On a generalization of the Gasca–Maeztu conjecture

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ABSTRACT. Denote the space of all bivariate polynomials of total degree not exceeding n by Π_n. The Gasca–Maeztu conjecture [Gasca M. and Maeztu J. I., On Lagrange and Hermite interpolation in $\mathbb{R}^k$, *Numer. Math.* **39** (1982), 1–14.] states that any Π_n-poised set of nodes, all fundamental polynomials of which are products of linear factors, possesses a maximal line, i.e., a line passing through $n + 1$ nodes. Till now it is proved to be true for $n \le 5$. The case $n = 5$ was proved recently in [Hakopian H., Jetter K. and Zimmermann G., The Gasca–Maeztu conjecture for $n = 5$, *Numer. Math.* **127** (2014), 685–713]. In an earlier paper the following generalized conjecture was proposed by the authors of the present paper: Any Π_n-poised set of nodes, all fundamental polynomials of which are reducible, possesses a maximal curve of some degree k, $1 \le k \le n-1$, i.e., an algebraic curve passing through $(1/2)k(2n - k + 3)$ nodes. Clearly the two above conjectures coincide in the case $n \le 2$. In this paper we prove that the generalized conjecture is true for $n = 3$.

CONTENTS

1. Introduction

The bivariate (multivariate) polynomial interpolation is much more involved than the respective univariate one. A poised interpolation with a set of nodes and a polynomial space means that there is, for arbitrary data at those nodes, exactly one polynomial from the space that matches the given information. In order to have an n — poised interpolation with Π_n, the space of bivariate algebraic polynomials of total degree not exceeding n, the number of interpolation nodes has to fit the dimension of the space:

$$N := \dim \Pi_n = \frac{(n+2)(n+1)}{2}.$$

In contrast with the univariate interpolation, even if this is the case, the poisedness of the bivariate interpolation essentially depends on the geometrical distribution of nodes. Thus, a new problem arises, which is permanently actual in the subject — the identification of simple constructions of poised node sets.

The Gasca–Maeztu conjecture concerns perhaps the simplest such construction — the so called GC_n construction, based on a geometric condition. Namely, in GC_n set every fundamental polynomial is a product of n linear factors, as it always takes place in the univariate case. Geometrically this means that for each fixed node there are n lines which pass through all the nodes of the set but the fixed one. The conjecture states that each GC_n set possesses a maximal line, i.e., line passing through $n+1$ nodes. The conjecture is equivalent to the statement that each GC_n set is a particular case of an extremely simple construction — the Berzolari–Radon construction (see Definition 1.0.6). So far the conjecture was proved to be true for $n \leq 5$. Let us mention that settling the conjecture for particular n enables also the classification of GC_n sets of the respective order (see [3]).

In this paper a generalization of the Gasca–Maeztu conjecture is considered in terms of higher order curves. We call a node set GC_n^* set if every fundamental polynomial is reducible, i.e., is the product of two or more nontrivial factors. The generalized conjecture claims that (see Conjecture 2.0.11), for each GC_n^* set, there is a maximal curve of some degree k, with $1 \leq k \leq n-1$, i.e., a curve containing as much as $d(n,k) := (1/2)k(2n+3-k)$ nodes. This is maximal possible number of nodes given that the node set is n-poised.

The generalized conjecture in its turn is equivalent to the statement that each GC_n^* set is a particular case of another simple known construction, which is a generalization of the Berzolari–Radon construction (see [15]).

The present paper settles the generalized conjecture for $n = 3$, which is the first interesting case, stating that under the assumption of reducibility, there is a maximal line (containing n + 1 nodes), or a maximal conic (containing 2n + 1 nodes).

Consider a set of nodes (points):

$$\mathcal{T}_s := \{(x_1, y_1), \ldots, (x_s, y_s)\}.$$

The interpolation problem consists of finding a polynomial $p \in \Pi_n$ such that

$$(1.0.1) \qquad p(x_i, y_i) = c_i, \quad i = 1, 2, \ldots, s.$$

Definition 1.0.1. A set of nodes $\mathcal{T}_s$ is called Π_n-*poised*, or briefly n-*poised*, if for any data $\bar{c} = \{c_1, \ldots, c_s\}$ there exists a unique polynomial $p \in \Pi_n$ satisfying the conditions (1.0.1).

A necessary condition for n-poisedness of $\mathcal{T}_s$ is: $s = N$.

Denote by $p|_\mathcal{T}$ the restriction of p to $\mathcal{T}$.

A polynomial $p \in \Pi_n$ is called n-*fundamental* for $T = (x_k, y_k) \in \mathcal{T}_s$ if

$$p|_{\mathcal{T}_s \setminus \{T\}} = 0, \quad p(T) = 1.$$

We denote by $p_T^\star := p_k^\star$ this fundamental polynomial.

Now let us consider an important concept for node sets.

Definition 1.0.2. A set of nodes $\mathcal{T}_s$ is called n-*independent*, if all its nodes have n-fundamental polynomials: $p_i^\star \in \Pi_n$, $i = 1, \ldots, s$.

The fundamental polynomials are linearly independent. Therefore a necessary condition of n-independence of $\mathcal{T}_s$ is: $s \leq N$. Clearly any n-poised set is n-independent. We also have that $\mathcal{T}_s$ is n-independent if and only if the interpolation problem (1.0.1) is solvable, meaning that for any data $\bar{c}$ there exists a polynomial $p \in \Pi_n$ (not unique, if $s < N$) satisfying the conditions (1.0.1).

In the sequel we will need the following proposition (see [6], Proposition 1, see also [9], Theorem 9, for the case of multiple nodes).

Proposition 1.0.3. *Any set of k nodes, with $k \leq 2n + 1$, in the plane, is n-independent if and only if no $n + 2$ of them are collinear.*

Next we present so called GC_n *sets* introduced by Chung and Yao [5].

