Left determined model categories

Philippe Gaucher

ABSTRACT. Several methods for constructing left determined model structures are expounded. The starting point is Olschok's work on locally presentable categories. We give sufficient conditions to obtain left determined model structures on a full reflective subcategory, on a full coreflective subcategory and on a comma category. An application is given by constructing a left determined model structure on star-shaped weak transition systems.

CONTENTS

1. Introduction

Summary. The notion of combinatorial model category is a powerful framework for doing homotopy [Bek00] [Ros09]. It consists of a locally presentable category equipped with a cofibrantly generated model structure. Among them, there are the left determined ones in the sense of [RT03], that is the combinatorial model categories such that the class of weak equivalences is minimal with respect to a given class of cofibrations. The interest of constructing left determined model structures is that, for a given class of cofibrations, all other ones are left Bousfield localizations of the left determined one. J. H. Smith conjectured that for any locally presentable category and any set of maps I, there exists a left determined combinatorial model category such that the class of cofibrations is generated by I. This statement is a consequence of Vopěnka's principle [RT03, Theorem 2.2]. To our

knowledge, the conjecture is still open without assuming this large-cardinal axiom.

A remarkable step towards a proof of this conjecture is Olschok's paper [Ols09b]. The latter paper generalizes Cisinski's work about the homotopy theory of toposes [Cis02] to the framework of locally presentable categories. It proves the existence of this left determined model structure under reasonable hypotheses.

Several model structures are constructed in [Gau11], [Gau14], [Gau15b] using Olschok's work. The common pattern of all these constructions is to start from an application of Oslchok's theorem and to restrict the model structure to reflective and coreflective full subcategories.

We expound here in full generality these methods. This paper is written for two reasons:

(1) We will use these methods repeatedly in our studies of higher dimensional transition systems,[1] in particular in the companion paper [Gau15a].

(2) We hope that some people will find these methods useful and maybe generalizable.

This paper is therefore designed to be a toolbox. Not only are methods for obtaining left determined model structures on reflective and coreflective subcategories given in this paper, but also sufficient conditions for the standard model structure on a comma category to be left determined as well. This paper ends with an application to star-shaped weak transition systems.

Outline of the paper. Section 2 recalls Olschok's work and introduces the notion of Olschok model structure. When the associated cartesian cylinder is very good, we obtain a left determined model structure by choosing an empty set of generating anodyne cofibrations. There is nothing new in this section except Proposition 2.6. Section 3 explains how to restrict an Olschok model category to a full reflective subcategory. Theorem 3.1 encompasses all constructions made in [Gau11], [Gau14] and [Gau15b] on reflective subcategories. Section 4 explains how to restrict an Olschok model category to a full coreflective subcategory (Theorem 4.1, Theorem 4.3 and Theorem 4.4). Theorem 4.1 is implicitly used in [Gau11] and [Gau15b]: we prove the statement in full generality. Section 5 explains how to obtain Olschok model categories on comma categories (Theorem 5.8). Finally, Section 6 is devoted to an application of Theorem 5.8 and Theorem 4.3 to star-shaped weak transition systems. The last section is the only one which is specific to the theory of higher transition systems.

Prerequisites and notations. All categories are locally small. The set of maps in a category $\mathcal{K}$ from X to Y is denoted by $\mathcal{K}(X,Y)$. The class of maps of a category $\mathcal{K}$ is denoted by $\mathrm{Mor}(\mathcal{K})$. The composite of two maps is

---

[1]This is a work in progress belonging to the interface between algebraic topology and concurrency theory in computer science

denoted by fg instead of $f \circ g$. The initial (final resp.) object, if it exists, is always denoted by $\varnothing$ ($\mathbf{1}$ resp.). The identity of an object X is denoted by Id_X. A subcategory is always isomorphism-closed. Let f and g be two maps of a locally presentable category $\mathcal{K}$. Write $f\square g$ when f satisfies the *left lifting property* (LLP) with respect to g, or equivalently g satisfies the *right lifting property* (RLP) with respect to f. Let us introduce the notations $\mathbf{inj}_{\mathcal{K}}(\mathcal{C}) = \{g \in \mathcal{K}, \forall f \in \mathcal{C}, f\square g\}$ and $\mathbf{cof}_{\mathcal{K}}(\mathcal{C}) = \{f \in \mathcal{K}, \forall g \in \mathbf{inj}_{\mathcal{K}}(\mathcal{C}), f\square g\}$ where $\mathcal{C}$ is a class of maps of $\mathcal{K}$. We refer to [AR94] for locally presentable categories, and to [Ros09] for combinatorial model categories. We refer to [Hov99] and to [Hir03] for model categories. For general facts about weak factorization systems, see also [KR05]. The reading of the first part of [Ols09a], published in [Ols09b], is recommended for any reference about good, cartesian, and very good cylinders.

2. Olschok model category

This is a section recalling Olschok's construction and introducing thereby the notion of Olschok model category. Note that Proposition 2.6 is new.

2.1. Notation. For every map $f : X \to Y$ and every natural transformation $\alpha : F \to F'$ between two endofunctors of a locally presentable category $\mathcal{K}$, the map $f \star \alpha$ is defined by the diagram:
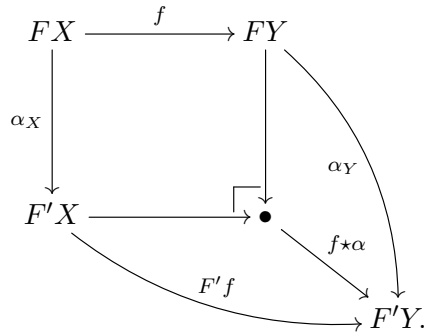
For a set of morphisms $\mathcal{A}$, let $\mathcal{A} \star \alpha = \{f \star \alpha, f \in \mathcal{A}\}$.

2.2. Definition. Let $\mathcal{K}$ be a locally presentable category. A cylinder is a triple $(\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}, \gamma : \mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Cyl}, \sigma : \mathrm{Cyl} \Rightarrow \mathrm{Id})$ consisting of a functor $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ and two natural transformations $\gamma = \gamma^0 \oplus \gamma^1 : \mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Cyl}$ and $\sigma : \mathrm{Cyl} \Rightarrow \mathrm{Id}$ such that the composite $\sigma\gamma$ is the codiagonal functor $\mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Id}$.

2.3. Definition. Let $\mathcal{K}$ be a locally presentable category. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a cofibrantly generated model structure on $\mathcal{K}$ where $\mathcal{C}$ is the class of cofibrations, $\mathcal{W}$ the class of weak equivalences and $\mathcal{F}$ the class of fibrations. A *cylinder* for $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a cylinder

$$(\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}, \gamma : \mathrm{Id} \oplus \mathrm{Id} \Rightarrow \mathrm{Cyl}, \sigma : \mathrm{Cyl} \Rightarrow \mathrm{Id})$$

such that the functorial map $\sigma_X : \mathrm{Cyl}(X) \to X$ belongs to $\mathcal{W}$ for every object X. The cylinder is *good* if the functorial map $\gamma_X : X \sqcup X \to \mathrm{Cyl}(X)$ is a cofibration for every object X. It is *very good* if moreover the map $\sigma_X : \mathrm{Cyl}(X) \to X$ is a trivial fibration for every object X. A good cylinder is *cartesian* if:

- The functor $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ has a right adjoint $\mathrm{Path} : \mathcal{K} \to \mathcal{K}$ called the *path functor*.
- If f is a cofibration, then so are $f \star \gamma^0$, $f \star \gamma^1$ and $f \star \gamma$.

The notions of Definition 2.3 can be adapted to a cofibrantly generated weak factorization system $(\mathcal{L}, \mathcal{R})$ by considering the combinatorial model structure $(\mathcal{L}, \mathrm{Mor}(\mathcal{K}), \mathcal{R})$. A cylinder with respect to a set of maps I is a cylinder for the weak factorization system $(\mathbf{cof}_{\mathcal{K}}(I), \mathbf{inj}_{\mathcal{K}}(I))$, i.e., for the model structure $(\mathbf{cof}_{\mathcal{K}}(I), \mathrm{Mor}(\mathcal{K}), \mathbf{inj}_{\mathcal{K}}(I))$.

2.4. Notation. Let I and S be two sets of maps of a locally presentable category $\mathcal{K}$. Let $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ be a cylinder with respect to I. Denote by $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$ the set of maps defined as follows:

- $\Lambda_{\mathcal{K}}^0(\mathrm{Cyl}, S, I) = S \cup (I \star \gamma^0) \cup (I \star \gamma^1)$
- $\Lambda_{\mathcal{K}}^{n+1}(\mathrm{Cyl}, S, I) = \Lambda_{\mathcal{K}}^n(\mathrm{Cyl}, S, I) \star \gamma$
- $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I) = \bigcup_{n \geqslant 0} \Lambda_{\mathcal{K}}^n(\mathrm{Cyl}, S, I)$.

Let us denote by $\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I)$ the class of maps $f : X \to Y$ of $\mathcal{K}$ such that for every object T which is $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$-injective, the induced set map $\mathcal{K}(Y, T)/\simeq \xrightarrow{\cong} \mathcal{K}(X, T)/\simeq$ is a bijection, where $\simeq$ means the homotopy relation associated with the cylinder $\mathrm{Cyl}(-)$, i.e., for all maps $f, g : X \to Y$, $f \simeq g$ is equivalent to the existence of a *homotopy* $H : \mathrm{Cyl}(X) \to Y$ with $H\gamma^0 = f$ and $H\gamma^1 = g$.

