

A distributional approach to Feynman’s operational calculus

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ABSTRACT. In this paper we will construct an operator-valued distribution that will extend Feynman’s operational calculus in the setting of Jefferies and Johnson, 2001–2003, and Johnson–Lapidus–Nielsen, 2014, from the disentangling of holomorphic functions of several variables to the disentangling of Schwartz functions on \mathbb{R}^n . It will be shown that the disentangled operator corresponding to a Schwartz function (i.e., the disentangling of a Schwartz function) can be realized as the limit of a sequence of operator-valued distributions of compact support in a ball of a certain radius centered at $0 \in \mathbb{R}^n$. In this way, we can extend the operational calculi to the Schwartz space.

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1. Introduction

The primary purpose of this paper is to use distributional methods to enable Feynman’s operational calculus to be applied to elements of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of tempered functions. As the reader will see, the distributional approach developed below that facilitates the use of Schwartz functions in the operational calculus will not lend itself to simple computation of the “disentangled operator”. On the other hand, we will see that the “disentangled operator” corresponding to a Schwartz function will be defined using a limit of a sequence of “disentangled operators” that result from distributions with compact support. Theorems 4.2 and 4.3 take care

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of the “heavy lifting” required to define the operational calculus on the Schwartz class. With these Theorems in hand, it is Definition 4.5 which enables our extension of the operational calculus to the Schwartz space of tempered functions. Indeed, Definition 4.5 is the ultimate goal of this paper. Following Definition 4.5, there is a brief discussion concerning the approach taken to obtain this definition, as well as the difficulties of using this definition in a practical way. We will also comment on the relation between the present paper and other work that has been done on determining an extension of the operational calculus to functions other than analytic functions of several variables.

It may be helpful to present, at this time, some background discussion on the operational calculus. Feynman’s operational calculus originated with the 1951 paper [2] and concerns itself with the formation of functions of noncommuting operators. Indeed, even with functions as simple as $f(x, y) = xy$ it is not clear how to define $f(A, B)$ if A and B do not commute — does one let $f(A, B) = AB$, $f(A, B) = BA$, $f(A, B) = \frac{1}{2}AB + \frac{1}{2}BA$, or some other expression involving sums of products of A and B ? One has to decide, usually with a particular problem in mind, how to form a given function of noncommuting operators. One approach to this problem (the approach followed in this paper) was developed by Jefferies and Johnson in the papers [6, 7, 8, 9]. This approach is expanded on in the papers [10, 11, 12, 13, 14, 17], and others. It is important to note that, in the setting of the original Jefferies–Johnson approach, measures on intervals $[0, T]$ are used to determine when a given operator will act in products. Furthermore, the measures used in the original papers are continuous measures. However, Johnson and the current author extended the operational calculus to measures with both continuous and discrete parts in the aforementioned paper [17].

The discussion just above begs the question of how measures can be used to determine the order of operators in products. Feynman’s heuristic rules for the formation of functions of noncommuting operators give us a starting point.

- (1) Attach time indices to the operators to specify the order of operators in products.
- (2) With time indices attached, form functions of these operators by treating them as though they were commuting.
- (3) Finally, “disentangle” the resulting expressions; i.e., restore the conventional ordering of the operators.

As is well known, the most difficult problem with the operational calculus is the disentangling process. Indeed in his 1951 paper, [2], Feynman points out that “The process is not always easy to perform and, in fact, is the central problem of this operator calculus.”

We first address rule (1) above. It is in the use of this rule that we will see measures used to track the action of operators in products. First, it may be that the operators involved may come with time indices naturally attached.

For example, we might have operators of multiplication by time dependent potentials. However, it is also commonly the case that the operators used are independent of time. Given such an operator A , we can (as Feynman most often did) attach time indices according to Lebesgue measure as follows:

$$A = \frac{1}{t} \int_0^t A(s) ds$$

where $A(s) := A$ for $0 \leq s \leq t$. This way of attaching time indices does appear a bit artificial but turns out to be extremely useful in many situations. We also note that mathematical or physical considerations may dictate that one use a measure different from Lebesgue measure. For example, if μ is a probability measure on the interval $[0, T]$, and if A is a linear operator, we can write

$$A = \int A(s) \mu(ds)$$

where once again $A(s) := A$ for $0 \leq s \leq T$. When we write A in this fashion, we are able to use the time variable to keep track of when the operator A acts. Indeed, if we have two operators A and B , consider the product $A(s)B(t)$ (here, time indices have been attached). If $t < s$, then we have $A(s)B(t) = AB$ since here we want B to act first (on the right). If, on the other hand, $s < t$, then $A(s)B(t) = BA$ since A has the earlier time index. In other words, the operator with the smaller (or earlier) time index, acts to the right of (or before) an operator with a larger (or later) time index. (It needs to be kept in mind that these equalities are heuristic in nature.) For a much more detailed discussion of using measures to attach time indices, see Chapter 14 of the book [15] as well as Chapters 2, 7 and 9 of the forthcoming book [16] and the references contained in these books.

Concerning the rules (2) and (3) above, we mention that, once we have attached time indices to the operators involved, we calculate functions of the noncommuting operators as if they actually do commute. These calculations are, of course, heuristic in nature but the idea is that with time indices attached, one carries out the necessary calculations giving no thought to the operator ordering problem; the time indices enable us to restore the desired ordering of the operators once the calculations are finished; this is the *disentangling* process and is typically the most difficult part of any given problem. While we will not go into detail concerning how to form functions of several, noncommuting, operators, we will record in Section 2 the essential notation and results concerning the disentangling process as it is done in the Jefferies–Johnson approach to the operational calculus. (For a thorough discussion of the operational calculus, we refer the interested reader to the book [15] and the forthcoming volume [16].)

Section 3 contains the necessary definitions and results concerning Fourier analysis and distribution theory that will be needed in Section 4. We closely follow the notation and definitions found in book [3] for much of this material. Also contained in Section 3 is the disentangled exponential function

that will play a crucial role in this paper. Furthermore, Section 3 contains a norm estimate for the disentangled exponential function that will be needed in Section 4.

The fourth section of this paper contains the main theorems, Theorems 4.2 and 4.3, that enable us to extend the domain of the operational calculus to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Along with the Fourier transform, we will make use of the disentangling of the exponential function

$$\exp(z_1 + \cdots + z_n),$$

an entire function of n complex variables. It is here that the Jefferies–Johnson formalism enters in to our discussion. With the disentangled exponential in hand, we choose a smooth ϕ from the appropriate function space and define an entire function F_ϕ that will, via the Paley–Wiener theorem for distributions, enable us to obtain a distribution T_ϕ of compact support acting on the Schwartz space of tempered functions. We will be able to associate the action of T_ϕ to an explicit disentangling series. Finally, using the ideas of approximate identities, we will be able to define the disentangling of a tempered function as a limit of a sequence in $\mathcal{L}(X)$, using the distributions T_ϕ . As mentioned previously, this is the main goal of the current paper.

Finally, Section 5 contains a brief discussion of the approach taken in this paper and its relation to [12] and [5].

2. Definitions and notation

We start with a brief outline of the operational calculus as developed in [6, 7, 8, 9] for the time independent setting and in [11] for the time-dependent setting. Both approaches are developed with considerably more detail in the forthcoming book [16].

