Full faithfulness theorem for torsion crystalline representations

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ABSTRACT. Mark Kisin proved that a certain "restriction functor" on crystalline p-adic representations is fully faithful. In this paper, we prove the torsion analogue of Kisin's theorem.

CONTENTS

1. Introduction

Let $p > 2$ be a prime number and $r, r' \geq 0$ integers. Let K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field and absolute ramification index e. Let $\pi = \pi_0$ be a uniformizer of K and π_n a p^n-th root of π such that $\pi_{n+1}^p = \pi_n$ for all $n \geq 0$. Put $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and denote by G_K and G_∞ absolute Galois groups of K and K_∞, respectively. In Theorem (0.2) of [Kis], Kisin proved that the functor "restriction to G_∞" from crystalline $\mathbb{Q}_p$-representations of G_K to $\mathbb{Q}_p$-representations of G_∞ is fully faithful, which was a conjecture of Breuil ([Br2]). Hence we may say that crystalline $\mathbb{Q}_p$-representations of G_K are characterized by their restriction to G_∞. It should be noted that there exists an established theory describing representations of G_∞ by easy linear algebra data, which is called étale φ-modules, introduced by Fontaine ([Fo1] A 1.2). In this paper, we are interested in the torsion analogue of the above Kisin's result. For example, Breuil proved in Theorem 3.4.3 of [Br3] that

the functor "restriction to G_∞" from finite flat representations of G_K to torsion $\mathbb{Z}_p$-representations of G_∞ is fully faithful (Remark 20(2)). Our main theorem is motivated by his result:

Theorem 1. *Suppose $er < p - 1$ and $e(r' - 1) < p - 1$. Let T (resp. T') be a torsion crystalline $\mathbb{Z}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$ (resp. $[0, r']$). Then any G_∞-equivariant morphism $T \to T'$ is in fact G_K-equivariant.*

In particular, the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge–Tate weights in $[0, r]$ to torsion $\mathbb{Z}_p$-representations of G_∞, obtained by restricting the action of G_K to G_∞, is fully faithful.

Here a torsion $\mathbb{Z}_p$-representation of G_K is said to be *torsion crystalline with Hodge–Tate weights in $[0, r]$* if it can be written as the quotient of two lattices in some crystalline $\mathbb{Q}_p$-representation of G_K with Hodge–Tate weights in $[0, r]$. For example, a torsion $\mathbb{Z}_p$-representation of G_K is finite flat if and only if it is torsion crystalline with Hodge–Tate weights in $[0, 1]$ (Remark 20(2)). If $e = 1$, the latter part of Theorem 1 has been proven by Breuil via Fontaine–Laffaille theory (Remark 20(3)). On the other hand, our proof is based on results on Kisin modules and $(\varphi, \hat{G})$-modules (the notion of $(\varphi, \hat{G})$-modules is introduced in [Li2]). More precisely, we use maximal models for Kisin modules introduced in [CL1] and results on "the range of monodromy" for $(\varphi, \hat{G})$-modules given in Section 4 of [GLS].

It seems natural to ask whether the condition "$er < p - 1$" in the latter part of Theorem 1 is necessary and sufficient for the full faithfulness. In fact, we know that the condition "$er < p - 1$" is not necessary since our restriction functor is fully faithful for any e when $r = 1$ (Remark 20(2)). (Maybe the necessary and sufficient condition for the full faithfulness is "$e(r - 1) < p - 1$" (Remark 20).) In addition, in the last section, we give some examples such that the restriction functor appeared in Theorem 1 is not full under some choices of K and r which do not satisfy "$er < p - 1$" (more precisely, "$e(r - 1) < p - 1$"). Examples are mainly given by using two methods: The first one is direct computations of Galois cohomologies, which is a purely local method. The second one is based on the classical Serre's modularity conjecture, which is a global method.

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2. Preliminaries

Throughout this paper, we fix a prime number $p > 2$. Let $r \geq 0$ be an integer. Let k be a perfect field of characteristic p, $W(k)$ its ring of Witt

vectors, $K_0 = W(k)[1/p]$, K a finite totally ramified extension of K_0, $\overline{K}$ a fixed algebraic closure of K and $G_K = \mathrm{Gal}(\overline{K}/K)$. Fix a uniformizer $\pi \in K$ and denote by $E(u)$ its Eisenstein polynomial over K_0. For any integer $n \geq 0$, let $\pi_n \in \overline{K}$ be a p^n-th root of π such that $\pi_{n+1}^p = \pi_n$. Let $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ and $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$.

For any topological group H, we denote by $\mathrm{Rep}_{\mathrm{tor}}(H)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}(H)$) the category of finite torsion $\mathbb{Z}_p$-representations of H (resp. the category of finite free $\mathbb{Z}_p$-representations of H). We denote by $\mathrm{Rep}_{\mathbb{Z}_p}^r(G_K)$ the category of lattices in crystalline $\mathbb{Q}_p$-representations of G_K with Hodge–Tate weights in $[0, r]$. We say that $T \in \mathrm{Rep}_{\mathrm{tor}}(G_K)$ is *torsion crystalline with Hodge–Tate weights in* $[0, r]$ if it can be written as the quotient of $L' \subset L$ in $\mathrm{Rep}_{\mathbb{Z}_p}^r(G_K)$, and denote by $\mathrm{Rep}_{\mathrm{tor}}^r(G_K)$ the category of them.

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$ where $\mathcal{O}_{\overline{K}}$ is the integer ring of $\overline{K}$ and the transition maps are given by the p-th power map. Write $\underline{\pi} = (\pi_n)_{n \geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative of $\underline{\pi}$. Let $\mathfrak{S} = W(k)[\![u]\!]$ equipped with a Frobenius endomorphism φ given by $u \mapsto u^p$ and the Frobenius on $W(k)$. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R)$ via the map $u \mapsto [\underline{\pi}]$. This embedding extends to an embedding $\mathfrak{S} \hookrightarrow W(R)$, which is compatible with Frobenius endomorphisms.

A *φ-module* (*over* $\mathfrak{S}$) is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi \colon \mathfrak{M} \to \mathfrak{M}$. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear map $\mathfrak{M}_1 \to \mathfrak{M}_2$ compatible with φ_1 and φ_2. Denote by $'\mathrm{Mod}_{/\mathfrak{S}}^r$ the category of φ-modules $(\mathfrak{M}, \varphi)$ *of height* $\leq r$ in the sense that $\mathfrak{M}$ is of finite type over $\mathfrak{S}$ and the cokernel of $1 \otimes \varphi \colon \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. Let $\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$ be the full subcategory of $'\mathrm{Mod}_{/\mathfrak{S}}^r$ consisting of finite $\mathfrak{S}$-modules which are killed by some power of p and have projective dimension 1 in the sense that $\mathfrak{M}$ has a two term resolution by finite free $\mathfrak{S}$-modules. Let $\mathrm{Mod}_{/\mathfrak{S}}^r$ be the full subcategory of $'\mathrm{Mod}_{/\mathfrak{S}}^r$ consisting of finite free $\mathfrak{S}$-modules. We call an object of $\mathrm{Mod}_{/\mathfrak{S}_\infty}^r$ (resp. $\mathrm{Mod}_{/\mathfrak{S}}^r$) a *torsion Kisin module* (resp. a *free Kisin module*). A *Kisin module* is a torsion Kisin module or a free Kisin module. For any Kisin module $\mathfrak{M}$, we define a $\mathbb{Z}_p$-representation $T_{\mathfrak{S}}(\mathfrak{M})$ of G_∞ by

$$T_{\mathfrak{S}}(\mathfrak{M}) = \begin{cases} \mathrm{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}}) & \text{if } \mathfrak{M} \text{ is torsion} \\ \mathrm{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) & \text{if } \mathfrak{M} \text{ is free.} \end{cases}$$

Here, a G_∞-action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(\sigma.f)(x) = \sigma(f(x))$ for $\sigma \in G_\infty, f \in T_{\mathfrak{S}}(\mathfrak{M}), x \in \mathfrak{M}$.

