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# An elementary approach to $C^*$-algebras associated to topological graphs

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Abstract. We develop notions of a representation of a topological graph $E$ and of a covariant representation of a topological graph $E$ which do not require the machinery of $C^*$-correspondences and Cuntz–Pimsner algebras. We show that the $C^*$-algebra generated by a universal representation of $E$ is isomorphic to the Toeplitz algebra of Katsura's topological-graph bimodule, and that the $C^*$-algebra generated by a universal covariant representation of $E$ is isomorphic to Katsura's topological graph $C^*$-algebra. We exhibit our results by constructing the isomorphism between the $C^*$-algebra of a row-finite directed graph $E$ with no sources and the $C^*$-algebra of the topological graph arising from the shift map acting on the infinite-path space $E^\infty$.

Contents



## 1. Introduction

Let $E$ be a countable directed graph with vertex set $E^0$, edge set $E^1$ and range and source maps $r, s : E^1 \to E^0$. The Toeplitz–Cuntz–Krieger algebra $\mathcal{T}C^*(E)$ is the universal $C^*$-algebra generated by a family of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a family of partial isometries $\{s_e : e \in E^1\}$ such that:

(1) $s_e^* s_e = p_{s(e)}$ for every $e \in E^1$.

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(2) $p_v \geq \sum_{e \in F} s_e s_e^*$ for every $v \in E^0$ and finite $F \subset r^{-1}(v)$.

The graph algebra $C^*(E)$ is universal for families as above satisfying the additional relation that $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $r^{-1}(v)$ is nonempty and finite.

These $C^*$-algebras have been studied very extensively over the last fifteen years, part of their appeal being precisely that they can be defined, as above, in the space of a paragraph. Combined with key structure theorems like the gauge-invariant uniqueness theorem [1] and the Cuntz–Krieger uniqueness theorem [11], or Fowler and Raeburn's analogue of Coburn's Theorem for the Toeplitz algebra of a directed graph [4], the above elementary presentation often makes it easy to verify that a given $C^*$-algebra is isomorphic to $C^*(E)$ or $\mathcal{T}C^*(E)$ as appropriate.

Some years after the introduction of graph algebras, Katsura introduced topological graphs and their $C^*$-algebras [5]. His construction is based on that of [6], which in turn was a modification of Pimsner's construction of $C^*$-algebras associated to $C^*$-correspondences in [9]. Roughly speaking, a topological graph is like a directed graph except that $E^0$ and $E^1$ are locally compact Hausdorff spaces rather than countable discrete sets, and $r$ and $s$ are required to be topologically well-behaved. Building on Fowler and Raeburn's construction of a $C^*$-correspondence from a directed graph given in [4], Katsura associated to each topological graph $E$ a $C^*$-correspondence $X(E)$, and defined the $C^*$-algebra of the topological graph $E$ to be the $C^*$-algebra $\mathcal{O}_{X(E)}$ he had associated to the module $X(E)$ in [6].

The drawback of this approach is that the relations defining the $C^*$-algebra are complicated and cannot be stated without first introducing at least the rudiments of $C^*$-correspondences, which is quite a bit of overhead — see Section 3. Clearly it would be handy to be able to circumvent all this technical overhead when defining and dealing with the $C^*$-algebras of topological graphs. This paper takes the first step in this direction. Our main theorems show that if $E$ is a topological graph, then the Toeplitz algebra $\mathcal{T}(E)$ and Katsura's topological graph algebra $\mathcal{O}(E)$ of [5] can be described as $C^*$-algebras universal for relatively elementary relations involving elements of $C_0(E^0)$ and $C_c(E^1)$. In particular, our presentations can be stated without any reference to $C^*$-correspondences. We state our results so that they apply to arbitrary topological graphs, but we also show how our crucial covariance condition simplifies when the range map $r : E^1 \to E^0$ is a local homeomorphism. Even this special situation is interesting: it includes, for example, Cantor minimal systems, and all crossed products of abelian $C^*$-algebras by $\mathbb{Z}$.

To emphasise how little background is needed to present our definition of a representation of a topological graph and our covariance condition, we begin by stating all our key definitions and main theorems in Section 2. In Section 3 we recall Katsura's construction of a $C^*$-correspondence from a topological graph and his definitions of the associated $C^*$-algebras. In

Section 4, we prove our main results. We finish in Section 5 by considering the topological graph $\widehat{E}$ obtained from a row-finite directed graph $E$ with no sources by taking $\widehat{E}^0 = \widehat{E}^1 = E^\infty$ and defining the range map to be the identity map, and the source map to be the left-shift map $\sigma$. We apply our results to provide a relatively elementary proof that $\mathcal{O}(\widehat{E})$ is canonically isomorphic to $C^*(E)$. This result could be recovered from [2, 3], but working out the details provides a good example of the efficacy of our results.

## 2. Main results

**Definition 2.1** ([5, Definition 2.1])**.** A quadruple $E = (E^0, E^1, r, s)$ is called a *topological graph* if $E^0$, $E^1$ are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism.

We think of $E^0$ as a space of vertices, and we think of each $e \in E^1$ as an arrow pointing from $s(e)$ to $r(e)$. If $E^0$, $E^1$ are both countable and discrete, then $E$ is a directed graph in the sense of [8, 10].

Given $x \in C_c(E^1)$ and $f \in C_0(E^0)$, we define $x \cdot f$ and $f \cdot x$ in $C_c(E^1)$ by

$$(x \cdot f)(e) = x(e)f(s(e)) \quad \text{and} \quad (f \cdot x)(e) = f(r(e))x(e). \tag{2.1}$$

To define our notion of a representation of a topological graph, we need a couple of preliminary ideas. An *s-section* in a topological graph $E$ is a subset $U \subset E^1$ such that $s|_U$ is a homeomorphism. An *r-section* is defined similarly, and a *bisection* is a set which is both an $s$-section and an $r$-section.

If $x \in C_c(E^1)$ then $\operatorname{osupp}(x)$ denotes the precompact open set

$$\{e \in E^1 : x(e) \neq 0\},$$

and $\operatorname{supp}(x)$ is the closure of $\operatorname{osupp}(x)$. If $x \in C_c(E^1)$ and $\operatorname{osupp}(x)$ is an $s$-section, we define $\widehat{x} : E^0 \to \mathbb{C}$ by

$$\widehat{x}(v) := \begin{cases} |x(e)|^2 & \text{if } v = s(e) \text{ for } e \in \operatorname{osupp}(x), \\ 0 & \text{for } v \notin s(\operatorname{osupp}(x)). \end{cases} \tag{2.2}$$

**Definition 2.2.** Let $E$ be a topological graph. A *Toeplitz representation of* $E$ in a $C^*$-algebra $B$ is a pair $(\psi, \pi)$ where $\psi : C_c(E^1) \to B$ is a linear map, $\pi : C_0(E^0) \to B$ is a homomorphism, and:

(1) $\psi(f \cdot x) = \pi(f)\psi(x)$, for all $x \in C_c(E^1)$, $f \in C_0(E^0)$;
(2) for $x \in C_c(E^1)$ such that $\operatorname{supp}(x)$ is contained in an open $s$-section, $\pi(\widehat{x}) = \psi(x)^*\psi(x)$;
(3) for $x$, $y \in C_c(E^1)$ such that $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ are contained in disjoint open $s$-sections, $\psi(x)^*\psi(y) = 0$.

We say that a Toeplitz representation $(\psi, \pi)$ of $E$ in $B$ is *universal* if for any Toeplitz representation $(\psi', \pi')$ of $E$ in $C$, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

**Remark 2.3.** Suppose that $x \in C_c(E^1)$ and $\operatorname{supp}(x)$ is contained in an open $s$-section $U$. Since $E^1$ is locally compact, we may cover $\operatorname{supp}(x)$ with precompact open sets $\{V_i : i \in I\}$, and since $\operatorname{supp}(x)$ is compact, we may pass to a finite subcover $\{V_i : i \in F\}$. Then each $V_i \cap U$ is precompact and open, so $\bigcup_{i\in F}(V_i \cap U)$ is a precompact open $s$-section containing $\operatorname{supp}(x)$. Thus we may assume in Conditions (2) and (3) that the open $s$-sections containing $\operatorname{supp}(x)$ and $\operatorname{supp}(y)$ are precompact.

To state our first main theorem, recall that if $E$ is a topological graph, then $\mathcal{T}(E)$ denotes the Toeplitz algebra of Katsura's topological-graph bimodule (see Notation 3.6).

**Theorem 2.4.** *Let $E$ be a topological graph. Then there is a universal Toeplitz representation $(i_1, i_0)$ of $E$ which generates $\mathcal{T}(E)$. Moreover the $C^*$-algebra generated by the image of any universal Toeplitz representation of $E$ is isomorphic to $\mathcal{T}(E)$.*

**Remark 2.5.** Since the map $\psi$ occurring in a Toeplitz representation of $E$ is not a homomorphism, it will not usually be norm-decreasing with respect to the supremum norm. So it is not clear that one can just extend by continuity a linear map $\psi_0$ defined on a dense subspace of $C_c(E^1)$. We show in Proposition 4.12 how to get around this difficulty: the map $\psi$ *is* norm-decreasing with respect to the supremum norm when applied to functions supported on $s$-sections.

We now describe the covariance condition for a Toeplitz representation of a topological graph. The condition is somewhat technical, but we will indicate how it simplifies under additional hypotheses in Corollary 2.15.

We first need a little notation from [5].

**Definition 2.6** ([5, Definition 2.6])**.** Let $E$ be a topological graph. We define:

(1) $E^0_{\mathrm{sce}} = E^0 \setminus \overline{r(E^1)}$.
(2) $E^0_{\mathrm{fin}} = \{v \in E^0 :$ there exists a neighbourhood $N$ of $v$ such that $r^{-1}(N)$ is compact$\}$.
(3) $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$.