Definition 1.0.4. An n-poised set of nodes $\mathcal{T}$ is called a GC_n set, if the n-fundamental polynomial of each its node is a product of n linear factors.

We say that a node $T \in \mathcal{T}$ *uses* an algebraic curve q of degree k if the latter divides the fundamental polynomial of T, i.e., $p_T^\star = qr$ for some $r \in \Pi_{n-k}$. Thus each node of a GC_n set uses n lines.

The Gasca–Maeztu conjecture is the following [7]:

Conjecture 1.0.5 (Gasca–Maeztu). *If $\mathcal{T}$ is a GC_n set, then there is at least one line l such that $\#(\mathcal{T} \cap l) = n + 1$.*

So far this conjecture has been verified only for $n \leq 5$ (see [1],[2],[11] for $n \leq 4$ and [12] for $n = 5$). In fact, the conjecture states that every GC_n set is a particular case of a very simple construction of n-poised sets, called *Berzolari–Radon* (see [4]):

Definition 1.0.6. A set of $N = 1 + \cdots + (n+1)$ nodes is called Berzolari–Radon set for degree n, or briefly B-R set, if there exist lines $l_1, l_2, \ldots, l_{n+1}$, such that the sets l_1, $l_2 \setminus l_1$, $l_3 \setminus (l_1 \cup l_2), \ldots,$ $l_{n+1} \setminus (l_1 \cup \cdots \cup l_n)$ contain exactly $(n+1), n, n-1, \ldots, 1$ nodes, respectively.

2. Maximal curves and the generalized conjecture

Let us start with the following well-known statement.

Proposition 2.0.7. *Assume that l is a line and $\mathcal{T}_{n+1}$ is any subset of l containing $n+1$ points. Then we have that*

$$p \in \Pi_n \quad and \quad p|_{\mathcal{T}_{n+1}} = 0 \;\Rightarrow\; p = lr, \; where \; r \in \Pi_{n-1}.$$

Denote

$$d := d(n,k) := \dim \Pi_n - \dim \Pi_{n-k} = (1/2)k(2n+3-k).$$

The following is a generalization of Proposition 2.0.7.

Proposition 2.0.8 ([15], Proposition 3.1)**.** *Let q be an algebraic curve of degree $k \leq n$ without multiple components. Then the following hold:*

(i) *Any subset of q containing more than $d(n,k)$ nodes is n-dependent.*
(ii) *Any subset $\mathcal{T}_d$ of q containing exactly $d(n,k)$ nodes is n-independent if and only if the following condition holds:*

$$p \in \Pi_n \quad and \quad p|_{\mathcal{T}_d} = 0 \Rightarrow p = qr, \; where \; r \in \Pi_{n-k}.$$

Suppose that $\mathcal{T}$ is an n-poised set of nodes and q is an algebraic curve of degree $k \leq n$. Then of course any subset of $\mathcal{T}$ is n-independent too. Therefore, according to Proposition 2.0.8(i), at most $d(n,k)$ nodes of $\mathcal{T}$ can lie in the curve q. Let us mention that a special case of this when q is a set of k lines is proved in [3]. This motivates the following (see [15], Def.3.1).

Definition 2.0.9. Given an n-poised set of nodes $\mathcal{T}$. A curve of degree $k \leq n$ is called *maximal* if it passes through $d(n,k)$ nodes of the set $\mathcal{T}$.

We have that $d(n,1) = n+1$, $d(n,2) = 2n+1$, $d(n,3) = 3n$. In view of Proposition 1.0.3, any set of $n+1$ nodes located in a line is n-independent. Note that a maximal line, as a line passing through $n+1$ nodes of $\mathcal{T}$, is defined in [2] (see also [10] for the case of general dimension). Any irreducible conic, i.e., conic which is not a pair of lines, contains at most two collinear points. Hence by Proposition 1.0.3, any set of $2n+1$ nodes located in an irreducible conic is n-independent. In the case of cubics (and similarly in the case of curves of higher degree) we already deal with a new phenomenon. Namely, not any set of $3n$ nodes in an irreducible cubic is n-independent (see [13]). Since $d(n,n) = N-1$ we have that each n-fundamental polynomial of any n-poised set $\mathcal{T}$ is a maximal curve of degree n.

Next we bring a characterization of maximal curves:

Proposition 2.0.10 ([15], Prop. 3.3). *Let a node set $\mathcal{T}$ be n-poised. Then a polynomial q of degree k, $k \leq n$, is a maximal curve if and only if it is used by any node in $\mathcal{T} \setminus q$.*

Note that one side of this statement follows from Proposition 2.0.8(ii). In the case of lines this was proved in ([2]). For other properties of maximal curves we refer reader to [15], where (in Conjecture 7.2) we propose the following generalized:

Conjecture 2.0.11 (H.H., L.R.). Suppose that $\mathcal{T}$ is an n-poised node set and the fundamental polynomial of each node is reducible. Then $\mathcal{T}$ possesses a maximal curve of some degree k, $1 \leq k \leq n-1$, i.e., a curve q such that $\#(\mathcal{T} \cap q) = d(n,k)$.

Note the degree of the maximal curve here does not exceed $n-1$ and the same estimate holds for the degrees of factors of fundamental polynomials. By taking into account this fact we put forward the following refined:

Conjecture 2.0.12. Suppose that $\mathcal{T}$ is an n-poised node set and the fundamental polynomial of each node is a product of factors whose degrees do not exceed m, where $1 \leq m \leq n-1$. Then $\mathcal{T}$ possesses a maximal curve of some degree k, $1 \leq k \leq m$.

Clearly this conjecture coincides with the Gasca–Maeztu conjecture and Conjecture 2.0.11 if $m = 1$ and $m = n-1$, respectively.

2.1. The generalized conjecture for $n = 3$. We start this subsection with the particular case $n = 3$ of Conjecture 2.0.11 (or, which is the same, case $n = 3$, $m = 2$ of Conjecture 2.0.12).