Definition 2.1. Given n nonnegative real numbers (in practice, these numbers are typically strictly positive) r_1, \dots, r_n , we define $\mathbb{A}(r_1, \dots, r_n)$ to be the family of functions $f(z_1, \dots, z_n)$ of n complex variables which are analytic on the open polydisk

$$\{(z_1, \dots, z_n) : |z_j| < r_j, j = 1, \dots, n\}$$

and continuous on its boundary

$$\{(z_1, \dots, z_n) : |z_j| = r_j, j = 1, \dots, n\}.$$

Given any $f \in \mathbb{A}(r_1, \dots, r_n)$, we can write its Taylor series centered at the origin in \mathbb{C}^n as

$$(2.1) \quad f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n},$$

where

$$a_{m_1, \dots, m_n} = \frac{1}{m_1! \cdots m_n!} \partial z_1^{m_1} \cdots \partial z_n^{m_n} f(0, \dots, 0)$$

and the series converges absolutely on the closed polydisk

$$\{(z_1, \dots, z_n) : |z_j| \leq r_j, j = 1, \dots, n\}.$$

We define a norm on $\mathbb{A}(r_1, \dots, r_n)$ by

$$\|f\|_{\mathbb{A}} := \sum_{m_1, \dots, m_n=0}^{\infty} |a_{m_1, \dots, m_n}| r_1^{m_1} \cdots r_n^{m_n}.$$

The proof that $\|\cdot\|_{\mathbb{A}}$ is indeed a norm can be found in [6] and in [16]. With this norm, and via point-wise operations, $\mathbb{A}(r_1, \dots, r_n)$ becomes a commutative Banach algebra (see [6], [16]). We will refer to this algebra below as \mathbb{A} .

Remark 2.2. Clearly, $\mathbb{A}(r_1, \dots, r_n)$ consists of functions analytic on the open polydisk with radii r_1, \dots, r_n and which are continuous on its boundary. Moreover, with the norm $\|\cdot\|_{\mathbb{A}}$, $\mathbb{A}(r_1, \dots, r_n)$ is a weighted ℓ^1 -space.

Next, given a Banach space X , we take the maps

$$A_j : [0, T] \rightarrow \mathcal{L}(X),$$

$j = 1, \dots, n$, to be measurable in the sense that $A_j^{-1}(E)$ is a Borel set in $[0, T]$ for all strongly open $E \subseteq \mathcal{L}(X)$. (For the definition of the strong operator topology, see, for example, page 182 of [18, Vol. 1].) For each $j = 1, \dots, n$, we associate to $A_j(\cdot)$ a Borel probability measure μ_j on $[0, T]$. As mentioned in the introduction, we will refer to μ_j as the *time-ordering measure* associated to $A_j(\cdot)$. We will assume that

$$\int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) < \infty,$$

for $j = 1, \dots, n$.

Remark 2.3. We are *not* assuming that our time-ordering measures are continuous (recall that a measure μ on a measurable space Ω is continuous if $\mu(\{x\}) = 0$ for all singleton sets $\{x\} \subset \Omega$). Continuous time-ordering measures are used [6, 7, 8, 9] and in much of the subsequent work on this approach to the operational calculus. However, the operational calculus in the presence of time-ordering measures with nonzero discrete parts has been developed in [17] and, more exhaustively in [16]. If a given time-ordering measure μ_j has a nonzero discrete part, we will assume that the support of the discrete part is finite (see [17], [16]). The presence of a nontrivial discrete part will not affect the results contained in this paper. Indeed, it is a strength of the approach taken in this paper that the presence of a discrete part in any (or all) of the time-ordering measure(s) will not affect this papers' results.

Given the operator-measure pairs $(A_j(\cdot), \mu_j)$, $j = 1, \dots, n$, we let

$$r_j := \int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds).$$

We use the numbers r_1, \dots, r_n to construct the commutative Banach algebra \mathbb{A} . We can now define the *disentangling algebra*. It is this commutative Banach algebra that supplies the “commutative world” in which Feynman’s rules can be applied in a mathematically rigorous fashion.

Definition 2.4. The *disentangling algebra* $\mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ associated to the algebra $\mathbb{A}(r_1, \dots, r_n)$ is defined as follows. Discard all operator-theoretic aspects (and time-dependence, if present) of the $\mathcal{L}(X)$ -valued functions $A_j(\cdot)$, $j = 1, \dots, n$, keeping only the associated real number r_j . We obtain *formal commuting objects* $\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)$. To define the disentangling algebra $\mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$, given $f \in \mathbb{A}(r_1, \dots, r_n)$, we replace the complex variables z_1, \dots, z_n with the objects $\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)$, obtaining expressions (obtained using the Taylor series expansion (2.1))

$$(2.2) \quad f\left(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)\right) = \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1, \dots, m_n} \left(\tilde{A}_1(\cdot)\right)^{m_1} \cdots \left(\tilde{A}_n(\cdot)\right)^{m_n}.$$

The disentangling algebra $\mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ is the collection of all such expressions for which

$$\|f\|_{\mathbb{D}} := \sum_{m_1, \dots, m_n=0}^{\infty} |a_{m_1, \dots, m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty.$$

$(\mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)), \|\cdot\|_{\mathbb{D}})$ is a commutative Banach algebra via pointwise operations (see [6], [16]). We will, below, refer to this algebra as \mathbb{D} .

Remark 2.5. It is shown in [6] and also in [16] that \mathbb{A} and \mathbb{D} are isometrically isomorphic.

With the algebra \mathbb{D} in hand, we can use Feynman’s rules to carry out the time-ordering calculations necessary for computing the disentangled version of a function $f \in \mathbb{D}$; we map the end result of these calculations into $\mathcal{L}(X)$ via the *disentangling map* $\mathcal{T}_{\mu_1, \dots, \mu_n} : \mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \rightarrow \mathcal{L}(X)$. It turns out that the essential ingredient in the definition of the disentangling map is the computation of the disentangling of the monomial

$$P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) := \left(\tilde{A}_1(\cdot)\right)^{m_1} \cdots \left(\tilde{A}_n(\cdot)\right)^{m_n},$$

where $m_1, \dots, m_n \in \mathbb{N} \cup \{0\}$. Before recording the disentangling of the monomial, some notation is necessary. First, given $m \in \mathbb{N}$, we let S_m be the group of permutations on m objects. For $m \in \mathbb{N}$ and $\pi \in S_m$, we let

$$(2.3) \quad \Delta_m(\pi) := \{(s_1, \dots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < T\}.$$

To accommodate the use of time-ordering measures with nontrivial (finitely supported) discrete parts, we need to modify (2.3) as follows. Let $m \in \mathbb{N}$ and suppose that $\tau_1, \dots, \tau_h \in (0, T)$ are such that $\tau_1 < \tau_2 < \cdots < \tau_h$. Choose

nonnegative integers $\theta_1, \dots, \theta_{h+1} \in \mathbb{N} \cup \{0\}$ such that $\theta_1 + \dots + \theta_{h+1} = m$ and define, for $\pi \in S_m$,

$$(2.4) \quad \Delta_{m;\theta_1, \dots, \theta_{h+1}}(\pi) := \left\{ (s_1, \dots, s_m) \in [0, T]^m : \begin{aligned} &0 < s_{\pi(1)} < \dots < s_{\pi(\theta_1)} \\ &< \tau_1 < s_{\pi(\theta_1+1)} < \dots < s_{\pi(\theta_1+\theta_2)} < \tau_2 \\ &< s_{\pi(\theta_1+\theta_2+1)} < \dots < s_{\pi(\theta_1+\dots+\theta_h)} < \tau_h \\ &< s_{\pi(\theta_1+\dots+\theta_{h+1})} < \dots < s_{\pi(m)} < T \end{aligned} \right\}.$$

The reader will note that, given the numbers τ_1, \dots, τ_h , the nonnegative integer θ_1 serves to count the number of time indices that occur before τ_1 , the nonnegative integer θ_2 counts the number of time indices that occur between τ_1 and τ_2 , etc.