**Remark 2.7.** The set $E^0_{\mathrm{fin}}$ is open in $E^0$. To see this, fix $v \in E^0_{\mathrm{fin}}$. Then there exists a neighbourhood $U$ of $v$ such that $r^{-1}(U)$ is compact and there exists an open neighbourhood $V$ of $v$ contained in $U$. So $V \subset E^0_{\mathrm{fin}}$, whence $E^0_{\mathrm{fin}}$ is open. It follows that $E^0_{\mathrm{sce}}$, $E^0_{\mathrm{fin}}$, $E^0_{\mathrm{rg}}$ are all open in $E^0$. Moreover $E^0_{\mathrm{rg}}$ is the intersection of $E^0_{\mathrm{fin}}$ with the interior of $\overline{r(E^1)}$. Finally, as proved by Katsura (see [5, Lemma 1.23]), for any compact subset $K \subset E^0_{\mathrm{fin}}$, the set $r^{-1}(K)$ is compact in $E^1$.

**Notation 2.8.** Let $X$ be a locally compact Hausdorff space and $U$ be an open subset of $X$. Then the standard embedding $\iota_U$ of $C_0(U)$ as an ideal of

$C_0(X)$ is given by

$$\iota_U(f)(x) := \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{otherwise.} \end{cases}$$

We will usually suppress the map $\iota_U$ and just identify each of $C_0(U)$ and $C_c(U)$ with their images in $C_0(X)$ under $\iota_U$. That is, we think of $C_0(U)$ as an ideal of $C_0(X)$, and we regard $C_c(U)$ as an (algebraic) ideal of $C_0(X)$.

**Remark 2.9.** Let $X$ be a Hausdorff space, and let $K \subset X$ be compact. Fix a finite cover $\mathcal{N}$ of $K$ by open subsets of $X$. By [7, Chapter 5.W], there exists a partition of unity $\{h_N : N \in \mathcal{N}\}$ on $K$ subordinate to $\{N \cap K : N \in \mathcal{N}\}$, where $h_N(x) \in [0,1]$, for all $N \in \mathcal{N}$, $x \in K$. Since $X$ may not be normal, we cannot necessarily extend this to a partition of unity on $X$. Nevertheless, if $f \in C_c(X)$ with $\operatorname{supp}(f) \subset K$, then the functions $f_N : X \to \mathbb{C}$ given by

$$f_N(x) := \begin{cases} f(x)h_N(x) & \text{if } x \in K \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

belong to $C_c(X)$. Hence each $\operatorname{osupp}(f_N) \subset N \cap K$, and $\sum_{N\in\mathcal{N}} f_N = f$.

**Definition 2.10.** Let $E$ be a topological graph, and let $(\psi, \pi)$ be a Toeplitz representation of $E$ in a $C^*$-algebra $B$. We call $(\psi, \pi)$ *covariant* if there exists a collection $\mathcal{G} \subset C_c(E^0_{\mathrm{rg}})$ of nonnegative functions which generates $C_0(E^0_{\mathrm{rg}})$ as a $C^*$-algebra, and for each $f \in \mathcal{G}$ there exist a finite cover $\mathcal{N}_f$ of $r^{-1}(\operatorname{supp}(f))$ by open $s$-sections, and a collection of nonnegative functions $\{f_N : N \in \mathcal{N}_f\} \in C_c(E^1)$ such that:

(1) $\operatorname{osupp}(f_N) \subset N \cap r^{-1}(\operatorname{supp}(f))$ for all $N \in \mathcal{N}_f$.
(2) $\sum_{N\in\mathcal{N}_f} f_N = f \circ r$.
(3) $\pi(f) = \sum_{N\in\mathcal{N}_f} \psi\big(\sqrt{f_N}\big)\psi\big(\sqrt{f_N}\big)^*$.

We say that a covariant Toeplitz representation $(\psi, \pi)$ of $E$ in $B$ is *universal* if for any covariant Toeplitz representation $(\psi', \pi')$ of $E$ in $C$, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$ and $h \circ \pi = \pi'$.

**Remark 2.11.** Definition 2.10 is formulated so as to make it easy to check that a given pair $(\psi, \pi)$ is covariant. However, when *using* covariance of a pair $(\psi, \pi)$ it is helpful to know that (3) of Definition 2.10 holds for every $f \in C_c(E^0_{\mathrm{rg}})$, every $\mathcal{N}_f$, and every $\{f_N : N \in \mathcal{N}_f\}$ satisfying (1) and (2) of Definition 2.10. We prove this in Proposition 4.8.

**Remark 2.12.** In Definition 2.10, we made implicit use of Remark 2.9 because the preimage $r^{-1}(\operatorname{supp}(f))$ is compact by Remark 2.7. Observe that the covariance condition for a Toeplitz representation of $E$ only involves functions in $C_0(E^0_{\mathrm{rg}}) \subset C_0(E^0)$.

Recall that if $E$ is a topological graph, then $\mathcal{O}(E)$ denotes Katsura's topological graph $C^*$-algebra (see Notation 3.6).

**Theorem 2.13.** *Let $E$ be a topological graph. Then there is a universal covariant Toeplitz representation $(j_1, j_0)$ of $E$ which generates $\mathcal{O}(E)$. Moreover the $C^*$-algebra generated by the image of any universal covariant Toeplitz representation of $E$ is isomorphic to $\mathcal{O}(E)$.*

Although Definition 2.10 looks complicated, the hypotheses are relatively easy to check in specific instances. To give some intuition, we indicate how the definition simplifies if the range map $r : E^1 \to E^0$ is a local homeomorphism. This situation still includes many interesting examples.

**Definition 2.14.** Let $E$ be a topological graph. A pair $(\mathcal{U}, V)$ is called a *local $r$-fibration* if $V$ is a subset of $E^0$, and $\mathcal{U}$ is a finite collection of mutually disjoint bisections such that $r^{-1}(V) = \bigcup_{U\in\mathcal{U}} U$, and $r(U) = V$ for each $U \in \mathcal{U}$. A local $r$-fibration is *precompact* if each $U \in \mathcal{U}$ is precompact and $V$ is precompact. A local $r$-fibration is *open* if each $U \in \mathcal{U}$ is open and $V$ is open.

Suppose that $(\mathcal{U}, V)$ is an open local $r$-fibration. Suppose that $U \in \mathcal{U}$ and that $f \in C_c(V)$. We write $r_U^* f \in C_c(U)$ for the function

$$r_U^* f : e \mapsto \begin{cases} f(r(e)) & \text{if } e \in U \\ 0 & \text{otherwise.} \end{cases}$$

**Corollary 2.15.** *Let $E$ be a topological graph. Suppose that $r$ is a local homeomorphism. Let $(\psi, \pi)$ be a Toeplitz representation of $E$. Then $(\psi, \pi)$ is covariant if and only if there exists a collection $\mathcal{G} \subset C_c(E^0_{rg})$ of nonnegative functions which generates $C_0(E^0_{rg})$ as a $C^*$-algebra, and for each $f \in \mathcal{G}$ there exists an open local $r$-fibration $(\mathcal{U}, V)$ such that* $\operatorname{supp}(f) \subset V$, *and*

$$\pi(f) = \sum_{U\in\mathcal{U}} \psi\big(\sqrt{r_U^* f}\big)\psi\big(\sqrt{r_U^* f}\big)^*. \tag{2.4}$$

*If $(\psi, \pi)$ is covariant, then Equation (2.4) holds for every $f \in C_c(E^0_{rg})$ and open local $r$-fibration $(\mathcal{U}, V)$ with* $\operatorname{supp}(f) \subset V$.

**Remark 2.16.** Let $E$ be a topological graph. Fix $f \in C_c(E^0_{\mathrm{rg}})$ and suppose that $(\mathcal{U}, V)$ is an open local $r$-fibration such that $\operatorname{supp}(f) \subset V$. By Remark 2.7, $r^{-1}(\operatorname{supp}(f))$ is compact. Since $\mathcal{U}$ is an open cover of $r^{-1}(\operatorname{supp}(f))$ by disjoint open bisections, Remark 2.9 gives functions $\{f_U : U \in \mathcal{U}\}$ such that $\operatorname{osupp}(f_U) \subset U \cap r^{-1}(\operatorname{supp}(f))$, and $\sum_{U\in\mathcal{U}} f_U = f \circ r$, for all $U \in \mathcal{U}$. We then have $r_U^* f = f_U$, for all $U \in \mathcal{U}$.

## 3. $C^*$-correspondences, $C^*$-algebras and Katsura's construction

We recall some background on Hilbert $C^*$-modules. For more detail see [12].

**Definition 3.1.** Let $A$ be a $C^*$-algebra. A *right Hilbert $A$-module* is a right $A$-module $X$ equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$ such that for $x, y \in X$, $a \in A$:

(1) $\langle x, x \rangle_A \geq 0$ with equality only if $x = 0$.
(2) $\langle x, y + z \rangle_A = \langle x, y \rangle_A + \langle x, z \rangle_A$.
(3) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$.
(4) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$.
(5) $X$ is complete in the norm[1] defined by $\|x\|_A^2 = \|\langle x, x \rangle_A\|$.

Recall from, for example, [10, Page 72], that a right Hilbert $A$-module $X$ is a *$C^*$-correspondence* over $A$ if there is a left action of $A$ on $X$ such that

$$\langle a \cdot y, x \rangle_A = \langle y, a^* \cdot x \rangle_A \quad \text{for all } a \in A,\ x,\ y \in X. \tag{3.1}$$

An operator $T : X \to X$ is *adjointable* if there exists $T^* : X \to X$ such that $\langle T y, x \rangle_A = \langle y, T^* x \rangle_A$, for all $x$, $y \in X$. The adjoint $T^*$ is unique, and $T$ is automatically bounded and linear. The set $\mathcal{L}(X)$ of adjointable operators on $X$ is a $C^*$-algebra. Equation (3.1) implies that there is a homomorphism $\phi : A \to \mathcal{L}(X)$ such that $\phi(a)x = a \cdot x$ for all $a \in A$ and $x \in X$.

**Definition 3.2.** Let $A$ be a $C^*$-algebra and let $X$ be a right Hilbert $A$-module. Fix $x$, $y \in X$, we define $\Theta_{x,y} : X \to X$ by $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A$, for all $z \in X$. We define $\mathcal{K}(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\}$.

