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The complementing condition and its role in a bifurcation theory applicable to nonlinear elasticity

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Abstract. The complementing condition (CC) is an algebraic compatibility requirement between the principal part of a linear elliptic partial differential operator and the principal part of the corresponding boundary operators. When the CC holds the linear boundary value problem has many important functional analytic properties. Recently it has been found that the CC plays a very important role in the construction of a generalized degree, with all the properties of the Leray–Schauder degree, applicable to a general class of problems in nonlinear elasticity. In this paper we discuss the role of the CC in such development and present some examples of boundary value problems in nonlinear elasticity to which this new degree is applicable. In addition, when the CC fails we present some recent results on the implications of this failure in the context of nonlinear elasticity.

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1. Introduction

The complementing condition (CC), also known as the Šapiro–Lopatinskiĭ condition ([16], [24]), was introduced by Agmon, Douglis, and Nirenberg in [1], [2]. It is an algebraic compatibility requirement between the principal part of a linear elliptic differential operator and the principal part of the corresponding boundary operators (cf. [17], [27], [28], [30] and the references therein). When the CC holds, the corresponding linear boundary value problem has many important functional analytic properties. For instance, if we assume enough regularity on the boundary of the domain and on the coefficients of the differential and boundary operators, the Fredholm properties for the linear operator and some apriori estimates on the solutions of the associated boundary value problem can be obtained from strong ellipticity and the complementing condition (cf. Theorem 4).

The application of global continuation methods via Leray–Schauder degree [15] to problems in nonlinear elasticity theory has been quite successful, but mostly limited to problems reducible to ordinary differential equations (cf. [3]) or displacement problems (cf. [8]). One of the main difficulties in studying local and global bifurcation in fully 2D and 3D nonlinear elasticity is the presence of nonlinear traction Neumann type boundary conditions. In particular, it is not possible to write the equations of elasticity as a compact perturbation of the identity operator. Thus, is not obvious how to apply the well known Krasnoselskii–Rabinowitz approach ([14], [21], [22]), using Leray–Schauder degree [15], to obtain existence of local branches of bifurcating solutions that can be continued globally. Only recently Healey and Simpson [11] obtained results along these lines for the displacement equations of equilibrium, together with boundary traction and displacements. Their approach is based upon the construction of a degree ([7], [13], [11]) which has the same important properties of the classical Leray–Schauder degree. The CC for the corresponding linearized boundary value problem plays an important role in the application of this new degree to problems in nonlinear elasticity with nonlinear traction boundary conditions. (See also [5] for an application of this new degree in the theory of water waves.) We should mention that as far as we know, the first who used the CC and considered some of its consequences in elasticity was Thompson [28] who studied the linearized traction boundary value problem of the equations of three dimensional elasticity. Thompson also showed that violations of the CC are equivalent to the existence of Rayleigh waves of certain type, thus providing a physical interpretation for violation of the CC from within a dynamical theory.

The main purpose of this paper is to review some of the above mentioned results and some of our own work on related model problems. Most of the proofs are omitted but we include references where they can be found. We hope that this paper serves as a road map for other researchers interested in the relation between the CC and bifurcation in nonlinear elasticity. In

Section 2 we give the basic definitions leading to that of the complementing condition. We discuss as well some elementary boundary value problems and whether or not the CC holds for these problems. In Section 3 we introduce the equations of elasticity and discuss some of the consequences of the CC in this context. Section 4 is devoted to a discussion of a local and global bifurcation theorem based on the new degree and applicable to the boundary value problems of elasticity with nonlinear traction boundary conditions. Also in this section we present two examples of problems from elasticity to which this bifurcation theorem applies. When the CC fails, we present in Section 5 some of our recent results on the implications of this failure in the context of nonlinear elasticity. In particular we discuss the relation between violation of the CC and the existence of possible bifurcation points accumulating at the point where the CC fails.

Notation. For any $\Omega \subset \mathbb{R}^n$, we let $C^k(\overline{\Omega}, \mathbb{R}^r)$ be the Banach space of continuous functions with derivatives up to order k continuous over $\overline{\Omega}$. The norm in $C^k(\overline{\Omega}, \mathbb{R}^r)$ is given by

$$\|\mathbf{u}\|_{C^k(\overline{\Omega},\mathbb{R}^r)} = \sum_{|\mathbf{s}|\le k} \max_{\mathbf{x}\in\overline{\Omega}} \|\mathrm{D}^{\mathbf{s}}\mathbf{u}(\mathbf{x})\| .$$

Here $\mathbf{s} = (s_1, \ldots, s_n)$ is a multi-index of length n so that $\mathrm{D}^{\mathbf{s}}\mathbf{u}$ represents a partial derivative of order $|\mathbf{s}| = s_1 + \cdots + s_n$.

