = 1$, consider the unitary

$$x_1 = \begin{cases} z^2 & \operatorname{Im}(z) \geq 0 \\ \overline{z}^2 & \operatorname{Im}(z) < 0 \end{cases}$$

which represents the generating class of $KO^u_1(C_0(S^1 \setminus \{\pm 1\}), \sigma)$. Note that $\operatorname{ev}_{-1}(x_1) = \operatorname{ev}_1(x_1) = 1$. However, as a class of $KO^u_1(C(S^1), \sigma)) \cong \mathbb{Z}$ we have $[x_1] = [-1]$ since there is a homotopy from x_1 to w_1 unitaries w_t satisfying $(w_t)^{\sharp \otimes \sigma} = w_t$. Indeed, note that x_1 restricted to the right half of the circle is a unitary-valued path from -1 to -1 which is homotopic to a constant through such paths. Also note that any such path on the right half of the circle can be extended to a function on the whole circle satisfying the proper symmetry. (For unitaries in $(C(S^1), \sigma)$ there is no requirement that $\operatorname{ev}_1 = 1$.).

For $i = 3$, we start with the unitary

$$x_3 = \begin{cases} \operatorname{diag}(z^2, 1) & \operatorname{Im}(z) \geq 0 \\ \operatorname{diag}(1, z^2) & \operatorname{Im}(z) < 0. \end{cases}$$

representing a class in $KO^u_1(C_0(S^1 \setminus \{\pm 1\}), \sigma)$. In $KO^u_3(C(S^1, \sigma)) \cong \mathbb{Z}$ we have $[x_3] = [w_3]$ where w_3 is as above. Indeed, if $f(z)$ is any unitary-valued function on the circle such that $f(1) = 1$, then $\left(\begin{smallmatrix} f(z) & 0 \\ 0 & f(-z) \end{smallmatrix}\right)$ is in the

appropriate symmetry class for $KO_3^u(C(S^1),\sigma)$. Now, the two choices

$$f(z)=z \quad \text{and} \quad f(z)=\begin{cases} z^2 & \operatorname{Im}(z)\geq 0 \\ 1 & \operatorname{Im}(z)<0 \end{cases}$$

yield the two unitaries x_3 and w_3 under consideration. Since these two choices of f are themselves homotopic, they yield a homotopy from x_3 to w_3.

In a similar way, we obtain the given generator

$$[w_5]\in KO_5^u(C(S^1,\sigma))\cong\mathbb{Z}_2.$$

Example 9.2. We now study the boundary maps for the short exact sequence

$$0\xrightarrow{\iota}(C_0(S^1\setminus\{\pm 1\}),\zeta)\to(C(S^1),\zeta)\xrightarrow{\pi}(\mathbb{C}\oplus\mathbb{C},\mathrm{id})\to 0 \tag{12}$$

where $\pi=(\mathrm{ev}_1,\mathrm{ev}_{-1})$ and $f^\zeta(z)=f(\overline{z})$. The associated real C^*-algebra to $(C(S^1),\zeta)$ is

$$A=\{f\colon S^1\to\mathbb{C}\mid f(\overline{z})=\overline{f(z)}\}.$$

There is a different split exact sequence involving A, namely

$$0\to 0\to S^{-1}\mathbb{R}\to A\to\mathbb{R}\to 0$$

which easily implies $KO_*(A)\cong KO_*(\mathbb{R})\oplus\Sigma^{-1}KO_*(\mathbb{R})$, with individual groups shown in the table below.

i	0	1	2	3	4	5	6	7
$KO_i(C(S^1),\zeta)$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0

However, we will independently calculate the boundary maps associated with the Sequence (12) using our methods, arrive at the same abstract groups, and identify explicit unitary generators.

To compute

$$\partial_i\colon KO_i^u(\mathbb{C}\oplus\mathbb{C},\mathrm{id})\to KO_{i-1}^u(C_0(S^1\setminus\{\pm 1\}),\zeta)$$

we first identify the relevant groups as

i	0	1	2	3	4	5	6	7
$KO_i^u(\mathbb{C}\oplus\mathbb{C},\mathrm{id})$	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}^2$	0	0	0
$KO_{i-1}^u(C_0(S^1\setminus\{\pm 1\}),\zeta)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

so we know right away that $\partial_i=0$ unless $i=0,4$. For $i=0,4$ we have $\partial_i\colon\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}$. We will show that $\partial_0(r,s)=r-s$ and $\partial_2(r,s)=2r-2s$ (with appropriate identifications).