A calculation shows that $T\Theta_{x,y} = \Theta_{Tx,y}$, and $\Theta_{x,y}^* = \Theta_{y,x}$, for all $x$, $y \in X$, $T \in \mathcal{L}(X)$. Hence $\mathcal{K}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Toeplitz representations and Toeplitz algebras of $C^*$-correspondences were introduced and studied in [9]. Cuntz–Pimsner algebras were also introduced in [9], but the definition was later modified by Katsura in [6, Definition 3.5] so as to include graph algebras as a special case [10, Example 8.13]. We use Katsura's definition in this paper.

**Definition 3.3** ([6, Definition 2.1])**.** Let $A$ be a $C^*$-algebra and let $X$ be a $C^*$-correspondence over $A$. A *Toeplitz representation* of $X$ in a $C^*$-algebra $B$ is a pair $(\psi, \pi)$, where $\psi : X \to B$ is a linear map, $\pi : A \to B$ is a homomorphism, and for any $x$, $y \in X$ and $a \in A$:

(1) $\psi(a \cdot x) = \pi(a)\psi(x)$.
(2) $\psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A)$.

We say that a Toeplitz representation $(\psi, \pi)$ of $X$ in $B$ is *universal* if for any Toeplitz representation $(\psi', \pi')$ of $X$ in $C$, there exists a homomorphism $h : B \to C$, such that $h \circ \psi = \psi'$, and $h \circ \pi = \pi'$.

**Remark 3.4** ([6, Page 370])**.** Let $X$ be a $C^*$-correspondence over a $C^*$-algebra $A$ and let $(\psi, \pi)$ be a Toeplitz representation of $X$. For $x \in X$, $a \in A$, a calculation using Definition 3.3(2) and that $\pi$ is a homomorphism

[1] It is not supposed to be immediately obvious that this defines a norm, but it is true (see [12]).

shows that $(\psi(xa)-\psi(x)\pi(a))^*(\psi(xa)-\psi(x)\pi(a))=0$, and hence that that $\psi(x\cdot a)=\psi(x)\pi(a)$. Definition 3.3(2) show that $\psi$ is bounded with $\|\psi\|\le 1$.

Proposition 1.3 of [4] implies that there exists a $C^*$-algebra $\mathcal{T}_X$ generated by the image of a universal Toeplitz representation $(i_X, i_A)$ of $X$. This $C^*$-algebra is unique up to canonical isomorphism and we call it the *Toeplitz algebra* of $X$. Given another Toeplitz representation $(\psi,\pi)$ of $X$ in a $C^*$-algebra $B$, we write $h_{\psi,\pi}$ for the induced homomorphism from $\mathcal{T}_X$ to $B$.

Recall from [9, Page 202] (see also [4, Proposition 1.6]) that if $(\psi,\pi)$ is a Toeplitz representation of a $C^*$-correspondence $X$ in a $C^*$-algebra $B$, then there is a unique homomorphism $\psi^{(1)}:\mathcal{K}(X)\to B$ such that $\psi^{(1)}(\Theta_{x,y})=\psi(x)\psi(y)^*$ for all $x,\ y\in X$. Recall also that given a $C^*$-algebra $A$ and a closed two-sided ideal $J$ of $A$, we define $J^\perp := \{a\in A : ab=0 \text{ for all } b\in J\}$. Then $J^\perp$ is also a closed two-sided ideal of $A$.

**Definition 3.5** ([6, Definitions 3.4 and 3.5])**.** Let $A$ be a $C^*$-algebra, let $X$ be a $C^*$-correspondence over $A$, and write $\phi : A\to\mathcal{L}(X)$ for the homomorphism implementing the left action. A Toeplitz representation $(\psi,\pi)$ of $X$ is *covariant* if $\psi^{(1)}(\phi(a))=\pi(a)$ for all $a\in\phi^{-1}(\mathcal{K}(X))\cap(\ker\phi)^\perp$.

A covariant Toeplitz representation $(\psi,\pi)$ of $X$ in $B$ is *universal* if for any covariant Toeplitz representation $(\psi',\pi')$ of $X$ in $C$, there exists a homomorphism $h$ from $B$ into $C$, such that $h\circ\psi=\psi'$, and $h\circ\pi=\pi'$.

Recall from [10, Page 75] that there exists a $C^*$-algebra $\mathcal{O}_X$ generated by the image of a universal covariant Toeplitz representation $(j_X, j_A)$ of $X$. This $C^*$-algebra is unique up to canonical isomorphism. Given another covariant Toeplitz representation $(\psi,\pi)$ of $X$ in a $C^*$-algebra $B$, we write $\psi\times\pi$ for the induced homomorphism from $\mathcal{O}_X$ to $B$.

Let $E$ be a topological graph. As in [5], define actions of $C_0(E^0)$ on $C_c(E^1)$ by Equation (2.1). For $x_1,\ x_2\in C_c(E^1)$, define $\langle x_1,x_2\rangle_{C_0(E^0)} : E^0\to\mathbb{C}$ by

$$\langle x_1,x_2\rangle_{C_0(E^0)}(v)=\sum_{s(e)=v}\overline{x_1(e)}x_2(e).$$

(By convention an empty sum is 0.) If $\operatorname{osupp}(x)$ is an $s$-section, then the function $\widehat{x}$ of Equation (2.2) is equal to $\langle x,x\rangle_{C_0(E^0)}$. As in [5] (see also [10, Page 79]), $\langle\cdot,\cdot\rangle_{C_0(E^0)}$ defines a $C_0(E^0)$-valued inner product on $C_c(E^1)$, and the completion $X(E)$ of $C_c(E^1)$ in the norm $\|x\|^2_{C_0(E^0)}=\|\langle x,x\rangle_{C_0(E^0)}\|$ is a $C^*$-correspondence over $C_0(E^0)$. We call $X(E)$ the *graph correspondence* of $E$.

**Notation 3.6.** We denote by $\mathcal{T}(E)$ [5, Definition 2.2] the Toeplitz algebra $\mathcal{T}_{X(E)}$, and we denote by $\mathcal{O}(E)$ [5, Definition 2.10] the $C^*$-algebra $\mathcal{O}_{X(E)}$.

## 4. Proofs of the main results

To prove Theorem 2.4, we must show that $\psi(x)^*\psi(y)=\pi(\langle x,y\rangle_{C_0(E^0)})$ for all $x,y\in C_c(E^1)$. We establish this formula for $x$ and $y$ supported on a common $s$-section in Lemma 4.1, and then extend it to arbitrary $x, y \in C_c(E^1)$ in Proposition 4.3.

Let $U$, $V$ be complex vector spaces. Then any sesquilinear form $\varphi : V \times V \to U$ which is conjugate linear in the first variable satisfies the polarisation identity

$$\varphi(v_1, v_2) = \frac{1}{4}\sum_{n=0}^{3}(-i)^n \varphi(v_1 + i^n v_2, v_1 + i^n v_2).$$

(To prove this, just expand the sum.)

**Lemma 4.1.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a Toeplitz representation of $E$. Fix $x_1$, $x_2 \in C_c(E^1)$. Suppose that $\operatorname{supp}(x_1) \cup \operatorname{supp}(x_2)$ is contained in an open $s$-section. Then $\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) = \psi(x_1)^*\psi(x_2)$.*

**Proof.** The polarisation identity for $\langle \cdot, \cdot \rangle_{C_0(E^0)}$ and Definition 2.2(2) give

$$\begin{aligned}\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) &= \frac{1}{4}\sum_{n=0}^{3}(-i)^n \pi(\langle x_1 + i^n x_2, x_1 + i^n x_2\rangle_{C_0(E^0)}) \\ &= \frac{1}{4}\sum_{n=0}^{3}(-i)^n \psi(x_1 + i^n x_2)^*\psi(x_1 + i^n x_2) \\ &= \psi(x_1)^*\psi(x_2). \end{aligned}$$

□

In the following and throughout the rest of the paper we write $U^c$ for the complement $X \setminus U$ of a subset $U$ of a set $X$.

**Remark 4.2.** Let $X$ be locally compact Hausdorff space. Fix a compact subset $K \subset X$ and an open neighbourhood $U$ of $K$. Since $U^c$ is closed and disjoint from $K$, there is a function $f \in C(X, [0, 1])$ which is identically 1 on $K$ and identically 0 on $U^c$ (see for example [14, Theorem 37.A]). So $V := f^{-1}\big((1/2, 3/2)\big)$ is open and satisfies $K \subset V \subset \overline{V} \subset U$.

**Proposition 4.3.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a Toeplitz representation of $E$. Fix open $s$-sections $U_1$, $U_2 \subset E^1$, and $x_1$, $x_2 \in C_c(E^1)$ with $\operatorname{supp}(x_i) \subset U_i$, for $i = 1$, 2. Then $\pi(\langle x_1, x_2\rangle_{C_0(E^0)}) = \psi(x_1)^*\psi(x_2)$.*

**Proof.** Let $K_i = \operatorname{supp}(x_i)$, for $i = 1$, 2, and $K = K_1 \cup K_2$. Since $E^1$ is locally compact and Hausdorff, Remark 4.2 implies that there exist open sets $U_i' \subset E^1$ such that $K_i \subset U_i' \subset \overline{U_i'} \subset U_i$, for $i = 1, 2$.

For each $e \in K$ and $W \in \{U_i, U_i', \overline{U_i'}^c : i = 1, 2\}$ such that $e \in W$, fix an open neighbourhood $V(e, W)$ of $e$ such that $\overline{V(e, W)} \subset W$. Define

$$N_e = \Big(\bigcap\nolimits_{e \in U_i} V(e, U_i)\Big) \cap \Big(\bigcap\nolimits_{e \in K_i} V(e, U_i')\Big) \cap \Big(\bigcap\nolimits_{e \in U_i^c} V(e, \overline{U_i'}^c)\Big).$$

Then for $i = 1$, 2:

(1) If $e \in U_i$, then $\overline{N_e} \subset U_i$.
(2) If $e \in K_i$, then $\overline{N_e} \subset U_i'$.

(3) If $e \notin U_i$, then $\overline{N_e} \cap \overline{U_i'} = \emptyset$.

Since $K$ is compact, there is a finite $F \subset K$ such that $\{N_e : e \in F\}$ covers $K$.