Theorem 2.1.1. *Suppose that a node set $\mathcal{T}$ is Π_3-poised and the fundamental polynomial of each node is reducible. Then $\mathcal{T}$ possesses a maximal curve of degree ≤ 2, i.e., a maximal line or a maximal conic.*

This is our main result and will be proved in Sections 3–4. Note that a Π_3-poised set contains 10 nodes, while a maximal line, in case $n = 3$, passes through 4 nodes and a maximal conic passes through 7 nodes.

The following three simple lemmas will be used frequently in the sequel.

Lemma 2.1.2. *Assume that a node set $\mathcal{T}$ is Π_3-poised and 2 nodes in $\mathcal{T}$ use the same line. Then $\mathcal{T}$ possesses a maximal line.*

Proof. Suppose that two nodes T_0 and $T_1 \in \mathcal{T}$ use a line l: $p_0^\star = lq_0$ $p_1^\star = lq_1$, where $q_0, q_1 \in \Pi_2$. Assume also that l passes through ≤ 3 nodes, since otherwise it is maximal. Then both q_0 and q_1 vanish at the set $\mathcal{S} := \mathcal{T} \setminus (\{T_0, T_1\} \cup l)$ containing ≥ 5 nodes. Now, if the nodes in $\mathcal{S}$ are 2-independent then q_0 and q_1 determine the same conic, which means that $p_0^\star$ and $p_1^\star$ are linearly dependent, leading to a contradiction. Otherwise the nodes are 2-dependent and by Proposition 1.0.3, four of them are collinear. $\square$

Lemma 2.1.3. *Assume that a node set $\mathcal{T}$ is Π_3-poised. If a node $T \in \mathcal{T}$ uses a line l passing through exactly 3 nodes and there exists a line l' passing through exactly 3 nodes in $\mathcal{T} \backslash (\{T\} \cup l)$, then the following hold:*

(i) *The node T uses the line l'.*
(ii) *The remaining 3 nodes in $\mathcal{T} \backslash (\{T\} \cup l \cup l')$ lie in a line l'' and T uses the line l'' too.*

Proof. We have $p_T^\star = lq$, where $q \in \Pi_2$ vanishes at the 3 nodes in l'. Thus, according to Proposition 2.0.7 we have that $q = l'l''$, with $l'' \in \Pi_1$. Therefore $p_T^\star = ll'l''$ implying that the remaining 3 nodes are in l'', since none of them lies in l or l' by assumption. $\qquad\square$

Lemma 2.1.4. *Assume that a node set $\mathcal{T}$ is Π_3-poised without a maximal line and a maximal conic. Then the following hold:*

(i) *Each used line passes through exactly three nodes.*
(ii) *If l and l' are two lines, both used by a node of $\mathcal{T}$, then $l \cap l' \cap \mathcal{T} = \emptyset$.*

Proof. Suppose, for (i), that a node $T \in \mathcal{T}$ uses a line l passing just through 2 nodes, then $p_T^\star = lq$, where the conic $q \in \Pi_2$ passes through 7 nodes of $\mathcal{T} \backslash (\{T\} \cup l)$, and hence is maximal, which contradicts our assumption.

For (ii), suppose that l and l' are two lines used by $T \in \mathcal{T}$, i.e., $p_T^\star = ll'l''$, where $l'' \in \Pi_1$. Now, assume by way of contradiction that $l \cap l' \in \mathcal{T}$, hence there are only 5 nodes in $l \cup l'$. Then l'' passes through the 4 nodes of $\mathcal{T} \backslash (\{T\} \cup l \cup l')$, and is maximal, which contradicts our assumption. $\qquad\square$

2.2. Alternatives 1 and 2. Let us start the proof of Theorem 2.1.1. From now on we shall assume that

(2.2.1) $\qquad\qquad\qquad \mathcal{T}$ has no maximal lines or conics,

in order to derive a contradiction.

Now let as present the following proposition which is important for the later consideration.

Proposition 2.2.1. *Assume that each fundamental polynomial of a Π_3-poised node set $\mathcal{T}$ with (2.2.1) is reducible. Assume also that no node of $\mathcal{T}$ is intersection point of 4 used lines. Then the following hold.*

(1) *There are exactly 10 used lines.*
(2) *On each used line there are exactly 3 nodes.*
(3) *Each node is an intersection point of exactly 3 used lines.*
(4) *Each node uses a line and an irreducible conic.*

Proof. Note that the reducibility of fundamental polynomials in the case of degree 3 means that each node uses either 3 lines, or a line and an irreducible conic. By taking onto account (2.2.1), we get from Lemma 2.1.2 that there are at least 10 distinct used lines. Also we get that each node uses a line and an irreducible conic, if there are exactly 10 used lines. According to Lemma 2.1.4(i) each used line passes through exactly 3 nodes. Therefore in
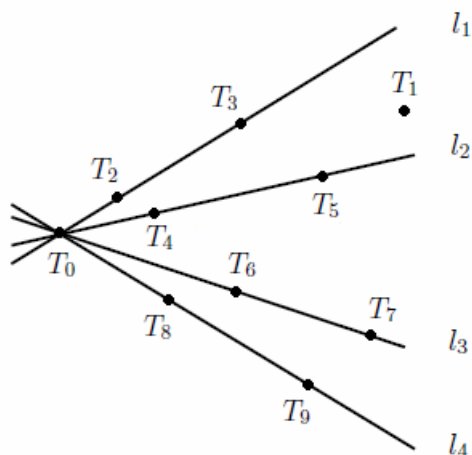
FIGURE 1. Four concurrent lines.

the case of exactly 10 distinct used lines the total number of nodes belonging to them equals 30. Thus we get that on average a node of $\mathcal{T}$ is in 3 used lines, i.e., is an intersection point of 3 used lines. Thus if there are more than 10 distinct used lines, or if a node is an intersection point of less than 3 used lines, then there is a node, which is an intersection point of four used lines, contrary to our assumption. □

In view of Proposition 2.2.1 we are to proceed in the following two alternative directions:

Alternative 1. Four used lines intersect at a node of $\mathcal{T}$.