Here we recall the theory of Liu's $(\varphi, \hat{G})$-modules (cf. [Li2]). Let S be the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. There exists a unique Frobenius map $\varphi \colon S \to S$ defined by $\varphi(u) = u^p$. Put $S_{K_0} = S[1/p] = K_0 \otimes_{W(k)} S$. The inclusion $W(k)[u] \hookrightarrow W(R)$ via the map $u \mapsto [\underline{\pi}]$ induces φ-compatible inclusions $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ and $S_{K_0} \hookrightarrow B_{\mathrm{cris}}^+$. Fix a choice of primitive p^i-root of

unity ζ_{p^i} for $i \geq 0$ such that $\zeta_{p^{i+1}}^p = \zeta_{p^i}$. Put $\underline{\varepsilon} = (\zeta_{p^i})_{i\geq 0} \in R^\times$ and $t = \log([\underline{\varepsilon}]) \in A_{\mathrm{cris}}$. Denote by $\nu\colon W(R) \to W(\overline{k})$ a unique lift of the projection $R \to \overline{k}$, which extends to a map $\nu\colon B_{\mathrm{cris}}^+ \to W(\overline{k})[1/p]$. For any subring $A \subset B_{\mathrm{cris}}^+$, we put $I_+A = \mathrm{Ker}(\nu \text{ on } B_{\mathrm{cris}}^+) \cap A$. For any integer $n \geq 0$, let $t^{\{n\}} = t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $\tilde{q}(n) \geq 0$, $0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. We define a subring $\mathcal{R}_{K_0}$ of B_{cris}^+ as below:

$$\mathcal{R}_{K_0} = \left\{ \sum_{i=0}^\infty f_i t^{\{i\}} \mid f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to \infty \right\}.$$

Put $\widehat{\mathcal{R}} = \mathcal{R}_{K_0} \cap W(R)$ and $I_+ = I_+\widehat{\mathcal{R}}$. Put $\hat{K} = \bigcup_{n\geq 0} K_\infty(\zeta_{p^n})$ and $\hat{G} = \mathrm{Gal}(\hat{K}/K)$. Lemma 2.2.1 in [Li2] shows that $\widehat{\mathcal{R}}$ (resp. $\mathcal{R}_{K_0}$) is a φ-stable $\mathfrak{S}$-algebra as a subring in $W(R)$ (resp. B_{cris}^+), and ν induces $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/I_+\mathfrak{S} \simeq W(k)$. Furthermore, $\widehat{\mathcal{R}}, I_+, \mathcal{R}_{K_0}$ and $I_+\mathcal{R}_{K_0}$ are G_K-stable, and G_K-actions on them factors through $\hat{G}$. For any Kisin module $\mathfrak{M}$, we equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with a Frobenius by $\varphi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$. It is known that the natural map $\mathfrak{M} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ given by $x \mapsto 1 \otimes x$ is an injection ([CL2], Section 3.1). By this injection, we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-stable submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

Definition 2. A $(\varphi, \hat{G})$-module (of height $\leq r$) is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ where:

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module (of height $\leq r$).
(2) $\hat{G}$ is a continuous $\widehat{\mathcal{R}}$-semilinear $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.
(3) the $\hat{G}$-action commutes with $\varphi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$.
(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^{H_K}$ where $H_K = \mathrm{Gal}(\hat{K}/K_\infty)$.
(5) $\hat{G}$ acts on the $W(k)$-module $(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})/I_+(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$ trivially.

If $\mathfrak{M}$ is a torsion (resp. free) Kisin module, we call $\hat{\mathfrak{M}}$ a *torsion* (resp. *free*) $(\varphi, \hat{G})$-*module*.

A morphism between two $(\varphi, \hat{G})$-modules $\hat{\mathfrak{M}}_1 = (\mathfrak{M}_1, \varphi_1, \hat{G})$ and $\hat{\mathfrak{M}}_2 = (\mathfrak{M}_2, \varphi_2, \hat{G})$ is a morphism $f\colon \mathfrak{M}_1 \to \mathfrak{M}_2$ of φ-modules such that

$$\widehat{\mathcal{R}} \otimes f\colon \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_1 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}_2$$

is $\hat{G}$-equivariant. We denote by $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\hat{G}}$ (resp. $\mathrm{Mod}_{/\mathfrak{S}}^{r,\hat{G}}$) the category of torsion $(\varphi, \hat{G})$-modules of height $\leq r$ (resp. free $(\varphi, \hat{G})$-modules of height $\leq r$). We often regard $\widehat{\mathcal{R}} \otimes_{\varphi,\varphi} \mathfrak{M}$ as a G_K-module via the projection $G_K \twoheadrightarrow \hat{G}$. A sequence $0 \to \hat{\mathfrak{M}}' \to \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'' \to 0$ of $(\varphi, \hat{G})$-modules is *exact* if it is exact as $\mathfrak{S}$-modules. For a $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$, we define a $\mathbb{Z}_p$-representation

$\hat{T}(\hat{\mathfrak{M}})$ of G_K by

$$\hat{T}(\hat{\mathfrak{M}}) = \begin{cases} \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathfrak{M} \text{ is torsion} \\ \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)) & \text{if } \mathfrak{M} \text{ is free.} \end{cases}$$

Here, G_K acts on $\hat{T}(\hat{\mathfrak{M}})$ by $(\sigma.f)(x) = \sigma(f(\sigma^{-1}(x)))$ for $\sigma \in G_K$, $f \in \hat{T}(\hat{\mathfrak{M}})$, $x \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then, there exists a natural G_∞-equivariant map

$$\theta \colon T_\mathfrak{S}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$$

defined by $\theta(f)(a \otimes m) = a\varphi(f(m))$ for $f \in T_\mathfrak{S}(\mathfrak{M})$, $a \in \widehat{\mathcal{R}}, m \in \mathfrak{M}$.

Fix a topological generator τ of $\mathrm{Gal}(\hat{K}/K_{p^\infty})$ where $K_{p^\infty} = \bigcup_{n \geq 0} K(\zeta_{p^n})$. We may suppose that $\zeta_{p^n} = \tau(\pi_n)/\pi_n$ for all n, and this implies $\tau(u) = [\varepsilon]u$ in $W(R)$. There exists $\mathfrak{t} \in W(R) \smallsetminus pW(R)$ such that $\varphi(\mathfrak{t}) = pE(0)^{-1}E(u)\mathfrak{t}$. Such $\mathfrak{t}$ is unique up to units of $\mathbb{Z}_p$ (cf. Example 2.3.5 of [Li1]). The following theorems play important rolls in the proof of Theorem 1.

Theorem 3 ([Li2])**.**

(1) *The map $\theta \colon T_\mathfrak{S}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ is an isomorphism.*
(2) *The contravariant functor $\hat{T}$ induces an anti-equivalence between the category $\mathrm{Mod}_\mathfrak{S}^{r,\hat{G}}$ of free $(\varphi, \hat{G})$-modules of height $\leq r$ and the category of G_K-stable $\mathbb{Z}_p$-lattices in semi-stable $\mathbb{Q}_p$-representations of G_K with Hodge–Tate weights in $[0, r]$.*

Theorem 4 ([CL2], Theorem 3.1.3 (4), [GLS], Proposition 5.9)**.** *Let $T \in \mathrm{Rep}_{\mathrm{tor}}^r(G_K)$ and take $L' \subset L$ in $\mathrm{Rep}_{\mathbb{Z}_p}^r(G_K)$ such that $T \simeq L/L'$.*

(1) *There exists an exact sequence $\mathcal{S} \colon 0 \to \hat{\mathfrak{L}} \to \hat{\mathfrak{L}}' \to \hat{\mathfrak{M}} \to 0$ of $(\varphi, \hat{G})$-modules such that:*
   (a) *$\hat{\mathfrak{L}}$ and $\hat{\mathfrak{L}}'$ are free $(\varphi, \hat{G})$-modules of height $\leq r$.*
   (b) *$\hat{\mathfrak{M}}$ is a torsion $(\varphi, \hat{G})$-module of height $\leq r$.*
   (c) *$\hat{T}(\mathcal{S})$ is isomorphic to the exact sequence $0 \to L' \to L \to T \to 0$ of $\mathbb{Z}_p[G_K]$-modules.*
(2) *Let $\hat{\mathfrak{M}}$ be as in (1). For any $x \in \mathfrak{M}$, we have*

$$\tau(x) - x \in u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}).$$

Proof. The assertion (2) is an easy consequence of [GLS], Proposition 5.9. Here is one remark: In *loc. cit*, K is assumed to be a finite extension of $\mathbb{Q}_p$, but arguments in Section 4.1 and 4.2 of *loc. cit.* proceed even if K is not only a finite extension of $\mathbb{Q}_p$ but also any complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field. $\qquad\square$

3. Proof of Theorem 1

For any integer $\alpha \geq 0$, we denote by $\mathfrak{m}_R^{\geq \alpha}$ the ideal of R consisting of $a \in R$ with $v_R(a) \geq \alpha$, where v_R is a valuation of R such that $v_R(\underline{\pi}) = \frac{1}{e}$.