The *Schauder space* $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^r)$, $0 < \alpha \le 1$, denotes the Banach space of functions $\mathbf{u} \in C^k(\overline{\Omega}, \mathbb{R}^r)$ such that

$$\sup_{\substack{\mathbf{x},\mathbf{y}\in\overline{\Omega}\\ \mathbf{x}\neq\mathbf{y}}} \frac{\|\mathrm{D}^{\mathbf{s}}\mathbf{u}(\mathbf{x}) - \mathrm{D}^{\mathbf{s}}\mathbf{u}(\mathbf{y})\|}{\|\mathbf{x}-\mathbf{y}\|^{\alpha}} < \infty,$$

for any multi-index $\mathbf{s}$ with $|\mathbf{s}| = k$. The norm in $C^{k,\alpha}(\overline{\Omega}, \mathbb{R}^r)$ is given by

$$\|\mathbf{u}\|_{k,\alpha} = \|\mathbf{u}\|_{C^k(\overline{\Omega},\mathbb{R}^r)} + \sum_{|\mathbf{s}|=k} \sup_{\substack{\mathbf{x},\mathbf{y}\in\overline{\Omega}\\ \mathbf{x}\neq\mathbf{y}}} \frac{\|\mathrm{D}^{\mathbf{s}}\mathbf{u}(\mathbf{x}) - \mathrm{D}^{\mathbf{s}}\mathbf{u}(\mathbf{y})\|}{\|\mathbf{x}-\mathbf{y}\|^{\alpha}}.$$

2. The complementing condition

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with a smooth boundary $\partial\Omega$. Let A be a linear partial differential operator given by:

$$A[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|\le 2m} a_{\mathbf{s}}(\mathbf{x})\mathrm{D}^{\mathbf{s}}u(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{2.1}$$

where $u : \Omega \longrightarrow \mathbb{R}$ and $m \ge 1$. The principal part of A is given by:

$$A^H[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x})\mathrm{D}^{\mathbf{s}}u(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.2}$$

The boundary operators are given by:

$$B_j[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|\leq m_j} b_{j\,\mathbf{s}}(\mathbf{x})\mathrm{D}^{\mathbf{s}}u(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega, \tag{2.3}$$

where $m_j \leq 2m-1$, for $j = 1,\dots,m$, with corresponding principal part:

$$B_j^H[u](\mathbf{x}) \equiv \sum_{|\mathbf{s}|= m_j} b_{j\,\mathbf{s}}(\mathbf{x})\mathrm{D}^{\mathbf{s}}u(\mathbf{x}), \quad \mathbf{x}\in\partial\Omega, \tag{2.4}$$

for $j = 1,\dots,m$. For any $f \in C(\overline{\Omega},\mathbb{R})$ and $\mathbf{g} \in C(\overline{\Omega},\mathbb{R}^m)$, we consider the boundary value problem:

$$A[u] = f \quad \text{in } \Omega, \tag{2.5a}$$

$$B[u] = \mathbf{g} \quad \text{over } \partial\Omega, \tag{2.5b}$$

where $B = (B_1,\dots,B_m)$.

For $\mathbf{x}_0 \in \partial\Omega$ and $\mathbf{n}(\mathbf{x}_0)$ the unit outer normal at $\mathbf{x}_0$, we define the half-space $\mathcal{H}$ by:

$$\mathcal{H} = \{\mathbf{x} \,:\, (\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}(\mathbf{x}_0) < 0\}\,. \tag{2.6}$$

For $\alpha \in \mathbb{R}$ consider the following *auxiliary problem* over $\mathcal{H}$:

$$A_0^H[v] = \alpha^2 v, \quad \text{in } \mathcal{H}, \tag{2.7a}$$

$$B_0^H[v] = \mathbf{0}, \quad \text{over } \partial\mathcal{H}, \tag{2.7b}$$

where

$$A_0^H[v](\mathbf{x}) \equiv \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0)\mathrm{D}^{\mathbf{s}}v(\mathbf{x}), \quad \mathbf{x}\in\mathcal{H}, \tag{2.8a}$$

$$B_{0,j}^H[v](\mathbf{x}) \equiv \sum_{|\mathbf{s}|=m_j} b_{j\,\mathbf{s}}(\mathbf{x}_0)\mathrm{D}^{\mathbf{s}}v(\mathbf{x}), \quad \mathbf{x}\in\partial\mathcal{H}, \tag{2.8b}$$

where $j = 1,\dots,m$. If $\boldsymbol{\xi}$ represents any nonzero vector in $\mathbb{R}^n$ with $\boldsymbol{\xi}\cdot\mathbf{n}(\mathbf{x}_0) = 0$, then we look for solutions of this auxiliary problem of the form

$$v(\mathbf{x}) = w((\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{n}(\mathbf{x}_0))\mathrm{e}^{i\boldsymbol{\xi}\cdot(\mathbf{x}-\mathbf{x}_0)}, \quad \mathbf{x}\in\mathcal{H}, \tag{2.9}$$

where $w(s) \longrightarrow 0$ as $s \longrightarrow -\infty$. These type of solutions are called *bounded exponential solutions.*

Definition 1. Consider the BVP (2.5), and for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$ and $\alpha \in \mathbb{R}$, the corresponding auxiliary problem (2.7).