Alternative 2. $\mathcal{T}$ satisfies the conditions (1)–(4) of Proposition 2.2.1.

We consider these two cases in the forthcoming Sections 3 and 4, respectively.

3. Alternative 1 — proof of the main result

Assume that we have a set of four used lines $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ that intersect at a node $T_0 \in \mathcal{T}$. Assume also, in view of Lemma 2.1.4(i) and assumption (2.2.1), that the two nodes in l_1, l_2, l_3, l_4, besides T_0, are T_2, T_3; T_4, T_5; T_6, T_7; T_8, T_9, respectively. Denote the node which does not belong to the lines of $\mathcal{L}$ by T_1 (see Figure 1).

Let us start with:

Lemma 3.0.2. *Suppose that $\mathcal{M}$ is a set of 3 or 4 lines from $\mathcal{L}$. Suppose also that a node T not belonging to the lines of $\mathcal{M}$ does not use any line of $\mathcal{L}$. Then the following hold:*

   (i) *The set $\mathcal{M}$ consists of 3 lines, i.e., $\#\mathcal{M} = 3$.*

    (ii) *Any line used by T intersects each of the three lines of $\mathcal{M}$ at a node, different from T_0.*

Proof. Suppose that the node T uses a line l. Since $l \notin \mathcal{L}$ we have, in view of Lemma 2.1.4(i), that l is not passing through T_0. Then, by Lemma 2.1.3(i), l passes through one node from each line of $\mathcal{M}$, since otherwise T uses it, contrary to our assumption. Now, in view of Lemma 2.1.4(i), we obtain that $\#\mathcal{M} = 3$. $\hfill\square$

Lemma 3.0.3. *The following hold:*
    (i) *No node of $\mathcal{T}$ can use 2 lines from $\mathcal{L}$.*
    (ii) *The node T_1 uses a line from $\mathcal{L}$.*

Proof. Statement (i) follows immediately from Lemma 2.1.4(ii). In order to verify statement (ii), assume that T_1 does not use any line of $\mathcal{L}$. Then by setting $\mathcal{M} = \mathcal{L}$, we obtain a contradiction in view of Lemma 3.0.2(i). $\hfill\square$

    Now we may assume without loss of generality:

$(\diamond)$ The node T_1 uses the line l_1.

With that assumption, $p_1^\star = l_1 q$, with q a quadratic polynomial that must vanish at the 6 nodes $T_4, \ldots, T_9$ at which $p_1^\star$ vanishes but l_1 does not. In other words:

(3.0.2)   The nodes $T_4, T_5, T_6, T_7, T_8, T_9$ are in a conic (possibly reducible).

Now, we turn to the nodes in the line l_1.

Lemma 3.0.4. *Each of the nodes T_2, T_3, lying in the line l_1, uses a line of $\mathcal{L}$.*

Proof. Suppose one of the nodes T_2, T_3, say T_2, does not use any line of $\mathcal{L}$. Assume that l is a line used by T_2, where $l \notin \mathcal{L}$. Then from Lemma 3.0.2(ii), with $\mathcal{M} = \{l_2, l_3, l_4\}$, we get that the line l intersects each of the lines l_i, $i = 2, 3, 4$, at a node, different from T_0. Hence 3 of 6 nodes mentioned in (3.0.2), which belong to a conic, are collinear. Thus the conic is reducible and the other 3 nodes, i.e., the 3 nodes in $\{T_4, T_5, T_6, T_7, T_8, T_9\} \setminus l$ are collinear too. Now, the remaining three nodes of ten: T_0, T_1, T_3, are not collinear, in contradiction with Lemma 2.1.3(ii). $\hfill\square$

    Thus there is no loss of generality in assuming:

$(\diamond)$ The nodes T_2 and T_3 use the lines l_2 and l_3, respectively.

    Next we consider the nodes in the line l_4.

Lemma 3.0.5. *The nodes T_8 and T_9, lying in the line l_4, use certain lines l' and l'', respectively, which intersect each of the three lines l_i, $i = 1, 2, 3$, at a node, different from T_0.*

Proof. We have that the line l_i is used by the node $T_i, i = 1, 2, 3$. Therefore, by Lemma 2.1.2 the nodes T_8 and T_9, do not use any line from $\mathcal{L}$. Now, it remains to set $\mathcal{M} = \{l_1, l_2, l_3\}$ and use Lemma 3.0.2(ii).   □

Furthermore, we have:

Lemma 3.0.6. *The lines l' and l'' of Lemma 3.0.5 have a common node T which belongs to $\mathcal{T} \backslash l_1$. Moreover T uses the line l_4.*

Proof. Assume by way of contradiction that there is no common node. Then, in view of Lemma 2.1.3(i) and Lemma 3.0.5, the nodes T_8 and T_9 both use the lines l' and l'', which is impossible by Lemma 2.1.2. Now suppose that l_4 is used by a node $T' \in \mathcal{T} \backslash l_4$, different from the common node T. Then T' does not belong to one of the lines l' and l'', say to l'. Next, by Lemma 2.1.3(i), the node T' uses the line l' already used by T_8. This contradicts Lemma 2.1.2. Finally, notice that, in view of Lemma 3.0.3(i), the common node T is not in the line l_1, i.e., it is not coinciding with the nodes T_2, T_3. Indeed, T uses the line $l_4 \in \mathcal{L}$, and the latter nodes use the lines $l_2, l_3 \in \mathcal{L}$, respectively.   □

Therefore the common node T is in the line l_2 or l_3. Hence, without loss of generality suppose that the lines l' and l'' intersect at T_4.