Fix $e, f \in F$. Suppose that $\overline{N_e} \cap \overline{N_f} \neq \emptyset$. We claim that

$$\text{either } \overline{N_e} \cup \overline{N_f} \subset U_1, \text{ or } \overline{N_e} \cup \overline{N_f} \subset U_2. \tag{4.1}$$

When $e = f$, this is trivial. Suppose $e \neq f$. Assume without loss of generality that $e \in K_1$. Then (2) forces $\overline{N_e} \subset U_1'$, so $\overline{N_e} \cap \overline{N_f} \neq \emptyset$ forces $\overline{N_f} \cap \overline{U_1'} \neq \emptyset$. Condition (3) then forces $f \in U_1$, so (1) forces $\overline{N_f} \subset U_1$ and hence $\overline{N_e} \cup \overline{N_f} \subset U_1$ as required.

Since $\{N_e : e \in F\}$ is a finite open cover of $K$, Remark 2.9 implies that there are finite collections of functions

$$\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$$

such that $\operatorname{osupp}(x_{i,e}) \subset N_e \cap K$ for all $e \in F$, $i = 1, 2$, and $\sum_{e \in F} x_{1,e} = x_1$, $\sum_{e \in F} x_{2,e} = x_2$. Linearity of $\psi$ gives $\psi(x_1)^*\psi(x_2) = \sum_{e,f \in F} \psi(x_{1,e})^*\psi(x_{2,f})$. Fix $e, f \in F$ such that $\overline{N_e} \cap \overline{N_f} = \emptyset$. By Remark 4.2 there are disjoint open $s$-sections $O_e, O_f \subset E^1$, such that $\overline{N_e} \subset O_e$, and $\overline{N_f} \subset O_f$. Thus condition (3) of Definition 2.2 gives $\psi(x_{1,e})^*\psi(x_{2,f}) = 0$ since $\operatorname{supp}(x_{i,e}) \subset \overline{N_e}$ for all $e \in F$, $i = 1, 2$. It follows that $\psi(x_1)^*\psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \psi(x_{1,e})^*\psi(x_{2,f})$. Whenever $\overline{N_e} \cap \overline{N_f} \neq \emptyset$, Equation (4.1) implies that $\overline{N_e} \cup \overline{N_f}$ is contained in an open $s$-section, so Lemma 4.1 gives

$$\psi(x_1)^*\psi(x_2) = \sum_{\overline{N_e} \cap \overline{N_f} \neq \emptyset} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) = \pi(\langle x_1, x_2 \rangle_{C_0(E^0)}). \qquad \square$$

**Proposition 4.4.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a Toeplitz representation of $E$ in $B$. Then*

$$\psi : \big(C_c(E^1), \|\cdot\|_{C_0(E^0)}\big) \to B$$

*is a bounded linear map. Let $\widetilde{\psi}$ be the unique extension of $\psi$ to $X(E)$. Then the pair $(\widetilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$.*

**Proof.** Fix $x_1, x_2 \in C_c(E^1)$. Let $K = \operatorname{supp}(x_1) \cup \operatorname{supp}(x_2)$. For each $e \in K$, there exists an open $s$-section $N_e$ containing $e$. Remark 4.2 yields an open neighbourhood $N_e'$ of $e$ such that $\overline{N_e'} \subset N_e$. Since $K$ is compact, there is a finite subset $F \subset K$, such that $\{N_e' : e \in F\}$ covers $K$. By Remark 2.9, there exist $\{x_{1,e} : e \in F\}, \{x_{2,e} : e \in F\} \subset C_c(E^1)$ such that $\operatorname{osupp}(x_{i,e}) \subset N_e' \cap K$ for all $e \in F$ and $i = 1, 2$, and such that $\sum_{e \in F} x_{1,e} = x_1$ and $\sum_{e \in F} x_{2,e} = x_2$. Proposition 4.3 implies that

$$\begin{aligned}\pi(\langle x_1, x_2 \rangle_{C_0(E^0)}) &= \sum_{e,f \in F} \pi(\langle x_{1,e}, x_{2,f} \rangle_{C_0(E^0)}) \\ &= \sum_{e,f \in F} \psi(x_{1,e})^*\psi(x_{2,f}) = \psi(x_1)^*\psi(x_2).\end{aligned}$$

Then

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x\rangle_{C_0(E^0)})\| \leq \|\langle x, x\rangle_{C_0(E^0)}\| = \|x\|^2_{C_0(E^0)}$$

for all $x \in C_c(E^1)$. So $\psi$ is bounded, and hence extends uniquely to a bounded linear map $\widetilde{\psi}$ on $X(E)$. By continuity, $(\widetilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$. □

**Remark 4.5.** Let $(\psi, \pi)$ be a Toeplitz representation of the graph correspondence $X(E)$. Then the pair $(\psi|_{C_c(E^1)}, \pi)$ is a Toeplitz representation of $E$. So Proposition 4.4 implies that $(\psi, \pi) \mapsto (\psi|_{C_c(E^1)}, \pi)$ is a bijection between Toeplitz representations of $X(E)$ and Toeplitz representations of $E$, with inverse described by Proposition 4.4.

**Proof of Theorem 2.4.** Let $(i_X, i_A)$ be the universal Toeplitz representation of $X(E)$ in $\mathcal{T}(E)$. Then $(i_1, i_0) := (i_X|_{C_c(E^1)}, i_A)$ is a Toeplitz representation of $E$. Fix another Toeplitz representation $(\psi, \pi)$ of $E$ in a $C^*$-algebra $B$. By Proposition 4.4, $\psi$ extends to $\widetilde{\psi} : X(E) \to B$ such that $(\widetilde{\psi}, \pi)$ is a Toeplitz representation of $X(E)$. By the universal property of $(i_X, i_A)$, there exists a homomorphism $h_{\widetilde{\psi},\pi} : \mathcal{T}(E) \to B$, such that $h_{\widetilde{\psi},\pi} \circ i_X = \widetilde{\psi}$, and $h_{\widetilde{\psi},\pi} \circ i_A = \pi$. In particular $h_{\widetilde{\psi},\pi} \circ i_1 = \psi$. Hence $(i_1, i_0)$ is a universal Toeplitz representation of $E$ which generates $\mathcal{T}(E)$. The second statement follows easily. □

Our next task is to prove Theorem 2.13. We first need some background results.

**Remark 4.6.** Let $E$ be a topological graph. Fix $v \in E^0_{\text{fin}}$. There exists a neighbourhood $U$ of $v$ such that $r^{-1}(U)$ is compact, and there exists an open neighbourhood $V$ of $v$ such that $V \subset U$. By Remark 4.2, there exists an open neighbourhood $W$ of $v$ such that $\overline{W} \subset V$. Since $r^{-1}(\overline{W})$ is closed and is contained in $r^{-1}(U)$, it is compact. Hence $v \in E^0_{\text{fin}}$ if and only if there exists an open neighbourhood $N$ of $v$ such that $r^{-1}(\overline{N})$ is compact.

Let $E$ be a topological graph. Recall from Definition 3.5 that Katsura's covariance condition for a Toeplitz representation of the correspondence $X(E)$ involves the ideal $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker\phi)^\perp$. Katsura computed this ideal in [5]. We quote his result and give a simple proof.

Observe that $\ker\phi = \{f \in C_0(E^0) : f(\overline{r(E^1)}) \equiv 0\}$. Hence

$$(\ker\phi)^\perp = \{f \in C_0(E^0) : f(\overline{E^0_{\text{sce}}}) \equiv 0\}.$$

**Proposition 4.7** ([5, Proposition 1.24])**.** *Let $E$ be a topological graph. Then*

$$\phi^{-1}(\mathcal{K}(X(E))) = C_0(E^0_{\text{fin}}).$$

*Moreover* $\phi^{-1}(\mathcal{K}(X(E))) \cap (\ker\phi)^\perp = C_0(E^0_{\text{rg}})$.

**Proof.** The final statement follows from the previous one and definition of $E^0_{\mathrm{rg}}$. So we just need to show that $\phi^{-1}(\mathcal{K}(X(E))) = C_0(E^0_{\mathrm{fin}})$.

Fix $f \in C_0(E^0) \setminus C_0(E^0_{\mathrm{fin}})$. We must show that $\phi(f) \notin \mathcal{K}(X(E))$. Fix $v_0 \in (E^0_{\mathrm{fin}})^c$, such that $f(v_0) \neq 0$. Let $m = |f(v_0)|$ and let

$$N_0 = \{v \in E^0 : |f(v)| > m/2\},$$

so $N_0$ is an open neighbourhood of $v_0$. By Remark 4.6, $r^{-1}(\overline{N_0})$ is not compact. Fix $x_1, \dots, x_n$ and $y_1, \dots, y_n$ in $C_c(E^1)$. Let

$$K = \bigcup_{i=1}^n \operatorname{supp}(x_i) \cup \operatorname{supp}(y_i).$$

Then $K$ is compact, so $r^{-1}(\overline{N_0})$ is not contained in $K$. So there exists $e_0 \in r^{-1}(\overline{N_0}) \setminus K$. By Remark 4.2 there exists $x_0 \in C_c(E^1)$ such that $x_0(e_0) = 1$. Hence

$$\begin{aligned}
&\left\|\phi(f) - \sum_{i=1}^n \Theta_{x_i,y_i}\right\| \\
&\geq \left\|\phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0)\right\|_{C_0(E^0)} \\
&\geq \left|\left\langle \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0),\ \phi(f)x_0 - \sum_{i=1}^n \Theta_{x_i,y_i}(x_0)\right\rangle_{C_0(E^0)}(s(e_0))\right|^{1/2} \\
&\geq m/2.
\end{aligned}$$

Thus $\|\phi(f) - a\| \geq m/2$ for all $a \in \operatorname{span}\{\Theta_{x,y} : x, y \in C_c(E^1)\}$. Since $\overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in C_c(E^1)\} = \mathcal{K}(X(E))$, it follows that $\phi(f) \notin \mathcal{K}(X(E))$.