(a) If for $\alpha = 0$, the only bounded exponential solution of the auxiliary problem is the one with $w \equiv 0$, then we say that the BVP (2.5) satisfy the *complementing condition* for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$.

(b) If for $\alpha \neq 0$, the only bounded exponential solution of the auxiliary problem is the one with $w \equiv 0$, then the BVP (2.5) satisfy *Agmon's condition* for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$.

If both, the complementing condition and Agmon's condition hold, then we say that the *strong complementing condition* holds.

Remark. The CC is actually an algebraic compatibility condition between the coefficients of the principal part of the differential operator and the coefficients of the principal part of the corresponding boundary value operator. This algebraic characterization of the complementing condition was the one originally used in [1]. In the Appendix A we show the equivalence of the algebraic characterization of the CC with the first part of Definition 1.

The case $\alpha = 0$ (cf. (2.7)) in the following two examples is from [30, Sec. 11.2, Pag. 157].

Example 2. For $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$, we consider the scalar operator:

$$A[u] = \Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}. \tag{2.10}$$

Any $\mathbf{x}_0 \in \partial\Omega$ is given by $(a, b, 0)$ with corresponding normal at $\mathbf{n}(\mathbf{x}_0) = (0, 0, -1)$. In this case (2.9) reduces to:

$$v(x_1, x_2, x_3) = w(s)\mathrm{e}^{i(\xi_1 y_1 + \xi_2 y_2)}, \quad (y_1, y_2) \in \mathbb{R}^2, \quad s < 0, \tag{2.11}$$

where $\xi_1^2 + \xi_2^2 \neq 0$, $y_1 = x_1 - a$, $y_2 = x_2 - b$, and $s = -x_3$. The equation (2.7a) is equivalent to:

$$w''(s) - (\xi_1^2 + \xi_2^2 + \alpha^2)w(s) = 0.$$

Thus $w(s) = c_1\mathrm{e}^{\omega s} + c_2\mathrm{e}^{-\omega s}$ where $\omega = \sqrt{\xi_1^2 + \xi_2^2 + \alpha^2} > 0$, even for $\alpha = 0$. The condition that $w(s) \longrightarrow 0$ as $s \longrightarrow -\infty$ implies that $c_2 = 0$. Hence $w(s) = c_1\mathrm{e}^{\omega s}$.

(a) For the case of a Dirichlet type boundary condition, the boundary operator is given by:

$$B[u](\mathbf{x}) = a_0(\mathbf{x})u(\mathbf{x}).$$

Thus for (2.11), the boundary condition $B_0^H[v] = 0$ reduces to

$$a_0(\mathbf{x}_0)w(0) = c_1 a_0(\mathbf{x}_0) = 0.$$

Thus provided $a_0(\mathbf{x}) \neq 0$ for all $\mathbf{x}$, the strong complementing condition is satisfied.

(b) For the Neumann type boundary operator:

$$B[u] = \frac{\partial u}{\partial \mathbf{n}},$$

the boundary condition $B_0^H[v] = 0$ reduces to $w'(0) = c_1\omega = 0$, i.e., $c_1 = 0$, and the strong complementing condition is satisfied.

(c) For the Robin type boundary operator:

$$B[u](\mathbf{x}) = b_0(\mathbf{x})u + b_r(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}},$$

the principal part of the operator is similar to the previous case for the Neumann boundary operator except for the multiplication by the function $b_r(\mathbf{x})$. Thus provided $b_r(\mathbf{x}) \neq 0$ for all $\mathbf{x}$, the strong complementing condition is satisfied.

Example 3. We consider the biharmonic operator:

$$\Delta^2 u = u_{x_1x_1x_1x_1} + 2u_{x_1x_1x_2x_2} + u_{x_2x_2x_2x_2}, \tag{2.12}$$

over the region $\Omega = \{(x_1, x_2) : x_2 > 0\}$, and boundary operators:

$$\Delta u = 0, \quad \frac{\partial \Delta u}{\partial x_2} = 0, \quad \text{over } \partial\Omega. \tag{2.13}$$

If $\mathbf{x}_0 \in \partial\Omega$, then $\mathbf{x}_0 = (a, 0)$ and $\mathbf{n}(\mathbf{x}_0) = (-1, 0)$. In this case (2.9) reduces to:

$$v(x_1, x_2) = w(s)\mathrm{e}^{i\xi y}, \quad y \in \mathbb{R}, \quad s < 0,$$

where $y = x_1 - a$, $s = -x_2$, and $\xi \neq 0$. The problem $\Delta^2 v = \alpha^2 v$ reduces to:

$$w^{(iv)}(s) - 2\xi^2 w''(s) + (\xi^4 - \alpha^2)w(s) = 0, \tag{2.14}$$

and the boundary conditions (2.13) to:

$$w''(0) - \xi^2 w(0) = 0, \quad w'''(0) - \xi^2 w'(0) = 0. \tag{2.15}$$

We consider several cases:

(a) Case $\alpha = 0$. The general solution of (2.14) satisfying $w(s) \longrightarrow 0$ as $s \longrightarrow -\infty$, is given by

$$w(s) = (c_1 + c_2 s)\mathrm{e}^{|\xi| s},$$

for some constants c_1, c_2. One can easily check that $w(s) = c_1\mathrm{e}^{|\xi|s}$ satisfies both boundary conditions (2.15) for any c_1. Thus (2.12) together with the boundary conditions (2.13), do not satisfy the complementing condition.

(b) Case $\alpha \neq 0$. If $|\alpha| < \xi^2$, then the general solution of (2.14) satisfying $w(s) \longrightarrow 0$ as $s \longrightarrow -\infty$ is given by

$$w(s) = c_1\mathrm{e}^{\omega_1 s} + c_2\mathrm{e}^{\omega_2 s},$$

where $\omega_1 = \sqrt{\xi^2 + |\alpha|}$, $\omega_2 = \sqrt{\xi^2 - |\alpha|}$. The boundary conditions (2.15) imply now that

$$c_1 - c_2 = 0, \quad \omega_1 c_1 - \omega_2 c_2 = 0.$$

Since $\omega_1 \neq \omega_2$, these equations imply that $c_1 = c_2 = 0$. Similar results are obtained when $|\alpha| \geq \xi^2$. Thus Agmon's condition holds for (2.12) together with the boundary conditions (2.13).

For ease of exposition we have stated the CC for the scalar operators (2.1) and (2.3). Definition 1 can be extended to systems of partial differential equations, where w in (2.9) is now replaced by a vector valued function $\mathbf{w}$ (see [2] for the details). In Section 3 we discuss the details of these notions for systems in the particular case of the linearized system of elasticity.

3. The CC in nonlinear elasticity

We consider a body which in its reference configuration occupies the region $\Omega \subset \mathbb{R}^3$. Let $\mathbf{f} : \Omega \to \mathbb{R}^3$ denote a deformation of the body and let its *deformation gradient*[1] be

$$\nabla \mathbf{f}(\mathbf{x}) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}(\mathbf{x}). \tag{3.1}$$

For smooth deformations, the requirement that $\mathbf{f}(\mathbf{x})$ is locally *invertible and preserves orientation* takes the form

$$\det \nabla \mathbf{f}(\mathbf{x}) > 0, \quad \mathbf{x} \in \Omega. \tag{3.2}$$

This condition, which is motivated by physical considerations, is one of the main source of technical difficulties in the study of the equations of elasticity. For example, the set of smooth deformations $\mathbf{f}$ satisfying (3.2) is not convex. Moreover, although it implies that the mapping $\mathbf{f}$ is locally 1-1, it does not imply so globally. We refer to [4] and its references therein for additional conditions which together with (3.2) imply the global invertibility of $\mathbf{f}$.

Let $W : \mathbb{R}^3 \times \mathrm{Lin}^+ \to \mathbb{R}$ be the *stored energy function* of the material of the body where $\mathrm{Lin}^+ = \{\mathbf{F} \in M^{3\times 3} \;:\; \det \mathbf{F} > 0\}$ and $M^{3\times 3}$ denotes the space of real 3×3 matrices. The *elastic potential energy* due to the deformation $\mathbf{f} : \Omega \longrightarrow \mathbb{R}^3$ is given by

$$I(\mathbf{f}) \equiv \int_\Omega W(\mathbf{x}, \nabla \mathbf{f}(\mathbf{x}))\, d\mathbf{x}\,. \tag{3.3}$$

We assume that $W(\mathbf{x}, \mathbf{F}) \to \infty$ as either $\det \mathbf{F} \to 0^+$ (cf. (3.2)) or as $\|\mathbf{F}\| \to \infty$ for any fixed $\mathbf{x} \in \Omega$.

The derivatives:

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) = \frac{\mathrm{d}}{\mathrm{d}\mathbf{F}} W(\mathbf{x}, \mathbf{F}), \quad \mathbf{C}(\mathbf{x}, \mathbf{F}) = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{F}^2} W(\mathbf{x}, \mathbf{F}), \tag{3.4}$$

are the *first (Piola–Kirchhoff) stress and elasticity tensors* respectively. We say that the elasticity tensor is *strongly elliptic* at a deformation $\mathbf{f}$ if for each $\mathbf{x} \in \bar{\Omega}$,

$$\mathbf{ab} : \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}(\mathbf{x}))[\mathbf{ab}] > 0, \tag{3.5}$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$.

Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $\mathbf{g}_\lambda$ be a given function where λ is some physical parameter of the problem. We consider the problem of minimizing the potential energy functional (3.3) over the admissible set

$$\mathrm{Def} = \left\{\mathbf{f} \in C^2(\overline{\Omega}, \mathbb{R}^3) \;:\; \det \nabla \mathbf{f} > 0, \quad \mathbf{f} = \mathbf{g}_\lambda \text{ on } \Gamma_1\right\}. \tag{3.6}$$

[1]The symbol ∇ is normally used to represent the column vector of the first order partial derivatives of a scalar function. In this paper we use the standard notation in elasticity theory in which $\nabla \mathbf{f}$ is the matrix representing the Fréchet derivative of $\mathbf{f}$.

The Euler–Lagrange equations for this problem are given by:

$$\operatorname{div} \mathbf{S}(\mathbf{x}, \nabla \mathbf{f}(\mathbf{x})) = \mathbf{0}, \quad \text{in } \Omega, \tag{3.7a}$$

$$\mathbf{f} = \mathbf{g}_\lambda \quad \text{on } \Gamma_1, \quad \mathbf{S}(\mathbf{x}, \nabla \mathbf{f}(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma_2, \tag{3.7b}$$

where $\mathbf{n}(\cdot)$ is the unit outer normal vector to $\partial\Omega$. This is a quasilinear system of pde's with a nonlinear (Neumann type) boundary condition.

Let us suppose we have a known branch of solutions $\{(\lambda, \mathbf{f}_\lambda) : \lambda \in \mathbb{R}\}$ of (3.7). We are interested in studying local and global bifurcation from this branch. For a fixed value of λ, the linearization of (3.7) about $\mathbf{f}_\lambda$ is formally given by:

$$\operatorname{div} \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}_\lambda(\mathbf{x}))[\nabla \mathbf{u}] = \mathbf{0} \quad \text{in } \Omega, \tag{3.8a}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}_\lambda(\mathbf{x}))[\nabla \mathbf{u}] \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma_2. \tag{3.8b}$$

This linear BVP corresponds or plays the role of (2.5) in the discussion of Section 2. Note that it is possible to check if the CC is satisfied for this BVP independently of a rigorous linearization.

Let $\mathbf{x}_0 \in \partial\Omega$. For the case $\mathbf{x}_0 \in \Gamma_2$, the auxiliary problem for the CC ($\alpha = 0$) is given by:

$$\operatorname{div} \mathbf{C}_{0,\lambda}[\nabla \mathbf{v}] = \mathbf{0} \quad \text{in } \mathcal{H}, \tag{3.9a}$$

$$\mathbf{C}_{0,\lambda}[\nabla \mathbf{v}] \cdot \mathbf{n}_0 = \mathbf{0} \quad \text{on } \partial\mathcal{H}. \tag{3.9b}$$

where $\mathbf{C}_{0,\lambda} = \mathbf{C}(\mathbf{x}_0, \nabla \mathbf{f}_\lambda(\mathbf{x}_0))$, $\mathbf{n}_0 = \mathbf{n}(\mathbf{x}_0)$. For the bounded exponential function (2.9), upon setting $s = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}_0)$, is easy to check that the boundary value problem (3.9) reduces to the initial value problem:

$$\begin{aligned} \mathbf{C}_{0,\lambda}\left[\mathbf{w}''(s) \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0 \;&+\; i\mathbf{C}_{0,\lambda}\left[\mathbf{w}'(s) \otimes \mathbf{n}_0\right] \cdot \boldsymbol{\xi} \,+ \\ i\mathbf{C}_{0,\lambda}\left[\mathbf{w}'(s) \otimes \boldsymbol{\xi}\right] \cdot \mathbf{n}_0 \;&-\; \mathbf{C}_{0,\lambda}\left[\mathbf{w}(s) \otimes \boldsymbol{\xi}\right] \cdot \boldsymbol{\xi} = \mathbf{0}, \quad s < 0, \end{aligned} \tag{3.10a}$$

$$\mathbf{C}_{0,\lambda}\left[\mathbf{w}'(0) \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0 \;+\; i\mathbf{C}_{0,\lambda}\left[\mathbf{w}(0) \otimes \boldsymbol{\xi}\right] \cdot \mathbf{n}_0 = \mathbf{0}. \tag{3.10b}$$