Thus, according to Lemma 3.0.6, we have (see Figure 2)

(◇)   The node $T_4 = l' \cap l''$ uses the line l_4.

Next, completely similarly to Lemma 3.0.5 we get:

Lemma 3.0.7. *The nodes T_6 and T_7, lying in the line l_3, use certain lines $l^\star$ and $l^{\star\star}$, respectively, which intersect each of the three lines l_i, $i = 1, 2, 4$, at a node, different from T_0.*

Now we are in a position to complete:

Proof of Theorem 2.1.1 in the case of Alternative 1. We have that the lines l' and l'' pass through the node T_4. In view of Lemma 3.0.5, we may assume without loss of generality that l' passes also through the node T_2 and one of T_6, T_7, while l'' passes through T_3 and another one of T_6, T_7. Next, we have, in view of Lemma 3.0.7, that the lines $l^\star$ and $l^{\star\star}$ pass through the node T_5 of the line l_2. Indeed, otherwise if one of them passes through the node $T_4 \in l_2$, then it coincides with one of the lines l' and l''. Again, in view of Lemma 3.0.7, we may assume without loss of generality that one of the lines $l^\star$ and $l^{\star\star}$, say $l^\star$, passes also through the node T_2 and one of T_8, T_9. Finally, let us turn to the node T_3 which uses the line l_3. By Lemma 2.1.3(i) it uses also $l^\star$, already used by T_6. This, in view of Lemma 2.1.2, is a contradiction.   □
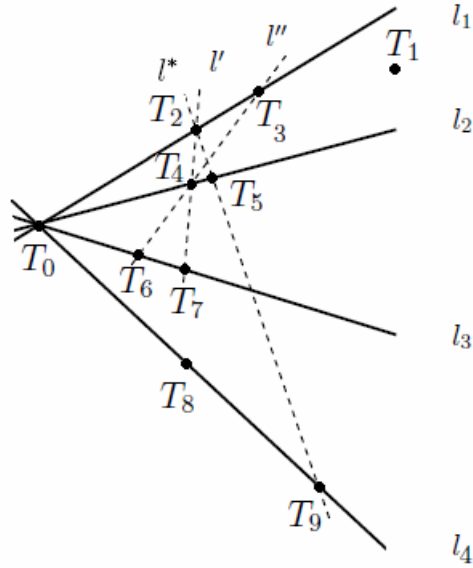
FIGURE 2. Alternative 1 — proof.

4. Alternative 2

4.1. The Desargues and Pascal theorems. Denote by l_{AB} the line passing through the points A and B.

For a set of points $A_1, A_2, A_3, B_1, B_2, B_3$ the following *cross- and v-type intersection points* (see Figure 3) will be considered in the sequel:

$$\dot{AB}_{1\vee 2} := l_{A_1 A_2} \cap l_{B_1 B_2} =: C_3^\vee, \qquad \dot{AB}_{3\vee 1} =: C_2^\vee, \qquad \dot{AB}_{2\vee 3} =: C_1^\vee,$$

$$\dot{AB}_{1\times 2} := l_{A_1 B_2} \cap l_{A_2 B_1} =: C_3^\times, \qquad \dot{AB}_{3\times 1} =: C_2^\times, \qquad \dot{AB}_{2\times 3} =: C_1^\times.$$

In the brief notation $C_i^\vee$ and $C_i^\times$ we take into account the fact that $\dot{AB}_{j\vee k} = \dot{AB}_{k\vee j}$ and $\dot{AB}_{j\times k} = \dot{AB}_{k\times j}$.

Remark 4.1.1. Notice that the intersection points $\dot{AB}_{1\vee 2}$ and $\dot{AB}_{1\times 2}$ will be interchanged if we interchange the points A_1, B_1 or A_2, B_2 (see Figure 4).

Let us now present the well-known Desargues and Pascal theorems in terms of the above cross- and v-type intersection points (see Figure 6).

Theorem 4.1.2 (Desargues). *Suppose the lines l_1, l_2, l_3 are concurrent and two points A_i, B_i are given on each line l_i, $i = 1, 2, 3$. Then the intersection points $C_1^\vee, C_2^\vee, C_3^\vee$ are collinear.*

Theorem 4.1.3 (Pascal). *Suppose six points: $A_1, A_2, A_3, B_1, B_2, B_3$ are given in a conic (i.e., in an algebraic curve of degree 2). Then the intersection points $C_1^\times, C_2^\times, C_3^\times$ are collinear.*
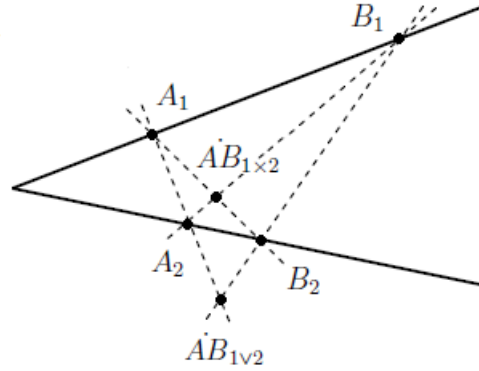
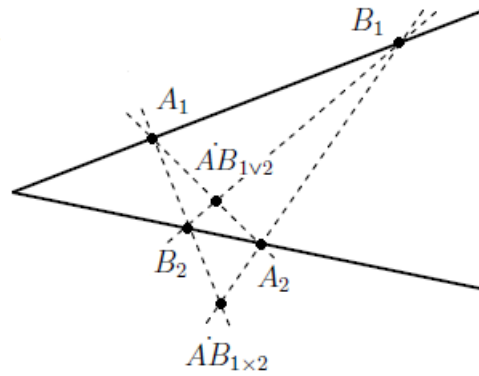FIGURE 3. Points $\dot{AB}_{1\vee2}$ and $\dot{AB}_{1\times2}$.

FIGURE 4. Points A_2 and B_2 interchanged.