2.5. Theorem (Olschok)**.** *Let $\mathcal{K}$ be a locally presentable category. Let I be a set of maps of $\mathcal{K}$. Let $S \subset \mathbf{cof}_{\mathcal{K}}(I)$ be a set of maps of $\mathcal{K}$. Let Cyl be a cartesian cylinder for the weak factorization system $(\mathbf{cof}_{\mathcal{K}}(I), \mathbf{inj}_{\mathcal{K}}(I))$. Suppose that the weak factorization system $(\mathbf{cof}_{\mathcal{K}}(I), \mathbf{inj}_{\mathcal{K}}(I))$ is cofibrant, i.e., for any object X of $\mathcal{K}$, the canonical map $\varnothing \to X$ belongs to $\mathbf{cof}_{\mathcal{K}}(I)$. Then there exists a unique combinatorial model category structure with class of cofibrations $\mathbf{cof}_{\mathcal{K}}(I)$ such that the fibrant objects are the $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$-injective objects. The class of weak equivalences is $\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I)$. All objects are cofibrant.*

Proof. The explanation is already given in [Gau14, Theorem 2.6]. This is a slight modification of Olschok's main theorem [Ols09b, Theorem 3.16] using the characterization of fibrant objects [Ols09b, Lemma 3.30(c)] and the fact that a model structure is characterized by its class of cofibrations and its class of fibrant objects: [Hir03, Theorem 7.8.6] works here since all objects are cofibrant; more generally [Joy, Proposition E.1.10] can be used. □

If the cylinder is very good in Theorem 2.5, then $\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I)$ is the Grothendieck localizer generated by S (with respect to the class of cofibrations $\mathbf{cof}_{\mathcal{K}}(I)$) by [Ols09b, Corollary 4.6]. In this case, $\mathcal{K}$ is left determined in the sense of [RT03] when $S = \varnothing$. And the model category we obtain for $S \neq \varnothing$ is the Bousfield localization $\underline{\mathbf{L}}_S \mathcal{K}$ of the left determined one by the set of maps S.

2.6. Proposition. *Let $\mathcal{K}$ be a combinatorial model category such that all objects are cofibrant. Let I be the set of generating cofibrations. Let*

$$\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$$

be a cartesian cylinder for the weak factorization system $(\mathbf{cof}_{\mathcal{K}}(I), \mathbf{inj}_{\mathcal{K}}(I))$. Let $S \subset \mathbf{cof}_{\mathcal{K}}(I)$ be a set of maps of $\mathcal{K}$. Then the following conditions are equivalent:

- *An object of $\mathcal{K}$ is fibrant if and only if it is $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$-injective.*
- *A map of $\mathcal{K}$ is a weak equivalence if and only if it belongs to*

$$\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I).$$

Proof. Let us suppose that the fibrant objects of $\mathcal{K}$ are the $\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I)$-injective ones. Then the model structure of $\mathcal{K}$ and the one given by Theorem 2.5 have the same class of cofibrations and the same class of fibrant objects. Since all objects are cofibrant, the class of weak equivalences is necessarily $\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I)$ by [Hir03, Theorem 7.8.6]. Conversely, let us suppose that a map of $\mathcal{K}$ is a weak equivalence if and only if it belongs to $\mathcal{W}_{\mathcal{K}}(\mathrm{Cyl}, S, I)$. Then the model structure of $\mathcal{K}$ and the one given by Theorem 2.5 have the same class of cofibrations and the same class of weak equivalences. The class of fibrations is determined by the class of trivial cofibrations. Therefore the two model structures are equal. So they have the same class of fibrant objects. $\square$

2.7. Definition. An Olschok model category is a combinatorial model category satisfying the conditions of Proposition 2.6 for some cartesian cylinder Cyl and some set of cofibrations S called the generating anodyne cofibrations.

The terminology "anodyne" comes from [Cis02] where the elements of the class

$$\mathbf{cof}_{\mathcal{K}}(\Lambda_{\mathcal{K}}(\mathrm{Cyl}, S, I))$$

are called, in French, "extensions anodines". When the class of generating anodyne cofibrations is not specified, it is supposed to be empty.

3. Restriction to a reflective subcategory

The following theorem gives a sufficient condition for the restriction to a full reflective subcategory of an Olschok model category to be an Olschok model category. It implies [Gau14, Theorem 9.3] and [Gau15b, Theorem 5.5] because in the latter cases the map $\eta_{\mathrm{Cyl}(X)}$ is an isomorphism.
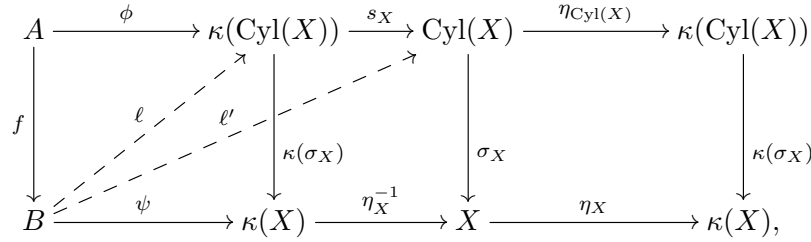
FIGURE 1. $\kappa \operatorname{Cyl}$ is very good.

3.1. Theorem. *Let $\mathcal{K}$ be an Olschok model category with generating cofibrations I, with generating anodyne cofibrations S and with cartesian cylinder Cyl. Let $\mathcal{A}$ be a full reflective locally presentable subcategory and let $\kappa : \mathcal{K} \to \mathcal{A}$ be the reflection. Suppose that $I = \kappa(I)$ (i.e., the source and targets of all maps of I belong to $\mathcal{A}$), that $\operatorname{Path}(\mathcal{A}) \subset \mathcal{A}$ where $\operatorname{Path} : \mathcal{A} \to \mathcal{A}$ is a right adjoint of $\operatorname{Cyl} : \mathcal{A} \to \mathcal{A}$, and that the unit map $\eta_{\operatorname{Cyl}(X)} : \operatorname{Cyl}(X) \to \kappa(\operatorname{Cyl}(X))$ has a section s_X (i.e., it is split epic) for all objects X of $\mathcal{A}$. Then:*

1. *The functor $\kappa \operatorname{Cyl} : \mathcal{A} \to \mathcal{A}$ is a cartesian cylinder with respect to $\kappa(I)$. Moreover if $\operatorname{Cyl} : \mathcal{K} \to \mathcal{K}$ is very good, then $\kappa \operatorname{Cyl} : \mathcal{A} \to \mathcal{A}$ is very good as well.*
2. *There exists a unique Olschok model structure on $\mathcal{A}$ with set of generating cofibrations $\kappa(I) = I$, with set of generating anodyne cofibrations $\kappa(S)$, such that an object of $\mathcal{A}$ is fibrant in $\mathcal{A}$ if and only if it is fibrant in $\mathcal{K}$. The cartesian cylinder of $\mathcal{A}$ is the functor $\kappa \operatorname{Cyl} : \mathcal{A} \to \mathcal{A}$. The reflection $\kappa : \mathcal{K} \to \mathcal{A}$ is a homotopically surjective (in the sense of [Dug01, Definition 3.1]) left Quillen adjoint.*

Note that the existence of the section is only used to prove the left-determinedness of the model structure of $\mathcal{A}$.

Proof. By [Ols09b, Lemma 5.2(c)], the functor $\kappa \operatorname{Cyl} : \mathcal{A} \to \mathcal{A}$ is a cartesian cylinder with respect to $\kappa(I) = I$. An object of $\mathcal{A}$ is fibrant in $\mathcal{A}$ if and only if it is fibrant in $\mathcal{K}$ by [Ols09b, Lemma 5.2(b)]. The existence of the Olschok model structure is then a consequence of Theorem 2.5. The proof of the fact that the reflection $\kappa : \mathcal{K} \to \mathcal{A}$ is a homotopically surjective left Quillen functor is mutatis mutandis the argument used for the same fact in [Gau14, Theorem 9.3].

Suppose now that Cyl is a very good cylinder with respect to I. Consider the diagram of solid arrows of $\mathcal{A}$ of Figure 1 where X is an object of $\mathcal{A}$ (this implies that η_X is invertible), where $f : A \to B$ belongs to I, and where the left-hand square is supposed to be commutative, i.e., $\kappa(\sigma_X)\phi = \psi f$. The right-hand square is commutative by naturality of the unit map of the

adjunction. One has

$$\begin{aligned}
\sigma_X s_X &= \eta_X^{-1}\eta_X\sigma_X s_X && \text{since } \eta_X \text{ is invertible} \\
&= \eta_X^{-1}\kappa(\sigma_X)\eta_{\mathrm{Cyl}(X)}s_X && \text{by naturality of the unit map} \\
&= \eta_X^{-1}\kappa(\sigma_X) && \text{by hypothesis on } s_X.
\end{aligned}$$

This means that the middle square is commutative as well. One deduces that the composite of the left-hand square and the middle square is a commutative square, i.e., $\sigma_X s_X \phi = \eta_X^{-1}\psi f$. Since Cyl is a very good cylinder of $\mathcal{K}$ with respect to I, there exists a lift $\ell' : B \to \mathrm{Cyl}(X)$ such that $\sigma_X \ell' = \eta_X^{-1}\psi$ and $\ell' f = s_X \phi$. Let $\ell = \eta_{\mathrm{Cyl}(X)}\ell'$. One has

$$\begin{aligned}
\kappa(\sigma_X)\ell &= \kappa(\sigma_X)\eta_{\mathrm{Cyl}(X)}\ell' && \text{by definition of } \ell \\
&= \eta_X \sigma_X \ell' && \text{since the right-hand square is commutative} \\
&= \eta_X \eta_X^{-1}\psi && \text{by definition of } \ell' \\
&= \psi && \text{by trivial simplification.}
\end{aligned}$$

