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Multiplication operators on the Bergman spaces of pseudoconvex domains

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Abstract. Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth pseudoconvex domain, and let $f = (f_1, \dots, f_n) : \overline{\Omega} \subset \mathbb{C}^n$ be an n-tuple of holomorphic functions on $\overline{\Omega}$. In this paper we study commutants of the corresponding multiplication operators $\{T_{f_1}, \dots, T_{f_n}\} = T_f$ on the Bergman space $A^2(\Omega)$. One of our main results is a geometric description of the algebra of commutants of $\{T_f, {T_f}^*\}$, generalizing a result by Douglas, Sun and Zheng, 2011.

Contents

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth pseudoconvex domain. The Bergman space of all square integrable holomorphic functions on Ω will be denoted by $A^2(\Omega)$, while the subspace of all bounded holomorphic functions on Ω will be denoted by $H^\infty(\Omega)$. Given a function $f \in L^\infty(\Omega)$, one defines the corresponding Toeplitz operator with the symbol $f : T_f : A^2(\Omega) \to A^2(\Omega)$, as the composition of the multiplication operator by f followed by the orthogonal projection from $L^2(\Omega)$ to $A^2(\Omega)$. If f is holomorphic, then $T_f = M_f$ is the multiplication operator by f. Questions related to commutants of Toeplitz operators have been of great interest for some time.

This paper is largely motivated by the following problem.

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Problem 1. *Let $f = (f_1, \dots, f_n) : \overline{\Omega} \to \mathbb{C}^n$ be a holomorphic mapping in a neighbourhood of $\overline{\Omega}$ with a nontrivial Jacobian determinant. Describe the algebra of commutants of $\{T_{f_i}, 1 \leq i \leq n\} = T_f$.*

It is of a special interests to describe the largest C^*-subalgebra of the above algebra, the algebra of commutants of $\{T_f, T_f^*\}$ (here and everywhere T_f^* denotes $\{T_{f_i}^*, 1 \leq i \leq n\}$). Indeed, reducing subspaces of T_f correspond to projections in this algebra.

Both of the above questions have been extensively studied for the past several decades when $n = 1$ and $\Omega = D$ is the unit disc. Indeed, by a result of Thomson [Th], it suffices to study the commutants of T_f when f is a finite Blaschke product. In this case it can be described it terms of the Riemann surface $f^{-1} \circ f(D')$, where D' is D with preimages of the critical values of f removed [[Co], Theorem 3] (although Cowen and Thomson worked in the Hardy space setting, their results easily carry over to the Bergman space).

In a recent important work by Douglas, Sun and Zheng [DSZ], the algebra of commutants of $\{T_f, T_f^*\}$ is explicitly described. In particular, they show that its dimension equals to the number of connected components of $f^{-1} \circ f(D')$ ([DSZ], Theorem 7.6). Also noteworthy are results of Guo and Huang, who under the assumption that $f : D \to f(D)$ is a covering map, described among other things the commutant of $\{T_f, T_f^*\}$ in terms of fundamental group of $f(D)$ [[GuoH2], Theorem 1.3].

Motivated by these results, we extend them to high dimensional domains. Namely, we introduce a certain n-dimensional complex manifold W_f (Definition 1)

$$W_f \subset (\Omega \setminus Z) \times_f (\Omega \setminus Z) = \{(z, w), f(z) = f(w), z, w \in \Omega \setminus Z\}$$

defined as the largest open subset of $(\Omega \setminus Z) \times_f (\Omega \setminus Z)$ such that the projection $p : W_f \to \Omega \setminus Z$ is a covering map, where Z is the preimage of all critical values of f on $\overline{\Omega}$. Under some mild assumptions on Ω, f (Assumptions 1, 2) we prove that the algebra of commutants of $\{T_f, {T_f}^*\}$ is isomorphic to the algebra of locally constant functions on W_f under convolution product (Theorem 6.1). This is a generalization of the above mentioned theorem by Douglas, Sun and Zheng [DSZ]. Our proof closely follows their ideas.

We also study the commutant of T_f in the Toeplitz algebra of Ω, the norm closed subalgebra of $B(A^2(\Omega))$ generated by all Toepltiz operators $T_h, h \in L^\infty(\Omega)$. Motivated by a result of Axler, Cuckovic and Rao [AxCR] on commutants of analytic Toeplitz operators in one variable, we prove that the commutant of T_f in the Toeplitz algebra of Ω consists of multiplication operators by bounded holomorphic functions on Ω, Theorem 5.7.

The paper is organized as follows. The first three sections have a preparatory nature. In Section 2 we establish some Nullstellensatz-type statements for the Bergman space $A^2(\Omega)$ that play a crucial role in studying the commutants of T_f. In Sections 3 and 4, we introduce some geometric objects attached to Ω, f (Section 3), and convolution algebras associated to them

(Section 4.1). Central parts of the paper are Sections 5 and 6, in which our main results about commutants of T_f and $\{T_f, T_f^*\}$ are established.

2. Nullstellensatz for the Bergman space

Throughout this paper given a holomorphic mapping $g : \Omega \to \mathbb{C}^n, \Omega \subset \mathbb{C}^n$, we will denote the determinant of the Jacobian of g by J_g.

In this section we will prove a (weak) version of Nullstellensatz for the Bergman space of a bounded pseudoconvex domain in $\mathbb{C}^n$ (Theorem 2.6). This result will be crucial for studying commutants of T_f. All the results in this section follow well-known approach of using Koszul and $\bar{\partial}$-complex for proving Nullstellensatz type statements on pseudoconvex domains and are essentially well-known (see for example [PS]). We include proofs for a reader's convenience.

As always, let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain. We will denote by $A^\infty(\Omega)$ the set of all holomorphic functions on Ω which are C^∞-smooth on $\overline{\Omega}$. Let $f = (f_1, \dots, f_m) : \Omega \to \mathbb{C}^m$ be an m-tuple of holomorphic functions from $A^\infty(\Omega)$, which will also be viewed as a holomorphic mapping to $\mathbb{C}^m$. Let us recall the definition of the Koszul double complex of f on Ω. Define the $\bar{\partial}$-Koszul double complex $(K, b_f, \bar{\partial})$ on Ω as follows

$$K = \bigoplus K_{i,j}, K_{i,j} = \Lambda^i(V) \otimes_{\mathbb{C}} C^\infty_{0,j}(\overline{\Omega})$$

where $V = \oplus_{i=1}^m \mathbb{C} v_i$, and $C^\infty_{0,j}(\overline{\Omega})$ denotes the space of all C^∞-smooth $(0,j)$-forms on $\overline{\Omega}$ There is a natural product on K defined as follows

$$(u \otimes \omega_1) \cdot (v \otimes \omega_2) = (u \wedge v) \otimes (\omega_1 \wedge \omega_2).$$

Differentials of this bicomplex are $\bar{\partial} : K_{i,j} \to K_{i,j+1}$ and the Koszul differential $b_f : K_{i,j} \to K_{i-1,j}$ defined as follows

$$b_f\left(\sum_i v_i \otimes \omega_i\right) = \sum_i f_i \omega,$$

$$b_f(x \cdot y) = b_f(x) \cdot y + (-1)^i x \cdot b_f(y), x \in K_{i,j}, \bar{\partial}(u \otimes \omega) = u \otimes \bar{\partial}(\omega)$$

Clearly $\bar{\partial} b_f = b_f \bar{\partial}$.