Below, in Remark 4.1.4, we bring two other equivalent formulations of the Pascal theorem. First one is well-known and second one will be used in the proof of the forthcoming Theorem 4.1.5.

Remark 4.1.4.

(i) If we apply the Pascal theorem for the 6 points $T_1, \ldots, T_6$ ordered as $\{T_1, T_5, T_3, T_4, T_2, T_6\} \equiv \{A_1, A_2, A_3, B_1, B_2, B_3\}$ then we get the following equivalent formulation of the Pascal theorem:

Suppose 6 points: $T_1, \ldots, T_6$ are given in a conic. Then the following three intersection points are collinear:

$$l_{12} \cap l_{45}, \quad l_{23} \cap l_{56}, \quad l_{34} \cap l_{61},$$

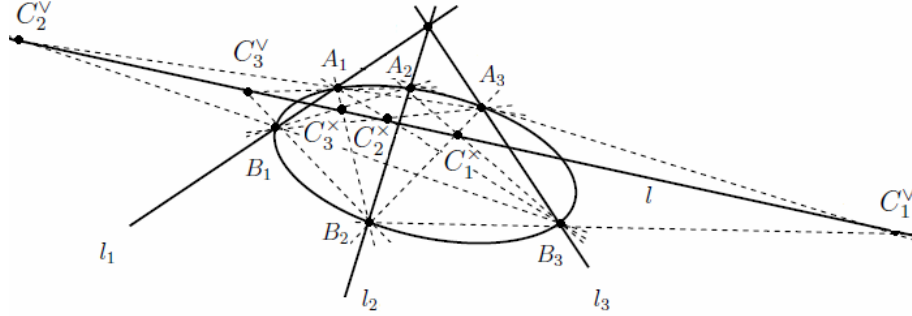*where l_{ij} is the line passing through T_i and T_j.*

FIGURE 5. The six collinear points.

(ii) If we interchange A_3 and B_3 in Theorem 4.1.3 then, in view of Remark 4.1.1, we get the following equivalent formulation of the Pascal theorem:

Suppose 6 *points:* $A_1, A_2, A_3, B_1, B_2, B_3$ *are given in a conic. Then the intersection points* $C_1^\vee, C_2^\vee, C_3^\times$ *are collinear.*

On the basis of the Desargues and Pascal theorems we get the following, interesting in itself:

Theorem 4.1.5. *Suppose the lines* l_1, l_2, l_3 *are concurrent and two points* A_i, B_i *are given on each line* l_i, $i = 1, 2, 3$, *such that the six points* A_1, A_2, A_3, B_1, B_2, B_3 *are in a conic. Then the following six intersection points:* $C_1^\vee$, $C_2^\vee$, $C_3^\vee$, $C_1^\times$, $C_2^\times$, $C_3^\times$ *are collinear (see Figure 5).*

Proof. In view of the Desargues theorem and the Pascal theorem formulated as in Remark 4.1.4(ii), we have that the points $C_1^\vee, C_2^\vee, C_3^\vee, C_3^\times$ are in a line l. Next we apply the Pascal theorem once more for the 6 points ordered in the following way: $A_3, A_1, A_2, B_3, B_1, B_2$, to get that the points $C_1^\vee, C_3^\vee, C_2^\times$ are collinear. Since first two of these 3 points are in l, also the third one: $C_2^\times$ is in l. To complete the proof we apply for the third time the Pascal theorem for the 6 points ordered in the following way: $A_2, A_3, A_1, B_2, B_3, B_1$, and get that the points $C_2^\vee, C_3^\vee, C_1^\times$ are collinear. Hence we obtain that $C_1^\times$ is in l. $\square$

Remark 4.1.6. Note that the inverses of the Desargues and Pascal theorems as well as Theorem 4.1.5 also hold true.

4.2. The construction of the node set. Now let us turn to the case of Alternative 2 described in Subsection 2.2. Before starting the proof of Theorem 2.1.1 in this case (in Subsection 4.3), we describe the construction of $\mathcal{T}$ and make some clarifications. Now, the conditions (1)–(4) of Proposition 2.2.1 hold. It is convenient to refer to these conditions, as conditions
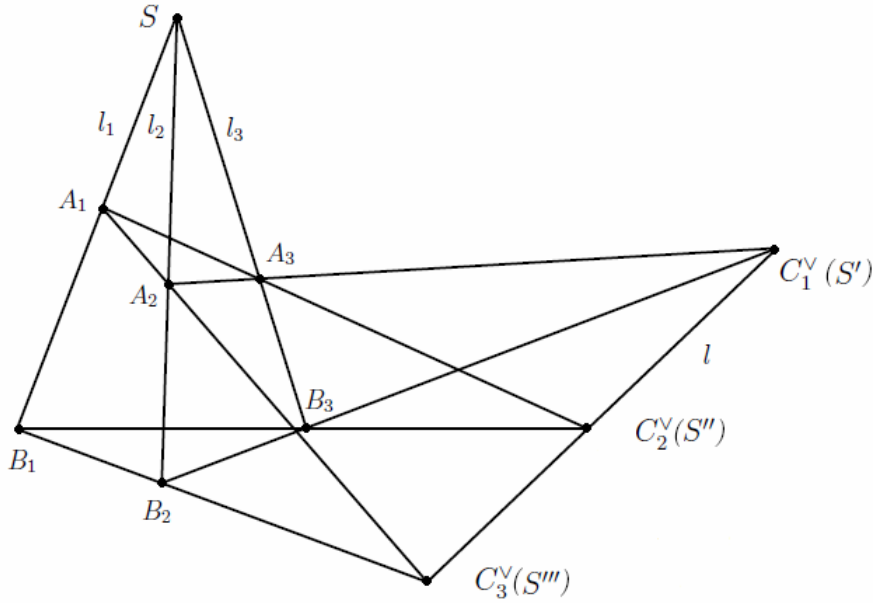
FIGURE 6. The construction of $\mathcal{T}$ with $C_3^\vee$.