Suppose $j=0$. Recall that $I^{(0)}=\mathrm{diag}(1,-1)$. Then generators of $KO_0^u(\mathbb{C}\oplus\mathbb{C},\mathrm{id})$ are $[w_1]$ and $[w_2]$ where

$$w_1=\left(1_2,I^{(0)}\right),w_2=\left(I^{(0)},1_2\right)\in M_2(\mathbb{C})\oplus M_2(\mathbb{C}).$$

We find self-adjoint lifts a_i of w_i to be

$$a_1(e^{2\pi it}) = \begin{pmatrix} 1 & 0 \\ 0 & f(t) \end{pmatrix} \qquad \text{and} \qquad a_2(e^{2\pi it}) = \begin{pmatrix} 1 & 0 \\ 0 & -f(t) \end{pmatrix}$$

where f is as in Example 9.1. Check that $a_i^\zeta = a_i$. Then $\partial_0([w_i]) = [u_i]$ where $u_i = -\exp(\pi i a_i)$ so

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \qquad \text{and} \qquad u_2 = \begin{pmatrix} 1 & 0 \\ 0 & v^* \end{pmatrix}$$

and v is as in Example 9.1 (check that $v^\zeta = v$). We have an isomorphism $(C_0(S^1 \setminus \{\pm 1\}), \zeta) \cong (C_0(0,1) \oplus C_0(0,1), \sigma)$. Since $[v]$ represents a generator of $KO^u_{-1}(C_0(S^1 \setminus \{\pm 1\}), \zeta) \cong \mathbb{Z}$ and $[u_1] = -[u_2]$, this proves our claim for ∂_0.

Now suppose $j = 4$. The generators of $KO^u_4(\mathbb{C} \oplus \mathbb{C}, \mathrm{id})$ are represented by the unitaries

$$w_1 = (1_4, I^{(4)}) \qquad \text{and} \qquad w_2 = (I^{(4)}, 1_4)$$

that satisfy $w_i^{\sharp \otimes \mathrm{id}} = w_i$. Lifts of w_i that satisfy $a_i^{\sharp \otimes \zeta} = a_i$ are

$$a_1(e^{2\pi it}) = \mathrm{diag}(1, 1, f(t), f(t)) \quad \text{and} \quad a_2(e^{2\pi it}) = \mathrm{diag}(1, 1, -f(t), -f(t)).$$

Therefore $\partial_1([w_i]) = [E(a_i)] = [u_i]$ where

$$u_1 = \mathrm{diag}(1, 1, v, v) \qquad \text{and} \qquad u_2 = \mathrm{diag}(1, 1, v^*, v^*).$$

Through the isomorphisms

$$\begin{aligned} \mathbb{Z} \cong KU^u_1(C_0(0,1), \mathrm{id}) &\cong KO^u_3(C_0(0,1) \oplus C_0(0,1), \sigma) \\ &\cong KO^u_3(S^1 \setminus \{\pm 1\}, \zeta) \end{aligned}$$

we conclude that the $KO^u_3(S^1 \setminus \{\pm 1\}, \zeta) \cong \mathbb{Z}$ class of a unitary is determined by the winding number of that unitary on the top half of the circle. Thus $[\mathrm{diag}(v, v)]$ is twice a generator. (A generator would be given for example by a unitary such as w_3 below). Since $[u_1] = -[u_2]$, This proves the claim for $i = 4$.

Now that the boundary maps are understood, the only group that is not fully determined up to isomorphism by the exact sequences is $KO^u_1(C(S^1), \zeta)$ which is an extension of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ by $\mathbb{Z}$. We will show that $KO^u_1(C(S^1), \zeta) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Note that the generator of $KO^u_1(C_0(S^1 \setminus \{\pm 1\}), \zeta)$ is given by $[v']$ where $v'(z) = z^2$ (check that $(v')^\zeta = (v')^*$). The image $\iota_*[v']$ is divisible by 2 in $KO^u_1(C(S^1), \zeta)$, since $[\iota(v')] = 2[v'']$ where $v''(z) = z$ (again, check that $(v'')^\zeta = (v'')^*$). This shows that the extension problem for

$$0 \to \mathbb{Z} \xrightarrow{\iota_*} KO^u_1(C(S^1), \zeta) \xrightarrow{\pi_*} \mathbb{Z}_2^2 \to 0$$

is solved by $KO^u_1(C(S^1), \zeta) \cong \mathbb{Z} \oplus \mathbb{Z}^2$.