Note that, if we put $\tilde{\mathfrak{t}} = \mathfrak{t} \bmod p \in R$, then $v_R(\tilde{\mathfrak{t}}) = \frac{1}{p-1}$ since $\varphi(\tilde{\mathfrak{t}}) \in \underline{\pi}^e \tilde{\mathfrak{t}} \cdot R^\times$ (recall the equation $\varphi(\mathfrak{t}) = pE(0)^{-1}E(u)\mathfrak{t}$).

We note that we have natural inclusions

$$\mathfrak{M} \subset \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

for any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Denote by $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ the full subcategory of $\mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty}$ consisting of torsion $(\varphi,\hat{G})$-modules $\hat{\mathfrak{M}}$ which satisfy the following; for any $x \in \mathfrak{M}$,

$$\tau(x) - x \in u^p \varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}).$$

We define the full subcategory $\mathrm{Rep}^{r,\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$ of $\mathrm{Rep}_{\mathrm{tor}}(G_K)$ to be the essential image of the functor $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \subset \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}_\infty} \xrightarrow{\hat{T}} \mathrm{Rep}_{\mathrm{tor}}(G_K)$, where $\hat{T}$ is defined in the previous section. By Theorem 4, we have

$$\mathrm{Rep}^r_{\mathrm{tor}}(G_K) \subset \mathrm{Rep}^{r,\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K).$$

Remark 5.

(1) The subscript "cris" of $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ is plausible since a *free $(\varphi,\hat{G})$-module* $\hat{\mathfrak{M}}$ satisfying the condition

$$\tau(x) - x \in u^p \varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$$

corresponds to a crystalline representation. See Theorem 21 in the appendix for more precise information.

(2) Note that objects of $\mathrm{Rep}^{r,\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$ are not necessarily torsion crystalline representations. In fact, we do not know whether torsion $(\varphi,\hat{G})$-modules lift to free $(\varphi,\hat{G})$-modules.

Theorem 1 follows from the following result.

Theorem 6. *Let $T \in \mathrm{Rep}^{r,\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$ and $T' \in \mathrm{Rep}^{r',\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$, and suppose $er < p - 1$ and $e(r' - 1) < p - 1$. Then any G_∞-equivariant morphism $T \to T'$ is in fact G_K-equivariant.*

Lemma 7. *Let $a \in W(R) \setminus pW(R)$. For any Kisin module $\mathfrak{M}$, the map*

$$W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \quad x \mapsto ax$$

is injective.

Proof. We may suppose that $\mathfrak{M}$ is a torsion Kisin module. By a dévissage argument ([Li1], Proposition 2.3.2 (4)), we may assume $p\mathfrak{M} = 0$. In this situation, the statement is clear since $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is a finite direct sum of R. □

The following is a key lemma for our proof of Theorem 6:

Lemma 8. *Let r and r' be nonnegative integers with $e(r-1) < p-1$ (without any assumption on r'). Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ be objects of $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\hat{G},\mathrm{cris}}$ and $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r',\hat{G},\mathrm{cris}}$, respectively. Then we have $\mathrm{Hom}(\hat{\mathfrak{M}},\hat{\mathfrak{N}}) = \mathrm{Hom}(\mathfrak{M},\mathfrak{N})$.*

In particular, if $e(r-1) < p-1$, then the forgetful functor

$$\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r,\hat{G},\mathrm{cris}} \to \mathrm{Mod}_{/\mathfrak{S}_\infty}^{r}$$

is fully faithful.

The condition $e(r-1) < p-1$ is essential. See Remark 11 below.

Proof. Let $f\colon \mathfrak{M} \to \mathfrak{N}$ be a morphism of Kisin modules and put

$$\hat{f} = W(R) \otimes f\colon W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}.$$

It suffices to prove that, for any $x \in \mathfrak{M}$, $\Delta(1 \otimes x) = 0$ where $\Delta = \tau \circ \hat{f} - \hat{f} \circ \tau$. We proceed by induction on n such that $p^n \mathfrak{N} = 0$.

Suppose $n = 1$, that is, $p\mathfrak{N} = 0$. We may identify $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ with $R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$. Since $\Delta(1 \otimes x) = (\tau - 1)(1 \otimes f(x)) - \hat{f}((\tau - 1)(1 \otimes x))$, we obtain the following implication

(0)   For any $x \in \mathfrak{M}$, $\quad \Delta(1 \otimes x) \in \mathfrak{m}_R^{\geq c(0)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N})$

where $c(0) = \frac{p}{p-1} + \frac{p}{e}$. Note that

$$\Delta(1 \otimes E(u)^r x) = \tau(\varphi(E(u)))^r \Delta(1 \otimes x) = (\varepsilon u)^{per} \Delta(1 \otimes x) \in R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}.$$

On the other hand, since $\mathfrak{M}$ is of height $\leq r$, we can write $E(u)^r x = \sum_{i \geq 0} a_i \varphi(y_i)$ for some $a_i \in \mathfrak{S}$ and $y_i \in \mathfrak{M}$. Then we obtain

$$\Delta(1 \otimes E(u)^r x) = \sum_{i \geq 0} \tau(\varphi(a_i)) \varphi(\Delta(1 \otimes y_i))$$

and it is contained in $\mathfrak{m}_R^{\geq pc(0)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N})$ by the implication (0). Since $R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ is free as an R-module, we obtain the implication

(1)   For any $x \in \mathfrak{M}$, $\quad \Delta(1 \otimes x) \in \mathfrak{m}_R^{\geq c(1)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N})$

where $c(1) = pc(0) - pr = \frac{p^2}{p-1} + \frac{p^2}{e} - pr$. By repeating the same argument, for any $s \geq 0$, we see the following implication

(s)   For any $x \in \mathfrak{M}$, $\quad \Delta(1 \otimes x) \in \mathfrak{m}_R^{\geq c(s)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N})$

where $c(s) = pc(s-1) - pr = \frac{p^{s+1}}{p-1} + \frac{p^{s+1}}{e} - p^s r - \cdots - pr$. Since $e(r-1) < p-1$, we know that $\mathfrak{m}_R^{\geq c(s)}$ goes to zero when $s \to \infty$ and then we obtain $\Delta(1 \otimes x) = 0$.

Suppose $n > 1$. Consider the exact sequence

(∗) $0 \to \mathrm{Ker}(p) \to \mathfrak{N} \xrightarrow{p} p\mathfrak{N} \to 0$

of φ-modules. By Lemma 2.3.1 and Proposition 2.3.2 of [Li1], we know that $\mathfrak{N}' := \mathrm{Ker}(p)$ and $\mathfrak{N}'' := p\mathfrak{N}$ are in $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r'}$. Equipping $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}''$ with

$\hat{G}$-action via the natural identification $p(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}) = \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}''$, we see that $\mathfrak{N}''$ has a structure as a $(\varphi, \hat{G})$-module. We can also equip $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'$ with $\hat{G}$-action via the exact sequence

$$0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}' \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'' \to 0$$

(for the exactness, see [CL2], Lemma 3.1.2). Since the sequence

$$0 \to \widehat{\mathcal{R}}/I_+ \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}' \to \widehat{\mathcal{R}}/I_+ \otimes_{\varphi\,\mathfrak{S}} \mathfrak{N} \to \widehat{\mathcal{R}}/I_+ \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'' \to 0$$

is also exact ([Oz], Corollary 2.11), we know that $\mathfrak{N}'$ also has a structure as a $(\varphi, \hat{G})$-module. Summary, we obtained an exact sequence

$$0 \to \hat{\mathfrak{N}}' \to \hat{\mathfrak{N}} \xrightarrow{p} \hat{\mathfrak{N}}'' \to 0$$

in $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r',\hat{G}}$ whose underlying sequence of φ-modules is $(*)$. Remark that $p\mathfrak{N}' = 0$ and $p^{n-1}\mathfrak{N}'' = 0$. It is clear that $\hat{\mathfrak{N}}'' \in \mathrm{Mod}_{/\mathfrak{S}_\infty}^{r',\hat{G},\mathrm{cris}}$. Since

$$0 \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}' \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N} \to \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'' \to 0$$

is exact and $p^{n-1}\mathfrak{N}'' = 0$, we obtain $\Delta(1 \otimes x) \in \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}' \subset W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'$ for any $x \in \mathfrak{M}$ by the induction hypothesis. Moreover, we have in fact $\Delta(1 \otimes x) \in u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}')$ since Lemma 7 implies

$$(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}') \cap u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}) = u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}').$$