Finally, define

$$(2.5) \quad \tilde{C}_i(s) := \begin{cases} \tilde{A}_1(s) & \text{if } i \in \{1, \dots, m_1\}, \\ \tilde{A}_2(s) & \text{if } i \in \{m_1 + 1, \dots, m_1 + m_2\}, \\ \vdots & \\ \tilde{A}_n(s) & \text{if } i \in \{m_1 + \dots + m_{n-1} + 1, \dots, m\} \end{cases}$$

and

$$(2.6) \quad C_i(s) := \begin{cases} A_1(s) & \text{if } i \in \{1, \dots, m_1\}, \\ A_2(s) & \text{if } i \in \{m_1 + 1, \dots, m_1 + m_2\}, \\ \vdots & \\ A_n(s) & \text{if } i \in \{m_1 + \dots + m_{n-1} + 1, \dots, m\}, \end{cases}$$

where $m := m_1 + \dots + m_n$. The tilded objects $\tilde{C}_i(s)$ give the appropriate formal commuting object $\tilde{A}_{(\cdot)}(s)$ (that replace the variables z_1, \dots, z_n), depending on the block $\{1, \dots, m_1\}, \dots, \{m_1 + \dots + m_{n-1} + 1, \dots, m\}$ to which the index i belongs. The same comment holds regarding the $C_i(s)$; the difference is that the tildes are erased, turning the tilded objects into the $\mathcal{L}(X)$ -valued functions.

With the notation introduced above, we are now ready to record the disentangling (in the disentangling algebra) of the monomial

$$P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) = (\tilde{A}_1(\cdot))^{m_1} \dots (\tilde{A}_n(\cdot))^{m_n}.$$

We will state the result for continuous measures and measures with finitely supported discrete parts separately. For complete details concerning how the following proposition is arrived at, see [6], [17], and especially [16].

Proposition 2.6.

- (1) *In the disentangling algebra \mathbb{D} , when the time-ordering measures μ_1, \dots, μ_n are continuous, the monomial $P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ can be written as*

(2.7)

$$P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) = \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \tilde{C}_{\pi(m)}(s_{\pi(m)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)}) \cdot (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \dots, ds_m).$$

(2) When each of the time-ordering measures μ_1, \dots, μ_n has a finitely supported discrete part, we write

$$\mu_j = \lambda_j + \eta_j,$$

where λ_j is continuous and η_j is purely discrete and finitely supported for $j = 1, \dots, n$. Let $\{\tau_1, \dots, \tau_h\}$, $0 < \tau_1 < \cdots < \tau_h < T$, be the union of the supports of η_1, \dots, η_n . Write $\eta_j = \sum_{i=1}^h p_{ji} \delta_{\tau_i}$, where $\sum_{i=1}^h p_{ji} = 1$ for $j = 1, \dots, n$. In the disentangling algebra \mathbb{D} , the monomial $P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ can be written as

(2.8) $P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$

$$\begin{aligned} &= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! \cdots q_{n1}! q_{n2}!} \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \\ &\quad \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!} \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \\ &\quad \sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+\cdots+q_{n1}; \theta_1, \dots, \theta_{h+1}}(\pi)} \tilde{C}_{\pi(q_{11}+\cdots+q_{n1})}(s_{\pi(q_{11}+\cdots+q_{n1})}) \\ &\quad \cdots \tilde{C}_{\pi(\theta_1+\cdots+\theta_{h+1})}(s_{\pi(\theta_1+\cdots+\theta_{h+1})}) \left[\prod_{\alpha=0}^{n-1} \left(p_{n-\alpha, h} \tilde{A}_{n-\alpha}(\tau_h) \right)^{j_{n-\alpha, h}} \right] \\ &\quad \tilde{C}_{\pi(\theta_1+\cdots+\theta_h)}(s_{\pi(\theta_1+\cdots+\theta_h)}) \cdots \tilde{C}_{\pi(\theta_1+1)}(s_{\pi(\theta_1+1)}) \\ &\quad \left[\prod_{\alpha=0}^{n-1} \left(p_{n-\alpha, 1} \tilde{A}_{n-\alpha}(\tau_1) \right)^{j_{n-\alpha, 1}} \right] \tilde{C}_{\pi(\theta_1)}(s_{\pi(\theta_1)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)}) \\ &\quad (\lambda_1^{q_{11}} \times \cdots \times \lambda_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}). \end{aligned}$$

The complexity seen in part (2) of the proposition above arises from the presence of the discrete measures η_1, \dots, η_n . See [17] or [16] for a complete discussion and derivation of the time-ordering of the monomial in the presence of time-ordering measures with nonzero discrete parts.

We now define the disentangling map that takes the time-ordered element of \mathbb{D} to an operator in $\mathcal{L}(X)$.

Definition 2.7.

(1) When the time-ordering measures μ_1, \dots, μ_n associated to the $\mathcal{L}(X)$ -valued functions $A_1(\cdot), \dots, A_n(\cdot)$, respectively, are continuous we define the image of the monomial $P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ under the

disentangling map $\mathcal{T}_{\mu_1, \dots, \mu_n} : \mathbb{D} \rightarrow \mathcal{L}(X)$ by

$$(2.9) \quad \begin{aligned} &\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \\ &:= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\ &\quad (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \dots, ds_m) \end{aligned}$$

where $C_i(\cdot)$ is defined by (2.6).

- (2) When the time-ordering measures μ_1, \dots, μ_n associated to the $\mathcal{L}(X)$ -valued functions $A_1(\cdot), \dots, A_n(\cdot)$, respectively, have nontrivial discrete parts, we define the image of the monomial

$$P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$$

under the disentangling map $\mathcal{T}_{\mu_1, \dots, \mu_n} : \mathbb{D} \rightarrow \mathcal{L}(X)$ as follows. For each $j = 1, \dots, n$, write $\mu_j = \lambda_j + \eta_j$ where η_j is purely discrete of finite support and is written as in Proposition 2.6. Define

$$(2.10) \quad \begin{aligned} &\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \\ &:= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \frac{m_1! \cdots m_n!}{q_{11}!q_{12}! \cdots q_{n1}!q_{n2}!} \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \\ &\quad \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!} \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \\ &\quad \sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+\cdots+q_{n1}; \theta_1, \dots, \theta_{h+1}}(\pi)} C_{\pi(q_{11}+\cdots+q_{n1})}(s_{\pi(q_{11}+\cdots+q_{n1})}) \\ &\quad \cdots C_{\pi(\theta_1+\cdots+\theta_{h+1})}(s_{\pi(\theta_1+\cdots+\theta_{h+1})}) \left[\prod_{\alpha=0}^{n-1} (p_{n-\alpha, h} A_{n-\alpha}(\tau_h))^{j_{n-\alpha, h}} \right] \\ &\quad C_{\pi(\theta_1+\cdots+\theta_h)}(s_{\pi(\theta_1+\cdots+\theta_h)}) \cdots C_{\pi(\theta_1+1)}(s_{\pi(\theta_1+1)}) \\ &\quad \left[\prod_{\alpha=0}^{n-1} (p_{n-\alpha, 1} A_{n-\alpha}(\tau_1))^{j_{n-\alpha, 1}} \right] C_{\pi(\theta_1)}(s_{\pi(\theta_1)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\ &\quad (\lambda_1^{q_{11}} \times \cdots \times \lambda_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}), \end{aligned}$$

where $C_i(\cdot)$ is defined by (2.6).