Now that we have determined the groups $KO^u_i(C(S^1), \zeta)$ up to isomorphism, we identify the generators and write down specific isomorphisms.

- $KO_0^u(C(S^1),\zeta) \cong \mathbb{Z}$ generated by $[w_0]$ where $w_0 = 1_2$. The isomorphism can be realized as $[w] \mapsto \frac{1}{2}\mathrm{trace}(w(1))$.
- $KO_1^u(C(S^1),\zeta) \cong \mathbb{Z}\oplus\mathbb{Z}_2$ generated by $[w_1]$ and $[w_1']$ where $w_1(z) = z$ and $w_1'(z) = -1$. Clearly, $[w_1']$ generates the element of order 2. The map
$$KO_1^u(C(S^1),\zeta) \to \mathbb{Z}\oplus\mathbb{Z}_2$$
is described by
$$[w] = \left(\mathrm{winding}(t \mapsto \det\left(w(e^{2\pi i t})\right)), \tfrac{1}{2} - \tfrac{1}{2}\det(w(1))\right)$$
where t is in $[0,1]$.
- $KO_2^u(C(S^1),\zeta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by $[w_2]$ and $[w_2']$ where
$$w_2(x+iy) = \begin{pmatrix} y & ix \\ -ix & y \end{pmatrix} \qquad \text{and} \qquad w_2'(x+iy) = \begin{pmatrix} y & -ix \\ ix & y \end{pmatrix}.$$
Moreover, a nice formula is that the isomorphism $KO_2^u(C(S^1),\zeta) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is given by
$$[w] \mapsto (\mathrm{sign}(\mathrm{Pf}(w(1)), \mathrm{sign}(\mathrm{Pf}(w(-1)))\,.$$
- $KO_3^u(C(S^1),\zeta) \cong \mathbb{Z}_2$ generated by $[w_3]$ where
$$w_3(z) = \begin{cases} \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} & \mathrm{Im}\, z \geq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & \overline{z}^2 \end{pmatrix} & \mathrm{Im}\, z < 0. \end{cases}$$
The class of any unitary w can be determined by looking at the winding number of w restricted to the top half of the circle (modulo 2).
- $KO_4^u(C(S^1),\zeta) \cong \mathbb{Z}$ generated by $[w_4]$ where $w_4 = 1_4$. The isomorphism can be realized as $[w] \mapsto \frac{1}{2}\mathrm{trace}(w(1))$.
- $KO_5^u(C(S^1),\zeta) \cong \mathbb{Z}$ generated by $[w_5]$ where
$$w_5(z) = \begin{cases} \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} & \mathrm{Im}\, z \geq 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} & \mathrm{Im}\, z < 0. \end{cases}$$
The class of any unitary w can be determined by looking at the winding number of w restricted to the top half of the circle.

Sketch of Proof. For $i = 0, 1, 2$ it suffices to show that the shown generators map via π_* to corresponding generators of $KO_i^u(\mathbb{C}\oplus\mathbb{C}, \mathrm{id})$. For $i = 4$, it suffices to show that the shown generator $[1_4]$ maps via π_* to the generator of the kernel of ∂_4. This generator corresponds to $(2,2)$ in our conventional isomorphism $KO_3^u(\mathbb{C}\oplus\mathbb{C}, \mathrm{id}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

For $i = 3, 5$ it suffices to show that the shown generators are the image of generators of $KO_i^u(C_0(S^1 \setminus \{\pm 1\}), \zeta)$.

For example, to identify the generators of $KO_2(C(S^1), \zeta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, first note that $(w_2)^\zeta = -w_2$ and similarly for w_2'. We know that π_* is an isomorphism on $KO_2^u(-)$ so it suffices to note that $[\pi(w_2)] = [I^{(2)}, -I^{(2)}]$ and $[\pi(w_2')] = [-I^{(2)}, I^{(2)}]$ which are the generators of $KO_2^u(\mathbb{C} \oplus \mathbb{C}, \mathrm{id}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Example 9.3. We will consider the exact sequence

$$0 \to C_0(U, \mathbb{R}) \to C(D, \mathbb{R}) \to C(S^1, \mathbb{R}) \to 0$$

where D is the unit disk and $U = D \setminus S^1$ is the interior of D. In terms of $C^{*,\tau}$-algebras, we have

$$0 \to (C_0(U), \mathrm{id}) \to (C(D), \mathrm{id}) \to (C(S^1), \mathrm{id}) \to 0.$$