Identifying $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'$ with $R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}'$, we obtain

$$\Delta(1 \otimes x) \in \mathfrak{m}_R^{\geq c(0)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}').$$

By an analogous argument of the case where $n = 1$, we obtain the implication

(s')   For any $x \in \mathfrak{M}$, $\Delta(1 \otimes x) \in \mathfrak{m}_R^{\geq c(s)}(R \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}')$

for any $s \geq 0$ and this implies $\Delta(1 \otimes x) = 0$.   □

Before giving the proof of Theorem 6, we have to recall the theory of *maximal* Kisin modules. Now we give a very rough sketch of it (for more precise information, see [CL1]. Our sketch here is the case where "$r = \infty$" in *loc. cit.*). For any $\mathfrak{M} \in \mathrm{Mod}_{/\mathfrak{S}_\infty}^r$, put $\mathfrak{M}[1/u] = \mathfrak{S}[1/u] \otimes_{\mathfrak{S}} \mathfrak{M}$ and denote by $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$ the (partially) ordered set (by inclusion) of torsion Kisin modules $\mathfrak{N}$ of finite height which is contained in $\mathfrak{M}[1/u]$ and $\mathfrak{N}[1/u] = \mathfrak{M}[1/u]$ as φ-modules. Here, a torsion Kisin module is called *of finite height* if it is of height $\leq s$ for some integer $s \geq 0$. The set $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$ has a greatest element (cf. *loc. cit.*, Corollary 3.2.6), which is denoted by $\mathrm{Max}(\mathfrak{M})$. We say that $\mathfrak{M}$ is *maximal* if it is the greatest element of $F_{\mathfrak{S}}(\mathfrak{M}[1/u])$. The implication $\mathfrak{M} \mapsto \mathrm{Max}(\mathfrak{M})$ defines a functor "Max" from the category of torsion Kisin modules of finite height into the category $\mathrm{Max}_{/\mathfrak{S}_\infty}$ of maximal torsion Kisin modules. Furthermore, the functor $T_{\mathfrak{S}} \colon \mathrm{Max}_{/\mathfrak{S}_\infty} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$, defined by $T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$, is fully faithful (cf. *loc.*

cit., Corollary 3.3.10). It is not difficult to check that $T_{\mathfrak{S}}(\mathrm{Max}(\mathfrak{M}))$ is canonically isomorphic to $T_{\mathfrak{S}}(\mathfrak{M})$ as representations of G_∞ for any torsion Kisin module $\mathfrak{M}$.

Lemma 9. *Suppose $er < p - 1$. Then any $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ is maximal.*

Proof. We prove by induction on n such that $p^n \mathfrak{M} = 0$. If $n = 1$, then the assertion follows by Lemma 3.3.4 of [CL1]. Suppose $n > 1$ and $p^n \mathfrak{M} = 0$. Take any $\mathfrak{N} \in F_{\mathfrak{S}}(\mathfrak{M}[1/u])$ such that $\mathfrak{M} \subset \mathfrak{N}$ and put $M = \mathfrak{M}[1/u] = \mathfrak{N}[1/u]$. Denote by pr the natural surjection $M \to M/pM$. Putting $\mathfrak{M}' = pM \cap \mathfrak{M}, \mathfrak{M}'' = \mathrm{pr}(\mathfrak{M}), \mathfrak{N}' = pM \cap \mathfrak{N}$ and $\mathfrak{N}'' = \mathrm{pr}(\mathfrak{N})$, we see that $\mathfrak{M}'$ and $\mathfrak{M}''$ are objects of $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, and $\mathfrak{N}'$ and $\mathfrak{N}''$ are torsion Kisin modules of finite height. Furthermore, we see that natural sequences

$$0 \to \mathfrak{M}' \to \mathfrak{M} \xrightarrow{\mathrm{pr}} \mathfrak{M}'' \to 0,$$

$$0 \to \mathfrak{N}' \to \mathfrak{N} \xrightarrow{\mathrm{pr}} \mathfrak{N}'' \to 0,$$

of φ-modules are exact. By the induction hypothesis, we know that $\mathfrak{M}'$ and $\mathfrak{M}''$ are maximal and thus $\mathfrak{N}' = \mathfrak{M}'$ and $\mathfrak{N}'' = \mathfrak{M}''$ (remark that $\mathfrak{M}'[1/u] = \mathfrak{N}'[1/u] = pM$ and $\mathfrak{M}''[1/u] = \mathfrak{N}''[1/u] = M/pM$). This implies $\mathfrak{N} = \mathfrak{M}$. $\square$

Proof of Theorem 6. Suppose that $er < p - 1$ and $e(r' - 1) < p - 1$. Let $T \in \mathrm{Rep}^{r,\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$ (resp. $T' \in \mathrm{Rep}^{r',\hat{G},\mathrm{cris}}_{\mathrm{tor}}(G_K)$) and take $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ (resp. $\hat{\mathfrak{M}}' \in \mathrm{Mod}^{r',\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$) such that $T = \hat{T}(\hat{\mathfrak{M}})$ (resp. $T' = \hat{T}(\hat{\mathfrak{M}}')$). Note that $\mathfrak{M} = \mathrm{Max}(\mathfrak{M})$ by Lemma 9. By Theorem 3(1), we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}_{G_K}(T, T') & \lhook\joinrel\xrightarrow{\hspace{5cm}} & \mathrm{Hom}_{G_\infty}(T, T') \\
{\scriptstyle \hat{T}} \big\uparrow & & \big\uparrow {\scriptstyle T_{\mathfrak{S}}} \\
\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}) \xrightarrow{\mathrm{forgetful}} \mathrm{Hom}(\mathfrak{M}', \mathfrak{M}) & \xrightarrow{\mathrm{Max}} & \mathrm{Hom}(\mathrm{Max}(\mathfrak{M}'), \mathfrak{M}).
\end{array}
$$

The first bottom horizontal arrow is bijective by Lemma 8 and the second is also by an easy argument. Since the right vertical arrow is bijective, the top horizontal arrow must be bijective. $\square$

Remark 10. By Lemma 8, we can prove the latter part of Theorem 1 directly without using the former part of Theorem 1 as below: Suppose that $er < p - 1$. Let $T \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$ (resp. $T' \in \mathrm{Rep}^r_{\mathrm{tor}}(G_K)$) and take $\hat{\mathfrak{M}}$ (resp. $\hat{\mathfrak{M}}'$) be as in Theorem 4, which is an object of $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$. By Theorem 3(1), we have the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_{G_K}(T, T') & \lhook\joinrel\longrightarrow & \mathrm{Hom}_{G_\infty}(T, T') \\
{\scriptstyle \hat{T}} \big\uparrow & & \big\uparrow {\scriptstyle T_{\mathfrak{S}}} \\
\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}) & \xrightarrow{\mathrm{forgetful}} & \mathrm{Hom}(\mathfrak{M}', \mathfrak{M})
\end{array}
$$

and then we obtain the desired result by an analogous argument before this remark.

Remark 11. The condition $e(r-1) < p-1$ in Lemma 8 is essential for the fullness of the forgetful functor $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ (note that this functor is always faithful). In fact, we have an example which implies that this forgetful functor is not full even if $e(r-1) = p-1$. We show below that the forgetful functor $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ is not full when $K = \mathbb{Q}_p$ and $r = p$.