If we look for solutions of (3.10) of the form $\mathbf{w}(s) = \mathrm{e}^{rs}\mathbf{p}$, where $\mathbf{p} \in \mathbb{R}^3$, then (3.10a) reduces to the matrix equation:

$$\left(r^2 M + ir(N + N^t) - Q\right) \mathbf{p} = \mathbf{0}, \tag{3.11}$$

where

$$\begin{aligned} M\mathbf{p} &= \mathbf{C}_{0,\lambda}\left[\mathbf{p} \otimes \mathbf{n}_0\right] \cdot \mathbf{n}_0, \\ N\mathbf{p} &= \mathbf{C}_{0,\lambda}\left[\mathbf{p} \otimes \mathbf{n}_0\right] \cdot \boldsymbol{\xi}, \\ Q\mathbf{p} &= \mathbf{C}_{0,\lambda}\left[\mathbf{p} \otimes \boldsymbol{\xi}\right] \cdot \boldsymbol{\xi}. \end{aligned} \tag{3.12}$$

The ellipticity condition (3.5) implies that equation (3.11) has exactly three roots r_1, r_2, r_3 with positive real part each, which for simplicity we assume to be different, and with corresponding nonzero vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. The general solution of (3.10a) satisfying that $\mathbf{w}(s) \longrightarrow \mathbf{0}$ as $s \longrightarrow -\infty$, is given by $\mathbf{w}(s) = \sum_k \alpha_k e^{r_k s} \mathbf{p}_k$. The substitution of this expression into (3.10b), leads to the matrix equation:

$$\sum_k \alpha_k \left(r_k M + i N^t \right) \mathbf{p}_k = \mathbf{0}. \tag{3.13}$$

Thus the complementing condition fails for the problem (3.8) at $\mathbf{x}_0 \in \Gamma_2$ with normal $\mathbf{n}_0$ if and only if:

$$g(\lambda, \boldsymbol{\xi}, \mathbf{x}_0) = 0, \tag{3.14}$$

for some nonzero $\boldsymbol{\xi}$, where

$$g(\lambda, \boldsymbol{\xi}, \mathbf{x}_0) \equiv \det \left[MPD + iN^t P \right], \tag{3.15}$$

and

$$P = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3], \quad D = \operatorname{diag}(r_1, r_2, r_3).$$

It follows that the existence of nontrivial bounded exponential solutions for the BVP (3.9), i.e., the failure of the complementing condition, reduces to the vanishing or nonvanishing of (3.15). A similar argument shows that if $\mathbf{x}_0 \in \Gamma_1$, then (3.8) satisfies the CC for the pair $(\mathbf{x}_0, \mathbf{n}(\mathbf{x}_0))$.

We now discuss some of the consequences of the CC. For $0 < \beta < 1$, let

$$X = \left\{ \mathbf{u} \in C^{2,\beta}(\overline{\Omega}, \mathbb{R}^3) \,:\, \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \right\}, \quad Z = C^{0,\beta}(\overline{\Omega}, \mathbb{R}^3), \tag{3.16a}$$

$$Y = C^{0,\beta}(\overline{\Omega}, \mathbb{R}^3) \times C^{1,\beta}(\Gamma_2, \mathbb{R}^3), \tag{3.16b}$$

and define the operator $L : X \longrightarrow Y$ by:

$$L[\mathbf{u}](\mathbf{x}) = (\operatorname{div} \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}_\lambda(\mathbf{x}))[\nabla \mathbf{u}], \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}_\lambda(\mathbf{x}))[\nabla \mathbf{u}] \cdot \mathbf{n}(\mathbf{x})).$$

We assume enough regularity on the stored energy function W so as to make the operator L continuous.

Theorem 4. *Suppose that $\partial\Omega$ is smooth, that $\mathbf{C}(\cdot, \nabla \mathbf{f}_\lambda(\cdot))$ is strongly elliptic, and that the linearized BVP (3.8) satisfy the CC. Then:*

- *The operator L is Fredholm.*
- *There exists a constant $c > 0$ such that for all $\mathbf{u} \in X$:*

$$\|\mathbf{u}\|_X \le c \left[\|L(\mathbf{u})\|_Y + \|\mathbf{u}\|_Z \right].$$

- *If Agmon's condition holds, then the spectrum of L is bounded below.*

The main ideas for the proof of this theorem can be found in [1] and [2], and for the specific context of elasticity in [11] and [27].

A sufficient condition for a solution $\mathbf{f}_\lambda$ of the nonlinear BVP (3.7) to be a weak local minimizer of the energy functional (3.3) is that the *second variation*:

$$\delta^2 I(\mathbf{f}_\lambda)[\mathbf{u}] = \int_\Omega \nabla \mathbf{u}(\mathbf{x}) \cdot \mathbf{C}(\mathbf{x}, \nabla \mathbf{f}_\lambda(\mathbf{x}))[\nabla \mathbf{u}(\mathbf{x})] \, \mathrm{d}\mathbf{x},$$

be uniformly positive, i.e., that there exists a constant $c > 0$ such that:

$$\delta^2 I(\mathbf{f}_\lambda)[\mathbf{u}] \ge c \left[\int_\Omega |\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}\mathbf{x} \right],$$

for all $\mathbf{u} \in C^1(\overline{\Omega})$, $\mathbf{u} = \mathbf{0}$ on Γ_1. Given that $\delta^2 I(\mathbf{f}_\lambda)$ is positive, uniform positivity is related to the CC. Specifically, Simpson and Spector in [27] proved the following.