- (3) Let $f \in \mathbb{D}$. Write the Taylor series for f as in (2.2). Define

$$(2.11) \quad \begin{aligned} &\mathcal{T}_{\mu_1, \dots, \mu_n} f(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \\ &:= \sum_{m_1, \dots, m_n=0}^{\infty} a_{m_1, \dots, m_n} \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)); \end{aligned}$$

i.e., we define the disentangling map for an arbitrary element of \mathbb{D} term-by-term in the Taylor series for f .

- If the time-ordering measures μ_1, \dots, μ_n are continuous, we use part (1) for $\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$.
- If the time-ordering measures have nonzero, finitely supported, discrete parts, we compute $\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ via part (2).

The disentangling map has the following properties.

Theorem 2.8. *The disentangling map $\mathcal{T}_{\mu_1, \dots, \mu_n} : \mathbb{D} \rightarrow \mathcal{L}(X)$ is:*

- (1) *linear;*
- (2) *a norm 1 contraction if $A_j(s) \equiv A_j$ for all $s \in [0, T]$ and $j = 1, \dots, n$, i.e., the operator-valued functions are constant-valued (the time independent setting);*
- (3) *a contraction in the time dependent setting, but not necessarily of norm 1.*

For the proofs of these statements, we refer the reader to [6], [17], [11] and [16].

3. Specifics needed for the distributional approach to the operational calculus

Now that we have a general outline of the ideas of the operational calculus in Jefferies–Johnson setting of the operational calculus [6, 7, 8, 9], we move on to the ideas needed to obtain the main results, Theorems 4.2 and 4.3, of the present paper. As in the previous section, we assume that $A_j : [0, T] \rightarrow \mathcal{L}(X)$, $j = 1, \dots, n$, are measurable in the sense that $A_j^{-1}(E)$ is a Borel set in $[0, T]$ for every strongly open $E \subseteq \mathcal{L}(X)$. Associate to each $A_j(\cdot)$, $j = 1, \dots, n$, a Borel probability measure μ_j ; μ_j may be continuous or it may have a nontrivial finitely supported discrete part. Define real numbers r_1, \dots, r_n by

$$(3.1) \quad r_j := \int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds).$$

We assume that each of these numbers is finite. Construct the commutative Banach algebra $\mathbb{A}(r_1, \dots, r_n)$ and the associated disentangling algebra $\mathbb{D}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$. Now, let

$$f(z_1, \dots, z_n) := \exp(z_1 + \dots + z_n).$$

It is clear that f is an entire function and so is an element of \mathbb{A} and $f(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot))$ is an element of \mathbb{D} . As is well known, we may write

$$f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \dots m_n!} z_1^{m_1} \dots z_n^{m_n}.$$

The disentangling of f is, then

$$\begin{aligned}
 (3.2) \quad & \mathcal{T}_{\mu_1, \dots, \mu_n} f(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \\
 &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \\
 &=: \exp_{\mu_1, \dots, \mu_n} \left(\sum_{j=1}^n \int_{[0, T]} A_j(s) \mu_j(ds) \right).
 \end{aligned}$$

Remark 3.1. If the time-ordering measures μ_1, \dots, μ_n are continuous, we use part (1) of Definition 2.7 to determine the disentangling of f . If the time-ordering measures μ_1, \dots, μ_n have nontrivial, finitely supported, discrete parts, we use part (2) of Definition 2.7 to determine the disentangling of f . However, the particular forms of the disentangling of the exponential will play no explicit role in what follows.

A norm estimate for (3.2) will play a crucial role below. It turns out that norm estimate does not depend on whether or not the time-ordering measures have discrete parts. Indeed, we have

$$\begin{aligned}
 (3.3) \quad & \left\| \exp_{\mu_1, \dots, \mu_n} \left(\sum_{j=1}^n \int_{[0, T]} A_j(s) \mu_j(ds) \right) \right\|_{\mathcal{L}(X)} \\
 & \leq \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} \left\| \mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) \right\|_{\mathcal{L}(X)} \\
 & \leq \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} \left(\int_{[0, T]} \|A_1(s)\|_{\mathcal{L}(X)} \mu_1(ds) \right)^{m_1} \cdots \\
 & \qquad \qquad \qquad \left(\int_{[0, T]} \|A_n(s)\|_{\mathcal{L}(X)} \mu_n(ds) \right)^{m_n} \\
 & = \exp \left(\int_{[0, T]} \|A_1(s)\|_{\mathcal{L}(X)} \mu_1(ds) + \cdots + \int_{[0, T]} \|A_n(s)\|_{\mathcal{L}(X)} \mu_n(ds) \right).
 \end{aligned}$$

This norm estimate is obtained by using the triangle inequality for the first inequality. Then, when computing an estimate for $\|\mathcal{T}_{\mu_1, \dots, \mu_n} P^{m_1, \dots, m_n}\|_{\mathcal{L}(X)}$, we again apply the triangle inequality followed by the standard Banach algebra inequality $\|AB\|_{\mathcal{L}(X)} \leq \|A\|_{\mathcal{L}(X)} \|B\|_{\mathcal{L}(X)}$ to the operator products. (See Equations (2.9) and (2.10) of Definition 2.7.) With the operators enclosed with norms, all of the terms in the integrands become real-valued and so commutative. We then “unravel” the disentangling computations and obtain the second inequality in (3.3). (See [17], [15], [16] for the details of obtaining such inequalities.)

We now define $F_{\text{exp}} : \mathbb{C}^n \rightarrow \mathcal{L}(X)$ by

$$(3.4) \quad F_{\text{exp}}(\xi_1, \dots, \xi_n) := \exp_{\mu_1, \dots, \mu_n} \left(2\pi i \sum_{j=1}^n \xi_j \int_{[0, T]} A_j(s) \mu_j(ds) \right).$$

Clearly, F_{exp} is an entire $\mathcal{L}(X)$ -valued function. From the norm estimate (3.3), it is clear that

$$(3.5) \quad \|F_{\text{exp}}(\xi_1, \dots, \xi_n)\|_{\mathcal{L}(X)} \leq \exp \left(2\pi \sum_{j=1}^n |\xi_j| \int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right).$$

For the reader's convenience, we now move on to sketch out the essential facts that we will need about the Schwartz space of test functions, the Fourier transform and the convolution product. We will follow Chapter 2 of [3].

Definition 3.2. A C^∞ complex-valued function f on \mathbb{R}^n is called a Schwartz function if, for every pair $\alpha, \beta \in \mathbb{N}^n \cup \{0, \dots, 0\}$ of multi-indices, there is a positive constant $C_{\alpha, \beta}$ such that

$$\rho_{\alpha, \beta}(f) := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| = C_{\alpha, \beta} < \infty.$$

The set of all Schwartz functions on \mathbb{R}^n will be denoted by $\mathcal{S}(\mathbb{R}^n)$.

Remark 3.3. In this definition we have used the standard notation concerning multi-indices — $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta := \frac{\partial^{\beta_1 + \cdots + \beta_n}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$, etc.

Definition 3.4. Given $f \in \mathcal{S}(\mathbb{R}^n)$, we define the Fourier transform \hat{f} of f by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) dx,$$

where $\xi \cdot x$ denotes the standard inner product on \mathbb{R}^n .