Now fix a nonnegative function $f \in C_c(E^0_{\mathrm{fin}})$. We must show that $\phi(f) \in \mathcal{K}(X(E))$. Let $K' = \operatorname{supp}(f)$. For any $e \in r^{-1}(K')$, there exists an open $s$-section $N_e$ containing $e$. Remark 2.7 shows that $r^{-1}(K')$ is compact. Hence there exists a finite subset $F \subset r^{-1}(K')$ such that $\{N_e\}_{e \in F}$ covers $r^{-1}(K')$. Since $\operatorname{supp}(f \circ r) \subset r^{-1}(K')$, Remark 2.9 yields a finite collection of functions $\{f_e : e \in F\} \subset C_c(E^1)$ such that each $\operatorname{osupp}(f_e) \subset N_e \cap r^{-1}(K')$ and $\sum_{e \in F} f_e = f \circ r$. Since the $f_e$ are supported on the $s$-sections $N_e$, we have $\theta_{\sqrt{f_e},\sqrt{f_e}}(x)(e') = f_e(e')x(e')$ for all $e' \in E^1$. Hence

$$\phi(f) = \sum_{e \in F} \Theta_{\sqrt{f_e},\sqrt{f_e}} \in \mathcal{K}(X(E)). \tag{4.2}$$ □

**Proposition 4.8.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a covariant Toeplitz representation of $E$. Then the Toeplitz representation $(\widetilde{\psi}, \pi)$ of $X(E)$ from Proposition 4.4 is also covariant. For any nonnegative function $f \in C_c(E^0_{\mathrm{rg}})$, any finite cover $\mathcal{N}$ of $r^{-1}(\operatorname{supp}(f))$ by open $s$-sections and* *any collection of nonnegative functions* $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$*, such that* $\mathrm{osupp}(f_N) \subset N \cap r^{-1}(\mathrm{supp}(f))$*, and* $\sum_{N\in\mathcal{N}} f_N = f \circ r$*, we have*

$$\pi(f) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*.$$

**Proof.** Choose collections $\mathcal{G}$, $\{\mathcal{N}_f : f \in \mathcal{G}\}$, and $\{f_N : N \in \mathcal{N}_f\}$ as in Definition 2.10. By Corollary 4.7, to prove that $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$, must show that $\widetilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$, for all $f \in \mathcal{G}$. Fix $f \in \mathcal{G}$. Since $\phi(f) = \sum_{N\in\mathcal{N}_f} \Theta_{\sqrt{f_N},\sqrt{f_N}}$, we have $\widetilde{\psi}^{(1)} \circ \phi(f) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*$. Hence $\widetilde{\psi}^{(1)} \circ \phi(f) = \pi(f)$ by Definition 2.10. Therefore $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$.

For the second statement observe that since $(\widetilde{\psi}, \pi)$ is a covariant Toeplitz representation of $X(E)$,

$$\pi(f) = \widetilde{\psi}^{(1)} \circ \phi(f) = \sum_{N\in\mathcal{N}_f} \widetilde{\psi}^{(1)}(\Theta_{\sqrt{f_N},\sqrt{f_N}}) = \sum_{N\in\mathcal{N}_f} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^*. \qquad \square$$

**Proposition 4.9.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a covariant Toeplitz representation of $X(E)$. Then $(\psi|_{C_c(E^1)}, \pi)$ is a covariant Toeplitz representation of $E$.*

**Proof.** Remark 4.5 implies that $(\psi|_{C_c(E^1)}, \pi)$ is a Toeplitz representation of $E$. Let $\mathcal{G}$ be the set of all nonnegative functions in $C_c(E^0_{\mathrm{rg}})$. Fix $f \in \mathcal{G}$. By Equation (4.2), there exists a finite cover $\mathcal{N}$ of $r^{-1}(\mathrm{supp}(f))$ by open $s$-sections and a finite collection of functions $\{f_N : N \in \mathcal{N}\} \subset C_c(E^1)$, such that $\mathrm{osupp}(f_N) \subset N \cap r^{-1}(\mathrm{supp}(f))$, $\sum_{N\in\mathcal{N}} f_N = f \circ r$, and $\phi(f) = \sum_{N\in\mathcal{N}} \Theta_{\sqrt{f_N},\sqrt{f_N}}$. Since $(\psi, \pi)$ is a covariant Toeplitz representation of $X(E)$, we have

$$\begin{aligned}\pi(f) = \psi^{(1)} \circ \phi(f) &= \sum_{N\in\mathcal{N}} \psi(\sqrt{f_N})\psi(\sqrt{f_N})^* \\ &= \sum_{N\in\mathcal{N}} \psi|_{C_c(E^1)}(\sqrt{f_N})\psi|_{C_c(E^1)}(\sqrt{f_N})^*.\end{aligned}$$

Hence $(\psi|_{C_c(E^1)}, \pi)$ is covariant. $\square$

**Proof of Theorem 2.13.** Propositions 4.8 and 4.9 provide a bijective map from covariant Toeplitz representations of $E$ to covariant Toeplitz representations of $X(E)$. The result now follows from the same argument as Theorem 2.4. $\square$

We still need to prove Corollary 2.15. We must first show that under the hypotheses of the corollary, there are plenty of local $r$-fibrations (see Definition 2.14).

**Lemma 4.10.** *Let $E$ be a topological graph. Suppose that $r$ is a local homeomorphism. Then for any $v \in E^0_{\mathrm{rg}}$, there exists a precompact open local $r$-fibration $(\mathcal{U}, V)$, such that $v \in V \subset \overline{V} \subset E^0_{\mathrm{rg}}$.*

**Proof.** Fix $v \in E^0_{\mathrm{rg}}$. Since $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}} \subset E^0_{\mathrm{fin}} \setminus E^0_{\mathrm{sce}}$, Lemma 1.22 of [5] shows that $r^{-1}(v)$ is nonempty. Lemma 1.23 of [5] shows that it is also compact. Since $r$ is a local homeomorphism, $r^{-1}(v)$ is finite. Since $E^1$ is locally compact Hausdorff, we can separate points in $r^{-1}(v)$ by mutually disjoint precompact open bisections $\{U_e : e \in r^{-1}(v)\}$. We can assume, by shrinking if necessary, that the $U_e$ have common range $N$ with $\overline{N} \subset E^0_{\mathrm{rg}}$. Now $|r^{-1}(w)| \geq |r^{-1}(v)|$ for all $w \in N$.

We claim there exists an open neighbourhood $V$ of $v$ such that $V \subset N$, and $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. Suppose for a contradiction there exists a convergent net $(v_\alpha)_{\alpha \in \Lambda} \subset N$ with limit $v$ satisfying $|r^{-1}(v_\alpha)| > |r^{-1}(v)|$ for all $\alpha \in \Lambda$. Then for any $\alpha \in \Lambda$, there exists $e_\alpha \notin \bigcup_{e \in r^{-1}(v)} U_e$ such that $r(e_\alpha) = v_\alpha$. Since $r^{-1}(\overline{N})$ is compact, [13, Theorem IV.3] implies that there exists a convergent subnet $(e_{h(a)})_{a \in A}$ of $(e_\alpha)_{\alpha \in \Lambda}$ with limit $e \notin \bigcup_{f \in r^{-1}(v)} U_f$. By continuity of $r$, we have $r(e) = v$, which is a contradiction.

Hence there exists an open neighbourhood $V$ of $v$ satisfying $V \subset N$, such that $|r^{-1}(w)| = |r^{-1}(v)|$ for all $w \in V$. So with

$$\mathcal{U} = \{U_e \cap r^{-1}(V) : e \in r^{-1}(v)\},$$

the pair $(\mathcal{U}, V)$ is a precompact open local $r$-fibration. □

**Lemma 4.11.** *Let $E$ be a topological graph. Suppose that $r$ is a local homeomorphism. Let $\mathcal{G}$ be the set of all nonnegative functions $f$ in $C_c(E^0_{\mathrm{rg}})$ such that $\operatorname{supp}(f) \subset V$ for some open local $r$-fibration $(\mathcal{U}, V)$. Then $\mathcal{G}$ generates $C_0(E^0_{\mathrm{rg}})$.*

**Proof.** By Lemma 4.10, for all $v \neq w \in E^0_{\mathrm{rg}}$, there is a local $r$-fibration $(\mathcal{U}, V)$ such that $v \in V$ and $w \notin V$. So there exists $f \in \mathcal{G}$ such that $f(v) = 1$ and $f(w) = 0$. Now the result follows from the Extended Stone–Weierstrass Theorem. □

**Proof of Corollary 2.15.** Let $(\psi, \pi)$ be a covariant Toeplitz representation of $E$. Let $\mathcal{G}$ be the set of all nonnegative functions $f$ in $C_c(E^0_{\mathrm{rg}})$ such that $\operatorname{supp}(f) \subset V$ for some open local $r$-fibration $(\mathcal{U}, V)$. Lemma 4.11 implies that $\mathcal{G}$ generates $C_0(E^0_{\mathrm{rg}})$. Fix $f \in \mathcal{G}$ and an open local $r$-fibration $(\mathcal{U}, V)$ with $\operatorname{supp}(f) \subset V$. Since $\mathcal{U}$ is a finite cover of $r^{-1}(\operatorname{supp}(f))$ by open $s$-sections, each $\operatorname{osupp}(r^*_U f) \subset U \cap r^{-1}(\operatorname{supp}(f))$, and $\sum_{U \in \mathcal{U}} r^*_U f = f \circ r$. By Proposition 4.8, we have $\pi(f) = \sum_{U \in \mathcal{U}} \psi(\sqrt{r^*_U f})\psi(\sqrt{r^*_U f})^*$. The converse of the first statement follows from Definition 2.10 and Remark 2.16. The second statement follows easily from the construction of $\mathcal{G}$. □

**4.1. The $C^*$-algebra generated by a Toeplitz representation.** In this subsection we provide some technical results which may prove useful in using our descriptions of the $C^*$-algebras associated to topological graphs. Proposition 4.12 aids in constructing representations; and Proposition 4.16 provides a well-behaved collection of spanning elements for the image of any Toeplitz representation of $E$, and also a formula for computing products of these spanning elements.

To construct Toeplitz representations of a topological graph, one needs to build linear maps $\psi : C_c(E^1) \to B$ that are $\|\cdot\|_{C_0(E^0)}$-bounded. The following technical result reduces the task to defining $\psi$ on functions that are dense in supremum norm on $C_0(U)$ for a suitable family of open $s$-sections $U$.