Suppose $K = \mathbb{Q}_p$. Let E_π be the Tate curve over $\mathbb{Q}_p$ associated with π. Lemma 18 in the next section says that the 2-dimensional $\mathbb{F}_p$-representation $E_\pi[p]$ of $G_{\mathbb{Q}_p}$ is torsion crystalline with Hodge–Tate weights in $[0,p]$. In particular, by Theorem 4, there exists a $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} \in \mathrm{Mod}^{p,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ such that $\hat{T}(\hat{\mathfrak{M}}) \simeq E_\pi[p]$. On the other hand, for any nonnegative integer ℓ, define the $(\varphi, \hat{G})$-module $\hat{\mathfrak{S}}_1(\ell) = (\mathfrak{S}_1(\ell), \varphi, \hat{G})$ as below: $\mathfrak{S}_1(\ell) = k[\![u]\!] \cdot \mathfrak{f}^\ell$ is the rank-1 free $k[\![u]\!]$-module equipped with the Frobenius $\varphi(\mathfrak{f}^\ell) = c_0^{-\ell} u^{e\ell} \cdot \mathfrak{f}^\ell$, and define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}_1(\ell)$ by $\tau(\mathfrak{f}^\ell) = \hat{c}^\ell \cdot \mathfrak{f}^\ell$. Here, $\hat{c} = \prod_{n=1}^\infty \varphi^n(\frac{E(u)}{\tau(E(u))})$, which is contained in $\widehat{\mathcal{R}}^\times$ (cf. Example 3.2.3 of [Li4]). Then Example 3.2.3 of *loc. cit.* says that $\hat{T}(\hat{\mathfrak{S}}_1(\ell)) \simeq \mathbb{F}_p(\ell)$. On the other hand, we define the $(\varphi, \hat{G})$-module $\hat{\mathfrak{S}}_1(\ell)_0 = (\mathfrak{S}_1(\ell)_0, \varphi, \hat{G})$ as below: Put $\ell_0 = \max\{\ell' \in \mathbb{Z}_{\geq 0}; e\ell - (p-1)\ell' \geq 0\}$. We denote by $\mathfrak{S}_1(\ell)_0 = k[\![u]\!] \cdot \mathfrak{g}^\ell$ the rank-1 free $k[\![u]\!]$-module equipped with the Frobenius $\varphi(\mathfrak{f}^\ell) = c_0^{-\ell} u^{e\ell - (p-1)\ell_0} \cdot \mathfrak{g}^\ell$, and define a $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}_1(\ell)$ by $\tau(\mathfrak{g}^\ell) = \underline{\varepsilon}^{-p\ell_0} \hat{c}^\ell \cdot \mathfrak{g}^\ell$. (The generator $\mathfrak{g}^\ell$ is taken to behave as $u^{-\ell_0} \mathfrak{f}^\ell$.) Then we see that $\mathrm{Max}(\mathfrak{S}_1(\ell)) = \mathfrak{S}_1(\ell)_0$, and $\hat{\mathfrak{S}}_1(\ell)_0$ (and $\hat{\mathfrak{S}}_1(\ell)$) are objects of $\mathrm{Mod}^{\ell,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$. We also see $\hat{T}(\hat{\mathfrak{S}}_1(\ell)_0) \simeq \hat{T}(\hat{\mathfrak{S}}_1(\ell)) \simeq \mathbb{F}_p(\ell)$. Now we consider the following commutative diagram (here we remark that $\mathfrak{S}_1(0) \oplus \mathfrak{S}_1(1)_0$ is maximal):

$$
\begin{array}{ccc}
\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbb{F}_p \oplus \mathbb{F}_p(1), E_\pi[p]) & \hookrightarrow & \mathrm{Hom}_{G_\infty}(\mathbb{F}_p \oplus \mathbb{F}_p(1), E_\pi[p]) \\
\uparrow{\scriptstyle \hat{T}} & & \uparrow{\scriptstyle T_\mathfrak{S}} \\
\mathrm{Hom}(\hat{\mathfrak{M}}, \hat{\mathfrak{S}}_1(0) \oplus \hat{\mathfrak{S}}_1(1)_0) & & \\
\downarrow{\scriptstyle \text{forgetful}} & & \\
\mathrm{Hom}(\mathfrak{M}, \mathfrak{S}_1(0) \oplus \mathfrak{S}_1(1)_0) & \xrightarrow{\mathrm{Max}} & \mathrm{Hom}(\mathrm{Max}(\mathfrak{M}), \mathfrak{S}_1(0) \oplus \mathfrak{S}_1(1)_0).
\end{array}
$$

The bottom horizontal arrow and the right vertical arrow are bijective since $\mathfrak{S}_1(0) \oplus \mathfrak{S}_1(1)_0$ is maximal. On the other hand, it is well-known that the inclusion $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\mathbb{F}_p \oplus \mathbb{F}_p(1), E_\pi[p]) \subset \mathrm{Hom}_{G_\infty}(\mathbb{F}_p \oplus \mathbb{F}_p(1), E_\pi[p])$ is not equal. Therefore,

$$
\mathrm{Hom}(\hat{\mathfrak{M}}, \hat{\mathfrak{S}}_1(0) \oplus \hat{\mathfrak{S}}_1(1)_0) \xrightarrow{\text{forgetful}} \mathrm{Hom}(\mathfrak{M}, \mathfrak{S}_1(0) \oplus \mathfrak{S}_1(1)_0)
$$

is not surjective. This implies that the forgetful functor

$$\mathrm{Mod}^{p,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^p_{/\mathfrak{S}_\infty}$$

is not full.

Remark 12. Combining Theorem (2.3.5) of [Kis], Theorem 4 and Lemma 8, we see that the forgetful functor $\mathrm{Mod}^{1,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^1_{/\mathfrak{S}_\infty}$ is an equivalence of categories.

4. Nonfullness: Examples

In the previous section, we showed that the restriction functor

$$\mathrm{Rep}^r_{\mathrm{tor}}(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$$

is fully faithful under the condition that $er < p - 1$. However, the full faithfulness may not hold if $er \geq p - 1$. In this section, we give some examples of this phenomenon. It should be noted that all our examples appearing in this section are given under the condition $e(r - 1) \geq p - 1$.

Let μ_{p^n} be the set of p^n-th roots of unity in $\overline{K}$, $\mu_{p^\infty} := \bigcup_{n \geq 0} \mu_{p^n}$ and denote by $G_1 \subset G_K$ the absolute Galois group of $K(\pi_1)$. Remark that, if the restriction functor $\mathcal{C} \to \mathrm{Rep}_{\mathrm{tor}}(G_1)$ is not fully faithful for a full subcategory $\mathcal{C}$ of $\mathrm{Rep}_{\mathrm{tor}}(G_K)$, then the restriction functor $\mathcal{C} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is not fully faithful. Furthermore, we also remark that restriction functors $\mathcal{C} \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ and $\mathcal{C} \to \mathrm{Rep}_{\mathrm{tor}}(G_1)$ are always faithful.

Proposition 13. *Let K be a finite extension of $\mathbb{Q}_p$. Let s be the largest integer n such that $\mu_{p^n} \subset K$. Suppose that $s \geq 1$ and $K(\mu_{p^{s+1}})/K$ is ramified. Then the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge–Tate weights in $[0, p + 1]$ to torsion $\mathbb{Z}_p$-representations of G_1, obtained by restricting the action of G_K to G_1, is not full.*

The lemma below follows from direct calculations.

Lemma 14. *Let $s \geq 1$ be an integer and $\psi \colon G_K \to \mathbb{Z}_p^\times$ an unramified character with the property that s is the largest integer n such that $\psi \bmod p^n$ is trivial. Define $\beta_\psi \colon G_K \to \mathbb{Z}_p$ by the relation $\psi = 1 + p^s \beta_\psi$ and put $\bar{\beta}_\psi = \beta \bmod p$. Denote by $\delta^0_\psi \colon H^0(G_K, \mathbb{Q}_p/\mathbb{Z}_p(\psi)) \to H^1(G_K, \mathbb{F}_p)$ the connection map coming from the exact sequence $0 \to \mathbb{F}_p \to \mathbb{Q}_p/\mathbb{Z}_p(\psi) \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p(\psi) \to 0$ of G_K-modules. Then $\bar{\beta}_\psi \in H^1(G_K, \mathbb{F}_p)$ and $\mathrm{Im}(\delta^0_\psi) = \mathbb{F}_p.\bar{\beta}_\psi$.*