We will also have use of the convolution.

Definition 3.5. For $f, g \in \mathcal{S}(\mathbb{R}^n)$, define the convolution $f * g$ of f with g by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x - y) dy.$$

Remark 3.6. Of course, if f and g are in $L^1(\mathbb{R}^n)$, the convolution is defined, but, as we will only use functions in the Schwartz space, we state the definition for Schwartz functions.

We will need the following theorem. (See [3, Proposition 2.2.11].)

Theorem 3.7. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ and let $\alpha \in \mathbb{N}^n \cup \{0, \dots, 0\}$. Then:

- (a) $\widehat{(\partial^\alpha f)}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi)$.
- (b) $\widehat{f * g} = \hat{f} \hat{g}$.
- (c) $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

We will have occasion to make use of the following as well. (See [3, Proposition 2.2.7].)

Proposition 3.8. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then fg and $f * g$ are in $\mathcal{S}(\mathbb{R}^n)$. Moreover,*

$$\partial^\alpha(f * g) = (\partial^\alpha f) * g = f * (\partial^\alpha g)$$

for all multi-indices $\alpha \in \mathbb{N}^n \cup \{0, \dots, 0\}$.

We next recall the inverse Fourier transform.

Definition 3.9. Given $f \in \mathcal{S}(\mathbb{R}^n)$, the inverse Fourier transform of f is

$$\check{f}(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

Finally, we will need functions in the Silva space \mathcal{G} of test functions for the so-called Fourier ultra-hyperfunctions of S. Silva and Morimoto–Park. The following characterization of elements of Silva space is found in Theorem 3.4 of [1].

Theorem 3.10. *The Silva space \mathcal{G} consists of all locally integrable functions ϕ on \mathbb{R}^n such that, for any $h > 0$,*

$$(3.6) \quad \sup_{x \in \mathbb{R}^n} |\phi(x)| e^{h\|x\|} < \infty, \text{ and } \sup_{\xi \in \mathbb{R}^n} |\hat{\phi}(\xi)| e^{h\|\xi\|} < \infty.$$

Here, $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^n .

Remark 3.11. Note that if $\phi \in C^\infty(\mathbb{R}^n)$ and if ϕ satisfies (3.6) for all $h > 0$, then certainly $\phi \in \mathcal{S}(\mathbb{R}^n)$ as well as being an element of \mathcal{G} .

It will be prudent to very briefly review some basic facts about operator-valued tempered distributions. An element of $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{L}(X))$ is a tempered $\mathcal{L}(X)$ -valued distribution (or operator-valued distribution). As usual, the Fourier transform \hat{T} of a $\mathcal{L}(X)$ -valued tempered distribution is defined by $\hat{T}(f) := T(\hat{f})$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Similarly, the inverse Fourier transform of T is defined by $\check{T}(g) := T(\check{g})$.

An element T of $\mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ is a distribution with compact support [19, Theorem 24.2]. (The topology of $C^\infty(\mathbb{R}^n)$ is the topology of uniform convergence of functions and their derivatives on compact subsets of \mathbb{R}^n .) For $T \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$, the *support* of T is the complement of the set of all points $x \in \mathbb{R}^n$ for which there exists an open neighborhood V such that $T(f) = 0$ for all smooth functions f supported in V . (Of course, every distribution with compact support also has finite order, but this will play no explicit role in this paper.)

Finally, we will present a vector-valued version of the Paley–Wiener theorem for distributions of compact support. The scalar version of the Paley–Wiener Theorem can be found in [19, Theorem 29.2]. The vector-valued version stated below follows easily from the scalar-valued version. Define $U(r) := \{x \in \mathbb{R}^n : \|x\| \leq r\}$. Given $\xi \in \mathbb{C}^n$, denote the real part of ξ by $\Re \xi$

and denote the imaginary part of ξ by $\Im\xi$. (Of course, $\Re\xi = (\Re\xi_1, \dots, \Re\xi_n)$ and $\Im\xi = (\Im\xi_1, \dots, \Im\xi_n)$, where $\xi = (\xi_1, \dots, \xi_n)$.)

Theorem 3.12 (Paley–Wiener Theorem). *Let X be a Banach space and let $T \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), X)$ be a tempered distribution. Then there exists an $r \geq 0$ such that T has compact support contained in the ball $U(r)$ if and only if T is the Fourier transform of an entire function $e : \mathbb{C}^n \rightarrow X$ for which there exists $C \geq 0, s \geq 0$ such that*

$$\|e(\xi)\|_B \leq C(1 + \|\xi\|)^s e^{r\|\Im\xi\|},$$

for all $\xi \in \mathbb{C}^n$.

4. Main results

It is in this section that we work towards a way to compute the disentangled operator $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$ corresponding to $f \in \mathcal{S}(\mathbb{R}^n)$. The reader will note that we will not make use of the commutative Banach algebras \mathbb{A} and \mathbb{D} in this section. The essential ingredient for our development will be the disentangled exponential function (see (3.2), above)

$$(4.1) \quad \mathcal{T}_{\mu_1, \dots, \mu_n} \exp \left\{ 2\pi i \left(\tilde{A}_1 + \dots + \tilde{A}_n \right) \right\} \\ =: \exp_{\mu_1, \dots, \mu_n} \left(2\pi i \sum_{j=1}^n \int_{[0, T]} A_j(s) \mu_j(ds) \right).$$

As we wish to find the disentangled operator corresponding to an arbitrary element $f \in \mathcal{S}(\mathbb{R}^n)$, it is, perhaps, not a surprise that we move away from explicit use of the disentangling algebra. However, it is important to note that the exponential function $\exp(z_1 + \dots + z_n)$ which leads to the disentangled exponential used below is, as an entire function, an element of every disentangling algebra, and furthermore, that the disentangling (4.1) results from the Jefferies–Johnson approach to the operational calculus that was outlined above. It is in the computation of the disentangled operator (4.1) that the machinery outlined in Section 2 comes into play.

In order to begin working towards our goal of a distributional representation of Feynman’s operational calculus, we start by defining, for $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$, the function $F_\phi : \mathbb{R}^n \rightarrow \mathcal{L}(X)$ by

$$(4.2) \quad F_\phi(\xi) := \hat{\phi}(\xi) \exp_{\mu_1, \dots, \mu_n} \left(2\pi i \sum_{j=1}^n \xi_j \int_{[0, T]} A_j(s) \mu_j(ds) \right).$$

Recalling (3.4), the definition of F_{exp} , we see that $F_\phi = \hat{\phi}F_{\text{exp}}$. Now, since the Fourier transform, as a function of $\xi \in \mathbb{C}^n$, is an entire function and

since the function

$$\xi \in \mathbb{C}^n \mapsto \exp_{\mu_1, \dots, \mu_n} \left(2\pi i \sum_{j=1}^{\infty} \xi_j \int_{[0, T]} A_j(s) \mu_j(ds) \right)$$

is an entire $\mathcal{L}(X)$ -valued function, it is evident that, as a function of $\xi \in \mathbb{C}^n$, F_ϕ is an entire $\mathcal{L}(X)$ -valued function. Furthermore, using (3.3), we obtain the following bound on $\|F_\phi(\xi)\|_{\mathcal{L}(X)}$:

(4.3)

$$\begin{aligned} & \|F_\phi(\xi)\|_{\mathcal{L}(X)} \\ & \leq |\hat{\phi}(\xi)| \exp \left\{ 2\pi \sum_{j=1}^n |\xi_j| \int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right\} \\ & \leq |\hat{\phi}(\xi)| \exp \left\{ 2\pi \sqrt{\sum_{j=1}^n |\xi_j|^2} \sqrt{\sum_{j=1}^n \left(\int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right)^2} \right\} \\ & \leq |\hat{\phi}(\xi)| \exp \left\{ 2\pi (\|\Re(\xi)\| + \|\Im(\xi)\|) \sqrt{\sum_{j=1}^n \left(\int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right)^2} \right\}. \end{aligned}$$

Let $R := \sqrt{\sum_{j=1}^n \left(\int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right)^2}$. Then (4.3) can be written as

$$(4.4) \quad \|F_\phi(\xi)\|_{\mathcal{L}(X)} \leq |\hat{\phi}(\xi)| \exp \{ 2\pi R \|\Re(\xi)\| + 2\pi R \|\Im(\xi)\| \}.$$

Also, with $h := 2\pi R$, there is a $C_0 > 0$ such that

$$|\hat{\phi}(\xi)| \leq C_0 e^{-2\pi R \|\xi\|}.$$

(See Theorem 3.10.) We have, finally,

$$(4.5) \quad \|F_\phi(\xi)\|_{\mathcal{L}(X)} \leq C_0 e^{2\pi R \|\Im(\xi)\|}.$$

Let

$$(4.6) \quad K := \{x \in \mathbb{R}^n : \|x\| \leq 2\pi R\}.$$

The Paley–Wiener Theorem, Theorem 3.12, then tells us that the operator-valued distribution

$$(4.7) \quad T_\phi(f) := \int_{\mathbb{R}^n} F_\phi(\xi) \hat{f}(x) dx \quad (f \in \mathcal{S}(\mathbb{R}^n))$$

has compact support contained in K .

We now proceed to investigate the distribution T_ϕ . Using the definition (4.2) of F_ϕ , and the series (4.1) for the disentangling of the exponential

function, we have, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} T_\phi(f) &= \int_{\mathbb{R}^n} F_\phi(x) \hat{f}(x) dx \\ &= \int_{\mathbb{R}^n} \left\{ \hat{\phi}(x) \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(2\pi i x_1)^{m_1} \dots (2\pi i x_n)^{m_n}}{m_1! \dots m_n!} \right. \\ &\quad \left. \cdot P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \right\} \hat{f}(x) dx \end{aligned}$$

By the obvious vector-valued version of Corollary 12.33 of [4], we may interchange the integral and the sum, leading to

(4.8)

$$\begin{aligned} T_\phi(f) &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \dots m_n!} \left\{ \int_{\mathbb{R}^n} (2\pi i x_1)^{m_1} \dots (2\pi i x_n)^{m_n} \hat{\phi}(x) \hat{f}(x) dx \right\} \\ &\quad P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \dots m_n!} \left\{ \int_{\mathbb{R}^n} (2\pi i x_1)^{m_1} \dots (2\pi i x_n)^{m_n} \widehat{\phi * f}(x) dx \right\} \\ &\quad P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \dots m_n!} \frac{\partial^{m_1 + \dots + m_n} (\phi * f)}{\partial x_1^{m_1} \dots \partial x_n^{m_n}}(0) P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \end{aligned}$$

where we've used Theorem 3.7. We are therefore able to write

$$(4.9) \quad T_\phi(f) = (\phi * f)_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot)),$$

provided that the series in (4.8) converges. However, since $\phi \in \mathcal{G}$ and since $f \in \mathcal{S}(\mathbb{R}^n)$, the series in (4.8) does indeed converge absolutely. This is most easily seen using the norm estimate (4.5) as well as the fact that $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

Remark 4.1. As is apparent, we are restricting F_ϕ to \mathbb{R}^n and so the series in (4.8) is a real-valued series. However, we do have the disentangled monomial

$$P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))$$

present in each term of the series and so, following Feynman's rules, the series derived for $T_\phi(f)$, $f \in \mathcal{S}(\mathbb{R}^n)$, is indeed a sum of time-ordered products; i.e., it is a disentangled operator. We will refer to $T_\phi(f)$ as the ϕ -weighted *disentangling* of $f \in \mathcal{S}(\mathbb{R}^n)$. It is these disentangling which will enable us to obtain the disentangled operator $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$, for $f \in \mathcal{S}(\mathbb{R}^n)$.

We have proven the following theorem.

Theorem 4.2. Let $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$. Define $F_\phi : \mathbb{R}^n \rightarrow \mathcal{L}(X)$ by

$$(4.10) \quad F_\phi(x) := \hat{\phi}(x) \exp_{\mu_1, \dots, \mu_n} \left\{ 2\pi i \sum_{j=1}^n x_j \int_{[0, T]} A_j(s) \mu_j(ds) \right\}.$$

(F_ϕ is the restriction to \mathbb{R}^n of the entire function defined in (4.2).) Then the distribution T_ϕ defined by

$$(4.11) \quad T_\phi(f) := \int_{\mathbb{R}^n} F_\phi(x) \hat{f}(x) dx \quad (f \in \mathcal{S}(\mathbb{R}^n)),$$

has compact support in $K = \{x \in \mathbb{R}^n : \|x\| \leq 2\pi R\}$, where

$$(4.12) \quad R := \sqrt{\sum_{j=1}^n \left(\int_{[0, T]} \|A_j(s)\|_{\mathcal{L}(X)} \mu_j(ds) \right)^2}.$$

Moreover, given $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$(4.13) \quad T_\phi(f) = (\phi * f)_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot)),$$

where $(\phi * f)_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$ is given by the series in the last line of Equation (4.8).

We now use the ϕ -weighted disentanglings developed above to define a disentangled operator $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We start by selecting $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$ with

$$\int_{\mathbb{R}^n} \phi(x) dx = 1.$$

Define, for each $k \in \mathbb{N}$,

$$(4.14) \quad \phi_{1/k}(x) = k^{-n} \phi\left(\frac{x}{k}\right).$$

As is easily seen,

$$\int_{\mathbb{R}^n} \phi_{1/k}(x) dx = 1,$$

for all $k \in \mathbb{N}$. Therefore, following [3, Example 1.2.16, page 24], $\{\phi_{1/k}\}_{k=1}^\infty$ is an approximate identity. Moreover, it is easy to see that $\phi_{1/k} \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$ for every $k \in \mathbb{N}$. Hence, for $f \in \mathcal{S}(\mathbb{R}^n)$ and every $k \in \mathbb{N}$, we have

$$T_{\phi_{1/k}}(f) = (\phi_{1/k} * f)_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot));$$

that is, we have sequence $\{T_{\phi_{1/k}}(f)\}_{k=1}^\infty$ of $\mathcal{L}(X)$ -valued distributions of compact support (Theorem 4.2).