Lemma 2.1. *In the above setting let $U \subset\subset \Omega$ be an open subset such that $f^{-1}(0) \cap \bar{U} = \emptyset$. Let $w \in K_{i,j}$ be such that $b_f(w) = \bar{\partial}(w) = 0$, and* $\operatorname{supp}(w) \subset U$. *Then there exists $w' \in K_{i+1,j}$ such that $w = b_f w', \bar{\partial}(w') = 0$.*

Proof. Let $w \in K_{i,j}$. We will proceed by the descending induction on i. We claim that there exists $y \in K_{i+1,j}$ supported on U such that $b_f y = w$. Indeed, let $g_i \in C^\infty(\overline{\Omega})$ be such that $(\sum_i f_i g_i)_U = 1$. Therefore

$$b_f\Big(\Big(\sum v_i \otimes g_i\Big) \cdot w\Big) = w.$$

Then $\bar\partial(y) \in K_{i+1,j+1}$ satisfies the inductive assumption, so there exists z such that $b_f(z) = \bar\partial(y)$ and $\bar\partial(z) = 0$. Let z_1 be such that $\bar\partial(z_1) = z$ (it exists by Kohn's theorem). Replacing y by $y - b_f(z_1)$ we are done. $\square$

Corollary 2.2. *Let $f_1, \dots, f_n \in A^\infty(\Omega)$ and let $U \subset\subset \Omega$ be an open subset of Ω such that $f^{-1}(0) \subset U$. If $g \in A^\infty(\Omega)$ such that $g \in \sum_i f_i A(U)$, then $g \in \sum_i f_i A^\infty(\Omega)$.*

Proof. Let $h_i \in C^\infty(\bar\Omega) \cap A(U)$ such that $g = \sum_i f_i h_i$. Then $bx = \bar\partial(x) = 0$ where $x = \sum v_i \otimes \bar\partial(h_i)$. Thus by the above there exists $z \in K_{2,0}$ such that $x = b(\bar\partial(z))$. Then

$$\bar\partial\Big(\sum v_i \otimes h_i - b(z)\Big) = 0, \quad b\Big(\sum v_i \otimes h_i - b_f(z)\Big) = g.$$

Let us write $\sum f_i \otimes h_i - b(z)$ as $\sum v_i \otimes h_i$. Then

$$g = \sum_i f_i h_i, \quad h_i \in A^\infty(\overline{\Omega}).$$ $\square$

For a subset $S \subset \overline{\Omega}$, we will denote by $I(S)$ the ideal of holomorphic functions on Ω which vanish on S.

The proof below directly follows the proofs of similar statements by Overlid [Øv], Hakim–Sibony [HS].

Corollary 2.3. *Let $f = f_1, \dots, f_m \in A^\infty(\Omega)$ be such that $f^{-1}(0)$ is a finite set. If the Jacobian of f has the full rank on each point of $f^{-1}(0)$, then*

$$I(f^{-1}(0)) \cap A^\infty(\overline{\Omega}) = \sum_i f_i A^\infty(\overline{\Omega}).$$

Proof. Let $h \in I(F^{-1}(0)) \cap A^\infty(\overline{\Omega})$. It follows from the Hilbert Nullstellensatz for the local complex analytic case [[GunR], III.A.7] that there exists an open neighbourhood U of $f^{-1}(0)$, and $g_i \in A(U)$, such that $h|_U = \sum_i f_i|_U g_i$. Now by Corollary 2.2 we are done. $\square$

We will need the following assumption on Ω. It was first introduced in [AgS], see also [PS].

Assumption 1. *$\Omega \subset \mathbb{C}^n$ is a connected smooth bounded pseudoconvex domain, such that for any $z \in \partial\Omega, A^\infty(\Omega) \cap I(z)$ is dense in $A^2(\Omega)$.*

Recall the following simple lemma.

Lemma 2.4. *Assumption 1 is satisfied for bounded smooth strongly pseudoconvex domains or star-shaped smooth pseudoconvex domains.*

Proof. Notice that to verify Assumption 1, it suffices to check the following: for a given $z \in \partial\Omega$, there exists a sequence $f_n \in A^\infty(\Omega)$ such that $f_n(z) = 1$ and $\lim_{n\to\infty} \|f_n\|_{A^2(\Omega)} = 0$. Indeed, let $g \in A^\infty(\Omega)$. Then $g - g(z)f_n \in I(z)$ and $\lim_{n\to\infty}(g - g(z)f_n) = g$ in $A^2(\Omega)$. Thus, $A^\infty(\Omega) \cap I(z)$ is dense in $A^\infty(\Omega)$, and since $A^\infty(\Omega)$ is dense in $A^2(\Omega)$ by a result of Catlin [[Ca], Theorem 3.1.4], it follows that $A^\infty(\Omega) \cap I(z)$ is dense in $A^2(\Omega)$.

Now let us suppose that Ω is a smooth strongly pseudoconvex domain. Let $z \in \partial\Omega$. It is well-known that z is a peak point. Let $f \in A^\infty(\Omega)$ be such that $f(z) = 1, |f(w)| < 1, w \in \overline{\Omega} \setminus z$. Then $\lim_{m\to\infty} \|f^m\|_2 = 0$.

Now let Ω be a star shaped smooth domain. Without loss of generality, we may assume that $r\Omega \subset \Omega, 0 \le r \le 1$. Let $\theta \in \partial\Omega$. Let $f \in A^2(\Omega)$ be such that $\lim_{w\to\theta}(f(w)) = \infty$. Existence of such f follows for example from [[Ca2], Lemma1, page 153]. Then $f_r(z) = f(rz) \in A^\infty(\Omega)$ and $\|f_r\|_2 \le r^{-2n}\|f\|_2$, while $\lim_{r\to 1} f_r(\theta) = \infty$. □

We have another:

Lemma 2.5. *If Ω satisfies Assumption 1, then for any finite set $S \subset \partial\Omega$, $A^\infty(\Omega) \cap I(S)$ is dense in $A^2(\Omega)$.*

Proof. Put $S = \{z_i\}_{1\le i\le m}$. Let $\epsilon > 0$. Let $g \in A^\infty(\Omega)$. Let $\phi_i \in A^\infty(\Omega)$ be such that $\phi_i(z_j) = \delta_{ij}$. Let $g_i \in A^\infty(\Omega)$ such that $g_i(z_i) = 1, \|g_i\| < \epsilon$ (such g_i exists by Assumption 1). Then $g - \sum_i g(z_i)\phi_i g_i \in I(S)$ and

$$\left\| \sum_i g(z_i)\phi_i g_i \right\|_2 < \|g\|_{L^\infty(\Omega)} \sum_i \|\phi_i\|_{A^2(\Omega)}\epsilon.$$

Thus, $A^\infty(\Omega)\cap I(S)$ is dense in $A^\infty(\Omega)$, and since $A^\infty(\Omega)$ is dense in $A^2(\Omega)$, we are done. □

For $w \in \Omega$, we will denote by $K_w \in A^2(\Omega)$ the reproducing kernel of the Bergman space $A^2(\Omega)$. Thus $\langle g, K_w\rangle = g(w)$ for any $g \in A^2(\Omega)$. Also, denote by k_w the normalized Bergman kernel $\frac{K_w}{\|K_w\|}$.