And one has

$$\begin{aligned}
\ell f &= \eta_{\mathrm{Cyl}(X)}\ell' f && \text{by definition of } \ell \\
&= \eta_{\mathrm{Cyl}(X)}s_X \phi && \text{by definition of } \ell' \\
&= \phi && \text{by hypothesis on } s_X.
\end{aligned}$$

Therefore ℓ is a lift for the left-hand square. Hence the cylinder $\kappa\,\mathrm{Cyl} : \mathcal{A} \to \mathcal{A}$ is very good with respect to I. The proof is complete. $\qquad\square$

3.2. Corollary. *With the notations of Theorem* 3.1*, there exists a Bousfield localization of $\mathcal{K}$ which is Quillen equivalent to $\mathcal{A}$.*

3.3. Corollary. *With the notations of Theorem* 3.1*, the inclusion $\mathcal{A} \subset \mathcal{K}$ reflects weak equivalences.*

Proof. Let $f : X \to Y$ be a map of $\mathcal{A}$ which is a weak equivalence of $\mathcal{K}$. Then for any fibrant object F of $\mathcal{K}$, the set map $\mathcal{K}(Y, F) \to \mathcal{K}(X, F)$ induced by composing by f gives rise to a bijection between the homotopy classes. Since the fibrant objects of $\mathcal{A}$ are the fibrant objects of $\mathcal{K}$ belonging to $\mathcal{A}$, this implies that f is a weak equivalence of $\mathcal{A}$. $\qquad\square$

4. Restriction to a coreflective subcategory

The following theorem is the general theorem behind the construction of the homotopy theory of cubical transition systems in [Gau11].

4.1. Theorem. *Let $\mathcal{K}$ be an Olschok model category with cartesian cylinder* Cyl *and with set of generating cofibrations I. Let $\mathcal{A}$ be a full coreflective locally presentable subcategory such that:*

- *There exists a set of maps J such that $\mathbf{cof}_{\mathcal{A}}(J) = \mathbf{cof}_{\mathcal{K}}(I) \cap \mathrm{Mor}(\mathcal{A})$.*
- $\mathrm{Cyl}(\mathcal{A}) \subset \mathcal{A}$.

Then there exists a structure of Olschok model category on $\mathcal{A}$ such that the cofibrations are the cofibrations of $\mathcal{K}$ between objects of $\mathcal{A}$ and such that the restriction to $\mathcal{A}$ of Cyl *is a cartesian cylinder for this model structure. Moreover, if* Cyl *is very good in $\mathcal{K}$, then its restriction to $\mathcal{A}$ gives rise to a very good cylinder in $\mathcal{A}$.*

Proof. The set J will be the set of generating cofibrations of the Olschok model category $\mathcal{A}$. Let A be an object of $\mathcal{A}$. Consider the factorization of the codiagonal of A given by this cylinder:

$$A \sqcup A \xrightarrow{\quad \gamma_A \quad} \mathrm{Cyl}(A) \xrightarrow{\quad \sigma_A \quad} A.$$

By hypothesis, $\mathrm{Cyl}(A)$ is an object of $\mathcal{A}$. Therefore γ_A is a cofibration of $\mathcal{A}$. So the restriction of Cyl to $\mathcal{A}$ gives rise to a good cylinder. Let $f : A \to B$ be a cofibration of $\mathcal{A}$. Then the maps $f \star \gamma^\epsilon : B \sqcup_A \mathrm{Cyl}(A) \longrightarrow B$ for $\epsilon = 0, 1$ and $f \star \gamma : (B \sqcup B) \sqcup_{A \sqcup A} \mathrm{Cyl}(A) \longrightarrow B \sqcup B$ are cofibrations of $\mathcal{K}$ since Cyl is a cartesian cylinder. The sources and the targets of these maps belong to $\mathcal{A}$ since $\mathcal{A}$ is a coreflective subcategory. So the maps $f \star \gamma^\epsilon$ for $\epsilon = 0, 1$ and $f \star \gamma$ are cofibrations of $\mathcal{A}$. Let A and B be two objects of $\mathcal{A}$. Then

$$
\begin{aligned}
\mathcal{A}(\mathrm{Cyl}(A), B) &= \mathcal{K}(\mathrm{Cyl}(A), B) && \text{since } \mathcal{A} \text{ is a full subcategory} \\
&= \mathcal{K}(A, \mathrm{Path}(B)) && \text{where Path is a right adjoint of Cyl} \\
&= \mathcal{A}(A, \xi(\mathrm{Path}(B))) && \text{where } \xi \text{ is the coreflection.}
\end{aligned}
$$

This implies that the restriction of Cyl to $\mathcal{A}$ gives rise to a cartesian cylinder. The proof of the existence of the model structure is complete thanks to Theorem 2.5.

Let us suppose now that Cyl is very good in $\mathcal{K}$. Then for every object A of $\mathcal{A}$, the map $\sigma_A : \mathrm{Cyl}(A) \to A$ is a trivial fibration of $\mathcal{K}$ which satisfies the RLP with respect to any cofibration of $\mathcal{K}$. Since the cofibrations of $\mathcal{A}$ are exactly the cofibrations of $\mathcal{K}$ between objects of $\mathcal{A}$, the map $\sigma_A : \mathrm{Cyl}(A) \to A$ is a trivial fibration of $\mathcal{A}$ as well. $\square$

4.2. Theorem. *With the notations and hypotheses of Theorem 4.1, assume that the set S of generating anodyne cofibrations of $\mathcal{K}$ belongs to $\mathcal{A}$. Let us equip $\mathcal{A}$ with the Olschok model structure having the same set of generating anodyne cofibrations. Then the inclusion functor $\mathcal{A} \to \mathcal{K}$ is a left Quillen functor.*

Proof. It is mutatis mutandis the proof of [Gau11, Theorem 6.3]. $\square$

Theorem 4.1 has the following corollaries:

4.3. Theorem. *Let $\mathcal{K}$ be an Olschok model category with cartesian cylinder* Cyl*. Let $\mathcal{A}$ be a full coreflective subcategory such that:*

- *$\mathcal{A}$ is a small cone-injectivity class with respect to a set of cofibrations of $\mathcal{K}$.*
- *$\mathrm{Cyl}(\mathcal{A}) \subset \mathcal{A}$.*

Then there exists a structure of Olschok model category on $\mathcal{A}$ such that the cofibrations are the cofibrations of $\mathcal{K}$ between objects of $\mathcal{A}$ and such that the restriction to $\mathcal{A}$ of Cyl *is a cartesian cylinder for this model structure. Moreover, if* Cyl *is very good in $\mathcal{K}$, then its restriction to $\mathcal{A}$ gives rise to a very good cylinder in $\mathcal{A}$.*

Note that this is the theorem used in [Gau11].

Proof. Since $\mathcal{A}$ is full coreflective, it is cocomplete. And since it is a small cone-injectivity class, it is accessible by [AR94, Proposition 4.16]. Therefore $\mathcal{A}$ is locally presentable. Let I be the set of generating cofibrations of $\mathcal{K}$. By [Gau11, Theorem A.5], there exists a set of maps J such that

$$\mathbf{cof}_{\mathcal{A}}(J) = \mathbf{cof}_{\mathcal{K}}(I) \cap \mathrm{Mor}(\mathcal{A}).$$

We can then apply Theorem 4.1.   □

4.4. Theorem. *Let $\mathcal{K}$ be an Olschok model category with cartesian cylinder* Cyl *with set of generating cofibrations I. Let $\mathcal{A}$ be a full coreflective locally presentable subcategory such that:*

- *I has a solution set $J \subset \mathbf{cof}_{\mathcal{K}}(I)$ with respect to $\mathcal{A}$, i.e., J is a set of maps of $\mathcal{A}$ such that every map $i \to w$ of $\mathrm{Mor}(\mathcal{K})$ from $i \in I$ to $w \in \mathrm{Mor}(\mathcal{A})$ factors as a composite $i \to j \to w$ with $j \in J$.*
- *$\mathrm{Cyl}(\mathcal{A}) \subset \mathcal{A}$.*

Then there exists a structure of Olschok model category on $\mathcal{A}$ such that the cofibrations are the cofibrations of $\mathcal{K}$ between objects of $\mathcal{A}$ and such that the restriction to $\mathcal{A}$ of Cyl *is a cartesian cylinder for this model structure. Moreover, if* Cyl *is very good in $\mathcal{K}$, then its restriction to $\mathcal{A}$ gives rise to a very good cylinder in $\mathcal{A}$.*

Proof. By [Gau11, Lemma A.3], there is the equality

$$\mathbf{cof}_{\mathcal{A}}(J) = \mathbf{cof}_{\mathcal{K}}(I) \cap \mathrm{Mor}(\mathcal{A}).$$

We can then apply Theorem 4.1.   □

5. Olschok model category and comma category

The following well-known proposition introduces some useful notations:

5.1. Proposition. *Let $\mathcal{K}$ be a locally presentable category. Let i be an object of $\mathcal{K}$. The forgetful functor $\omega^i : i{\downarrow}\mathcal{K} \to \mathcal{K}$ defined on objects by $\omega^i(i \to X) = X$ and on maps by $\omega^i(i \to f) = f$ is a right adjoint. In particular, it is limit-preserving. A colimit in the comma category $i{\downarrow}\mathcal{K}$ is obtained by taking the colimit in $\mathcal{K}$ of the cone with top the object i and with basis the diagram of underlying objects of $\mathcal{K}$. The forgetful functor $\omega^i : i{\downarrow}\mathcal{K} \to \mathcal{K}$ commutes with colimits of connected diagrams (and in particular, it is accessible).*

Note that the forgetful functor $\omega^i : (i{\downarrow}\mathcal{K}) \to \mathcal{K}$ does not preserve binary coproducts. Indeed, the binary coproduct of $i \to X$ and $i \to Y$ is the amalgamated sum $i \to X \sqcup_i Y$.