(1)–(4) of Alternative 2, or briefly, as Alt2.1–Alt2.4. Note that conditions (1)–(3) mean that the 10 nodes of $\mathcal{T}$ and 10 used lines form a 10_3 configuration (see [14], Chapter 3, Section 19). Recall that by condition (4) each node of $\mathcal{T}$ uses exactly one line and one irreducible conic. Below we describe the construction of $\mathcal{T}$, starting with any node (see Figure 6).

Proposition 4.2.1. *Let $S \in \mathcal{T}$ be any (starting) node. Assume that l_1, l_2, l_3 are the three used lines passing through S (Alt2.3) Assume also that A_i, B_i are the 2 nodes, besides S, in the line l_i, $i = 1, 2, 3$ (Alt2.2). Then the following hold.*

   (i) *The 6 nodes A_1, A_2, A_3, B_1, B_2, B_3 are in the irreducible conic used by S.*
  (ii) *The line l used by S passes through the remaining three nodes of $\mathcal{T}$, i.e., the nodes of $\mathcal{T} \setminus \{S, A_1, A_2, A_3, B_1, B_2, B_3\}$. These three nodes can be identified as $C_1^\vee, C_2^\vee$, and $C_3^\vee$ or $C_3^\times$ (for this we may interchange the nodes A_i, B_i in the lines l_i, $i = 3, 1$, if necessary).*
 (iii) *The three nodes in l, i.e., $C_1^\vee, C_2^\vee, C_3^\vee$ or $C_3^\times$, use the lines l_1, l_2, l_3, respectively.*
  (iv) *The 6 intersection points $C_1^\vee, C_2^\vee, C_3^\vee, C_1^\times, C_2^\times, C_3^\times$ belong to the line l.*

Proof. Consider the line l used by S. In view of conditions Alt2.2 and Alt2.3, there are 3 nodes: S', S'', S''' in l and through each node there pass

2 used lines, besides l. Thus, we have identified 7 used lines: l and the other 6 ones intersecting l at a node.

Now, let us verify that the remaining 3 used lines of 10, which already do not intersect l at a node, pass through S, i.e., coincide with l_1, l_2, l_3. Otherwise, according to Lemma 2.1.3(i), each line not passing through S and not intersecting l at a node, will be used by S, which already uses the line l. But, this contradicts condition Alt2.4. Thus, the 6 nodes A_1, A_2, A_3, B_1, B_2, B_3 are outside the line l. Therefore, in view of condition Alt2.4, they are in the irreducible conic used by S, and (i) is proved.

Next, we observe that the line l used by the node S can be determined by each of the following two criteria: Take the 3 used lines l_1, l_2, l_3, passing through the node S and then the line used by this node is

(a) the line passing through the 3 nodes of $\mathcal{T}$ not belonging to l_1, l_2, l_3,
(b) the line not intersecting the 3 lines l_1, l_2, l_3, at a node.

Then, in view of (b), we get that each of the 3 nodes in l uses one of the concurrent lines l_i, $i = 1, 2, 3$, since all other 6 lines intersect l at a node. Now let us fix one of the 3 nodes, say S''', and suppose that it uses l_3 (see Figure 6). According to (a), all the nodes except the 3 in the line l_3, i.e., S, A_3, B_3, are in the used lines passing through S'''. Thus the nodes A_1, A_2, B_1, B_2 are in the two used lines passing through S''', since the third line through S''' is l, which passes through S' and S''. Therefore we have two possibilities. The two used lines pass either through the pairs of nodes $\{A_1, A_2\}$, $\{B_1, B_2\}$ or $\{A_1, B_2\}$, $\{A_2, B_1\}$. In other words, we get that S''' is either $C_3^\vee$ or $C_3^\times$. In the same way, by supposing that S' and S'' use l_1 and l_2, respectively, we get that S' is either $C_1^\vee$ or $C_1^\times$ and S'' is either $C_2^\vee$ or $C_2^\times$. Now, we interchange the nodes A_3, B_3 in the line l_3 and then the nodes A_1, B_1 in the line l_1, if necessary, to fix these nodes as they are mentioned in (ii). Thus (ii) and (iii) are proved. (Note that we cannot interchange the nodes in the line l_2 to fix one of the two possibilities also for the third node: $C_3^\vee$ or $C_3^\times$, since then the already fixed intersection point $C_1^\vee$ will turn into $C_1^\times$.) Finally, we get (iv) by using (i) and Theorem 4.1.5. $\square$

The following proposition along with Proposition 4.2.1 is a main tool in proving Theorem 2.1.1 for the case of Alternative 2.

Proposition 4.2.2. *Suppose that two used lines l_1 and l_2 pass through a node $S \in \mathcal{T}$. Suppose also that A_i, B_i are the 2 nodes, besides S, in the line l_i, $i = 1, 2$. Then one of the points $C_3^\vee, C_3^\times$ (i.e., $\dot{A}\dot{B}_{1\vee 2}, \dot{A}\dot{B}_{1\times 2}$) is a node in $\mathcal{T}$ and another coincides with the intersection point $l_3 \cap l$, where l_3 is the third used line through S and l is the line used by S.*