Now we are redady to prove the main result of this section.

Theorem 2.6. *Suppose that domain $\Omega \subset \mathbb{C}^n$ satisfies Assumption 1. Let $f = (f_1, \dots, f_n) : \overline{\Omega} \to \mathbb{C}^n$ be a holomorphic mapping such that J_f is not identically 0. If J_f is nonzero on $f^{-1}(0) \cap \overline{\Omega}$, then*

$$\left(\sum f_i A^2(\Omega)\right)^\perp = \sum_{w\in f^{-1}(0)} \mathbb{C}K_w.$$

Proof. Let us put $S = f^{-1}(0) \cap \Omega = \{w_1, \dots, w_m\}$ and $S' = f^{-1}(0) \cap \partial\Omega$. It follows from Corollary 2.3 that

$$\sum_i f_i A^\infty(\Omega) = I(f^{-1}(0)) \cap A^\infty(\Omega).$$

Now we claim that

$$(A^2(\Omega) \cap I(S))^\perp = \sum_{w\in f^{-1}(0)} \mathbb{C}K_w.$$

Indeed, it is clear that $K_w \perp (A^2(\Omega)\cap I(S))$ for all $w \in S$. On the other hand, since $K_w, w \in S$ are linearly independent and codimension of $(A^2(\Omega)\cap I(S))$ in $A^2(\Omega)$ is at most $m = |S|$, we obtain the desired equality.

Thus it suffices to show that $\sum f_i A^2(\Omega)$ is dense in $A^2(\Omega) \cap I(S)$. It suffices to check that $I(f^{-1}(0)) \cap A^\infty(\bar{\Omega})$ is dense in $A^2(\Omega) \cap I(S)$ by Corollary 2.3. Let $f \in A^2(\Omega) \cap I(S)$, and let $f_n \in A^\infty(\bar{\Omega}) \cap I(S')$ be such that $\lim_{n\to\infty} f_n = f$ in $A^2(\Omega)$. Let $g_i, i = 1, \dots, m$ be polynomials such that $g_i(w_j) = \delta_{ij}, g_i(S') = 0$. Put $\phi_n = f_n - \sum_{i=1}^m f_n(w_i)g_i$. Then $\phi_n(w_j) = 0$ for all j, n. Also, for any $i, \lim_{n\to\infty} f_n(w_i) = 0$. Therefore, $\lim_{n\to\infty} \phi_n = f$ and $\phi_n \in I(f^{-1}(0)) \cap A^\infty(\bar{\Omega})$. So, $I(f^{-1}(0)) \cap A^\infty(\bar{\Omega})$ is dense in $A^2(\Omega) \cap I(S)$. $\square$

3. Some geometry related to Ω, f

In the rest of the paper, we will fix once and for all a domain $\Omega \subset \mathbb{C}^n$ satisfying Assumption 1 and a holomorphic mapping

$$f = (f_1, \dots, f_n) : \overline{\Omega} \to \mathbb{C}^n$$

in a neighbourhood of $\overline{\Omega}$ such that determinant of its Jacobian J_f is not identically 0. The goal of this section is to define a certain complex manofold W_f (Definition 3.2) and establish some of its basic properties in relation to the mapping f. It will play a crucial role in studying commutants of T_f.

Given a function $g : X \to Y$, we will denote by $X \times_g X$ the set

$$\{(z, w) \in X \times X | g(z) = g(w)\}.$$

Let us fix once and for all several notations related with Ω, f.

Notation 1. *Put*

$$Z = f^{-1}(f(V(J_f))), \quad \Omega' = \Omega \setminus Z,$$

where $V(J_f)$ is the zero locus of J_f in $\overline{\Omega}$. We will also put

$$\Omega'' = \Omega' \setminus f^{-1}(f(\partial\Omega)).$$

Thus, $\Omega'' \times_f \Omega'' \subset \Omega' \times_f \Omega'$ are n-dimensional complex manifolds. As usual $p_1, p_2 : \Omega' \times_f \Omega' \to \Omega'$ denote the projections on the first, second coordinate respectively. Clearly both p_1, p_2 are surjective finite-to-one locally biholomorphic mappings.

Remark that $f : \Omega'' \to f(\Omega'')$ is a proper locally biholomorphic mapping. Therefore it is a covering map. Also, Ω' is connected while Ω'' might not be.

In this setting we have the following result.

Lemma 3.1. *Let W be an open subset of $\Omega' \times_f \Omega'$ such that $p_1|_W : W \to \Omega'$ is a covering. Then $p_2|_W : W \to \Omega'$ is also a covering. In particular, $\partial(W) \subset \partial(\Omega') \times_f \partial(\Omega')$, and $p_1|_{\overline{W}}, p_2|_{\overline{W}} : \overline{W} \to \overline{\Omega} \setminus Z$ are coverings, where $\overline{W}$ denotes the closure of W in $(\overline{\Omega} \setminus Z) \times_f (\overline{\Omega} \setminus Z)$.*

Proof. Let $z \in \Omega'$. Let $X \subset \overline{\Omega}$ be a closed set of measure 0 such that $X \cap f^{-1}(f(z)) = \emptyset$ and $\Omega' \setminus X$ is simply connected. Since by assumption $p_1|_W \to \Omega'$ is a covering, Then the projection

$$p_1 : W \setminus p_1^{-1}(X) \to \Omega' \setminus X$$

is an m-fold trivial covering for some m. So there exist holomorphic embeddings $\rho_i : \Omega' \setminus X \to \Omega', 1 \le i \le m$ such that for any $u \in \Omega' \setminus X$ we have

$$p_1^{-1}(u) \cap W = \{(u, \rho_i(u)), 1 \le i \le m\}.$$

Put $U = \Omega' \setminus f^{-1}(f(X))$. Then $z \in U, f^{-1}(f(U)) \cap \Omega = U$ and $\Omega' \setminus U$ has measure 0. Since ρ_i induces a bijection on $f^{-1}(f(u)) \cap \Omega$ for all $u \in U$, it follows that $\rho_i : U \to U$ is a bijection for all $1 \le i \le m$. Remark that the set of bijections $\{\rho_i\}_{1 \le i \le m}$ is not closed under taking compositions or inverses.

Therefore

$$p_2 : p_2^{-1}(U) \cap W = \{(\rho_i^{-1}(z), z), z \in U, 1 \le i \le m\} \to U$$

is an m-fold trivial covering. Since U is a neighbourhood of z, we conclude that $p_2|_W : W \to \Omega'$ is a covering map.