Theorem 5. *Let $\delta^2 I(\mathbf{f}_\lambda)[\mathbf{u}] > 0$ for all nonzero $\mathbf{u} \in C^1(\Omega)$ vanishing over Γ_1. Then $\delta^2 I(\mathbf{f}_\lambda)$ is uniformly positive if and only if $\mathbf{C}(\cdot, \nabla \mathbf{f}_\lambda(\cdot))$ is strongly elliptic and the corresponding linearization of the nonlinear BVP (3.7) about $\mathbf{f}_\lambda$ satisfies the complementing condition over Γ_2.*

4. Global bifurcation and the CC

As we will show later in this section, our elasticity problem can be formulated abstractly as follows:

$$G(\lambda, \mathbf{u}) = \mathbf{0}\,,$$

where $\lambda \in \mathbb{R}$ is some physical parameter, and G is a differentiable nonlinear operator between appropriate Banach spaces. In many problems G has the additional property that:

$$G(\lambda, \mathbf{0}) = \mathbf{0}, \quad \lambda \in \mathbb{R}.$$

The set $\{(\lambda, \mathbf{0}) \,:\, \lambda \in \mathbb{R}\}$ is called the *trivial branch* of solutions. A necessary condition to have bifurcation from the trivial branch at $(\lambda_*, \mathbf{0})$, is that the linearized problem

$$G_{\mathbf{u}}(\lambda_*, \mathbf{0})[\mathbf{v}] = \mathbf{0}$$

has nontrivial solutions. One of the main difficulties in studying local and global bifurcation in 2D and 3D nonlinear elasticity problems non reducible to ode's, is the presence of the nonlinear traction boundary condition $(3.7\text{b})_2$. In particular, it is not possible to write the operator G as a compact perturbation of the identity. Thus, is not obvious how to apply the well known Krasnoselskii–Rabinowitz approach ([14], [21], [22]), using Leray–Schauder degree [15], to obtain existence of local branches of bifurcating solutions from the trivial branch that can be continued globally.

When using the Leray–Schauder topological degree to get existence of local branches of bifurcating solutions that can be continued globally, two essential ingredients are:

(a) The linearization is a Fredholm operator of index zero possessing only a finite number of negative eigenvalues, each of finite algebraic multiplicity.
(b) Properness of the corresponding nonlinear operator, which in their case is a compact perturbation of identity.

We recall that a mapping $G : X \to Y$ is *proper* if $G^{-1}(K)$ is compact in X for any compact set $K \subset Y$.

Fesnke [7] and Kielhöfer [13] defined a degree more general than the Leray–Schauder degree for maps satisfying (a) and (b) above. Healey and Simpson [11] improved upon the work of Fenske and Kielhöfer and constructed a degree specifically designed to overcome the difficulties associated to the general class of nonlinear BVP's in 3D elasticity. The presence of the more general boundary conditions forces the consideration of the complementing and Agmon's conditions for the corresponding linearized problems.

As before, we assume that $\{(\lambda, \mathbf{f}_\lambda) : \lambda \in \mathbb{R}\}$ is a known solution branch of the (nonlinear) problem (3.7) and define $\mathbf{u} = \mathbf{f} - \mathbf{f}_\lambda$. Then we can write the BVP (3.7) as:

$$\text{(4.1a)} \qquad \operatorname{div} \mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) = \mathbf{0}, \quad \text{in } \Omega,$$

$$\text{(4.1b)} \qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad \mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma_2,$$

where

$$\mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) = \mathbf{S}(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}) + \nabla \mathbf{f}_\lambda(\mathbf{x})).$$

Using the spaces (3.16), and in reference to (3.2), (3.5), and (3.14), we define the set:

$$\begin{aligned} \mathcal{U} = \Big\{ (\lambda, \mathbf{u}) \in \mathbb{R} \times X \ : \ & \det(\nabla \mathbf{u}(\mathbf{x}) + \nabla \mathbf{f}_\lambda(\mathbf{x})) > 0, \ \forall\, \mathbf{x} \in \overline{\Omega}, \\ & |g(\lambda, \boldsymbol{\xi}, \mathbf{x})| > 0, \ \forall\ \boldsymbol{\xi} \cdot \mathbf{n}(\mathbf{x}) = 0, \ \|\boldsymbol{\xi}\| \neq 0, \ \mathbf{x} \in \Gamma_2, \\ & \mathbf{ab} : \mathbf{C}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))[\mathbf{ab}] > 0, \ \forall\, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}, \ \mathbf{x} \in \overline{\Omega} \Big\}, \end{aligned}$$

and the operator $G : \mathcal{U} \longrightarrow Y$ by:

$$\text{(4.2)} \qquad G(\lambda, \mathbf{u})(\mathbf{x}) = \Big(\operatorname{div} \mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})), \mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) \Big).$$

The BVP (4.1) is now equivalent to:

$$G(\lambda, \mathbf{u}) = \mathbf{0}.$$

Note that

$$\text{(4.3)} \qquad G(\lambda, \mathbf{0}) = \mathbf{0}, \quad \forall\, \lambda,$$

which is just a restatement that $\mathbf{f}_\lambda$ is a solution of (3.7).