Proof. Assume, in view of Proposition 4.2.1(ii) and Remark 4.1.1, without loss of generality, that $C_3^\vee \in \mathcal{T}$. Then we need to prove that
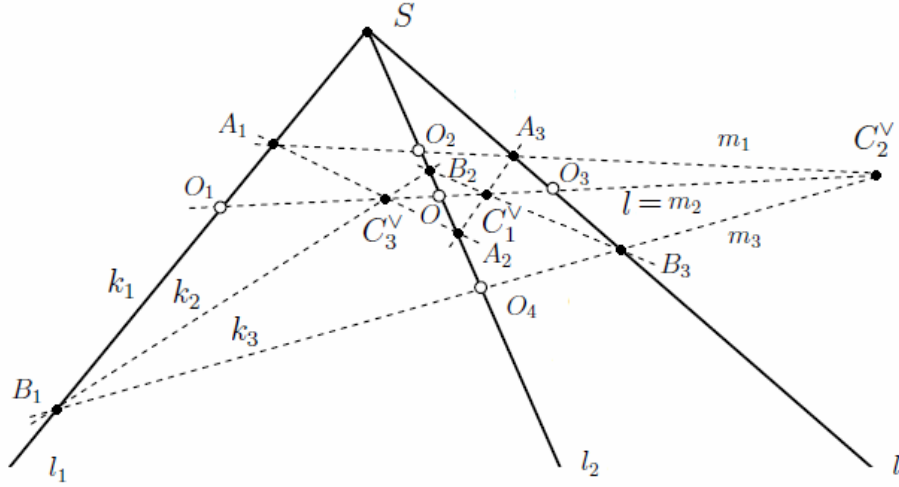
$$C_3^\times = l_3 \cap l.$$

FIGURE 7. Alternative 2 — the proof.

First notice that by Proposition 4.2.1(iv) $C_3^\times \in l$. Next, we are going to use Proposition 4.2.1(iii) with the starting node $S = C_3^\vee$. We have that this node uses the line l_3. On the other hand the pairs of nodes $\{A_1, A_2\}, \{B_1, B_2\}$ are in the two used lines passing through the starting node $C_3^\vee$, respectively. Therefore, by using Proposition 4.2.1(iv), we conclude that $C_3^\times \in l_3$ (see Figure 3).   □

4.3. Alternative 2 — proof of the main result. Consider a node $S \in \mathcal{T}$ satisfying the condition

$$(4.3.1) \qquad\qquad\qquad S \notin \mathrm{conv}\{\mathcal{T} \setminus S\}.$$

Suppose the 3 used lines passing through S are l_1, l_2, l_3. Suppose also that the 2 nodes in l_i, besides S, are A_i, B_i, $i = 1, 2, 3$. These nodes, in view of the condition (4.3.1) are on the rays starting with S, which we denote by l_i^+, $i = 1, 2, 3$. Assume that A_1, B_2, A_3 are the middle nodes in the rays, i.e., they lie in the segments SB_1, SA_2, SB_3, respectively. (see Figure 7). Without loss of generality assume that l_2^+ is between l_1^+ and l_3^+, meaning that

$$(4.3.2) \qquad\qquad\qquad l_2^+ \text{ belongs to } \angle\alpha,$$

where $\angle\alpha$ is the angle $(< \pi)$ with the sides l_1^+, l_3^+.

Then it is easily seen that

$$(4.3.3) \qquad\qquad C_1^\vee := \dot{A}B_{2\vee3} \in \mathrm{conv}\{B_2, A_2, B_3, A_3\},$$

$$(4.3.4) \qquad\qquad C_3^\vee := \dot{A}B_{1\vee2} \in \mathrm{conv}\{A_1, B_1, A_2, B_2\},$$

and

$$(4.3.5) \qquad\qquad C_2^\vee := \dot{A}B_{3\vee1} \notin \angle\alpha \cup \angle\alpha^-,$$

where $\angle\alpha^-$ is the opposite angle of $\angle\alpha$.

Now let us verify that the points $C_1^\vee, C_2^\vee$ and $C_3^\vee$ are nodes of $\mathcal{T}$. Indeed, otherwise, by Proposition 4.2.2, they belong to the lines l_1, l_2, and l_3, respectively, contrary to (4.3.3–4.3.5).

Next, notice that one of the nodes B_1 or B_3 has the property (4.3.1) of S, depending on which one is an end point in the triple B_1, B_3, S. Suppose it is the node B_1 (as in Figure 7). Note that the 3 used lines through B_1 are the lines $k_1 := l_{B_1S}, k_2 := l_{B_1B_2}$ and $k_3 := l_{B_1B_3}$. Then the 2 nodes in k_i, besides B_1, are in the respective rays starting with B_1, which we denote by k_i^+, $i = 1, 2, 3$, respectively.

Now, let us show that

(4.3.6) k_2^+ is between k_1^+ and k_3^+.

For this notice that, in view of Proposition 4.2.1(ii), S uses the line l passing through the nodes $C_1^\vee, C_2^\vee$ and $C_3^\vee$. Hence, according to Proposition 4.2.2, $C_2^\times := \dot{A}B_{3\times1}$, i.e., the intersection point of diagonals of the quadrangle A_1, A_3, B_3, B_1, coincides with the point $O := l_2 \cap l$. From here we get that

(4.3.7) m_2^+ is between m_1^+ and m_3^+,

where m_1^+, m_2^+, m_3^+ are the rays starting with $C_2^\vee$ and passing through $A_1, C_1^\vee, B_1$, respectively.

Therefore

(4.3.8) $O_1 \in \text{conv}\{A_1, B_1\}, \quad O \in \text{conv}\{O_2, O_4\},$

where $O_1 := m_2 \cap l_1$, $O_2 := m_1 \cap l_2$, $O_4 = m_3 \cap l_2$.

On the other hand, by the condition (4.3.4), and the fact that $C_3^\vee \in l$, we conclude that

$$C_3^\vee \in \text{conv}\{O_1, O\}.$$

This, in view of (4.3.8), establishes (4.3.6), since the ray k_2^+ intersects l at $C_3^\vee$.

Now notice that the relations (4.3.4) and (4.3.3) proved for the starting node S, in the case of the starting node B_1 imply that $\text{conv}\{A_1, S, B_2, C_3^\vee\}$ contains in its interior one of the 3 nodes in the line used by B_1, i.e., one of $A_2, C_1^\vee, A_3$. But it is easily seen that the later nodes belong to the angle with sides l_2^+, l_3^+, which is a contradiction. □

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