For $k, l \in \mathbb{N}$, we have, using (4.8),

$$\begin{aligned}
& \left\| T_{\phi_{1/k}}(f) - T_{\phi_{1/l}}(f) \right\|_{\mathcal{L}(X)} \\
& \leq \sum_{m_1, \dots, m_n=0}^{\infty} \frac{1}{m_1! \cdots m_n!} \\
& \quad \cdot \left| \frac{\partial^{m_1+\dots+m_n}(\phi_{1/k} * f)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(0) - \frac{\partial^{m_1+\dots+m_n}(\phi_{1/l} * f)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(0) \right| \\
& \quad \cdot \left\| P_{\mu_1, \dots, \mu_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \right\|_{\mathcal{L}(X)}.
\end{aligned}$$

Next, using Proposition 3.8 and the fact that $\phi \in \mathcal{G}$, we have, for $k \in \mathbb{N}$,

$$\frac{\partial^{m_1+\dots+m_n}(\phi_{1/k} * f)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} = \phi_{1/k} * \left\{ \frac{\partial^{m_1+\dots+m_n} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \right\}.$$

It follows from Theorem 1.2.19 of [3] (concerning approximate identities), that

$$\begin{aligned}
(4.15) \quad & \left| \frac{\partial^{m_1+\dots+m_n}(\phi_{1/k} * f)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(0) - \frac{\partial^{m_1+\dots+m_n}(\phi_{1/l} * f)}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}}(0) \right| \\
& = \left| \left(\phi_{1/k} * \frac{\partial^{m_1+\dots+m_n} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \right) (0) - \left(\phi_{1/l} * \frac{\partial^{m_1+\dots+m_n} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \right) (0) \right| \\
& \leq \left\| \phi_{1/k} * \frac{\partial^{m_1+\dots+m_n} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} - \phi_{1/l} * \frac{\partial^{m_1+\dots+m_n} f}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} \right\|_{L^\infty(B(0,R))} \rightarrow 0
\end{aligned}$$

as $k, l \rightarrow \infty$, where $B(0, R)$ is the ball of radius R centered at 0 in \mathbb{R}^n . Finally, an application of the dominated convergence theorem tells us that

$$\left\| T_{\phi_{1/k}}(f) - T_{\phi_{1/l}}(f) \right\|_{\mathcal{L}(X)} \rightarrow 0$$

as $k \rightarrow \infty$; i.e., the sequence $\left\{ T_{\phi_{1/k}}(f) \right\}_{k=1}^{\infty}$ is a Cauchy sequence of operators in $\mathcal{L}(X)$. It follows at once that there is an element $T_{\mu_1, \dots, \mu_n} f \in \mathcal{L}(X)$ such that

$$T_{\phi_{1/k}}(f) \rightarrow T_{\mu_1, \dots, \mu_n} f$$

in $\mathcal{L}(X)$ -norm as $k \rightarrow \infty$. We have obtained the following theorem. It is this theorem that spells out how we can determine the disentangling of $f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 4.3. *Let $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$ be such that*

$$(4.16) \quad \int_{\mathbb{R}^n} \phi \, dx = 1.$$

For $k \in \mathbb{N}$, define

$$\phi_{1/k}(x) := k^{-n} \phi\left(\frac{x}{k}\right).$$

Then $\phi_{1/k} \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$ for all $k \in \mathbb{N}$ and is an approximate identity. For each $f \in \mathcal{S}(\mathbb{R}^n)$, the sequence $\left\{T_{\phi_{1/k}}(f)\right\}_{k=1}^\infty$ of $\phi_{1/k}$ -weighted disentanglings is a Cauchy sequence in $\mathcal{L}(X)$ and so has a (norm) limit $T_{\mu_1, \dots, \mu_n} f$; this limit is independent of the choice of approximate identity $\phi_{1/k}$, $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$ and satisfying (4.16).

Remark 4.4. The independence of the choice of ϕ results from the limit in (4.15), the definition of an approximate identity and Theorem 1.2.19 of [3].

In view of Theorems 4.2 and 4.3, we can now accomplish the main goal of this paper, the disentangling of an arbitrary element $f \in \mathcal{S}(\mathbb{R}^n)$. We do this using the corresponding sequence $\left\{T_{\phi_{1/k}}(f)\right\}_{k=1}^\infty$ of compactly supported operator-valued distributions.

Definition 4.5 (Definition of the disentangling of $f \in \mathcal{S}(\mathbb{R}^n)$). The disentangling of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$(T_{\mu_1, \dots, \mu_n} f)(\tilde{A}_1(\cdot), \dots, \tilde{A}_n(\cdot)) := \lim_{k \rightarrow \infty} T_{\phi_{1/k}}(f),$$

for any $\phi \in C^\infty(\mathbb{R}^n) \cap \mathcal{G}$.

With this definition, we have extended the operational calculi from functions of n complex variables which are analytic on a polydisk to the space of Schwartz functions on \mathbb{R}^n .

Remark 4.6. As is clear from Theorem 4.3, the disentangling

$$f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$$

is obtained via a limit of $\mathcal{L}(X)$ -valued distributions of compact support. However, it may not be that $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$ has compact support.

5. Discussion

We have, in Theorems 4.2 and 4.3, shown that it is possible to obtain, for $f \in \mathcal{S}(\mathbb{R}^n)$, a disentangled operator $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$. Moreover, this operator is determined via a limit of a sequence of compactly supported operator-valued distributions. Of course, while each distribution $T_{\phi_{1/k}}$ has compact support, the limit as $k \rightarrow \infty$ may not. On the other hand, while we have derived an explicit disentangling series for the ϕ -weighted disentanglings $(\phi * f)_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$ (see Equation (4.8)), we did not record a series expansion for $f_{\mu_1, \dots, \mu_n}(A_1(\cdot), \dots, A_n(\cdot))$; indeed, due to the fact that we would need to compute, for a given Silva function ϕ , the convolution $\phi_{1/k} * f$, this would be difficult to compute explicitly. However, the interest here is in the construction of a version of the operational calculus for Schwartz functions, as opposed to functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ which are analytic on a particular polydisk. That is, we extend the operational calculi to the Schwartz space.

Our extension of the domain of Feynman's operational calculi to the Schwartz class is similar in some ways to the approach taken in the paper [12]. Let $\mu = (\mu_1, \dots, \mu_n)$ be an n -tuple of continuous Borel probability measures on $[0, 1]$ (or $[a, b]$). In [12], an n -tuple $\mathbf{A} = (A_1, \dots, A_n)$ of bounded linear operators on a Banach space is said to be of *Paley–Wiener type* (s, r, μ) if there exists $C, r, s \geq 0$ such that

$$\|\mathcal{T}_{\mu_1, \dots, \mu_n} \left(e^{i\langle \zeta, \tilde{\mathbf{A}} \rangle} \right)\|_{\mathcal{L}(X)} \leq C (1 + |\zeta|)^s e^{r|\Im \zeta|}, \quad \text{for all } \zeta \in \mathbb{C}^n.$$

Of course, if this estimate holds, then there is a unique $\mathcal{L}(X)$ - valued distribution $\mathcal{F}_{\mu, \mathbf{A}} \in \mathcal{L}(C^\infty(\mathbb{R}^n), \mathcal{L}(X))$ such that

$$\mathcal{F}_{\mu, \mathbf{A}}(f) = \int_{\mathbb{R}^n} \mathcal{T}_{\mu_1, \dots, \mu_n} \left(e^{i\langle \zeta, \tilde{\mathbf{A}} \rangle} \right) \hat{f}(\zeta) d\zeta,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Moreover, the distribution so defined is compactly supported. The support $\gamma_\mu(\mathbf{A})$ of $\mathcal{F}_{\mu, \mathbf{A}}$ is referred to in [12] as the μ -joint spectrum of \mathbf{A} . (A similar result can be found in [5].) The main result of [12] is that any n -tuple of bounded self-adjoint operators on a Hilbert space is of Paley–Wiener type $(0, r, \mu)$ with $r = \sqrt{\sum_{j=1}^n \|A_j\|^2}$. It is, then, apparent that the approach taken in this paper has some similarities with the approach taken in [12]. Indeed, recall that the distributions $T_{\phi_{1/k}}$, $k \in \mathbb{N}$, have compact support and are reminiscent of the distribution $\mathcal{F}_{\mu, \mathbf{A}}$. Also, the distribution $\mathcal{F}_{\mu, \mathbf{A}}$ is defined by using the disentangling of the exponential $e^{i\langle \zeta, \tilde{\mathbf{A}} \rangle}$; what we have done above is in the same spirit, although here we have “weighted” the disentangling of the exponential with a Silva function $\phi_{1/k}$.