Proof of Proposition 13. Let $\varepsilon \colon G_K \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and $\bar{\varepsilon} := \varepsilon \bmod p$ the mod p cyclotomic character. Let K and $s \geq 1$ be as in Proposition 13. Let $\chi \colon G_K \to \mathbb{Z}_p^\times$ be an unramified character such that $\chi \bmod p^s$ is trivial. It suffices to show that, for some choice of χ, there exist $\rho \colon G_K \to GL_2(\mathbb{Z}_p)$ and $2 \leq r \leq p + 1$ with an exact sequence $0 \to \chi\varepsilon^r \to \rho \to 1 \to 0$ of representations of G_K such that $\rho \bmod p$ is not trivial on G_K but is trivial on G_1. Here, 1 in the

above exact sequence means the trivial character. Note that such ρ is always crystalline (cf. [BK, Example 3.9]). Since $\mu_p \subset K$, we can define $f_0 \in H^1(G_K, \mathbb{F}_p)$ such that f_0 factors through $\hat{G}$, $f_0(\tau) = 1$ and $f_0|_{H_K} = 0$, where H_K is defined in Definition 2. The kernel of the restriction map $H^1(G_K, \mathbb{F}_p) \to H^1(G_1, \mathbb{F}_p)$ is a one dimensional $\mathbb{F}_p$-vector space which is generated by f_0. Let $H \subset H^1(G_K, \mathbb{F}_p)$ be an annihilator of f_0 under the Tate paring. For any integer ℓ, denote by $\delta^1_{\chi,\ell} \colon H^1(G_K, \mathbb{F}_p) \to H^2(G_K, \mathbb{Z}_p(\chi\varepsilon^\ell))$ (resp. $\delta^0_{\chi,\ell} \colon H^0(G_K, \mathbb{Q}_p/\mathbb{Z}_p(\chi^{-1}\varepsilon^{1-\ell})) \to H^1(G_K, \mathbb{F}_p)$) the connection map coming from the exact sequence $0 \to \mathbb{Z}_p(\chi\varepsilon^\ell) \xrightarrow{p} \mathbb{Z}_p(\chi\varepsilon^\ell) \to \mathbb{F}_p \to 0$ (resp. $0 \to \mathbb{F}_p \to \mathbb{Q}_p/\mathbb{Z}_p(\chi^{-1}\varepsilon^{1-\ell}) \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p(\chi^{-1}\varepsilon^{1-\ell}) \to 0$) of G_K-modules. By Tate local duality, the condition that f_0 lifts to $H^1(G_K, \mathbb{Z}_p(\chi\varepsilon^\ell))$ is equivalent to the condition that $\mathrm{Im}(\delta^0_{\chi,\ell}) \subset H$. Hence it is enough to choose χ which satisfies the latter condition for some $2 \le \ell \le p+1$.

Since $K(\mu_{p^{s+1}})/K$ is ramified, we know that s is the largest integer n such that $\chi^{-1}\varepsilon^{-1} \bmod p^n$ is trivial. Take $\beta_{\chi^{-1}\varepsilon^{-1}}$ and $\bar{\beta}_{\chi^{-1}\varepsilon^{-1}}$ as in Lemma 14. For simplicity, we write $\alpha_\chi := \beta_{\chi^{-1}\varepsilon^{-1}}$ and $\bar{\alpha}_\chi := \bar{\beta}_{\chi^{-1}\varepsilon^{-1}}$. By Lemma 14, $\mathrm{Im}(\delta^0_{\chi,2})$ is generated by $\bar{\alpha}_\chi$. If $\bar{\alpha}_1$ is contained in H, then we finish the proof (choose χ as the trivial character 1). Suppose $\bar{\alpha}_1$ is not contained in H. From now on, we fix χ as follows; χ is the unramified character $G_K \to \mathbb{Z}_p^\times$ with $\chi(\mathrm{Frob}_K) = (1+p^s)^{-1}$, where Frob_K is the arithmetic Frobenius of K. Let $u_1 \colon G_K \to \mathbb{F}_p$ be the unramified homomorphism with $u_1(\mathrm{Frob}_K) = 1$. Then we obtain $\bar{\alpha}_\chi = u_1 + \bar{\alpha}_1$. Since $K(\mu_{p^{s+1}})/K$ is ramified, we see that $\bar{\alpha}_1|_{I_K}$ is not zero where I_K is the inertia subgroup of G_K. This implies $u_1 \notin \mathbb{F}_p.\bar{\alpha}_1$. Noting that $H^1(G_K, \mathbb{F}_p) = H \oplus \mathbb{F}_p.\bar{\alpha}_1$, we have $\bar{\alpha}_\chi + \bar{a}\bar{\alpha}_1 \in H$ for some $\bar{a} \in \mathbb{F}_p$. Let $0 \le a \le p-1$ be the integer such that $a \bmod p$ is $\bar{a}$. Under the modulo p^{2s}, we have $\chi^{-1}\varepsilon^{-(1+a)} = \chi^{-1}\varepsilon^{-1} \cdot \varepsilon^{-a} = (1 + p^s\alpha_\chi)(1 + p^s a\alpha_1) = 1 + p^s(\alpha_\chi + a\alpha_1)$. Since $\bar{\alpha}_\chi + \bar{a}\bar{\alpha}_1 = u_1 + (\bar{a}+1)\bar{\alpha}_1 \ne 0$, we see that s is the largest integer n such that $\chi^{-1}\varepsilon^{-(1+a)} \bmod p^n$ is trivial. Hence, defining $\beta_{\chi^{-1}\varepsilon^{-(1+a)}}$ as in Lemma 14, we obtain $\bar{\beta}_{\chi^{-1}\varepsilon^{-(1+a)}} = \bar{\alpha}_\chi + \bar{a}\bar{\alpha}_1$. Therefore, we obtain that $\mathrm{Im}(\delta^0_{\chi,2+a}) = \mathbb{F}_p.\bar{\beta}_{\chi^{-1}\varepsilon^{-(1+a)}} \subset H$ and we are done. $\square$

Unfortunately, Proposition 13 can not be applied even when $K = \mathbb{Q}_p$. On the other hand, the following proposition is effective for $K = \mathbb{Q}_p$, but we need a certain restriction on the choice of the uniformizer π. Let L be the unique degree p extension of K which is contained in $K(\mu_{p^\infty})$.

Proposition 15. *Let K be a finite extension of $\mathbb{Q}_p$. Suppose that π is contained in $\mathrm{Norm}_{L/K}(L^\times)$. (Thus the extension L/K must be totally ramified in this case.) Then the functor from torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge–Tate weights in $[0, p]$ to torsion $\mathbb{Z}_p$-representations of G_1, obtained by restricting the action of G_K to G_1, is not full.*

Proof. Let s be the largest integer n such that $\mu_{p^n} \subset K$. Then we can write $\varepsilon^{1-p} = 1 + p\psi$ with some map $\psi \colon G_K \to \mathbb{Z}_p$. Putting $\bar{\psi} = \psi \bmod p \colon G_K \to$

$\mathbb{F}_p$, we see that $\bar{\psi}$ is nontrivial homomorphism with kernel $\mathrm{Gal}(\overline{K}/L)$. Let $\delta^0 \colon H^0(G_K, \mathbb{Q}_p/\mathbb{Z}_p(1-p)) \to H^1(G_K, \mathbb{F}_p))$ be the connection map arising from the exact sequence $0 \to \mathbb{F}_p \to \mathbb{Q}_p/\mathbb{Z}_p(1-p) \xrightarrow{p} \mathbb{Q}_p/\mathbb{Z}_p(1-p) \to 0$. Under the isomorphism $K^\times/(K^\times)^p \simeq H^1(G_K, \mathbb{F}_p(1))$ via Kummer theory, π mod $(K^\times)^p$ corresponds to the 1-cocycle $[\pi]$ defined by $\sigma \mapsto \frac{\sigma(\pi_1)}{\pi_1}$ for $\sigma \in G_K$, which is clearly trivial on G_1. By Tate local duality and the fact that the image of δ^0 is generated by $\bar{\psi}$ (cf. Lemma 14), it suffices to show that $([\pi], \bar{\psi})$ maps to zero under the Tate pairing $H^1(G_K, \mathbb{F}_p(1)) \times H^1(G_K, \mathbb{F}_p) \to \mathbb{Q}/\mathbb{Z}$ (in fact, this implies that $[\pi]$ lifts to $H^1(G_K, \mathbb{Z}_p(p))$ and we obtain the desired result). Let $\phi_{L/K} \colon K^\times/\mathrm{Norm}_{L/K}(L^\times) \xrightarrow{\sim} \mathrm{Gal}(L/K)$ be the isomorphism of local class field theory. It is enough to show that $\bar{\psi}(\phi_{L/K}(\pi)) = 0$. Our assumption of π implies that this equality certainly holds. $\square$

Now we give an example for the nonfullness of our restriction functor without any assumption on the choice of the uniformizer π.