Proof. The left adjoint $\rho^i : \mathcal{K} \to i{\downarrow}\mathcal{K}$ is defined on objects by $\rho^i(X) = (i \to i \sqcup X)$ and on morphisms by $\rho^i(f) = \mathrm{Id}_i \sqcup f$. The last assertions are clear. $\qquad\square$

Let $\mathcal{K}$ be a locally presentable category. Let i be an object of $\mathcal{K}$. Then the comma category $i{\downarrow}\mathcal{K}$ is locally presentable by [AR94, Proposition 1.57]. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a cofibrantly generated model structure on $\mathcal{K}$. Then the triple

$$((\omega^i)^{-1}(\mathcal{C}), (\omega^i)^{-1}(\mathcal{W}), (\omega^i)^{-1}(\mathcal{F}))$$

is a cofibrantly generated model structure on $i{\downarrow}\mathcal{K}$ by [Hir15, Theorem 2.7]. If I is the set of generating cofibrations of $\mathcal{K}$, then the set of generating cofibrations of the comma category $i{\downarrow}\mathcal{K}$ is the set $\rho^i(I)$ where $\rho^i : \mathcal{K} \to i{\downarrow}\mathcal{K}$ is the left adjoint of the functor ω^i above defined.

5.2. Lemma. *Let* $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ *be a cylinder functor of a locally presentable category* $\mathcal{K}$. *Assume that it has a right adjoint* $\mathrm{Path} : \mathcal{K} \to \mathcal{K}$. *Let i be an object of $\mathcal{K}$. Define the functor* $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ *by the natural pushout diagram*

$$
\begin{array}{ccc}
\mathrm{Cyl}(i) & \xrightarrow{\quad\sigma_i\quad} & i \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Cyl}_i(i\to X)} \\
\mathrm{Cyl}(X) & \longrightarrow & \omega^i(\mathrm{Cyl}_i(i \to X))
\end{array}
$$

for every object X of $\mathcal{K}$ and $\mathrm{Path}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ *by the natural diagram*

$$\mathrm{Path}_i(i \to Y) := i \longrightarrow \mathrm{Path}(i) \longrightarrow \mathrm{Path}(Y)$$

for every object $i \to Y$ of $i{\downarrow}\mathcal{K}$ where $i \longrightarrow \mathrm{Path}(i)$ is the map corresponding to $\sigma_i : \mathrm{Cyl}(i) \to i$ by the adjunction. Then $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ *is left adjoint to* $\mathrm{Path}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$.

Note that it can be easily checked that the functor $\mathrm{Path}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ is accessible and limit-preserving. Therefore, by [AR94, Theorem 1.66], it has a left adjoint since the category $i{\downarrow}\mathcal{K}$ is locally presentable.

Proof. Let $i \to X$ and $i \to Y$ be fixed. By definition of Cyl_i, there is a bijection between the sets of commutative diagrams

$$
\left\{
\begin{array}{ccc}
i & == & i \\
\downarrow & & \downarrow \\
\omega^i(\mathrm{Cyl}_i(i \to X)) & \longrightarrow & Y
\end{array}
\right\}
\cong
\left\{
\begin{array}{ccc}
\mathrm{Cyl}(i) & \longrightarrow & i \\
\downarrow & & \downarrow \\
\mathrm{Cyl}(X) & \longrightarrow & Y
\end{array}
\right\}.
$$

By adjunction, there is a bijection between the sets of commutative diagrams

$$\left\{\begin{array}{ccc} \mathrm{Cyl}(i) & \longrightarrow & i \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(X) & \longrightarrow & Y \end{array}\right\} \cong \left\{\begin{array}{ccc} i & \longrightarrow & \mathrm{Path}(i) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Path}(Y) \end{array}\right\}.$$

Finally, by the definition of Path_i, there is a bijection between the sets of commutative diagrams

$$\left\{\begin{array}{ccc} i & \longrightarrow & \mathrm{Path}(i) \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{Path}(Y) \end{array}\right\} \cong \left\{\begin{array}{ccc} i & == & i \\ \downarrow & & \downarrow \\ X & \longrightarrow & \omega^i(\mathrm{Path}_i(Y)) \end{array}\right\}. \qquad \square$$

5.3. Lemma. *Let $\mathcal{K}$ be a locally presentable category. Let i be an object of $\mathcal{K}$. Let $s : A \to B$ be a map of $\mathcal{K}$. Let $i \to X$ be an object of the comma category $i{\downarrow}\mathcal{K}$. Then $i \to X$ is injective with respect to $\rho^i(s) : i \sqcup A \to i \sqcup B$ if and only if $X = \omega^i(i \to X)$ is injective with respect to s.*

Proof. One has the commutative diagram of sets:

$$\begin{array}{ccccc}
\mathcal{K}(B,X) & == & \mathcal{K}(B,\omega^i(i \to X)) & \xrightarrow{\;\cong\;} & (i{\downarrow}\mathcal{K})(\rho^i(B),X) \\
\downarrow{\scriptstyle f \mapsto fs} & & \downarrow{\scriptstyle f \mapsto fs} & & \downarrow{\scriptstyle g \mapsto g\rho^i(s)} \\
\mathcal{K}(A,X) & == & \mathcal{K}(A,\omega^i(i \to X)) & \xrightarrow{\;\cong\;} & (i{\downarrow}\mathcal{K})(\rho^i(A),X).
\end{array}$$

Therefore the left vertical arrow is onto if and only if the right vertical arrow is onto as well. $\qquad \square$

5.4. Corollary. *Let Λ be a set of maps of a locally presentable category $\mathcal{K}$. Let i be an object of $\mathcal{K}$. Then an object $i \to X$ of $i{\downarrow}\mathcal{K}$ is $\rho^i(\Lambda)$-injective if and only if X is Λ-injective.*

5.5. Lemma. *With the notations and hypotheses of Lemma 5.2, let A be an object of $\mathcal{K}$. Then there is the natural isomorphism*

$$\mathrm{Cyl}_i(\rho^i(A)) \cong \rho^i(\mathrm{Cyl}(A)).$$

Proof. One has the bijections

$$\begin{aligned}
(i{\downarrow}\mathcal{K})(\mathrm{Cyl}_i(\rho^i(A)), i \to B) &\cong (i{\downarrow}\mathcal{K})(\rho^i(A), \mathrm{Path}_i(i \to B)) \\
&\cong \mathcal{K}(A, \omega^i(\mathrm{Path}_i(i \to B))) \\
&\cong \mathcal{K}(A, \mathrm{Path}(B)) \\
&\cong \mathcal{K}(\mathrm{Cyl}(A), B) \\
&\cong \mathcal{K}(\mathrm{Cyl}(A), \omega^i(i \to B)) \\
&\cong (i{\downarrow}\mathcal{K})(\rho^i(\mathrm{Cyl}(A)), i \to B).
\end{aligned}$$

1104

Here, the first, second, fourth and sixth bijections are by adjunction, the third is by definition of Path_i, and the fifth by definition of ω^i. The result now follows by Yoneda. $\square$
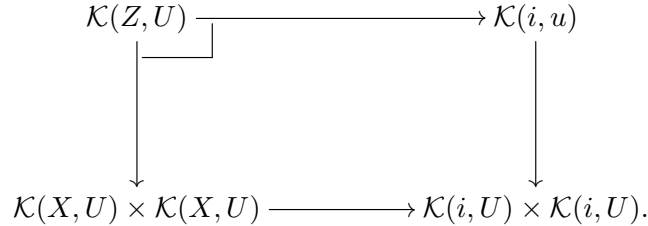
5.6. Lemma. *Let $\mathcal{K}$ be a locally presentable category. Let $i \to X$ be an object of $i{\downarrow}\mathcal{K}$. Then one has the pushout diagrams*

$$
\begin{array}{ccc}
i \sqcup i & \xrightarrow{\ \text{Id}_i \sqcup \text{Id}_i\ } & i \\
\downarrow & & \downarrow \\
X \sqcup X & \longrightarrow & X \sqcup_i X,
\end{array}
\qquad
\begin{array}{ccc}
i \sqcup i & \xrightarrow{\ \text{Id}_i \sqcup \text{Id}_i\ } & i \\
\downarrow & & \downarrow \\
X & =\!=\!=\!= & X.
\end{array}
$$
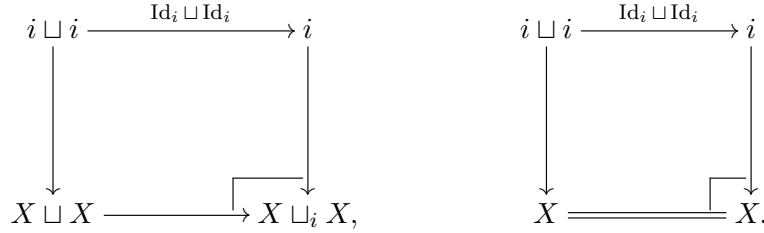
Proof. Consider the pushout diagrams of $\mathcal{K}$:

$$
\begin{array}{ccc}
i \sqcup i & \xrightarrow{\ \text{Id}_i \sqcup \text{Id}_i\ } & i \\
\downarrow & & \downarrow \\
X \sqcup X & \longrightarrow & Z,
\end{array}
\qquad
\begin{array}{ccc}
i \sqcup i & \xrightarrow{\ \text{Id}_i \sqcup \text{Id}_i\ } & i \\
\downarrow & & \downarrow \\
X & \longrightarrow & T.
\end{array}
$$
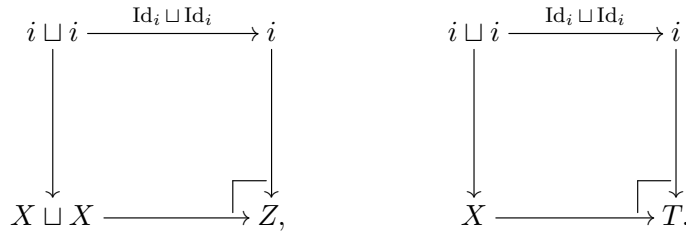
Let U be an object of $\mathcal{K}$. One has the pullback diagram of sets

$$
\begin{array}{ccc}
\mathcal{K}(Z,U) & \longrightarrow & \mathcal{K}(i,u) \\
\downarrow & & \downarrow \\
\mathcal{K}(X,U) \times \mathcal{K}(X,U) & \longrightarrow & \mathcal{K}(i,U) \times \mathcal{K}(i,U).
\end{array}
$$

Therefore one obtains the bijections of sets

$$
\mathcal{K}(Z,U) \cong (\mathcal{K}(X,U) \times \mathcal{K}(X,U)) \times_{\mathcal{K}(i,U) \times \mathcal{K}(i,U)} \mathcal{K}(i,U)
$$
$$
\cong \mathcal{K}(X,U) \times_{\mathcal{K}(i,U)} \mathcal{K}(X,U) \cong \mathcal{K}(X \sqcup_i X, U).
$$

By Yoneda, one obtains the isomorphism $Z \cong X \sqcup_i X$. And one has the pullback of sets

$$
\begin{array}{ccc}
\mathcal{K}(T,U) & \longrightarrow & \mathcal{K}(i,U) \\
\downarrow & & \downarrow \\
\mathcal{K}(X,U) & \longrightarrow & \mathcal{K}(i,U) \times \mathcal{K}(i,U).
\end{array}
$$

$$
\begin{array}{ccc}
i \sqcup i & \xrightarrow{\mathrm{Id}_i \sqcup \mathrm{Id}_i} & i \\
\downarrow{\scriptstyle e_X \sqcup e_X} & {\scriptstyle \gamma_i} \searrow & \| \\
X \sqcup X & \mathrm{Cyl}(i) \xrightarrow{\sigma_i} i & \| \\
\downarrow{\scriptstyle \gamma_X} & \downarrow{\scriptstyle \mathrm{Cyl}(e_X)} & \downarrow \\
\mathrm{Cyl}(X) & =\!=\!= \mathrm{Cyl}(X) \longrightarrow & Y.
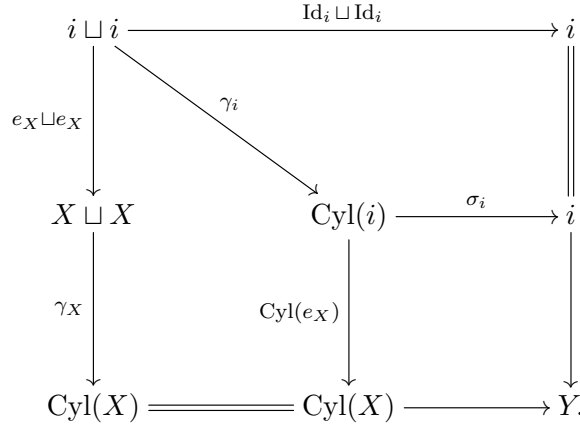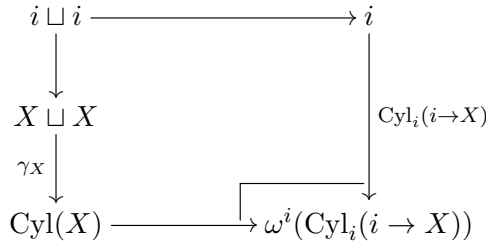\end{array}
$$
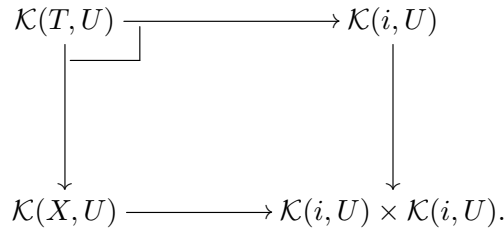
FIGURE 2. Isomorphism between two categories of commutative diagrams.

Therefore one obtains the bijections of sets

$$
\mathcal{K}(T,U) \cong \mathcal{K}(X,U) \times_{\mathcal{K}(i,U) \times \mathcal{K}(i,U)} \mathcal{K}(i,U) \cong \mathcal{K}(X,U).
$$

By Yoneda, one obtains the isomorphism $T \cong X$. □

5.7. Lemma. *Let* $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ *be a cylinder functor of a locally presentable category* $\mathcal{K}$. *Assume that it has a right adjoint* $\mathrm{Path} : \mathcal{K} \to \mathcal{K}$. *Let* i *be an object of* $\mathcal{K}$ *such that the map* $\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ *is epic. Then there is a pushout diagram*

$$
\begin{array}{ccc}
i \sqcup i & \longrightarrow & i \\
\downarrow & & \\
X \sqcup X & & \downarrow{\scriptstyle \mathrm{Cyl}_i(i \to X)} \\
\downarrow{\scriptstyle \gamma_X} & & \\
\mathrm{Cyl}(X) & \longrightarrow & \omega^i(\mathrm{Cyl}_i(i \to X))
\end{array}
$$

in $\mathcal{K}$ *for every object* $i \to X$ *of* $i{\downarrow}\mathcal{K}$.

Proof. Let $e_X : i \to X$ be a fixed object of $i{\downarrow}\mathcal{K}$. Consider a diagram of the form of Figure 2. We obtain a map F between the set of squares

$$
\left\{ \begin{array}{ccc} \mathrm{Cyl}(i) & \longrightarrow & i \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(X) & \longrightarrow & Y \end{array} \right\} \xrightarrow{F} \left\{ \begin{array}{ccc} i \sqcup i & \longrightarrow & i \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(X) & \longrightarrow & Y \end{array} \right\}.
$$

If D is a commutative square, then $F(D)$ is a commutative square. Since the map $\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ is epic, if $F(D)$ is a commutative square, then D

is a commutative square as well: D is commutative if and only if $F(D)$ is commutative. We have obtained a bijection between the sets of commutative diagrams

$$\left\{\begin{array}{ccc} \mathrm{Cyl}(i) & \longrightarrow & i \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(X) & \longrightarrow & Y \end{array}\right\} \cong \left\{\begin{array}{ccc} i \sqcup i & \longrightarrow & i \\ \downarrow & & \downarrow \\ \mathrm{Cyl}(X) & \longrightarrow & Y \end{array}\right\}$$

which gives rise to an isomorphism between the corresponding categories of commutative diagrams. The initial objects are the pushout diagrams. $\square$

The main theorem of this section is the following one:

5.8. Theorem. *Let $\mathcal{K}$ be an Olschok model category with the set of generating cofibrations I, the set of generating anodyne cofibration S, and the cartesian cylinder $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$. Let i be an object of $\mathcal{K}$ such that every map of $\mathcal{K}$ with source i is a cofibration and such that the map $\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ is epic. Then the combinatorial model category $i{\downarrow}\mathcal{K}$ is Olschok as well. The set of generating cofibrations of $i{\downarrow}\mathcal{K}$ is $\rho^i(I)$. The set of generating anodyne cofibrations of $i{\downarrow}\mathcal{K}$ is $\rho^i(S)$. The cartesian cylinder is the functor $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ defined in Lemma 5.2.*

Note that the condition "$\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ epic" is not satisfied for the model category of topological spaces: the inclusion map $\{0, 1\} \subset [0, 1]$ is not an epimorphism. We will see an example of such a situation in Section 6. Other examples of such a situation can be obtained by using the category of labelled symmetric precubical sets [Gau14], the category of flows [Gau03] or the category of multipointed d-spaces [Gau09] with $i = \{0\}$. We have for all these examples $\mathrm{Cyl}(i) = i$. The map $\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ is then the epimorphism $R : \{0, 1\} \to \{0\}$. Note that the model categories of topological spaces, of flows and of multipointed d-spaces are not Olschok model categories since they contain non-cofibrant objects. But it can be proved that they are left determined. The model category of labelled symmetric precubical sets of [Gau14] is an Olschok model category. However, it is not known if the latter is left determined.