We shall disregard the summands of $KO_*^u(C(D), \mathrm{id})$ and $KO_*^u(C(S^1), \mathrm{id})$ associated with the unit, so we consider the reduced K-theory

$$0 \to C_0(U, \mathbb{R}) \to C_0(D \setminus \{1\}, \mathbb{R}) \to C_0(S^1 \setminus \{1\}, \mathbb{R}) \to 0.$$

Then the boundary map is an isomorphism

$$\partial_i \colon KO_i^u(C_0(S^1 \setminus \{1\}), \mathrm{id}) \to KO_{i-1}^u(C_0(U), \mathrm{id})$$

with the groups as shown.

i	-2	-1	0	1	2	3	4	5
$KO_*^u(C(S^1 \setminus \{1\}), \mathrm{id})$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$KO_i^u(C_0(U), \mathrm{id})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

We will focus on the case when $i = -1$. The free abelian generator of $KO_{-1}^u(C_0(S^1 \setminus \{1\}), \mathrm{id})$ is $[w_{-1}]$ where $w_{-1}(x, y) = x + iy$, clearly satisfying $w_{-1}^{\mathrm{id}} = w_{-1}$. Then an appropriate lift of z in $C(D)$ is $a(x, y) = x + iy$, so $\partial_{-1}[w_{-1}] = [W_2 B(a) W_2^*] \in KO_{-2}(C_0(U), \mathrm{id})$ where

$$\begin{aligned} B(a) &= \begin{pmatrix} 2aa^* - 1 & 2a\sqrt{1 - a^*a} \\ 2a^*\sqrt{1 - aa^*} & 1 - 2a^*a \end{pmatrix} \\ &= \begin{pmatrix} 2(x^2 + y^2) - 1 & 2(x + iy)\sqrt{1 - (x^2 + y^2)} \\ 2(x - iy)\sqrt{1 - (x^2 + y^2)} & 1 - 2(x^2 + y^2) \end{pmatrix}. \end{aligned}$$

Notice that on the boundary of D, $B(a) = \mathrm{diag}(1, -1)$ so $\lambda(W_2 B(a) W_2^*) = I^{(2)}$ as expected.

There is a continuous map from $S^2 \setminus \{(0, 0, 1)\}$ to U given by

$$(x, y, z) \mapsto \sqrt{\frac{z + 1}{2(x^2 + y^2)}}(x, y)$$

which gives an isomorphism $C_0(S^2 \setminus \{(0,0,1)\})$ to $C_0(U)$. Under this transformation, we have

$$B(a) = \begin{pmatrix} z & x+iy \\ x-iy & -z \end{pmatrix} \in M_2(C(S^2)).$$

Then

$$W_2 B(a) W_2^* = \begin{pmatrix} -y & x+iz \\ x-iz & y \end{pmatrix}$$

which is equivalent (via a rigid automorphism of the sphere) to the generator of $KO^u_{-2}(C_0(S^2 \setminus \{1\})) \cong \mathbb{Z}$ that was identified in Example 7.13.

Our final example uses our machinery to deal with index maps applied to Fredholm operators with various symmetries. For a more direct approach, and one that allows for a $\mathbb{Z}_2$-graded in addition to a real structure, see [18].

Example 9.4. Let $\mathbb{B}^{\mathbb{R}}$ be the real C^*-algebra of bounded operators on a separable infinite dimensional real Hilbert space and let $\mathbb{K}^{\mathbb{R}}$ be the ideal of compact operators. Then we have a short exact sequence

$$0 \to \mathbb{K}^{\mathbb{R}} \to \mathbb{B}^{\mathbb{R}} \xrightarrow{\pi} \mathcal{Q}^{\mathbb{R}} \to 0$$

where $Q^{\mathbb{R}}$ is the real Calkin algebra. For any real C^*-algebra A, it follows from Theorem 1.12 and Proposition 1.15 of [5] that $KO_*(A) = 0$ if and only if $KU_*(A) = 0$. Therefore $KO_*(\mathbb{B}^{\mathbb{R}}) = 0$ and $\partial_i \colon KO_i(\mathcal{Q}^{\mathbb{R}}) \to KO_{i-1}(\mathbb{K}^{\mathbb{R}})$ is an isomorphism for each i. Therefore,

i	0	1	2	3	4	5	6	7
$KO_i(\mathcal{Q}^{\mathbb{R}})$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0
$KO_i(\mathbb{K}^{\mathbb{R}})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

We will identify the generators of $KO^u_i(\mathbb{K}^{\mathbb{R}})$ and, working backwards, the generators of $KO^u_i(\mathcal{Q}^{\mathbb{R}})$ for all i.