Let $(a_n) = (z_n, w_n) \in W$ be a sequence in W converging to the boundary $\partial(W)$. Since $p_1|_W, p_2|_W : W \to \Omega'$ are proper mappings as shows above, we get that both $(z_n), (w_n)$ converge to $\partial(\Omega')$. Therefore,

$$\partial(W) \subset \partial(\Omega') \times_f \partial(\Omega').$$

Let $z' \in \partial(\Omega) \setminus Z$. Let $Y \subset \Omega'$ be a simply connected open subset such that $\overline{Y}$ contains a neighbourhood of z' in $\overline{\Omega}$. Just as above,
let $\rho_i : Y \to \Omega', 1 \le i \le m$ be holomorphic embeddings such that

$$p_1^{-1}(Y) \cap W = \{(y, \rho_i(y)), 1 \le i \le m, y \in Y\}.$$

Without loss of generality $\overline{\rho_i(Y)} \cap \overline{\rho_j(Y)} = \emptyset, i \ne j$. Thus, $(z', \rho_i(z'))$, $1 \le i \le m$ are distinct points in $p_1^{-1}(z') \cap \partial(W)$. By shrinking Y further we may assume that each ρ_i extends to a holomorphic embedding from a neighbourhood of $\overline{Y}$ into a neighbourhood of $\overline{\Omega}$. Now let $w \in \partial(\Omega) \setminus Z$ be such that $(z, w) \in \partial(W)$. Then, there is a sequence $(z_n, w_n) \in W$ converging to (z', w). We may assume that $z_n \in Y$ and $w_n = \rho_i(z_n)$ for a fixed i. So $w = \rho_i(z')$. Therefore

$$\overline{W} \cap p_1^{-1}(\overline{Y}) = \{(y, \rho_i(y)), y \in \overline{Y}, 1 \le i \le m\}$$

Hence $p_1|_{\overline{W}} : \overline{W} \to \overline{\Omega} \setminus Z$ is an m-fold covering. □

Now we are ready define a certain open subset $W_f \subset \Omega' \times_f \Omega'$ which is the main object of this section.

Definition 3.2. Let $W_f \subset \Omega' \times_f \Omega'$ be the union of all connected components W of $\Omega' \times_f \Omega'$ such that the projection $p_1|_W : W \to \Omega'$ is a covering map.

In particular, the diagonal $\Delta(\Omega') = \{(z, z), z \in \Omega'\}$ is a connected component of W_f.

The following lemma summarizes properties of W_f that will be used later.

Lemma 3.3. *W_f is symmetric and transitive: if $(z, w) \in W_f$ then $(w, z) \in W_f$, if $(z, t) \in W_f$ and $(t, s) \in W_f$, then $(z, w) \in W_f$.*

Proof. It follows directly from Lemma 3.1 that W_f is symmetric. Put

$$U = \{(z,w) \in \Omega' \times_f \Omega' | \exists t \in \Omega' \quad s.t(z,t) \in W_f, (t,w) \in W_f\}.$$

Clearly $p_1 : U \to \Omega'$ is locally a biholomorphic mapping. We will show that it is also a proper mapping. Indeed, let K be a compact subset of Ω'. Then $p_1|_{W_f}((p_2|W_f)^{-1}(K)) = K'$ is also compact. Hence $p_1|_{W_f}((p_2|W_f)^{-1}(K')) = K''$ is compact too. Therefore $(p_2|_U)^{-1}(K) \subset K'' \times K$ is compact. So $p_2 : U \to \Omega'$ is proper, therefore it is a covering map. Hence $U = W_f$. □

Remark that if $f : \Omega \to f(\Omega)$ is a proper mapping, then $p_1 : \Omega' \times_f \Omega' \to \Omega'$ is a covering, thus in this case $W_f = \Omega' \times_f \Omega'$.

4. Convolution algebras

The purpose of this section is to fix some notations and recall basics results related to convolution algebras of finite covering maps.

Let $f : X \to Y$ be a finite a finite covering map of topological spaces. Recall the standard notation

$$X \times_f X = \{(z,w) \in X \times X, f(z) = f(w)\}.$$

We have two projections

$$p_1, p_2 : X \times_f X \to X, p_1(z,w) = z, p_2(z,w) = w.$$

Let W be a symmetric, reflexive subset of $X \times_f X$ (see Lemma 3.3) such that $p_1|_W : W \to X$ is a covering map. Recall that in this setting $\mathbb{C}[W]$ ($\mathbb{C}$-valued continuous functions on W) is an associative algebra under the convolution product $\star$:

$$\phi \star \psi(z,w) = \sum_{(z,t),(t,w) \in W} \phi(z,t)\psi(t,w), \phi, \psi \in \mathbb{C}[W].$$

Given $g \in \mathbb{C}[W]$, one defines the corresponding weighted composition operator $S_g : \mathbb{C}[X] \to \mathbb{C}[X]$ as follows

$$S_g(\phi)(x) = \sum_{(x,w) \in W} g(x,w)\phi(w), \phi \in \mathbb{C}[X], x \in X.$$

This way $\mathbb{C}[X]$ becomes a left $(\mathbb{C}[W], \star)$-module. It is straightforward to check that S_g commutes with T_f, where $T_f : \mathbb{C}[X] \to \mathbb{C}[X]$ is the multiplication operator by f.

If in addition X, Y, W are complex manifolds and f is locally biholomorphic mapping, then $A(W)$ (the space of all holomorphic functions on W) is a subalgebra of $(\mathbb{C}[W], \star)$.

Definition 4.1. Let $f : X \to Y, W \subset X \times_f X$ be as above. We will denote by $\mathcal{A}(W)$ the algebra of all locally constant functions on W under the convolution product. If $f : X \to Y$ is a finite covering, then we will denote $\mathcal{A}(X \times_f X)$ by $\mathcal{A}(X, f)$.

If $f : X \to Y$ is a finite covering, and X, Y are path connected, locally simply connected spaces, then $\mathcal{A}(X, f)$ can be naturally identified with the Hecke algebra of all bi -$\pi_1(X)$-invariant $\mathbb{C}$-valued functions on $\pi_1(Y)$ under the convolution product. In particular, if $f : X \to Y$ is a normal covering, then $\mathcal{A}(X, f)$ is isomorphic to the group algebra $\mathbb{C}[\pi_1(Y)/f_*\pi_1(X)]$.

Let $Y' \subset Y$. Then $f : X' = f^{-1}(Y') \to Y'$ is a covering map, and we have an algebra homomorphism $\mathcal{A}(X, f) \to \mathcal{A}(X', f')$ given by the restriction of elements of $\mathcal{A}(X, f)$ on $X' \times_f X'$.

Let $f : M \to N$ be a finite covering map of connected real manifolds with boundaries. Then we get restrictions of f which are again covering maps

$$f : M \setminus \partial(M) \to N \setminus \partial(N), \quad f : \partial(M) \to \partial(N).$$

In this setting we have the following easy but useful lemma.