Proof. Since every map with source i is a cofibration and since the identity of i is the initial object of $i{\downarrow}\mathcal{K}$, all objects of the model category $i{\downarrow}\mathcal{K}$ are cofibrant. Let $i \to X$ be an object of $i{\downarrow}\mathcal{K}$. Consider the composite diagram of $\mathcal{K}$ of Figure 3. By Lemma 5.7 and Lemma 5.6, the three squares above are pushout squares: in particular, they are commutative. The commutativity of Figure 3 implies that the natural map $X \sqcup_i X \to \mathrm{Cyl}_i(X) \to X$ is the codiagonal of $i \to X$ in $i{\downarrow}\mathcal{K}$. Since the functor $\mathrm{Cyl} : \mathcal{K} \to \mathcal{K}$ is a good cylinder, the map $X \sqcup_i X \to \omega^i(\mathrm{Cyl}_i(i \to X))$ is a cofibration of $\mathcal{K}$. Therefore
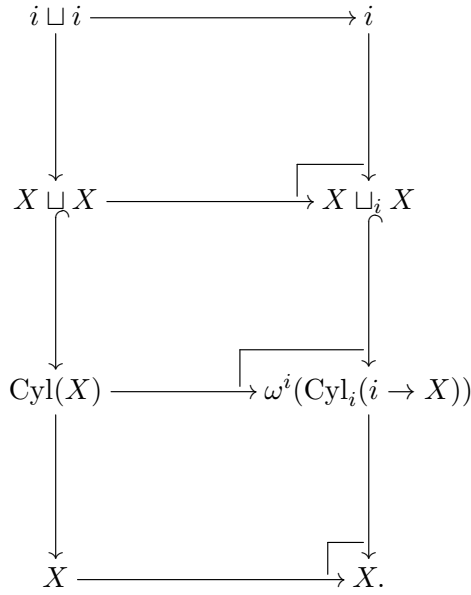
$$i \sqcup i \longrightarrow i$$
$$\downarrow \qquad\qquad \downarrow$$
$$X \sqcup X \longrightarrow X \sqcup_i X$$
$$\downarrow \qquad\qquad \downarrow$$
$$\mathrm{Cyl}(X) \longrightarrow \omega^i(\mathrm{Cyl}_i(i \to X))$$
$$\downarrow \qquad\qquad \downarrow$$
$$X \longrightarrow X.$$

FIGURE 3. Composite of three pushout squares (under the hypothesis $\gamma_i : i \sqcup i \to \mathrm{Cyl}(i)$ epic).

$$i \sqcup i \longrightarrow i$$

$$X \xhookrightarrow{\gamma^\epsilon_X} \mathrm{Cyl}(X) \longrightarrow \omega^i(\mathrm{Cyl}_i(i \to X))$$

$$Y \hookrightarrow \bullet \longrightarrow \bullet$$

$$Y \xhookrightarrow{\gamma^\epsilon_Y} \mathrm{Cyl}(Y) \longrightarrow Z,$$

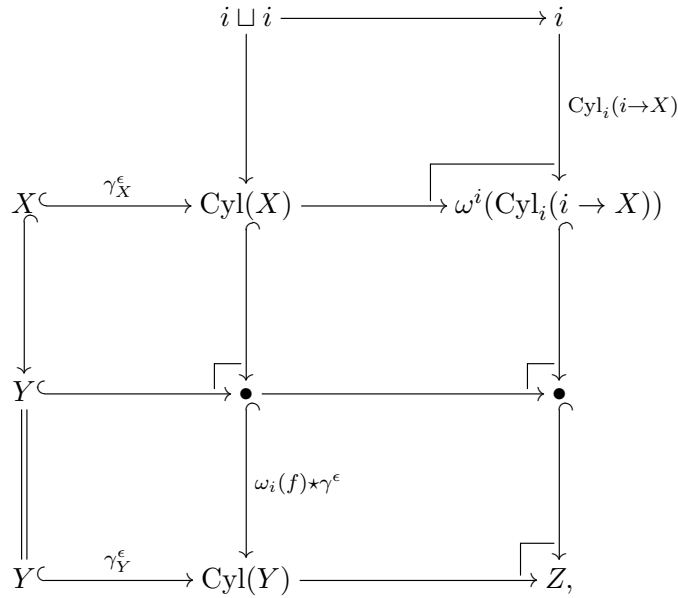with labels $\mathrm{Cyl}_i(i \to X)$ and $\omega_i(f) \star \gamma^\epsilon$.

FIGURE 4. Cyl_i is cartesian.

the functor $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ is a good cylinder for the set of maps $\rho^i(I)$.[2] By Lemma 5.2, the functor $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ has a right adjoint. Let $f : X \to Y$ be a cofibration of the comma category $i{\downarrow}\mathcal{K}$. Then $\omega^i(f)$ is a cofibration by definition of the model category $i{\downarrow}\mathcal{K}$. One has the commutative diagram of $\mathcal{K}$ of Figure 4, with $\epsilon = 0, 1$ where Z is defined as the pushout of the right-bottom square. Since we have the pushout diagram

$$
\begin{array}{ccc}
i \sqcup i & \longrightarrow & i \\
\downarrow & & \downarrow \\
\mathrm{Cyl}(Y) & \longrightarrow & Z,
\end{array}
$$

one deduces that $\omega^i(\mathrm{Cyl}_i(i \to Y)) = Z$ and we obtain the pushout diagram

$$
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
{\scriptstyle \omega_i(f) \star \gamma^\epsilon} \downarrow & & \downarrow {\scriptstyle f \star \gamma^\epsilon} \\
\mathrm{Cyl}(Y) & \longrightarrow & \omega^i(\mathrm{Cyl}_i(i \to Y)).
\end{array}
$$

Therefore the map $f \star \gamma^\epsilon$ is a cofibration of the comma category $i{\downarrow}\mathcal{K}$. We prove in the same way that $f \star \gamma$ is a cofibration. Hence the functor $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ is a cartesian cylinder for $\rho^i(I)$. By Theorem 2.5, we deduce that there exists a unique Olschok model category structure on $i{\downarrow}\mathcal{K}$ with the set of generating cofibrations $\rho^i(I)$, with the set of generating anodyne cofibrations $\rho^i(S)$, with the cartesian cylinder $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ and such that an object is fibrant if and only if it is $\Lambda_{i{\downarrow}\mathcal{K}}(\mathrm{Cyl}_i, \rho^i(S), \rho^i(I))$-injective.

Let $f : A \to B$ be a map of $\mathcal{K}$. Since the functor $\rho^i : \mathcal{K} \to i{\downarrow}\mathcal{K}$ preserves colimits, one has the commutative diagram of solid arrows of $i{\downarrow}\mathcal{K}$

$$
\begin{array}{ccc}
\rho^i(A) & \xrightarrow{\rho^i(f)} & \rho^i(B) \\
{\scriptstyle \rho^i(\gamma_A^\epsilon)} \downarrow & & \downarrow {\scriptstyle \rho^i(\gamma_B^\epsilon)} \\
\rho^i(\mathrm{Cyl}(A)) & \longrightarrow & \bullet \\
& {\scriptstyle \rho^i(\mathrm{Cyl}(f))} & \searrow {\scriptstyle \rho^i(f \star \gamma^\epsilon)} \\
& & \rho^i(\mathrm{Cyl}(B))
\end{array}
$$

---

[2]Note that we do not know yet that the map $\mathrm{Cyl}_i(X) \to X$ is a weak equivalence; this fact will be a consequence of this theorem. So we cannot yet say that $\mathrm{Cyl}_i : i{\downarrow}\mathcal{K} \to i{\downarrow}\mathcal{K}$ is a good cylinder for the model category $i{\downarrow}\mathcal{K}$.

for $\epsilon = 0, 1$. By Lemma 5.5, one deduces that $\rho^i(f) \star \gamma^\epsilon = \rho^i(f \star \gamma^\epsilon)$ for $\epsilon = 0, 1$. For the same reason, one has the commutative diagram of solid arrows of $i{\downarrow}\mathcal{K}$
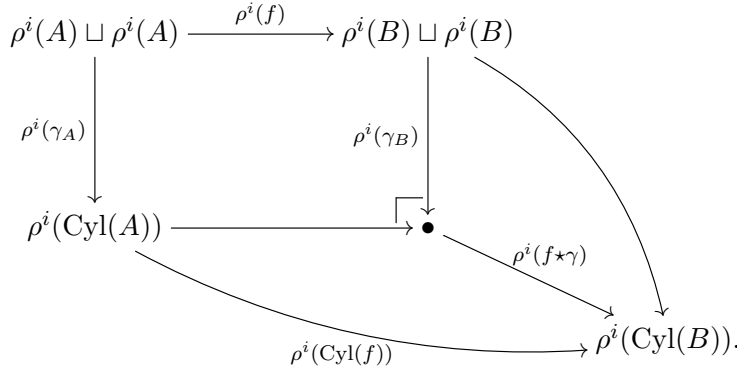

By Lemma 5.5, one deduces that $\rho^i(f) \star \gamma = \rho^i(f \star \gamma)$. So by Corollary 5.4, an object $i \to X$ of the comma category $i{\downarrow}\mathcal{K}$ is $\Lambda_{i{\downarrow}\mathcal{K}}(\mathrm{Cyl}_i, \rho^i(S), \rho^i(I))$-injective if and only if X is $\Lambda_\mathcal{K}(\mathrm{Cyl}, S, I)$-injective, i.e., if and only if X is fibrant in $\mathcal{K}$.