**Proposition 4.12.** *Let $E$ be a topological graph, let $\mathcal{B}$ be an open base for the topology on $E^1$ consisting of $s$-sections, and let $\mathcal{F} \subset C_c(E^1)$ be a collection of nonnegative functions such that* $\operatorname{osupp}(x)$ *is an $s$-section for all $x \in \mathcal{F}$. Suppose that for each $U \in \mathcal{B}$,*

$$\operatorname{span}\{x \in \mathcal{F} : \operatorname{osupp}(x) \subset U\} \text{ is } \|\cdot\|_\infty\text{-dense in } C_0(U). \tag{4.3}$$

*Then $X_0 := \operatorname{span}\mathcal{F}$ is dense in $X(E)$. Let $B$ be a $C^*$-algebra. Suppose that $\psi_0 : X_0 \to B$ is a linear map, that $\pi : C_0(E^0) \to B$ is a homomorphism, and that the pointwise product $xy$ satisfies*

$$\pi(\widehat{\sqrt{xy}}) = \psi_0(x)^*\psi_0(y) \quad \text{for all } x, y \in \mathcal{F}. \tag{4.4}$$

*Then $\psi_0$ extends uniquely to a bounded linear map $\psi$ on $C_c(E^1)$. Suppose that $\mathcal{G} \subset C_0(E^0)$ generates $C_0(E^0)$. If the extension $\psi$ satisfies*

$$\psi(f \cdot x) = \pi(f)\psi_0(x) \quad \text{for all } f \in \mathcal{G} \text{ and } x \in \mathcal{F}, \tag{4.5}$$

*then $(\psi, \pi)$ is a Toeplitz representation of $E$.*

**Proof.** Fix $x \in C_c(E^1)$. Let $K = \operatorname{supp}(x)$. For each $e \in K$, there exists an open $s$-section $N_e$ containing $e$, such that $N_e \in \mathcal{B}$. Since $K$ is compact, there is a finite subset $F \subset K$, such that $\{N_e : e \in F\}$ covers $K$. By Remark 2.9, there exists a finite collection of functions $\{x_e : e \in F\} \subset C_c(E^1)$, such that $\operatorname{osupp}(x_e) \subset N_e \cap K$ for all $e \in F$, and $\sum_{e\in F} x_e = x$. Fix $e \in F$. Since $\operatorname{osupp}(x_e) \subset N_e$, there exists a sequence $(x_{e,n}) \subset X_0 \cap C_0(N_e)$ converging to $x_e$ in supremum norm. That $(x_{e,n})$ and $x_e$ vanish off the $s$-section $N_e$ imply that $\|x_{e,n} - x_e\|_{C_0(E^0)} = \sup_{e\in E^1} |x_{e,n} - x_e|$. Hence $\sum_{e\in F} x_{e,n} \to x$ in $\|\cdot\|_{C_0(E^0)}$ norm. Therefore $X_0$ is dense in $X(E)$.

Fix $x, y \in \mathcal{F}$. Since $x, y$ are nonnegative, $\widehat{\sqrt{xy}} = \langle x, y\rangle_{C_0(E^0)}$. Hence (4.4) implies that $\pi(\langle x, y\rangle_{C_0(E^0)}) = \psi_0(x)^*\psi_0(y)$. Linearity of $\psi_0$ and $\pi$ gives $\pi(\langle x, y\rangle_{C_0(E^0)}) = \psi_0(x)^*\psi_0(y)$, for all $x$, $y \in X_0$. By Remark 3.4, $\psi_0$ is bounded, so extends uniquely to $\widetilde{\psi} : X(E) \to B$. Then $\psi := \widetilde{\psi}|_{C_c(E^1)}$ is the required map. Continuity and (4.5) show that $(\psi, \pi)$ is a Toeplitz representation of $E$. □

**Remark 4.13.** To prove Proposition 4.12 we showed that Equation (4.3) implies that $X_0$ is dense in $X(E)$ under the $\|\cdot\|_{C_0(E^0)}$ norm, and then deduced that $(\psi, \pi)$ extends to a Toeplitz representation of $E$. So replacing Equation (4.3) with the hypothesis that $X_0$ is dense in $X(E)$ would yield a formally stronger result. However, Equation (4.3) is in many instances easier to check.

Our next proposition provides a description of the $C^*$-algebra generated by a Toeplitz representation of $E$ in terms of a spanning family which captures many of the key properties of the usual spanning family in the Toeplitz algebra of a directed graph. We first need some notation and two technical lemmas.

Recall that $E^n$ denotes the space

$$\{\mu = \mu_1 \dots \mu_n : \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$$

of paths of length $n$ in $E$. We define $r, s : E^n \to E^0$ by $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$, and we give $E^n$ the relative topology inherited from the product space $\prod_{i=1}^n E^1$. For $x \in C_c(E^n)$ and $f \in C_0(E^0)$ we define $f \cdot x, x \cdot f \in C_c(E^n)$ by $(f \cdot x)(\mu) = f(r(\mu))x(\mu)$ and $(x \cdot f)(\mu) = x(\mu)f(s(\mu))$.

For $x_1, \dots, x_n \in C_c(E^1)$, we define $x_1 \diamond \cdots \diamond x_n \in C_c(E^n)$ by

$$(x_1 \diamond \cdots \diamond x_n)(\mu) = \prod_{i=1}^n x_i(\mu_i) \quad \text{for } \mu = \mu_1 \dots \mu_n \in E^n.$$

We use the symbol $\diamond$ for this operation to distinguish it from the pointwise product of elements of $C_c(E^1)$ appearing in, for example, Equation (4.4).

The second assertion of the following technical result follows from the discussion preceding [9, Proposition 3.3] together with [5, Proposition 1.27] (see also [10, Proposition 9.7]). We include the result and a simple proof here for completeness.

Suppose that $x, y \in C_c(E^n)$ are supported on $s$-sections. Then there is a unique $H(x, y) \in C_c(E^0)$ that vanishes on $E^0 \setminus \{s(\mu) : x(\mu)y(\mu) \neq 0\}$ and satisfies

$$H(x, y)(s(\mu)) = \overline{x(\mu)}y(\mu) \text{ whenever } x(\mu)y(\mu) \neq 0.$$

**Lemma 4.14.** *Let $E$ be a topological graph. Suppose that $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are supported on $s$-sections. Let $(\psi, \pi)$ be a Toeplitz representation of $E$. Let $x = x_1 \diamond \cdots \diamond x_n$ and $y = y_1 \diamond \cdots \diamond y_n$. Then*

$$\pi(H(x, y)) = \psi(x_n)^* \cdots \psi(x_1)^* \psi(y_1) \cdots \psi(y_n).$$

*If $x = y$ then $\prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(y_i)$.*

**Proof.** By Proposition 4.4, $(\psi, \pi)$ extends to a Toeplitz representation of $X(E)$. We have $H(x, y) = \langle x_n, H(x_1 \diamond \cdots \diamond x_{n-1}, y_1 \diamond \cdots \diamond y_{n-1}) \cdot y_n \rangle$, and hence

$$\pi(H(x, y)) = \psi(x_n)^* \pi(H(x_1 \diamond \cdots \diamond x_{n-1}, y_1 \diamond \cdots \diamond y_{n-1})\psi(y_n).$$

The first assertion now follows by induction. For the second assertion, we use the first to see that

$$\left(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i)\right)^* \left(\prod_{i=1}^n \psi(x_i) - \prod_{i=1}^n \psi(y_i)\right) = \pi(H(x,x) - H(x,y) - H(y,x) + H(y,y)),$$

which is equal to zero since $x = y$. □

**Lemma 4.15.** *Let $E$ be a topological graph. Suppose that $x_1, \dots, x_n \in C_c(E^1)$ are supported on $s$-sections, and fix $f \in C_0(E^0)$. Then there exists $\widetilde{f} \in C_c(E^0)$ such that*

$$\widetilde{f}(s(\mu)) = f(r(\mu)) \quad \text{whenever } (x_1 \diamond \cdots \diamond x_n)(\mu) \neq 0. \tag{4.6}$$

*For any such $\widetilde{f}$, we have $f \cdot (x_1 \diamond \cdots \diamond x_n) = (x_1 \diamond \cdots \diamond x_n) \cdot \widetilde{f}$, and $\pi(f)\prod_{i=1}^n \psi(x_i) = \prod_{i=1}^n \psi(x_i)\pi(\widetilde{f})$ for any Toeplitz representation $(\psi, \pi)$ of $E$.*

**Proof.** The second assertion will follow from the first by definition of $f \cdot (x_1 \diamond \cdots \diamond x_n)$ and $(x_1 \diamond \cdots \diamond x_n) \cdot \widetilde{f}$. The final assertion will then follow from Lemma 4.14. So we just need to prove the first assertion. Let $x := x_1 \diamond \cdots \diamond x_n$. Fix $f \in C_0(E^0)$. Since $K := \operatorname{supp}(x) \subseteq E^n$ is an $s$-section, there is a well-defined continuous function from $s(K)$ to $r(K)$ given by $s(\mu) \mapsto r(\mu)$ for $\mu \in K$. So there is a continuous function $\widetilde{f_0} \in C(s(K))$ given by $\widetilde{f_0}(s(\mu)) = f(r(\mu))$ for all $\mu \in K$. Since $s(K)$ is compact, an application of the Tietze extension theorem shows that $\widetilde{f}$ has an extension $\widetilde{f} \in C_0(E^0)$, which satisfies (4.6) by definition. □

**Proposition 4.16.** *Let $E$ be a topological graph and let $(\psi, \pi)$ be a Toeplitz representation of $E$. Let $\mathcal{B}$ be a base for the topology on $E^1$ consisting of open $s$-sections, and let $\mathcal{F} \subseteq C_c(E^1)$ be a collection of nonnegative functions such that $\operatorname{supp}(f)$ is an $s$-section for each $f \in \mathcal{F}$. Then:*