Let e be a rank 1 projection in $\mathbb{K}^{\mathbb{R}}$ and let $\iota \colon \mathbb{R} \to \mathbb{K}^{\mathbb{R}}$ be the homomorphism given by $t \mapsto te$, which induces an isomorphism on $KO_*(-)$. The generators of $KO^u_i(\mathbb{R})$ are given in Example 7.12. Recall that the generator of $KO^u_0(\mathbb{R})$ was identified as $[1_2]$ where $1_2 \in M_2(\mathbb{R})$. Working in the unitization $\widetilde{\mathbb{R}}$ this corresponds to the unitary $w = \operatorname{diag}(0,2) + \operatorname{diag}(\mathbb{1}, -\mathbb{1}) \in M_2(\widetilde{\mathbb{R}})$, which satisfies $[\lambda(w)] = [\operatorname{diag}(\mathbb{1}, -\mathbb{1})] = 0$ (see Remark 7.4). Then the generator of $KO^u_0(\mathbb{K}^{\mathbb{R}})$ is given by $[\widetilde{\iota}(w)] = [\operatorname{diag}(\mathbb{1}, 2e - \mathbb{1})]$, as shown below. The rest of the generators of $KO_i(\mathbb{K}^{\mathbb{R}})$ for $i = 1, 2, 4$ are worked out similarly. Note that these unitary representatives are given in terms of the $C^{*,\tau}$-algebra $(\mathbb{K}, \tau)$ where τ is the associated involution on $\mathbb{K}$.

- The generator of $KO^u_0(\mathbb{K}^{\mathbb{R}})$ is given by $[w_0]$ where
$$w_0 = \operatorname{diag}(\mathbb{1}, 2e - \mathbb{1},) \in M_2(\widetilde{\mathbb{K}}).$$
- The generator of $KO^u_1(\mathbb{K}^{\mathbb{R}})$ is given by $[w_1]$ where $w_1 = -2e + \mathbb{1} \in \widetilde{\mathbb{K}}$.

- The generator of $KO_2^u(\mathbb{K}^{\mathbb{R}})$ is given by $[w_2]$ where
$$w_2 = \begin{pmatrix} 0 & i(-2e+\mathbb{1}) \\ i(2e-\mathbb{1}) & 0 \end{pmatrix} \in M_2(\widetilde{\mathbb{K}}).$$
- The generator of $KO_4^u(\mathbb{K}^{\mathbb{R}})$ is given by $[w_4]$ where
$$w_4 = \operatorname{diag}(\mathbb{1}, \mathbb{1}, 2e-\mathbb{1}, 2e-\mathbb{1}) \in M_4(\widetilde{\mathbb{K}}).$$

In each case, an element generating $KO_i^u(\mathbb{R})$ corresponds to an element in $KO_i^u(\mathbb{K}^{\mathbb{R}})$ via $[u] \mapsto \left[(u - I_n^{(i)}) \otimes e + I_n^{(i)} \otimes \mathbb{1}_{\mathbb{K}}\right]$.

Let $s \in \mathbb{B}^{\mathbb{R}}$ be the one-sided shift operator defined by $s(e_i) = e_{i+1}$ where $\{e_i\}$ is a given basis of the underlying (real) Hilbert space. Then s satisfies $s^\tau = s^*$. Let $u = \pi(s) \in \mathcal{Q}^{\mathbb{R}}$, which is a unitary and also satisfies $u^\tau = u^*$. We claim that that generators of $KO_i^u(\mathcal{Q}^{\mathbb{R}})$ are the following.

- The generator of $KO_1^u(\mathcal{Q}^{\mathbb{R}})$ is given by $[v_1]$ where $v_1 = u \in M_1(\widetilde{\mathcal{Q}})$.
- The generator of $KO_2^u(\mathcal{Q}^{\mathbb{R}})$ is given by $[v_2]$ where
$$v_2 = \begin{pmatrix} 0 & iu \\ -iu^* & 0 \end{pmatrix} \in M_2(\widetilde{\mathcal{Q}}).$$
- The generator of $KO_3^u(\mathcal{Q}^{\mathbb{R}})$ is given by $[v_3]$ where
$$v_3 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in M_2(\widetilde{\mathcal{Q}}),$$
- The generator of $KO_5^u(\mathcal{Q}^{\mathbb{R}})$ is given by $[v_5]$ where $v_5 = \operatorname{diag}(u, u) \in M_2(\widetilde{\mathcal{Q}})$.