We deduce that the model category constructed in this proof has the same cofibrations and the same fibrant objects as the model category $i{\downarrow}\mathcal{K}$. Hence they are equal by [Hir03, Theorem 7.8.6] since all objects are cofibrant.[3] $\qquad\square$

It is not clear how to prove without additional hypothesis that if Cyl is very good, then Cyl_i is very good as well. In the situations one wants to use this construction, the map $\mathrm{Cyl}(X) \to \mathrm{Cyl}_i(X)$ is always split epic. In this case, one has:

5.9. Corollary. *With the same notations and hypotheses as in Theorem* 5.8, *if the map $p_X : \mathrm{Cyl}(X) \to \omega^i(\mathrm{Cyl}_i(X))$ is split epic for every X, then if* Cyl *is a very good cylinder of $\mathcal{K}$, then* Cyl_i *is a very good cylinder of $i{\downarrow}\mathcal{K}$.*

Proof. We start from a commutative diagram of $\mathcal{K}$ where f is a map of I:

---
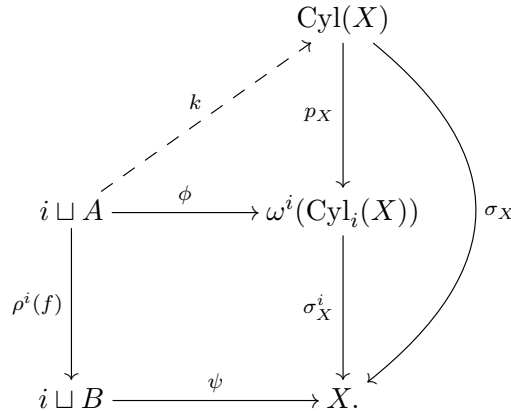
[3]Moreover, we can say now that the map $\mathrm{Cyl}_i(X) \to X$ is a weak equivalence of $\mathcal{K}$ as well, which was not possible earlier.

Let $k = s_X\phi$ where $s_X : \omega^i(\mathrm{Cyl}_i(X)) \to \mathrm{Cyl}(X)$ is a section of the split epic $\mathrm{Cyl}(X) \to \omega^i(\mathrm{Cyl}_i(X))$. Since i is cofibrant, $\rho^i(f)$ is a cofibration of $\mathcal{K}$. Since Cyl is very good, there exists a lift $\ell : i \sqcup B \to \mathrm{Cyl}(X)$ such that $\ell\rho^i(f) = k$ and $\sigma_X\ell = \psi$. Then one has $(p_X\ell)\rho^i(f) = p_X k = p_X s_X \phi = \phi$ and $\sigma_X^i(p_X\ell) = \sigma_X\ell = \psi$. Hence the cylinder Cyl_i is a very good cylinder of $i{\downarrow}\mathcal{K}$. $\qquad\square$

6. The homotopy theory of star-shaped weak transition systems

Weak transition systems are introduced in [Gau10] as a rewording of Cattani-Sassone's notion of higher dimensional transition system [CS96]. The purpose of these combinatorial objects is to model the concurrent execution of n actions by a transition between two states labelled by a multiset $\{u_1, \ldots, u_n\}$ of actions. The category of weak transition systems is a convenient category to study these objects from a categorical and homotopical point of view [Gau10] [Gau11] [Gau15b].

6.1. Notation. Let Σ be a fixed nonempty set of labels.

6.2. Definition. A weak transition system consists of a triple

$$X = \left(S, \mu : L \to \Sigma, T = \bigcup_{n \geqslant 1} T_n \right)$$

where S is a set of states, where L is a set of actions, where $\mu : L \to \Sigma$ is a set map called the labelling map, and finally where $T_n \subset S \times L^n \times S$ for $n \geqslant 1$ is a set of n-transitions or n-dimensional transitions such that one has:

- (Multiset axiom). For every permutation σ of $\{1, \ldots, n\}$ with $n \geqslant 2$, if the tuple $(\alpha, u_1, \ldots, u_n, \beta)$ is a transition, then the tuple

$$(\alpha, u_{\sigma(1)}, \ldots, u_{\sigma(n)}, \beta)$$

  is a transition as well.
- (Patching axiom[4]). For every $(n + 2)$-tuple $(\alpha, u_1, \ldots, u_n, \beta)$ with $n \geqslant 3$, for every $p, q \geqslant 1$ with $p + q < n$, if the five tuples

$$(\alpha, u_1, \ldots, u_n, \beta),$$
$$(\alpha, u_1, \ldots, u_p, \nu_1), (\nu_1, u_{p+1}, \ldots, u_n, \beta),$$
$$(\alpha, u_1, \ldots, u_{p+q}, \nu_2), (\nu_2, u_{p+q+1}, \ldots, u_n, \beta)$$

  are transitions, then the $(q + 2)$-tuple $(\nu_1, u_{p+1}, \ldots, u_{p+q}, \nu_2)$ is a transition as well.

---

[4]This axiom is called the Coherence axiom in [Gau10] and [Gau11], and the composition axiom in [Gau15b].

A map of weak transition systems

$$f : (S, \mu : L \to \Sigma, (T_n)_{n \geqslant 1}) \to (S', \mu' : L' \to \Sigma, (T'_n)_{n \geqslant 1})$$

consists of a set map $f_0 : S \to S'$ and a commutative square

$$
\begin{array}{ccc}
L & \xrightarrow{\mu} & \Sigma \\
\widetilde{f} \downarrow & & \| \\
L' & \xrightarrow{\mu'} & \Sigma
\end{array}
$$

such that if $(\alpha, u_1, \ldots, u_n, \beta)$ is a transition, then

$$(f_0(\alpha), \widetilde{f}(u_1), \ldots, \widetilde{f}(u_n), f_0(\beta))$$

is a transition. The corresponding category is denoted by $\mathcal{WTS}$. The n-transition $(\alpha, u_1, \ldots, u_n, \beta)$ is also called a transition from α to β. The maps f_0 and $\widetilde{f}$ will be also denoted by f.

Every set X may be identified with the weak transition system having the set of states X, with no actions and no transitions.

It is usual in computer science to work in the comma category $\{\iota\} \downarrow \mathcal{WTS}$ where the image of the state ι represents the initial state of the process which is modeled. It then makes sense to restrict to the states which are reachable from this initial state by a path of transitions. Hence we introduce the following definitions:

6.3. Definition. Let X be a weak transition system and let ι be a state of X. A state α of X is reachable from ι if it is equal to ι or if there exists a finite sequence of transitions t_i of X from α_i to α_{i+1} for $0 \leqslant i \leqslant n$ with $n \geqslant 0$, $\alpha_0 = \iota$ and $\alpha_{n+1} = \alpha$.

6.4. Definition. A star-shaped weak transition system is an object $\{\iota\} \to X$ of the comma category $\{\iota\} \downarrow \mathcal{WTS}$ such that every state of the underlying weak transition system X is reachable from ι. The full subcategory of $\{\iota\} \downarrow \mathcal{WTS}$ of star-shaped weak transition systems is denoted by $\mathcal{WTS}_\bullet$.

6.5. Proposition. *The category $\mathcal{WTS}_\bullet$ is a full isomorphism-closed coreflective subcategory of $\{\iota\} \downarrow \mathcal{WTS}$.*

Proof. Let $\{\iota\} \to X$ be an object of $\{\iota\} \downarrow \mathcal{WTS}$. Let $\underline{T}_\iota(X)$ be the set of transitions $(\alpha, u_1, \ldots, u_n, \beta)$ of X such that the initial state α is reachable from ι. Note that this implies that β is reachable from ι as well. The set $\underline{T}_\iota(X)$ satisfies the multiset axiom since permuting the actions does not change the initial state of a transition. It also satisfies the patching axiom because, with the notations of the patching axiom in Definition 6.2, ν_1 is reachable from ι. Therefore the triple consisting of the set of states of X which are reachable from ι, the set of actions of X with the same labelling map μ, and the set of transitions $\underline{T}_\iota(X)$ yields a well-defined weak transition

system $\underline{C}_\iota(X)$. By construction, the map $\iota \to \underline{C}_\iota(X)$ is a star-shaped weak transition system. Consider a commutative square

$$
\begin{array}{ccc}
\{\iota\} & =\!=\!=\!=\!= & \{\iota\} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\quad f \quad} & X
\end{array}
$$

where $\{\iota\} \to Y$ is a star-shaped weak transition system. By construction, for every state α of Y, $f(\alpha)$ is reachable from ι and every transition $(\alpha, u_1, \ldots, u_n, \beta)$ of Y is therefore mapped to a transition

$$(f(\alpha), f(u_1), \ldots, f(u_n), f(\beta))$$

of $\underline{T}_\iota(X)$. Therefore the commutative square above factors uniquely as the composite of commutative squares

$$
\begin{array}{ccccc}
\{\iota\} & =\!=\!=\!=\!= & \{\iota\} & =\!=\!=\!=\!= & \{\iota\} \\
\downarrow & & \downarrow & & \downarrow \\
Y & \longrightarrow & \underline{C}_\iota(X) & \xrightarrow{\ \subset\ } & X.
\end{array}
$$
$$f$$

$\square$

6.6. Proposition. *The category $\mathcal{WTS}_\bullet$ is a small cone-injectivity class of $\{\iota\}{\downarrow}\mathcal{WTS}$ and all maps of the cone can be chosen to be cofibrations.*

Proof. The category $\mathcal{WTS}_\bullet$ is a small cone-injectivity class with respect to the small cone formed by the inclusions of the weak transition system $\{\iota, \alpha\}$ in the weak transition systems

$$\iota \xrightarrow{\ t_1\ } \bullet \longrightarrow \ldots \longrightarrow \bullet \xrightarrow{\ t_n\ } \alpha$$

for all $n \geqslant 0$ and all transitions $t_1, \ldots, t_n$ with the labelling map Id_Σ (note that we must include in the cone the set map $\{\iota, \alpha\} \to \{\iota\}$). The cone is small because there is a *set* of labels Σ. Finally, all maps of the cone are cofibrations of weak transition systems because on the sets of actions, they are all of them the inclusion of the empty set in some set. $\square$