Lemma 4.2. *Suppose that $\partial(M)$(hence $\partial(N)$) is connected and $\pi_1(\partial(N))$ is Abelian. Then $\mathcal{A}(M, f) = \mathcal{A}(M \setminus \partial(M), f)$ is a commutative algebra.*

Proof. We have $\partial(M \times_f M) = \partial(M) \times_f \partial(M)$. Let X' be a connected component of $M \times_f M$. Then $p_1 : X' \to M$ is a covering map, hence $\partial(X')$ is a nonempty component of $\partial(M) \times_f \partial(M)$. Hence, if $\phi \in \mathcal{A}(M, f)$ is such that $\phi_{|}X' \neq 0$ then the image of ϕ in $\mathcal{A}(\partial(N), f)$ is nonzero on $\partial(X')$. So, $\mathcal{A}(M, f)$ embeds into $\mathcal{A}(\partial(M), f)$. Since $X' \setminus \partial(X') = X' \setminus (\partial(M) \times_f \partial(M))$ is connected, we obtain that $\mathcal{A}(M, f) = \mathcal{A}(M \setminus \partial(M), f)$. Since $\pi_1(\partial(N))$ is Abelian, $\partial M \to \partial N$ is a normal covering. Therefore

$$\mathcal{A}(\partial(M), f) = \mathbb{C}[\pi_1\partial(N)/\pi_1\partial(M)].$$

Hence $\mathcal{A}(\partial(M), f)$ is commutative. This implies that $\mathcal{A}(M \setminus \partial(M), f)$ is also commutative. □

5. Commutants of T_f

The goal of this section is to relate commutants of T_f to holomorphic functions on W_f (Definition 3.2). This will be achieved by Theorem 5.2. As an application we will show that there are no nonzero compact operators in the commutant of T_f (Theorem 5.4).

Recall notations from Notation 1. At first we will show the following preliminary

Lemma 5.1. *Let $S : A^2(\Omega) \to A^2(\Omega)$ be a bounded linear operator which commutes with T_f. Then there exists a function Φ on $\Omega' \times_f \Omega'$ such that for any $g \in A^2(\Omega)$ we have*

$$S(g)(z) = \sum_{w \in f^{-1}(f(z)) \cap \Omega} \Phi(z, w) g(w), \quad z \in \Omega'.$$

Moreover, Φ is holomorphic on $\Omega'' \times_f \Omega''$.

Proof. We claim that for any $z \in \Omega'$, we have

$$S^*(K_z) \in \sum_{w \in f^{-1}(f(z))} \mathbb{C}K_w.$$

Indeed, given $g_i \in A^\infty(\Omega)$, then

$$\bigcap Ker(T^*_{g_i}) = (\sum g_i A^2(\Omega))^\perp.$$

Applying this to $g_i = f_i - f_i(z)$, and using Theorem 2.6 we get that

$$\bigcap_i T^*_{f_i - f_i(z)} = \sum_{w \in f^{-1}(f(z))} \mathbb{C}K_w$$

and S^* preserves this space. In particular we may write

$$S^*(K_z) = \sum_{w \in f^{-1}(f(z))} \overline{\Phi(z,w)} K_w$$

for some $\Phi(z,w) \in \mathbb{C}$. Thus for any $g \in A^2(\Omega)$, we have

$$\langle g, S^*(K_w)\rangle = \langle S(g), K_w\rangle = S(g)(w) = \sum_{w \in f^{-1}(f(z))} \Phi(z,w) g(w).$$

Recall that $\Omega'' \to f(\Omega'')$ is a covering map. Thus, for any $z \in \Omega''$, there exists an open neighbourhood $z \in U \subset \Omega''$ and holomorphic embeddings $\rho_1, \dots, \rho_m : U \to \Omega$ such that

$$f^{-1}(f(z)) = \{\rho_1(z), \dots, \rho_m(z)\}, z \in U.$$

Denote $\Phi(z, \rho_i(z))$ by $\phi_i(z)$. Thus,

$$S(g)(z) = \sum_i \phi_i(z) g(\rho_i(z)), g \in A^2(\Omega), z \in U.$$

Fix $z \in U$. Let us choose polynomials $g_1, \dots, g_m \in \mathbb{C}[z_1, \dots, z_n]$ such that the matrix $A = g_i(\rho_j(z))$ is nondegenerate. Thus, its inverse is a holomorphic matrix in a neighbourhood of z. Therefore,

$$(\psi_i)_{1 \le i \le m} = A^{-1}(S(g_i)_{1 \le i \le m})$$

is holomorphic. So, Φ is holomorphic on $\Omega'' \times_f \Omega''$. □

The following is the main result of this section, which is well-known when Ω is a unit disc in $\mathbb{C}$ and f is a finite Blaschke product.

Theorem 5.2. *Suppose that a bounded linear operator $S : A^2(\Omega) \to A^2(\Omega)$ commutes with T_f. Then there exists a holomorphic function Φ on W_f such that for any $z \in \Omega', g \in A^2(\Omega)$ one has*

$$S(g)(z) = \sum_{(z,w) \in W_f} \Phi(z,w) g(w).$$

Proof. We know from Lemma 5.1 that there exists a function Φ on $\Omega' \times_f \Omega'$ such that

$$S(g)(z) = \sum_{w \in f^{-1}(f(z))} \Phi(z,w)g(w), z \in \Omega', g \in A^2(\Omega).$$

Moreover, Φ is holomorphic on $\Omega'' \times_f \Omega''$, where recall that

$$\Omega'' = \Omega' \setminus f^{-1}(f(\partial\Omega))).$$

Let us denote by W' the support of Φ in $\Omega' \times_f \Omega'$. We will prove that $p_1|_{W'} : W' \to \Omega'$ is a covering map.

Let $z \in \Omega'$. Let Ω_1 be a neighbourhood of $\overline{\Omega}$ such that f is extends to a holomorphic mapping on it. We will follow very closely Thomson's argument [Th]. Let $Y \subset \Omega'$ be a small neighbourhood of z, and let $\rho_1, \ldots, \rho_l : Y \to \Omega_1$ be holomorphic embeddings such that

$$f(\rho_i(w)) = w, \quad f^{-1}(f(w)) \cap \overline{\Omega} \subset \{\rho_i(z)_{1 \le i \le l}\}.$$

Let $P_z \subset \{1, \ldots, l\}$ be defined ass follows: $i \in P_z$ if there exists $w \in Y$ so that $\rho_i(w) \in \Omega$ and $\Phi(w, \rho_i(w)) \neq 0$. By making Y smaller if necessary, we may assume that $\overline{\rho_i(Y)} \cap \overline{\rho_j(Y)} = \emptyset$ for $i \neq j$. We claim that for all $i \in P_z, \rho_i(Y) \subset \Omega$. Indeed, suppose that for some i, $\rho_i(Y)$ is not a subset of Ω. Let $\epsilon > 0$ be such that

$$\epsilon < \frac{d(\rho_i(Y), \rho_j(Y))}{\sqrt{n}}, j \neq i.$$

For each $j \neq i$ let us pick k such that $|z_k - w_k| > \epsilon$ for all $z \in \rho_i(Y), w \in \rho_j(Y)$. For $w \in Y$, put

$$h_i^w(z) = \prod_{j \neq i} (z_k - \rho_j(w)_k) \in \mathbb{C}[z_1, \ldots, z_n].$$

Then $h_i^w(z)$ vanishes on $\rho_j(w), j \neq i$ and $h_i^w(\rho_i(w)) \neq 0$. It follows that $S(h_j^w(z))(w) = \langle h_j^w, S^* K_w \rangle$ is a holomorphic function on U. Then the function $S(h_i^w(z))(w) = \Phi(w, \rho_i(w)) h_i^w(\rho_i(w))$ is not identically 0, but vanishes on $\rho_i^{-1}(\Omega_1 \setminus \Omega)$, which contains a nonempty open subset by the assumption (recall that ρ_i is an open mapping). Hence $S(h_i^w(z))(w) = 0$ for all $w \in Y$, a contradiction.