Nevertheless, what has been done in the current paper differs markedly in several respects from the approach taken in [12]:

- (1) There is no requirement that the operators be time independent; i.e., fixed operators. As the reader will recall, we work with measurable $\mathcal{L}(X)$ -valued functions.
- (2) There is no requirement in the present paper that the operators be self-adjoint (or act on a Hilbert space).
- (3) There is no need in the current paper to require that the n -tuple of operators is of Paley–Wiener type.
- (4) The time-ordering measures involved do not need to be continuous measures. The results above will accommodate both continuous measures and measures with nonzero (finitely supported) discrete parts.
- (5) We define the operational calculi on the Schwartz space by defining the disentangling of $f \in \mathcal{S}$ as the limit of a sequence of ϕ -weighted disentanglings. (Because of the quite general setting of the current paper, it might have been expected that obtaining f_{μ_1, \dots, μ_n} , $f \in \mathcal{S}$, would be more difficult.)

It is hoped, then, that the approach taken in this paper to extend the domain of Feynman's operational calculus to the Schwartz space will allow new directions to be taken in the study of the operational calculus.

References

- [1] CHUNG, SOON-YEONG; KIM, DOHAN; LEE, SUNGJIN. Characterization for Beurling–Björck space and Schwartz space. *Proc. Amer. Math. Soc.* **125** (1997), no. 11, 3229–3234. MR1443817 (98c:46074), Zbl 0887.46013, doi: 10.1090/S0002-9939-97-04221-4.
- [2] FEYNMAN, RICHARD P. An operator calculus having applications in quantum electrodynamics. *Physical. Rev. (2)* **84** (1951), 108–128. MR0044379 (13, 410e), Zbl 0044.23304, doi: 10.1103/PhysRev.84.108.
- [3] GRAFAKOS, LOUKAS. Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008. xvi+489 pp. ISBN: 978-0-387-09431-1. MR2445437 (2011c:42001), Zbl 1220.42001, doi: 10.1007/978-0-387-09432-8.
- [4] HEWITT, EDWIN; STROMBERG, KARL. Real and abstract analysis. A modern treatment of the theory of functions of a real variable. Springer-Verlag, New York, 1965. vii+476 pp. MR188387 (32#5826), Zbl 0137.03202.
- [5] JEFFERIES, BRIAN. Spectral properties of noncommuting operators. Lecture Notes in Mathematics, 1843. Springer-Verlag, Berlin, 2004. viii+184 pp. ISBN: 3-540-21923-4. MR2069293 (2007a:47007), Zbl 1056.47002, doi: 10.1007/978-3-540-70746-2_1.
- [6] JEFFERIES, B.; JOHNSON, G. W. Feynman's operational calculi for noncommuting operators: definitions and elementary properties. *Russ. J. Math. Phys.* **8** (2001), no. 2, 153–171. MR1928726 (2003i:47019), Zbl 1186.81082.
- [7] JEFFERIES, B.; JOHNSON, G. W. Feynman's operational calculus for families of noncommuting operators: tensor products, ordered supports, and the disentangling of an exponential factor. *Mat. Zametki* **70** (2001), no. 6, 815–838; translation in *Math. Notes* **70** (2001), no. 5–6, 744–764. MR1887249 (2003a:47037), Zbl 1030.47008, doi: 10.1023/A:1012903732597.
- [8] JEFFERIES, B.; JOHNSON, G. W. Feynman's operational calculi for noncommuting operators: spectral theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002), no. 2, 171–199. MR1914834 (2003g:47031), Zbl 1096.47502, doi: 10.1142/S021902570200078X.
- [9] JEFFERIES, B.; JOHNSON, G. W. Feynman's operational calculi for noncommuting operators: the monogenic calculus. *Adv. Appl. Clifford Algebras* **11** (2001), no. 2, 239–265. MR1956400 (2003k:47011), Zbl 1096.47501, doi: 10.1007/BF03042315.
- [10] JEFFERIES, B.; JOHNSON, G. W.; KIM, B. S. Feynman's operational calculi: methods for iterative disentangling. *Acta Appl. Math.* **92** (2006), no. 3, 293–309. MR2266490 (2007m:81103), Zbl 1156.81403, doi: 10.1007/s10440-006-9061-2.
- [11] JEFFERIES, BRIAN; JOHNSON, G. W.; NIELSEN, LANCE. Feynman's operational calculi for time dependent noncommuting operators. *J. Korean Math. Soc.* **38** (2001), no. 2, 193–226. MR1817617 (2001m:81154), Zbl 0984.47013.
- [12] JEFFERIES, BRIAN; JOHNSON, GERALD W.; NIELSEN, LANCE. Feynman's operational calculi: spectral theory for noncommuting self-adjoint operators. *Math. Phys. Anal. Geom.* **10** (2007), no. 1, 65–80. MR2340533 (2009e:47018), Zbl 1180.47013, doi: 10.1007/s11040-007-9021-8.
- [13] JOHNSON, G. W.; KIM, B. S. Derivational derivatives and Feynman's operational calculi. *Houston J. Math.* **35** (2009), no. 2, 647–664. MR2519553, Zbl 1185.47005.
- [14] JOHNSON, G. W.; KIM, B. S. Extracting linear and bilinear factors in Feynman's operational calculi. *Math. Phys. Anal. Geom.* **6** (2003), no. 2, 181–200. MR1987042 (2004d:47017), Zbl 1051.47013, doi: 10.1023/A:1024049113118.

- [15] JOHNSON, GERALD W.; LAPIDUS, MICHEL L. The Feynman integral and Feynman's operational calculus. Oxford Mathematical Monographs. Oxford Science Publications. *The Clarendon Press, Oxford University Press, New York*, 2000. xviii+771 pp. ISBN: 0-19-853574-0. MR1771173 (2001i:58015), Zbl 0952.46044, doi: 10.1063/1.1404851.
- [16] Johnson, G. W.; Lapidus, M. L.; Nielsen, L. Feynman's operational calculus and beyond: noncommutativity and time ordering. To be published by Oxford University Press.
- [17] JOHNSON, G. W.; NIELSEN, LANCE. Feynman's operational calculi blending instantaneous and continuous phenomena in Feynman's operational calculi. *Stochastic analysis and mathematical physics (SAMP/ANESTOC 2002)*, 229–254. *World Sci. Publ., River Edge, NJ*, 2004. MR2115939 (2005i:81096), Zbl 1079.60010, doi: 10.1142/9789812702364_0012.
- [18] REED, MICHAEL; SIMON, BARRY. Methods of modern mathematical physics. I. Functional analysis. *Academic Press, New York-London*, 1972. xvii+325 pp. MR0493419, Zbl 0459.46001.
- [19] TRÉVES, FRANÇOIS. Introduction to pseudodifferential and Fourier integral operators. Vol. 1. Pseudodifferential operators. The University Series in Mathematics. *Plenum Press, New York-London*, 1980. xxvii+299+xi pp. ISBN: 0-306-40403-6. MR0597144, Zbl 0453.47027.

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