6.7. Corollary. *The category $\mathcal{WTS}_\bullet$ is locally presentable.*

Proof. The category $\mathcal{WTS}_\bullet$ is accessible by [AR94, Proposition 4.16]. It is cocomplete by Proposition 6.5. Therefore it is locally presentable. $\square$

We can now conclude this paper with the following application:

6.8. Theorem. *There exists a left determined model structure on the category* $\mathcal{WTS}_\bullet$ *of star-shaped weak transition systems with respect to the class of maps such that the underlying maps of weak transition systems are one-to-one on actions.*

Sketch of proof. The category $\mathcal{WTS}_\bullet$ is bicomplete by Corollary 6.7. By [Gau11, Theorem 5.11], there exists an Olschok model structure on the category of weak transition systems such that the cofibrations are the maps which are one-to-one on actions. Let $\mathrm{Cyl} : \mathcal{WTS} \to \mathcal{WTS}$ be the cylinder functor which is described in the proof of [Gau11, Proposition 5.8]. The map $\{\iota\} \sqcup \{\iota\} \to \mathrm{Cyl}(\{\iota\})$ is epic because by [Gau11, Proposition 5.8], one has $\mathrm{Cyl}(\{\iota\}) = \{\iota\}$. Hence we can apply Theorem 5.8. We obtain an Olschok model category on the comma category $\{\iota\}{\downarrow}\mathcal{WTS}$. By Lemma 5.2, the cylinder of $\{\iota\}{\downarrow}\mathcal{WTS}$ is obtained by identifying two states in $\mathrm{Cyl}(X)$. Since the set of states of $\mathrm{Cyl}(X)$ is equal to the set of states of X by the calculations made in the proof of [Gau11, Proposition 5.8], the two states identified are actually equal. This means that the underlying weak transition system of $\mathrm{Cyl}_{\{\iota\}}(\{\iota\} \to X)$ is $\mathrm{Cyl}(X)$. Therefore by Corollary 5.9, and since Cyl is very good by [Gau11, Proposition 5.7], the cylinder $\mathrm{Cyl}_{\{\iota\}}$ is very good and the Olschok model category $\{\iota\}{\downarrow}\mathcal{WTS}$ is left determined. Let $\{\iota\} \to X$ be a star-shaped weak transition system. By the calculations made in the proof of [Gau11, Proposition 5.8] again, the set of actions of $\mathrm{Cyl}(X)$ is $L \times \{0,1\}$ where L is the set of actions of X and a tuple $(\alpha, (u_1, \epsilon_1), \ldots, (u_n, \epsilon_n), \beta)$ is a transition of $\mathrm{Cyl}(X)$ if and only if $(\alpha, u_1, \ldots, u_n, \beta)$ is a transition of X. Therefore a state of $\mathrm{Cyl}(X)$ is reachable from $\{\iota\}$ if and only if it is reachable from $\{\iota\}$ in X (choose $\epsilon_i = 0$ for all intermediate transitions). One deduces that if $\{\iota\} \to X$ is a star-shaped weak transition system, then $\mathrm{Cyl}_{\{\iota\}}(\{\iota\} \to X)$ is a star-shaped weak transition system as well. Using Proposition 6.6, we can now apply Theorem 4.3: we have obtained an Olschok model structure which is left determined. The proof is complete. $\square$

References

[AR94] ADÁMEK, JAVIER; ROSICKÝ, JIŘÍ. Locally presentable and accessible categories. London Mathematical Society Lecture Note Series, 189. *Cambridge University Press, Cambridge,* 1994. xiv+316 pp. ISBN: 0-521-42261-2. MR1294136 (95j:18001), Zbl 0795.18007, doi: 10.1017/CBO9780511600579.

[Bek00] BEKE, TIBOR. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.* **129** (2000), no. 3, 447–475. MR1780498 (2001i:18015), Zbl 0964.55018, arXiv:math/0102087, doi: 10.1017/S0305004100004722.

[CS96] CATTANI, GIAN LUCA; SASSONE, VLADIMIRO. Higher-dimensional transition systems. *11th Annual IEEE Symposium on Logic in Computer Science* (New Brunswick, NJ, 1996), 55–62. *IEEE Comput. Soc. Press, Los Alamitos, CA,* 1996. MR1461821, doi: 10.1109/LICS.1996.561303.

[Cis02] CISINSKI, DENIS-CHARLES. Théories homotopiques dans les topos. *J. Pure Appl. Algebra* **174** (2002), no. 1, 43–82. MR1924082 (2003i:18021), Zbl 1015.18009, doi: 10.1016/S0022-4049(01)00176-1.

[Dug01] DUGGER, DANIEL. Combinatorial model categories have presentations. *Adv. Math.* **164** (2001), no. 1, 177–201. MR1870516 (2002k:18022), Zbl 1001.18001, arXiv:math/0007068, doi: 10.1006/aima.2001.2015.

[Gau03] GAUCHER, PHILIPPE. A model category for the homotopy theory of concurrency. *Homology, Homotopy Appl.* **5** (2003), no. 1, 549–599. MR2072345 (2005g:55029), Zbl 1069.55008, arXiv:math/0308054, doi: 10.4310/HHA.2003.v5.n1.a20.

[Gau09] GAUCHER, PHILIPPE. Homotopical interpretation of globular complex by multipointed *d*-space. *Theory Appl. Categ.* **22** (2009), 588–621. MR2591950 (2011b:55018), Zbl 1191.55013, arXiv:0710.3553.

[Gau10] GAUCHER, PHILIPPE. Directed algebraic topology and higher dimensional transition systems. *New York J. Math.* **16** (2010), 409–461. MR2740585 (2011k:18002), Zbl 1250.18004, arXiv:0903.4276.

[Gau11] GAUCHER, PHILIPPE. Towards a homotopy theory of higher dimensional transition systems. *Theory Appl. Categ.* **25** (2011), no. 12, 295–341. MR2861110, Zbl 1235.18004, arXiv:1011.0918.

[Gau14] GAUCHER, PHILIPPE. Homotopy theory of labelled symmetric precubical sets. *New York J. Math.* **20** (2014), 93–131. MR3159398, Zbl 06270244, arXiv:1208.4494.

[Gau15a] GAUCHER, PHILIPPE. The choice of cofibrations of higher dimensional transition systems. *New York J. Math.* **21** (2015), 1117–1151. arXiv:1507.07345.

[Gau15b] GAUCHER, PHILIPPE. The geometry of cubical and regular transition systems. *Cah. Topol. Géom. Différ. Catég.*, LVI-4, 2015. arXiv:1405.5184.

[Hir03] HIRSCHHORN, PHILIP S. Model categories and their localizations. Mathematical Surveys and Monographs, 99. *American Mathematical Society, Providence, RI*, 2003. xvi+457 pp. ISBN: 0-8218-3279-4. MR1944041 (2003j:18018), Zbl 1017.55001, doi: 10.1090/surv/099.

[Hir15] HIRSCHHORN, PHILIP S. Overcategories and undercategories of model categories. Preprint, 2015. arXiv:1507.01624.

[Hov99] HOVEY, M. Model categories. Mathematical Surveys and Monographs, 63. *American Mathematical Society, Providence, RI*, 1999. xii+209 pp. ISBN: 0-8218-1359-5. MR1650134 (99h:55031), Zbl 0909.55001.

[Joy] JOYAL, ANDRÉ. Notes on quasi-categories. Preprint, 2008. `http://www.math.uchicago.edu/~may/IMA/Joyal.pdf`.

[KR05] KURZ, ALEXANDER; ROSICKÝ, JIŘÍ. Weak factorizations, fractions and homotopies. *Appl. Categ. Structures* **13** (2005), no. 2, 141–160. MR2141595 (2006c:18001), Zbl 1077.18002, doi: 10.1007/s10485-004-6730-z.

[Ols09a] OLSCHOK, MARC. On constructions of left determined model structures. PhD thesis, Masaryk University, Faculty of Science, 2009. `http://is.muni.cz/th/183259/prif_d/diss.pdf`.

[Ols09b] OLSCHOK, MARC. Left determined model structures for locally presentable categories. *Appl. Categ. Structures* **19** (2011), no. 6, 901–938. MR2861071 (2012k:18008), Zbl 1251.18003, arXiv:0901.1627, doi: 10.1007/s10485-009-9207-2.

[Ros09] ROSICKÝ, JIŘÍ. On combinatorial model categories. *Appl. Categ. Structures* **17** (2009), no. 3, 303–316. MR2506258 (2010f:55023), Zbl 1175.55013, arXiv:0708.2185, doi: 10.1007/s10485-008-9171-2.

[RT03] ROSICKÝ, JIŘÍ; THOLEN, WALTER. Left-determined model categories and universal homotopy theories. *Trans. Amer. Math. Soc.* **355** (2003), no. 9, 3611–3623. MR1990164 (2004e:55023), Zbl 1030.55015, doi: 10.1090/S0002-9947-03-03322-1.

(Philippe Gaucher) Laboratoire PPS (CNRS UMR 7126), Univ Paris Diderot, Sorbonne Paris Cité, Case 7014, 75205 PARIS Cedex 13, France
http://www.pps.univ-paris-diderot.fr/~gaucher/