To summarize, we have holomorphic embeddings $\rho_i : Y \to \Omega_1, 1 \le i \le l$ and a subset $P_z \subset \{1, \ldots, l\}$, such that $f(\rho_i(w)) = F(w), w \in Y$, and for any $i \in P_z, \rho_i(Y) \subset \Omega'$, there exists $w \in Y$, so that $\Phi(w, \rho_i(w)) \neq 0$. Moreover, $\Phi(w, \rho_j(w)) = 0$ for all $j \notin P_z$. Thus, for any $w \in Y$ we have

$$\{(w, \rho_i(w))_{i \in P_z}\} = p_1^{-1}(w) \cap W'.$$

Therefore $p_1|_{W'} : W' \to \Omega'$ is a covering. Hence, W' is a union of connected components of W_f. Let us extend Φ to W by 0 on $W \setminus W'$. Then for any

$g \in A^2(\Omega), z \in \Omega'$ we have

$$S(g)(z) = \sum_{(z,w)\in W_f} \Phi(z,w)g(w).$$

It can be shown that Φ is holomorphic exactly as in the end of the proof of Lemma 5.1. □

Before proceeding further, let us summarize various choices that we have made in relation to f, W_f.

Proposition 5.3.

(1) *There is an open subset $Y \subset \Omega'$ such that $\partial Y \cap \partial\Omega$ contains a nonempty subset of $\partial\Omega$. There are holomorphic embeddings*

$$\rho_i : \bar{Y} \to \overline{\Omega} \setminus Z, 1 \leq i \leq m$$

such that

$$p_1^{-1}(Y) \cap W_f = \{(y, \rho_i(y)), y \in Y, 1 \leq i \leq m\},$$
$$\rho_i(\partial(Y) \cap \partial\Omega) = \partial\Omega \cap \partial(\rho_i(Y)),$$
$$\rho_i(\bar{Y}) \cap \rho_j(\bar{Y}) = \emptyset, \quad i \neq j.$$

(2) *There is an open subset $U \subset \Omega'$, such that $\Omega \setminus U$ has measure 0 and biholomorphic mappings $\rho_i : U \to U, 1 \leq i \leq m$ such that*

$$p_1^{-1}(U) \cap W_f = \{(z, \rho_i(z)), z \in U, 1 \leq i \leq m\}.$$

Proof. Let $Y \subset \Omega \setminus Z$ be a an open subset such that $\bar{Y}$ is simply connected and $\partial(Y) \cap \partial\Omega$ contains an open subset of $\partial\Omega$. Thus $p_1 : p_1^{-1}(\bar{Y}) \cap \overline{W_f} \to \bar{Y}$ is a trivial covering. Therefore there exist holomorphic mappings

$$\rho_i : \bar{Y} \to \overline{\Omega} \setminus Z, 1 \leq i \leq m$$

such that

$$p_1^{-1}(Y) \cap W_f = \{(y, \rho_i(y)), y \in Y, 1 \leq i \leq m\}.$$

Recall that $\partial W_f \subset \partial\Omega \times_f \partial\Omega$. Therefore, $\rho_i(\bar{Y} \cap \partial\Omega) = \rho_i(\bar{Y}) \cap \partial\Omega$. By shrinking Y further, we get that $\rho_i(\bar{Y}) \cap \rho_j(\bar{Y}) = \emptyset, i \neq j$.

Part (2) follows directly from the proof of Lemma 3.2. □

Our next goal is to prove the following theorem.

Theorem 5.4. *Let $S : A^2(\Omega) \to A^2(\Omega)$ be a compact operator such that it commutes with T_f. Then $S = 0$.*

Before proving the theorem we will need to recall some facts about the asymptotic behaviour of the Bergman kernel function K_w as w approached the boundary of Ω.

The following statement follows immediately from the well-known localization property of the Bergman kernel [[Oh], Localization Lemma, page 2], combined with the transformation formula of the Bergman kernel function under a biholomorphic mapping.

Proposition 5.5. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain. Let $z^1, z^2 \in \partial\Omega$ and $z^1 \in U_1, z^2 \in U_2$ be open neighbourhoods, such that there exists a biholomorphic mapping $\rho : \overline{\Omega} \cap U_1 \to \overline{\Omega} \cap U_2$, so that $\rho(z^1) = z^2$. Then $\|K_w\| = O(\|K_{\rho(w)}\|), w \in U_1 \cap \Omega$ and $\lim_{w \to \partial\Omega} \|K_w\| = \infty$.*

We will also need the following standard fact. We include its proof for a reader's convenience. Recall that k_w denotes the normalized Bergman kernel function at w.

Lemma 5.6. *Let $\Omega \subset \mathbb{C}^n$ be a smooth bounded pseudoconvex domain. Then $k_w \to 0$ weakly as $w \to \partial\Omega$*

Proof. [1] Let $g \in A^2(\Omega)$. For $\epsilon > 0$ let $g^\epsilon \in A^\infty(\overline{\Omega})$ be such that

$$\|g - g^\epsilon\|_{A^2(\Omega)} < \epsilon.$$

Then we have

$$|\langle g, k_w \rangle| < \epsilon + \langle g^\epsilon, k_w \rangle \leq \epsilon + \|g^\epsilon\|_{L^\infty(\Omega)} / \|K_w\|_{A^2(\Omega)}$$

Therefore, $\limsup |\langle g, k_w \rangle| \leq \epsilon$ as $w \to \partial\Omega$. □

Now we are ready to prove Theorem 5.4.

Proof of Theorem 5.4. We will use notations from Proposition 5.3. It follows from Theorem 5.2 and its proof that there are holomorphic functions $\phi_i \in A(Y)$ such that

$$S(g(w)) = \sum_i \phi(w) g(\rho_i(w)), \quad w \in Y.$$

Next we will look at the two variable Berezin transform of S. Since S is a compact operator and since by Lemma 5.6 k_w weakly as $w \to \partial\Omega$, we have

$$\lim_{w_1, w_2 \to \partial\Omega} \frac{\langle S(K_{w_1}), K_{w_2} \rangle}{\|K_{w_1}\| \|K_{w_2}\|} = 0.$$

Recall $\epsilon > 0$, and functions $h_i^w(z) = \prod_{j \neq i} h_{ij}(z, w)$, from the proof of Theorem 5.2: here $h_{ij}(z, w) = (z_k - \rho_j(w)_k)$ is linear in z such that

$$|h_{ij}(z, w)| \geq \epsilon, \quad z \in \rho_i(Y), \quad w \in Y, \quad i \neq j.$$

Since Ω is bounded, there exists $M > 0$ such that $\|h_i^w(z)\| < M$ for all $i, z \in \Omega, w \in Y$. Thus, for all $w \in Y$.

$$|\langle S(h_i^w), K_w \rangle| \leq M \|S\| \|K_w\|.$$

Then,

$$\langle S(h_i^w), K_w \rangle = \sum_j \phi_j(w) h_i^w(\rho_i(w)) = \phi_i(w) \prod_{j \neq i} h_{ji}(\rho_j(w), \rho_i(w)).$$

[1]Communicated to us by S. Sahutoglu.