Proposition 16. *The functor from torsion crystalline $\mathbb{Z}_p$-representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights in $[0, p]$ to torsion $\mathbb{Z}_p$-representations of G_1, obtained by restricting the action of $G_{\mathbb{Q}_p}$ to G_1, is not full.*

Lemma 17. *Let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p$. Then any 2-dimensional irreducible F-representation of $G_{\mathbb{Q}}$ whose determinant is the p-adic cyclotomic character is absolutely irreducible.*

Proof. Let $\rho \colon G_{\mathbb{Q}} \to GL_2(F)$ be as in the statement and denote by V the underlying F-vector space. Suppose that, for some finite extension F' over F, there exists a $G_{\mathbb{Q}}$-stable F'-subvector space W of $F' \otimes_F V$ of dimension 1. If we denote by $c \in G_{\mathbb{Q}}$ the complex conjugation, then $\rho(c)^2$ is the identity matrix and $\det(\rho)(c) = -1$. Hence it follows that $\rho(c)$ is conjugate (over F) with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (note that p is odd). By this fact and the fact that $\rho(c)$ preserves W, we see that W is defined over F. This is a contradiction. $\square$

Lemma 18. *Let K be a finite extension of $\mathbb{Q}_p$ and $q \in \mathbb{Q}_p^\times (K^\times)^p$. Let $E_q[p]$ be the Tate curve over K associated with q. If $p \nmid v_K(q)$, then $E_q[p]$ is torsion crystalline with Hodge–Tate weights in $[0, p]$.*

Proof. We have a decomposition $q = q'q''$, where $q' \in \mathbb{Q}_p^\times, v_K(q') > 0$ and $q'' \in (K^\times)^p$. Let $E_{q'}$ be the Tate curve over $\mathbb{Q}_p$ associated with q'. Then $E_{q'}[p]$ is a representation of $G_{\mathbb{Q}_p}$ and we have an isomorphism $E_q[p] \simeq (E_{q'}[p])|_{G_K}$. Hence we can reduce the case where $K = \mathbb{Q}_p$. Let $\ell > 3$ be a prime number different from p such that $-\ell$ is not a square in $\mathbb{F}_p^\times$ (recall that p is odd). Choose an elliptic curve $E_{(\ell)}$ over $\mathbb{Q}_\ell$ which has good supersingular reduction. Since $\ell > 3$, we have $\#E_{(\ell)}(\mathbb{F}_\ell) = 1 + \ell$. Thus the characteristic polynomial of $E_{(\ell)}[p]$ for the arithmetic Frobenius of ℓ is $X^2 + \ell \in \mathbb{F}_p[X]$, which does not have a root in $\mathbb{F}_p$. Hence $E_{(\ell)}[p]$ is an irreducible representation of $G_{\mathbb{Q}_\ell}$ where $G_{\mathbb{Q}_\ell}$ is the absolute Galois group of

$\mathbb{Q}_\ell$. We define $\mathcal{S}$ to be the set of $\mathbb{Q}$-isomorphism classes of elliptic curves E defined over $\mathbb{Q}$ which satisfy the following conditions:

(a) E has multiplicative reduction at p and

$$v_p(j(E)) = v_p(j(E_q))(= -v_p(q))$$

where $j(E)$ is the j-invariant of E.

(b) $E[p] \simeq E_q[p]$ as $\mathbb{F}_p$-representations of $G_{\mathbb{Q}_p}$.

(c) $E[p] \simeq E_{(\ell)}[p]$ as $\mathbb{F}_p$-representations of $G_{\mathbb{Q}_\ell}$.

The set $\mathcal{S}$ is infinite since elliptic curves over $\mathbb{Q}$, whose coefficients of their defining equations are p-adically close enough to that of E_π and also ℓ-adically close enough to that of $E_{(\ell)}$, are contained in $\mathcal{S}$. Now we take any elliptic curve E over $\mathbb{Q}$ whose $\mathbb{Q}$-isomorphism class is in the set $\mathcal{S}$. By the condition (c), $E[p]$ is irreducible as a representation of $G_{\mathbb{Q}}$. It is moreover absolutely irreducible by Lemma 17. By the classical Serre's modularity conjecture (proved by Khare and Wintenberger) and the well-known fact that p-adic representations arising from Hecke eigencusp forms of level prime to p are crystalline, we know that $(E[p] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p)|_{G_{\mathbb{Q}_p}}$ is the reduction of a lattice in some crystalline $\overline{\mathbb{Q}}_p$-representation. Furthermore, by the condition (a) and Proposition 5(2) of [Se], we know that $(E[p])|_{G_{\mathbb{Q}_p}}$ is torsion crystalline with Hodge–Tate weights in $[0, p]$. Therefore, so is $E_q[p]$ by (b). $\square$

Proof of Proposition 16. Put $T = E_\pi[p]$ and $T' = \mathbb{F}_p \oplus \mathbb{F}_p(1)$. We know that T and T' are in $\mathrm{Rep}_{\mathrm{tor}}^p(G_{\mathbb{Q}_p})$ by Lemma 18. They are not isomorphic as representations of $G_{\mathbb{Q}_p}$ but isomorphic as representations of G_1. This gives the desired result. $\square$

Here we suggest the following question.

Question 19. *What is the necessary and sufficient condition for that the functor*

$$\mathrm{Rep}_{\mathrm{tor}}^r(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty), \quad T \mapsto T|_{G_\infty}$$

is fully faithful? Furthermore, does this condition depend only on e and r?

Remark 20.

(1) We do not know whether the full faithfulness of the functor in Question 19 depends on the choice of the system $(\pi_n)_{n \geq 0}$ or not (see Proposition 15). However, it is not difficult to see the following: Take two systems $(\pi_n)_{n \geq 0}$ and $(\pi_n')_{n \geq 0}$ of p^n-th roots of a fixed uniformizer π of K (thus we have $\pi_0 = \pi_0' = \pi$). Put $K_\infty = \bigcup_{n \geq 0} K(\pi_n)$ (resp. $K_\infty' = \bigcup_{n \geq 0} K(\pi_n')$) and $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$ (resp. $G_\infty' = \mathrm{Gal}(\overline{K}/K_\infty')$). Then, the restriction functor

$$\mathrm{Rep}_{\mathrm{tor}}^r(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$$

is fully faithful if and only if the restriction functor

$$\mathrm{Rep}_{\mathrm{tor}}^r(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty')$$

is. In fact, we can check this from the fact that G_∞ and G'_∞ are conjugate with each other by some element of G_K.

(2) A torsion $\mathbb{Z}_p$-representation of G_K is called *finite flat* if it is isomorphic to $G(\overline{K})$ as $\mathbb{Z}_p$-representations of G_K for some p-power order finite flat commutative group scheme G over the integer ring of K. If $r = 1$, then the category $\mathrm{Rep}^r_{\mathrm{tor}}(G_K) = \mathrm{Rep}^1_{\mathrm{tor}}(G_K)$ coincides with the category of finite flat representations of G_K (this can be checked by, for example, Theorem 3.1.1 of [BBM]). Breuil proved in Theorem 3.4.3 of [Br3] that the restriction functor

$$\mathrm{Rep}^1_{\mathrm{tor}}(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$$

is fully faithful for any K without any restriction on e. In fact, this assertion is true even if $p = 2$ (cf. [Kim], [La], [Li4], proved independently. Explicitly, see Corollary 4.4 of [Kim]).

(3) If $e = 1$ and $r < p - 1$, then the fact that the restriction functor $\mathrm{Rep}^r_{\mathrm{tor}}(G_K) \xrightarrow{\mathrm{res}} \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is fully faithful has been already known ([Br2], the proof of Théorèm 5.2).

(4) Observing known results as above and results shown in this paper, it seems that the answer of Question 19 should be "$e(r-1) < p-1$".

Appendix A. $(\varphi, \hat{G})$-modules associated with crystalline representations

In Proposition 5.9 of [GLS], a necessary condition for representations arising from free $(\varphi, \hat{G})$-modules to be crystalline is given. In this appendix, we show that the converse holds. The result here justifies the subscript "cris" of the category $\mathrm{Mod}^{r,\hat{G},\mathrm{cris}}_{/\mathfrak{S}_\infty}$ defined in Section 3.