(1) *$C^*(\psi, \pi)$ is densely spanned by elements of the form*

$$\psi(x_1) \cdots \psi(x_n)\pi(f)\psi(y_m)^* \cdots \psi(y_1)^*$$

*where $m, n \geq 0$, $f \in C_c(E^0)$, the $x_i, y_j$ all belong to $\mathcal{F}$, each of the sets $s(\operatorname{osupp}(x_i)) \cap r(\operatorname{osupp}(x_{i+1}))$ and $s(\operatorname{osupp}(y_i)) \cap r(\operatorname{osupp}(y_{i+1}))$ is nonempty, and $s(\operatorname{osupp}(x_n)) \cap s(\operatorname{osupp}(y_m)) \cap \operatorname{osupp}(f) \neq \emptyset$.*

(2) *Consider a pair of spanning elements*

$$\psi(w_1) \dots \psi(w_m)\pi(f)\psi(x_n)^* \dots \psi(x_1)^*,$$
$$\psi(y_1) \dots \psi(y_p)\pi(g)\psi(z_q)^* \dots \psi(z_1)^*$$

*as in (1) with $p \geq n$. Let $x = x_1 \diamond \cdots \diamond x_n$ and $y = y_1 \diamond \cdots \diamond y_n$. Fix $k \in C_0(E^0)$ such that $(fH(x,y)) \cdot (y_{n+1} \diamond \cdots \diamond y_p) = (y_{n+1} \diamond \cdots \diamond y_p) \cdot k$*

*as in Lemma 4.15. Then*

$$\begin{aligned}\big(\psi(w_1)\dots\psi(w_m)\pi(f)\,\psi(x_n)^*\dots\psi(x_1)^*\big)&\\ \cdot\big(\psi(y_1)\dots\psi(y_p)\,\pi(g)\,\psi(z_q)^*\dots\psi(z_1)^*\big)&\\ =\psi(w_1)\dots\psi(w_m)\psi(y_{n+1})\dots\psi(y_p)\,\pi(kg)\,\psi(z_q)^*\dots\psi(z_1)^*.&\end{aligned}$$

**Remark 4.17.** Consider the situation of Proposition 4.16(2) but with $p < n$. Let $x = x_1 \diamond \cdots \diamond x_p$ and $y = y_1 \diamond \cdots \diamond y_p$, and fix $k' \in C_0(E^0)$ such that $(H(x,y)g)\cdot(x_{p+1}\diamond\cdots\diamond x_n) = (x_{p+1}\diamond\cdots\diamond x_n)\cdot k'$. Taking adjoints in Proposition 4.16(2) gives

$$\begin{aligned}\big(\psi(w_1)\dots\psi(w_m)\,\pi(f)\,\psi(x_n)^*\dots\psi(x_1)^*\big)&\\ \cdot\big(\psi(y_1)\dots\psi(y_p)\,\pi(g)\,\psi(z_q)^*\dots\psi(z_1)^*\big)&\\ =\psi(w_1)\dots\psi(w_m)\,\pi(fk')\,\psi(x_n)^*\cdots\psi(x_{p+1})\psi(z_q)^*\dots\psi(z_1)^*.&\end{aligned}$$

**Proof of Proposition 4.16.** (1) By Proposition 4.4, $(\widetilde{\psi},\pi)$ is a Toeplitz representation of $X(E)$, where $\widetilde{\psi}$ is the unique extension of $\psi$ to $X(E)$. The argument of [6, Proposition 2.7] shows that $C^*(\psi,\pi)$ is densely spanned by elements of the form $\psi(x_1)\cdots\psi(x_n)\,\pi(f)\,\psi(y_m)^*\cdots\psi(y_1)^*$ where each $x_i, y_i \in C_c(E_1)$ and $f \in C_c(E^0)$. Fix $x_1$, $x_2 \in C_c(E^1)$ with

$$s(\mathrm{osupp}(x_1)) \cap r(\mathrm{osupp}(x_2)) = \emptyset.$$

Then

$$\begin{aligned}\|\psi(x_1)\psi(x_2)\|^2 &= \|\psi(x_2)^*\psi(x_1)^*\psi(x_1)\psi(x_2)\|\\ &= \|\psi(x_2)^*\pi(\langle x_1,x_1\rangle_{C_0(E^0)})\psi(x_2)\|\\ &= \|\psi(x_2)^*\psi(\langle x_1,x_1\rangle_{C_0(E^0)}\cdot x_2)\| = 0.\end{aligned}$$

Similarly, $\psi(x_1)\psi(x_2)^* = 0$ whenever $s(\mathrm{osupp}(x_1)) \cap s(\mathrm{osupp}(x_2)) = \emptyset$. So elements of the form

$$\psi(x_1)\dots\psi(x_m)\pi(f)\psi(y_m)^*\dots\psi(y_1)^*$$

where

- $f \in C_c(E^0)$,
- each $s(\mathrm{osupp}(x_i)) \cap r(\mathrm{osupp}(x_{i+1})) \neq \emptyset$,
- each $s(\mathrm{osupp}(y_i)) \cap r(\mathrm{osupp}(y_{i+1})) \neq \emptyset$, and
- $s(\mathrm{osupp}(x_n)) \cap s(\mathrm{osupp}(y_m)) \cap \mathrm{osupp}(f) \neq \emptyset$

span a dense subspace of $C^*(\psi,\pi)$. Since Proposition 4.12 implies that $X_0 = \operatorname{span}\mathcal{F}$ is dense in $X(E)$ and hence in $C_c(E^1)$, the first assertion follows.

(2) Lemma 4.14 implies that

$$\pi(f)\psi(x_n)^*\dots\psi(x_1)^*\psi(y_1)\dots\psi(y_p) = \pi(fH(x,y))\psi(y_{n+1})\dots\psi(y_p).$$

The result now follows from Lemma 4.15. □

**Example 4.18.** Let $E$ be a directed graph regarded as a topological graph under the discrete topology. Let $\mathcal{G} = \{\delta_v : v \in E^0\}$ and $\mathcal{F} = \{\delta_e : e \in E^1\}$. Then $\mathcal{G}$ and $\mathcal{F}$ satisfy the hypotheses of Proposition 4.12, so we recover as an immediate consequence the isomorphism of the Toeplitz algebra $\mathcal{T}(E)$ of the graph bimodule (see [4] and [5]) with the Toeplitz algebra $\mathcal{T}C^*(E)$ of the graph $E$. The usual spanning family and multiplication rule for $\mathcal{T}C^*(E)$ follows from Proposition 4.16 applied to the same $\mathcal{F}$.

**Remark 4.19.** The multiplication formula of Proposition 4.16(2) has the drawback that the element $k$ has no explicit formula in terms of the $x_i$ the $y_i$ and the function $f$; it is obtained by an application of the Tietze extension theorem (see Lemma 4.15). However, in practise there will frequently be a natural choice for $k$. Suppose, for example, that $E^1$ is totally disconnected. Then $\mathcal{F}$ can be taken to consist of characteristic functions of compact open $s$-sections. We can then take $k$ to be the function that is identically zero off $s(\operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ and satisfies $k(s(\mu)) = f(r(\mu))H(x, y)(r(\mu))$ whenever $\mu \in \operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p)$; this $k$ is continuous because $s(\operatorname{supp}(y_{n+1} \diamond \cdots \diamond y_p))$ is clopen.

## 5. The topological graph arising from the shift map on the infinite-path space

In this section we discuss how our results apply to the topological graph $\widehat{E}$ arising from the shift map on the infinite-path space $E^\infty$ of a row-finite directed graph $E$ with no sources. It is known that $\mathcal{O}_{X(\widehat{E})}$ is isomorphic to $C^*(E)$ (it could be recovered from [2, 3]) but existing proofs use the universal property of $C^*(E)$ to induce a homomorphism from $C^*(E)$ to $\mathcal{O}_{X(\widehat{E})}$, invoke the gauge-invariant uniqueness theorem for $C^*(E)$ to establish injectivity, and then argue surjectivity by hand. It takes some work to show using the universal property of $\mathcal{O}_{X(\widehat{E})}$ that there is a homomorphism going in the other way.

Let $E$ be a row-finite directed graph with no sources. Let $E^* = \bigcup_{n \geq 0} E^n$ and let $E^\infty = \big\{z \in \prod_{i=1}^\infty E^1 : s(z_i) = r(z_{i+1}), \text{ for all } i = 1, 2, \dots\big\}$. We view $E^\infty$ as a topological space under the subspace topology coming from the ambient space $\prod_{i=1}^\infty E^1$. For any $\mu \in E^* \setminus E^0$, we define the cylinder set $Z(\mu) = \{z \in E^\infty : z_1 = \mu_1, \dots, z_{|\mu|} = \mu_{|\mu|}\}$. For $v \in E^0$ we define $Z(v) = \{z \in E^\infty : r(z_1) = v\}$. Since $E$ has no sources, each $Z(\mu)$ is nonempty. The space $E^\infty$ is a locally compact Hausdorff space with a base of compact open sets $\{Z(\mu) : \mu \in E^*\}$ ([8, Corollary 2.2]).

We construct a topological graph $\widehat{E} = (\widehat{E}^0, \widehat{E}^1, \widehat{r}, \widehat{s})$. Let $\widehat{E}^0 = \widehat{E}^1 = E^\infty$. Define $\widehat{r}$ to be the identity map, and define $\widehat{s}(z) = (z_2, z_3, \dots)$ for all $z \in \widehat{E}^1$. Since $Z(\mu)$ is a compact open $\widehat{s}$-section whenever $\mu \notin E^0$, the map $\widehat{s}$ is a local homeomorphism and hence $\widehat{E}$ is a topological graph.

For the following result recall that a Cuntz–Krieger $E$-family is a family $\{p_v, s_e\}$ satisfying the relations described in the first paragraph of the introduction: that is, the numbered relations (1) and (2) and the additional relation $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever $0 < |r^{-1}(v)| < \infty$.