By our assumption

$$\prod_{j\neq i}|h_{ji}(\rho_j(w),\rho_i(w))|\geq \epsilon^{m-1}.$$

This implies that there is N such that $|K_{\rho_i(w)}(\rho_j(w))|<N$ for all $i\neq j, w\in Y$. Thus, there exists $L>0$, such that $\phi_i(w)\leq L||K_w||$ for all $i, w\in Y$.

We have

$$\langle S(K_{\rho_i(w)}),K_w\rangle=\sum_j \phi_j(w)K_{\rho_i(w)}(\rho_j(w)).$$

So, for $i\neq j$ we have

$$\lim_{w\to\partial\Omega\cap\partial U}\frac{\phi_j(w)K_{\rho_i(w)}(\rho_j(w))}{\|K_w\|\|K_{\rho_i(w)}\|}=0.$$

Therefore,

$$\lim_{w\to\partial\Omega\cap\partial Y}\frac{\phi_i(w)\|K_{\rho_i(w)}\|}{\|K_w\|}=0,$$

which by Proposition 5.5 implies that $\lim_{w\to\partial\Omega\cap\partial Y}\phi_i(w)=0$ for all i. This implies that $\phi_i=0$ for all i by the Boundary uniqueness theorem [Ch, p. 289]. □

As a consequence of Theorem 5.4 we have the following result about the commutant of T_f in the Toeplitz algebra of Ω.

Corollary 5.7. *Let $\Omega\subset\mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain. If S is an element of the Toeplitz algebra of Ω which commutes with T_f, then S is a multiplication operator by a bounded holomorphic function on Ω.*

The proof of Corollary 5.7 is based on compactness of the Hankel operators $H_{\bar\phi}, \phi\in A^\infty(\Omega)$ (follows from [[Pe], Theorem 1.2]), and the following well known identity relating Toeplitz and Hankel operators

$$[T_g,T_\phi]=H^*_{\bar g}H_\phi,\quad g\in H^\infty(\Omega),\phi\in L^\infty(\Omega).$$

Proof of Corollary 5.7. It follows from the preceding discussion that for any $g\in L^\infty(\Omega)$, the commutator $[T_{z_i},T_g]$ is compact for all $1\leq i\leq n$. Thus for any element S of the Toeplitz algebra of Ω, operators $[T_{z_i},S], 1\leq i\leq n$ are compact. If in addition S commutes with T_f, then $[T_{z_i},S], 1\leq i\leq n$ are compact operators in the commutant of T_f. Thus by Theorem 5.4 $[T_{z_i},S]=0$ for all i. Now by [SSU] $S=T_h$ for some $h\in H^\infty(\Omega)$. □

6. Commutants of $\{T_f,T_f^*\}$

In this section we will relate the commutant algebra of $\{T_f,T_f^*\}$ with the algebra $\mathcal{A}(W_f)$ (Definition 4.1).

The following assumption on the mapping f will play a key role.

Assumption 2. *Assume that $Z = f^{-1}(f(V(J_f)))$ is not dense in the Zariski topology of $\Omega:$ There exists a nonzero $g \in A^\infty(\Omega)$ such that $g(Z) = 0$.*

This assumption is satisfied if f is a rational mapping, if $n = 1$, or $f : \Omega \to f(\Omega)$ is a proper mapping [Ru].

The following is the main result of the paper.

Theorem 6.1. *Assume that Assumption 1 holds for Ω. Then the algebra of commutants of $\{T_f, T_f^*\}$ is isomorphic to a subalgebra of $\mathcal{A}(W_f)$. If in addition mapping f satisfies Assumption 2, then these algebras are isomorphic.*

Proof. Recall that $p_1|W_f : W_f \to \Omega'$ is a covering. From now on we will denote $p_1|W_f$ by p_1 for simplicity. Similarly, $p_2|_{W_f}$ will be abbreviated to p_2. We will define an algebra homomorphism

$$\iota : A(W_f) \to \mathrm{Hom}_{\mathbb{C}}(A(\Omega'), A(\Omega'))$$

as follows. Let $c \in A(W_f), \phi \in A(\Omega')$. We will define a holomorphic function $\iota_c(\phi) \in A(\Omega')$ in the following way. We put

$$\iota_c(\phi)(z) = \sum_{(z,w)\in W} c(z,w)\frac{J_f(z)}{J_f(w)}\phi(w), \quad z \in \Omega'.$$

Clearly $\iota_c(\phi) \in A(\Omega')$. It is straightforward to check that ι is an algebra homomorphism. To define $\iota_c(\phi)$ more explicitly we will use notations from Proposition 5.3 Recall that by the chain rule

$$J_{\rho_i}(z) = \frac{J_f(z)}{J_f(\rho_i(z))}.$$

Therefore

$$\iota_c(\phi)(z) = \sum_i c(z, \rho_i(z))J_{\rho_i}(z)\phi(\rho_i(z)), \quad z \in \Omega'.$$

In what follows given $g \in A(\Omega'), z \in \Omega'$, by $J_\rho g(\rho(z))$ we will denote the column vector $(J_{\rho_i}(z)g(\rho_i(z)))_{1\le i\le m}$ in $\mathbb{C}^m$. Now we follow very closely Guo–Huang [[GuoH], the proof of Proposition 3.4].

Lemma 6.2. *Suppose that $S : A^2(\Omega) \to A^2(\Omega)$ commutes with T_f. Let $U \subset \Omega'$ be as above. Then there exists a holomorphic mapping $\Phi : U \to gl_m(\mathbb{C})$ such that $J_\rho S(g)(\rho(z)) = \Phi(z)J_\rho g(\rho)(z)$.*

Proof. Using Theorem 5.2, there exists $c \in A(W)$ such that

$$S(g)(z) = \sum_i J_{\rho_i}(z)c(z, \rho_i(z))g(\rho_i(z)) = \sum_{(z,w)\in W} c(z,w)\frac{J_f(z)}{J_f(w)}g(w).$$

Then the i-th coordinate of the vector$J_\rho S(g)(\rho(z))$ is

$$\frac{J_f(z)}{J_f(w)}\sum_{\tau\in p_1^{-1}(w)}\frac{J_f(w)}{J_f(\tau)}c(w,\tau)g(\tau), \quad w = \rho_i(z).$$