We continue to use the same notation as in Section 2. For any integer $n \geq 0$, we define ideals of $W(R)$ as below:

$$I^{[n]}W(R) := \{a \in W(R); \varphi^m(a) \in \mathrm{Fil}^n A_{\mathrm{cris}} \text{ for every } m \geq 0\},$$

$$I^{[n^+]}W(R) := I^{[n]}W(R)I_+W(R)$$

(see Section 5 of [Fo2] for more precise information). The proof of Lemma 3.2.2 of [Li2] shows that $I^{[n]}W(R)$ is a principal ideal of $W(R)$ generated by $\varphi(\mathfrak{t})^n$. In particular we see that $u^p\varphi(\mathfrak{t})$ is contained in

$$I^{[1^+]}W(R) = \varphi(\mathfrak{t})I_+W(R).$$

Recall that $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a semi-stable $\mathbb{Q}_p$-representation of G_K (Theorem 3(2)) and $\tau(x) - x \in I_+W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ for any $x \in \mathfrak{M}$. The main purpose of this appendix is to prove the following:

Theorem 21. *Let* $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ *be a* $(\varphi, \hat{G})$*-module. The following are equivalent:*

(1) $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *is crystalline.*

(2) *For any $x \in \mathfrak{M}$, we have $\tau(x) - x \in I^{[1^+]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.*
(3) *For any $x \in \mathfrak{M}$, we have $\tau(x) - x \in u^p \varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$.*

Before giving a proof of this theorem, we shall recall some known facts about $(\varphi, \hat{G})$-modules. Let $\hat{\mathfrak{M}} \in \mathrm{Mod}^{r,\hat{G}}_{/\mathfrak{S}}$ be a $(\varphi, \hat{G})$-module, and put $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Then $\mathcal{D}$ has a structure as a Breuil module which corresponds to the semi-stable representation $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of G_K. (Breuil modules here are objects of "$\mathcal{MF}_S(\varphi, N)$" defined in Section 6.1 of [Br1]. It is useful for the reader to refer also Section 5 of [Li1].) Denote by $N_{\mathcal{D}}$ the monodromy operator of $\mathcal{D}$ and define a G_K-action on $B^+_{\mathrm{cris}} \otimes_S \mathcal{D} = B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ by

$$g(a \otimes x) = \sum_{i=0}^{\infty} g(a)\gamma_i(-\log([\underline{\varepsilon}])) \otimes N_{\mathcal{D}}(x)$$

for $g \in G_K, a \in B^+_{\mathrm{cris}}, x \in \mathcal{D}$. By the construction of the quasi-inverse of the functor $\hat{T}$ of Theorem 3(2) ([Li2], Section 3.2), this G_K-action is stable on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset B^+_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and it factors through $\hat{G}$, which gives the original $\hat{G}$-action of the $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$. For any $n \geq 0$ and any $x \in \mathcal{D}$, an induction on n shows that

$$(\tau - 1)^n(x) = \sum_{m=n}^{\infty} \left\{ \sum_{i_1+\cdots i_n=m, i_j \geq 0} \frac{m!}{i_1! \cdots i_n!} \right\} \gamma_m(t) \otimes N_{\mathcal{D}}^m(x) \in B^+_{\mathrm{cris}} \otimes_S \mathcal{D}$$

and in particular $\frac{(\tau-1)^n}{n}(x) \to 0$ p-adically as $n \to \infty$. Hence we can define

$$\log(\tau)(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\tau - 1)^n}{n}(x) \in B^+_{\mathrm{cris}} \otimes_S \mathcal{D}.$$

It is not difficult to check the equation $\log(\tau)(x) = t \otimes N_{\mathcal{D}}(x)$. Consequently the monodromy operator $N_{\mathcal{D}}$ can be reconstructed from the τ-action of $\hat{\mathfrak{M}}$ by the relation $\frac{1}{t}\log(\tau)(x) = N_{\mathcal{D}}(x)$. Put $D = \mathcal{D}/I_+ S_{K_0}\mathcal{D}$. Then D has a structure as a filtered (φ, N)-module over K_0 which corresponds to $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and the monodromy operator N_D of D is given by $N_{\mathcal{D}}$ mod $I_+ S_{K_0}\mathcal{D}$ ([Br1], Section 6). Hence $\hat{T}(\hat{\mathfrak{M}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline if and only if $N_{\mathcal{D}}$ mod $I_+ S_{K_0}\mathcal{D}$ is zero.

Proof of Theorem 21. The implication $(1) \Rightarrow (3)$ follows from Proposition 5.9 of [GLS]. It is clear that (3) implies (2). Thus it suffices to show the implication $(2) \Rightarrow (1)$. Assume the condition (2). We use the same notation $\mathcal{D}, N_{\mathcal{D}}, D, N_D$ as the above. We often regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathcal{D}$. Let $x \in \mathfrak{M}$. For any integer $n > 0$, it is shown in the proof of Proposition 2.4.1 of [Li3] that:

(A) $(\tau - 1)^n(x) \in I^{[n]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

(B) $\frac{(\tau-1)^n}{nt}(x)$ is well-defined in $A_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ and $\frac{(\tau-1)^n}{nt}(x) \to 0$ p-adically as $n \to \infty$. Therefore, we have

$$\frac{1}{t}\log(\tau)(x) \in A_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \subset B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

By (A), we can take $y_n \in W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ such that $(\tau-1)^n(x) = \varphi(\mathfrak{t})^n y_n$. Then we have the equation

$$(*) \qquad cN_{\mathcal{D}}(x) = c \cdot \frac{1}{t}\log(\tau)(x) = \frac{\tau-1}{\varphi(\mathfrak{t})}(x) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{\varphi(\mathfrak{t})^{n-1}}{n}y_n.$$

Here $c = \frac{t}{\varphi(\mathfrak{t})}$, which is a unit of A_{cris} ([Li2], Example 3.2.3). Note that $\frac{\tau-1}{\varphi(\mathfrak{t})}(x)$ is contained in $I_+ W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ by the assumption (2).

Now we claim that there exists an integer $n_0 > 1$ such that $\frac{(n-2)!}{n}$ is in $\mathbb{Z}_p$ for any $n > n_0$. Admitting this claim, we proceed a proof of Theorem 21. Consider the decomposition

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\varphi(\mathfrak{t})^{n-1}}{n}y_n =$$

$$\varphi(\mathfrak{t})\sum_{n=2}^{n_0}(-1)^{n-1}\frac{\varphi(\mathfrak{t})^{n-2}}{n}y_n + \varphi(\mathfrak{t})\sum_{n=n_0+1}^{\infty}(-1)^{n-1}\frac{\varphi(\mathfrak{t})^{n-2}}{n}y_n.$$

By the claim, we see that $\frac{\varphi(\mathfrak{t})^{n-2}}{n} = \frac{\varphi(\mathfrak{t})^{n-2}}{(n-2)!}\cdot\frac{(n-2)!}{n} = c^{-(n-2)}\gamma_{n-2}(t)\frac{(n-2)!}{n}$ is contained in A_{cris} for any $n > n_0$ and it goes to zero p-adically as $n \to \infty$. In particular, (the first term and) the second term of the above decomposition are contained in $\varphi(\mathfrak{t})(B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})$, which is contained in $I_+ B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Hence $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\varphi(\mathfrak{t})^{n-1}}{n}y_n$ is also contained in $I_+ B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}}\mathfrak{M}$. Note that $\nu(c) = 1$ since $c = \frac{t}{\varphi(\mathfrak{t})} = \prod_{n=0}^{\infty}\varphi^n(\frac{c_0^{-1}E(u)}{p})$ and $\nu(u) = 0$, and furthermore $\nu(\mathfrak{t}) = 0$. Therefore, by $(*)$ modulo $I_+ B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$, we obtain the relation $N_D(\bar{x}) = 0$ in $D = \mathcal{D}/I_+ S_{K_0}\mathcal{D} \subset (B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})/(I_+ B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})$ where $\bar{x}$ is the residue class of x. Since the image of $\mathfrak{M}$ in $D = \mathcal{D}/I_+ S_{K_0}\mathcal{D}$ generates D as a K_0-vector space, we obtain that $N_D = 0$. This implies (1). Hence it suffices to show the claim. Let v_p be the p-adic valuation with $v_p(p) = 1$. For any positive integer n, write $n = p^s m$ with $p \nmid m$. If $s = 0$, it is clear that $\frac{(n-2)!}{n} \in \mathbb{Z}_p$. Suppose $s \geq 1$. If $m \geq 2$, we have $v_p((n-2)!) \geq v_p((2p^s-2)!) \geq v_p(p^s!) \geq s = v_p(n)$. If $m = 1$ and $s \geq 3$, we have

$$v_p((n-2)!) \geq v_p(p^{s-1}!) = \frac{1}{2}s(s-1) \geq s = v_p(n).$$

This finishes the proof. $\hfill\square$

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