**Proposition 5.1.** *Let $E$ be a row-finite directed graph with no sources, and let $\widehat{E}$ be the topological graph described above.*

1. *Let $(\psi, \pi)$ be a covariant Toeplitz representation of $\widehat{E}$. For $v \in E^0$ define $q_v := \pi(\chi_{Z(v)})$ and for $e \in E^1$ define $t_e := \psi(\chi_{Z(e)})$. Then the $q_v$ and the $t_e$ form a Cuntz–Krieger $E$-family.*
2. *Let $\{q_v : v \in E^0\}$, $\{t_e : e \in E^1\}$ be a Cuntz–Krieger $E$-family in a $C^*$-algebra $B$. Then there is a unique covariant Toeplitz representation $(\psi, \pi)$ of $\widehat{E}$ such that $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_\mu^*$, and $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ for all $e \in E^1$, $\mu \in E^*$.*
3. *Let $(j_1, j_0)$ be the universal covariant Toeplitz representation of $\widehat{E}$ in $\mathcal{O}(\widehat{E})$, and let $\{p_v, s_e : v \in E^0, e \in E^1\}$ be the Cuntz–Krieger $E$-family generating $C^*(E)$. There is an isomorphism $\mathcal{O}(\widehat{E}) \cong C^*(E)$ which carries each $j_1(\chi_{Z(e\mu)})$ to $s_{e\mu} s_\mu^*$, and each $j_0(\chi_{Z(\mu)})$ to $s_\mu s_\mu^*$.*

**Proof of Proposition 5.1(1).** The $q_v$ are mutually orthogonal projections because the $\chi_{Z(v)}$ are. For $\mu \in E^* \setminus E^0$, the set $Z(\mu)$ is a compact open $s$-section. Thus for $e \in E^1$, relation (2) of Definition 2.2 implies that $t_e^* t_e = \pi(\widehat{\chi_{Z(e)}}) = \pi(\chi_{Z(s(e))}) = q_{s(e)}$. For $\mu \in E^* \setminus E^0$, $(\{Z(\mu)\}, Z(\mu))$ is an open local $\widehat{r}$-fibration. Since $\operatorname{supp}(\chi_{Z(\mu)}) = Z(\mu)$ and $(\psi, \pi)$ is covariant, Corollary 2.15 implies that

$$\pi(\chi_{Z(\mu)}) = \psi\Big(\sqrt{r^*_{Z(\mu)} \chi_{Z(\mu)}}\Big)\psi\Big(\sqrt{r^*_{Z(\mu)} \chi_{Z(\mu)}}\Big)^* = \psi(\chi_{Z(\mu)})\psi(\chi_{Z(\mu)})^*. \tag{5.1}$$

So for $v \in E^0$, we have

$$q_v = \pi(\chi_{Z(v)}) = \sum_{r(e)=v} \pi(\chi_{Z(e)}) = \sum_{r(e)=v} \psi(\chi_{Z(e)})\psi(\chi_{Z(e)})^* = \sum_{r(e)=v} t_e t_e^*. \quad \square$$

**Proof of Proposition 5.1(2).** Let $\mathcal{G} := \{\chi_{Z(\mu)} : \mu \in E^* \setminus E^0\} \subseteq C_0(\widehat{E}^0)$. Then $\operatorname{span} \mathcal{G}$ is a dense $*$-subalgebra of $C_0(\widehat{E}^0)$. We aim to define a map $\pi_0 : \operatorname{span} \mathcal{G} \to B$ by $\pi_0(\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)}) = \sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^*$. We check that $\pi_0$ is well-defined. It suffices to prove that $\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)} = 0$ implies $\sum_{i=1}^n \alpha_i t_{\mu_i} t_{\mu_i}^* = 0$, where the $\mu_i$ are distinct. Since $E$ is row-finite and has no sources,

$$\begin{aligned}\pi_0\Bigg(\sum_{e \in r^{-1}(s(\mu))} \chi_{Z(\mu e)}\Bigg) &= \sum_{e \in r^{-1}(s(\mu))} t_{\mu e} t_{\mu e}^* \\ &= t_\mu \Bigg(\sum_{e \in r^{-1}(s(\mu))} t_e t_e^*\Bigg) t_\mu^* = t_\mu t_\mu^* = \pi_0(\chi_{Z(\mu)}),\end{aligned}$$

so we can assume that the $\mu_i$ have the same length. It follows that the $\chi_{Z(\mu_i)}$ are mutually orthogonal nonzero projections and hence each $\alpha_i = 0$. So $\pi_0$ is well-defined. It is obvious that $\pi_0$ is a linear map preserving the involution. [10, Corollary 1.15] implies that $\pi_0$ is a homomorphism. Now we show that $\pi_0$ is norm decreasing. Fix $\mu_1, \ldots, \mu_n$. We can assume that the $\mu_i$ are distinct and have the same length. Then

$$\begin{aligned}\left\|\pi_0\left(\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)}\right)\right\|^2 &= \left\|\sum_{i=1}^n |\alpha_i|^2 t_{\mu_i} t_{\mu_i}^*\right\| \\ &\le (\max_i |\alpha_i|^2)\left\|\sum_{i=1}^n t_{\mu_i} t_{\mu_i}^*\right\| \\ &\le \max_i |\alpha_i|^2 = \left\|\sum_{i=1}^n \alpha_i \chi_{Z(\mu_i)}\right\|^2,\end{aligned}$$

so $\pi_0$ is norm decreasing. Thus we obtain a unique homomorphism

$$\pi : C_0(\widehat{E}^0) \to C^*(E)$$

by $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$ for all $\mu \in E^*$.

We next aim to define a linear map $\psi : C_c(\widehat{E}^1) \to B$ by extension of the formula $\psi(\chi_{Z(e\mu)}) = t_{e\mu} t_\mu^*$, and to show that the pair $(\psi, \pi)$ is a Toeplitz representation of $\widehat{E}$. To do so, we will apply Proposition 4.12, so we need to set up the rest of the elements of the statement. Let

$$\mathcal{B} := \{Z(\mu) : \mu \in E^* \setminus E^0\}$$

and let $\mathcal{F} := \{\chi_{Z(e\mu)} : e \in E^1, \mu \in E^*\} \subseteq C_c(E^1)$. Certainly $\mathcal{F}$ and $\mathcal{B}$ satisfy Equation (4.3). Similarly to the construction of $\pi_0$, there is a well-defined linear map $\psi_0 : \operatorname{span} \mathcal{F} \to B$ satisfying

$$\psi_0\left(\sum_{i=1}^n \alpha_i \chi_{Z(e_i\mu_i)}\right) = \sum_{i=1}^n \alpha_i t_{e_i\mu_i} t_{\mu_i}^*.$$

Fix $x = \chi_{Z(e\mu)}$ and $y = \chi_{Z(f\nu)}$ in $\mathcal{F}$. We verify Equation (4.4). For this, observe that

$$\widehat{\sqrt{xy}} = \begin{cases} \chi_{Z(\mu)} & \text{if } e\mu = f\nu\mu' \\ \chi_{Z(\nu)} & \text{if } f\nu = e\mu\nu' \\ 0 & \text{otherwise.}\end{cases}$$

Then calculate:

$$(t_{e\mu} t_\mu^*)^* t_{f\nu} t_\nu^* = t_\mu t_{e\mu}^* t_{f\nu} t_\nu^* = \begin{cases} t_\nu t_\nu^* & \text{if } f\nu = e\mu\nu' \\ t_\mu t_\mu^* & \text{if } e\mu = f\nu\mu' \\ 0 & \text{otherwise.}\end{cases}$$

This establishes Equation (4.4). So Proposition 4.12 shows that $\psi_0$ extends uniquely to a linear map $\psi : C_c(\widehat{E}^1) \to B$. A similar calculation establishes

Equation (4.5). Proposition 4.12 implies that $(\psi, \pi)$ is a Toeplitz representation of $E$.

It remains to check covariance. Since the range map is a homeomorphism onto $\widehat{E}^0$ we can apply Corollary 2.15 with $\mathcal{G}$ as in the proof of part (2) and the local $\widehat{r}$-fibrations $\big\{(\{Z(\mu)\}, Z(\mu)) : \mu \in E^* \setminus E^0\big\}$ to see that $(\psi, \pi)$ is covariant. □

**Proof of Proposition 5.1(3).** We show $C^*(E)$ has the universal property of $\mathcal{O}(\widehat{E})$ and then invoke Theorem 2.13. Proposition 5.1(2) yields a covariant Toeplitz representation $(\theta_1, \theta_0)$ of $\widehat{E}$ in $C^*(E)$ such that

$$\theta_1(\chi_{Z(e\mu)}) = s_{e\mu}s_\mu^* \qquad \text{for all } e \in E^1,\ \mu \in E^*,$$
$$\theta_0(\chi_{Z(\mu)}) = s_\mu s_\mu^* \qquad \text{for all } \mu \in E^*.$$

Fix a covariant Toeplitz representation $(\psi, \pi)$ of $\widehat{E}$ in a $C^*$-algebra $B$. Then Proposition 5.1(1) gives a Cuntz–Krieger $E$-family

$$\{q_v := \pi(\chi_{Z(v)}), t_e := \psi(\chi_{Z(e)}) : v \in E^0, e \in E^1\}$$

in $B$. So [10, Proposition 1.21] gives a homomorphism $\rho : C^*(E) \to B$ such that $\rho(p_v) = q_v$, and $\rho(s_e) = t_e$. An induction on the length of $\mu$ using Equation (5.1) shows that $\pi(\chi_{Z(\mu)}) = t_\mu t_\mu^*$. For $e \in E^1$, and $\mu \in E^*$, we have $\rho\circ\theta_1(\chi_{Z(e\mu)}) = t_e t_\mu t_\mu^* = \psi(\chi_{Z(e)})\pi(\chi_{Z(\mu)}) = \psi(\chi_{Z(e\mu)})$. For $\mu \in E^* \setminus E^0$, we have $\rho\circ\theta_0(\chi_{Z(\mu)}) = t_\mu t_\mu^* = \pi(\chi_{Z(\mu)})$. Hence $\rho\circ\theta_1 = \psi$, and $\rho\circ\theta_0 = \pi$. Since the image of $(\theta_1, \theta_0)$ generates $C^*(E)$, Theorem 2.13 implies that there is an isomorphism $\mathcal{O}(E) \cong C^*(E)$ which carries each $j_1(\chi_{Z(e\mu)})$ to $s_{e\mu}s_\mu^*$, and each $j_0(\chi_{Z(\mu)})$ to $s_\mu s_\mu^*$. □

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