Proof. In each case, we verify that $\partial_i([v_i]) = [w_{i-1}]$ and then the result follows since ∂_i is an isomorphism.

For the first statement, we calculate $\partial_1([v_1]) = \partial_1([u])$. First lift u back to the partial isometry $s \in \mathbb{B}^{\mathbb{R}}$. Using the formulas $s^*s = 1$ and $ss^* = 1 - e$ we have
$$B(s) = \begin{pmatrix} 2ss^* - \mathbb{1} & 2s\sqrt{\mathbb{1} - s^*s} \\ 2s^*\sqrt{\mathbb{1} - ss^*} & \mathbb{1} - 2s^*s \end{pmatrix} = \begin{pmatrix} \mathbb{1} - 2e & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$
Then, noting that $Y_2^{(1)} = 1_2$, we have
$$\partial_1[u] = [Y_2^{(1)} B(s) Y_2^{(1)}] = [\operatorname{diag}(\mathbb{1} - 2e, -\mathbb{1})] = [\operatorname{diag}(\mathbb{1}, 2e - \mathbb{1})],$$
which is the generator of $KO_0^u(\mathbb{K}^{\mathbb{R}})$.

For the second statement, first notice that v_2 is a self-adjoint unitary and that $v_2^\tau = -v_2$. The appropriate lift to $\widetilde{\mathbb{K}}$ is
$$a = \begin{pmatrix} 0 & is \\ -is^* & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
&\partial_2([v_2]) \\
&\quad = \left[-\exp\left(\pi \begin{pmatrix} 0 & -s \\ s^* & 0 \end{pmatrix}\right)\right] \\
&\quad = \left[-\left(\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \pi \begin{pmatrix} 0 & -s \\ s^* & 0 \end{pmatrix} + \frac{\pi^2}{2!}\begin{pmatrix} -\mathbb{1}+e & 0 \\ 0 & -\mathbb{1} \end{pmatrix} + \frac{\pi^3}{3!}\begin{pmatrix} 0 & s \\ -s^* & 0 \end{pmatrix} + \cdots\right)\right] \\
&\quad = \left[-\left(\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} + \cos\left(\pi \cdot \begin{pmatrix} \mathbb{1}-e & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right) + \sin\left(\pi \cdot \begin{pmatrix} 0 & -s \\ s^* & 0 \end{pmatrix}\right)\right)\right] \\
&\quad = \left[\begin{pmatrix} \mathbb{1}-2e & 0 \\ 0 & \mathbb{1} \end{pmatrix}\right] \\
&\quad = [w_1]
\end{aligned}$$

which is the desired generator of $KO_1^u(\mathbb{K}^{\mathbb{R}})$.

Now consider v_3, which is a unitary that satisfies $v_3^{\sharp\otimes\tau} = v_3$. The obvious lift is $\mathrm{diag}(s, s^*) \in \mathbb{B}$. We calculate

$$B\left(\begin{pmatrix} s & 0 \\ 0 & s^* \end{pmatrix}\right) = \begin{pmatrix} \mathbb{1}-2e & 0 & 0 & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ 0 & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1}+2e \end{pmatrix}$$

and then conjugate by $Y_4^{(3)} = V_4 Q_4 W_4$ to obtain

$$B' = \begin{pmatrix} 0 & i(\mathbb{1}-e) & 0 & -ie \\ -i(\mathbb{1}-e) & 0 & ie & 0 \\ 0 & -ie & 0 & i(\mathbb{1}-e) \\ ie & 0 & -i(\mathbb{1}-e) & 0 \end{pmatrix}.$$

Notice that $\lambda(B') = I_2^{(2)}$ as expected. The class $[B']$ corresponds to the class in $KO_2^u(\mathbb{R})$ given by the unitary

$$C' = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}).$$

The Pfaffian of C' distinguishes it from the trivial element represented by

$$I_2^{(2)} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Therefore, $[B'] = [w_2]$ is the non-trivial class in $KO_2^u(\mathbb{K}^{\mathbb{R}})$.

Finally, the proof for v_5 is similar to that for v_1. The lift for v_5 is $a = \mathrm{diag}(s, s)$ and then

$$\partial_5([v_5]) = [B(a)] = [\mathrm{diag}(\mathbb{1}-2e, \mathbb{1}-2e, -\mathbb{1}, -\mathbb{1})] = [w_4]$$

as desired. (The conjugation matrix in this case is $Y_4^{(4)} = X_4 = 1_4$.) $\square$

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