Let us put $\Phi(z)_{jk} = c(\rho_j(z), \rho_k(z))$. Now it follows easily that

$$J_\rho S(g)(\rho(z)) = \Phi(z) J_\rho g(\rho)(z). \qquad \square$$

Now let assume that both S, S^* commute with T_f. Then by the above lemma there exist holomorphic mappings $\Phi, \Psi : U \to gl_m(\mathbb{C})$ such that

$$J_\rho S(g)(\rho(z)) = \Phi(z) J_\rho g(\rho)(z), J_\rho S^*(g)(\rho(z)) = \Psi(z) J_\rho g(\rho)(z).$$

Let $\lambda, \mu \in \Omega$. Given two polynomials $P, Q \in \mathbb{C}[x_1, \dots, x_n]$ we have

$$\langle P(T_f) S(K_\lambda), Q(T_f) K_\mu \rangle = \langle P(T_f)(K_\lambda), Q(T_f) S^*(K_\mu) \rangle.$$

So

$$\int_U P\bar{Q}(f)(z) S(K_\lambda) \bar{K_\mu} dV(z) = \int_U P\bar{Q}(F)(z)(K_\lambda) \overline{S^*(K_\mu)} dV(z)$$

Using the Stone–Weierstrass approximation, we see that for any $g \in C(\overline{F(\Omega)})$ one has

$$\int_U g(F(z)) S(K_\lambda) \bar{K_\mu} d_z V = \int_U g(F(z))(K_\lambda) \overline{S^*(K_\mu)} d_z V.$$

Thus the same equality holds for any $g \in L^\infty(\overline{F(\Omega)})$. This implies using change of variables that for all $z \in U$

$$\sum_j |J_{\rho_j}(z)|^2 S(K_\lambda)(\rho_j(z)) \overline{K_\mu(\rho_j(z))}$$
$$= \sum_j |J_{\rho_j}(z))|^2 K_\lambda(\rho_j(z)) \overline{S^*(K_\mu)(\rho_j(z))},$$

the latter equality can be rewritten as

$$\langle \Phi(z) J_\rho(z) K_\lambda(\rho(z)), J_\rho(z) K_\mu(\rho(z)) \rangle$$
$$= \langle J_\rho(z) K_\lambda(\rho(z)), \Psi(z) J_\rho(z) K_\mu(\rho(z)) \rangle,$$

where inner product is the standard one in $\mathbb{C}^m$. Next we will use the following simple lemma.

Lemma 6.3. *For any $z \in \Omega'$ vectors $\{J_\rho(z) K_\lambda(\rho(z))\}_{\lambda \in \Omega}$ span $\mathbb{C}^m$.*

Proof. Let vector $a = (a_i)_{i=1}^m \in \mathbb{C}^m$ be perpendicular to

$$\{J_\rho(z) K_\lambda(\rho(z))\}_{\lambda \in \Omega}.$$

Thus for all $\lambda \in \Omega$

$$0 = \sum_{i=1}^m a_i \overline{J_{\rho_i}(z) K_\lambda(\rho_i(z))} = \sum_{i=1}^m a_i \overline{J_{\rho_i}(z)} K_{\rho_i(z)}(\lambda).$$

Since $J_{\rho_i}(z) \neq 0$ and $K_{\rho_i(z)}, 1 \leq i \leq m$ are linearly independent, it follows that $a = 0$. $\square$

Now it follows from the above Lemma that $\Psi(z)$ is the adjoint of $\Phi(z)$. Since Φ, Ψ are holomorphic, it follows that Φ, Ψ are locally constant functions on U.

Thus, we conclude that if $S : A^2(\Omega) \to A^2(\Omega)$ is a bounded linear operator such that S, S^* commute with T_f, then there exists a locally constant function c on W_f, such that $S = \iota_c$. This implies that the algebra of commutants of $\{T_f, T_f^*\}$ is isomorphic to a subalgebra of $\mathcal{A}(W_f)$.

Now let us assume that Assumption 2 is satisfied. Therefore, by Bell's result $A^2(\Omega') = A^2(\Omega)$ [[Be], Removable singularity theorem]. Next, suppose that $c \in H^\infty(W_f)$ is bounded holomorphic function on W and $\phi \in A^2(\Omega)$. Then we claim that $\iota_c(\phi) \in A^2(\Omega)$. Indeed, it follows from the change of variables that for all $1 \le i \le m$

$$\|c(z, \rho_i(z)) J_{\rho_i}(z) \phi(\rho_i(z))\|_{L^2(U)} \le \|c\|_{L^\infty(W)} \|\phi\|_{L^2(\Omega')}.$$

Therefore,

$$\|\iota_c(\phi)\|_{A^2(\Omega')} \le m \|c\|_{L^\infty(W)} \|\phi\|_{L^2(\Omega')}.$$

Hence $\iota_c(\phi) \in A^2(\Omega)$.

Let $c \in \mathcal{A}(W)$. Put $c^*(z, w) = \overline{c(w, z)}, (z, w) \in W$. Let $\phi, \psi \in A^2(\Omega)$. We have

$$\begin{aligned}\langle \iota_c(\phi), \psi \rangle_{A^2(\Omega)} &= \sum_j \int_U c(z, \rho_j(z)) J_{\rho_j}(z) \phi(\rho_j(z)) \overline{\psi(z)} dV(z) \\ &= \sum_j \int_{\rho_j(U)} \phi(w) c({\rho_j}^{-1}(w), w) \overline{J_{{\rho_j}^{-1}}}(w) \overline{\psi({\rho_j}^{-1}(w))} dV(w),\end{aligned}$$

the latter equals to $\langle \phi, \iota_{c^*}(\psi) \rangle_{A^2(\Omega)}$. Thus, we have shown that for any $c \in \mathcal{A}(W)$, $\iota_c : A^2(\Omega) \to A^2(\Omega)$ is a bounded linear operator commuting with T_f. Moreover $(\iota_c)^* = \iota_{c^*}$. This concludes the proof of Theorem 6.1. □

As a consequence, we can reprove the following theorem of Douglas, Putinar and Wang [[DPW], Theorem 2.3].

Theorem 6.4. *Let $f \in A^\infty(D)$ be a finite Blaschke product on the unit disc D. Then the algebra of commutants of $\{T_f, T_f^*\}$ is isomorphic to*

$$\mathbb{C} \underbrace{\oplus \cdots \oplus}_{q} \mathbb{C},$$

where q equals the number of irreducible components of $D' \times_f D'$.

Proof. It follows from Definition 4.1 that $\dim_{\mathbb{C}} \mathcal{A}(D', f) = q$. The algebra of commutants of $\{T_f, T_f^*\}$ is isomorphic to $\mathcal{A}(D', f)$ by Theorem 6.1. But $\mathcal{A}(D', f)$ is isomorphic to a subalgebra of $\mathcal{A}(\partial(D), f)$ by Lemma 4.2, which is commutative since $\pi_1(\partial D) = \mathbb{Z}$ is Abelian. Thus, the algebra of commutants of $\{T_f, T_f^*\}$ is a q-dimensional commutative Von Neumann algebra, hence it must be isomorphic to $\mathbb{C} \underbrace{\oplus \cdots \oplus}_{q} \mathbb{C}$. □

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