If we assume enough regularity on the stored energy function W, on the boundary $\partial\Omega$, and on $\mathbf{f}_\lambda$ as a function of λ, then we get that $G \in C^2$. (See [29].) Moreover

$$G_{\mathbf{u}}(\lambda, \mathbf{u})[\mathbf{h}](\mathbf{x}) = \Big(\operatorname{div} \mathbf{C}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))[\nabla \mathbf{h}], \mathbf{C}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))[\nabla \mathbf{h}] \cdot \mathbf{n}(\mathbf{x}) \Big),$$

where $G_{\mathbf{u}}(\lambda, \cdot)$ is the Fréchet derivative of G with respect to its second argument. We now have (see Figure 1):

Theorem 6 ([11], [9], [10])**.** *Let $G : \mathcal{U} \longrightarrow Y$ be a C^2 operator and assume that $G(\lambda, \mathbf{0}) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. Let λ_* be such that:*

(H1) $\dim \ker G_{\mathbf{u}}(\lambda_*, \mathbf{0}) = 1$.

(H2) *The linearized problem $G_{\mathbf{u}}(\lambda_*, \mathbf{0})[\mathbf{h}] = \mathbf{f}$ satisfies the CC.*

(H3) *If $\ker G_{\mathbf{u}}(\lambda_*, \mathbf{0}) = \operatorname{span}\{\mathbf{h}_*\}$, and $M \equiv G_{\mathbf{u},\lambda}(\lambda_*, \mathbf{0})$, then*

$$M\mathbf{h}_* \notin \operatorname{range} G_{\mathbf{u}}(\lambda_*, \mathbf{0}), \quad \textit{(crossing condition).}$$

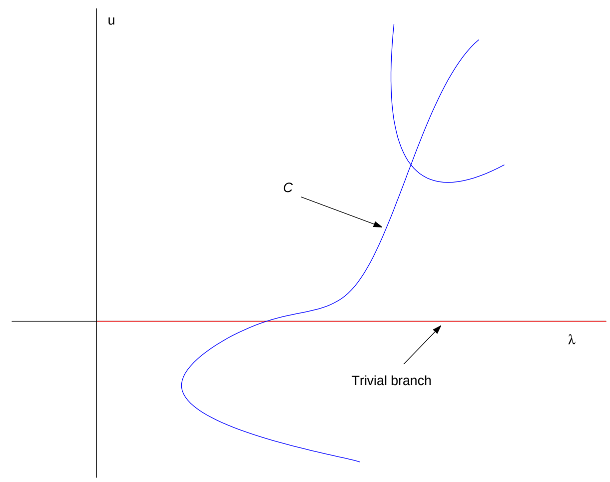

FIGURE 1. A possible sketch for the set $\mathcal{C}$ in Theorem 6.

Then, $(\lambda_*, \mathbf{0})$ *is a bifurcation point of a local continuous branch of nontrivial solutions of* $G(\lambda, \mathbf{u}) = \mathbf{0}$*. If in addition to* (H2)*, the linearized problem* $G_{\mathbf{u}}(\lambda_*, \mathbf{0})[\mathbf{h}] = \mathbf{f}$ *satisfies Agmon's condition, then this branch can be globally extended to a set* $\mathcal{C}$ *satisfying at least one of the following alternatives:*

(A1) $\mathcal{C}$ *is unbounded in* $\mathbb{R} \times X$*.*
(A2) $\mathcal{C}$ *comes back to the trivial branch* $\{(\lambda, \mathbf{0}) : \lambda \in \mathbb{R}\}$*.*
(A3) *Local injectivity (cf.* (3.2)*) is violated at some point of* $\mathcal{C}$*.*
(A4) *Ellipticity fails at some point of* $\mathcal{C}$*.*
(A5) *The complementing condition fails at some point of* $\mathcal{C}$*.*

Remarks.

- The hypotheses (H1)–(H3) of the theorem are in general hard to verify in elasticity even for specific stored energy functions W.
- Both (A3) and (A4) can be ruled out by making physically reasonable constitutive assumptions on the stored energy function W. (See [12].) However (A5) can not be ruled out in general by pointwise constitutive assumptions alone.

Example 7. The following presentation is based mostly on the results in [10]. We consider the axial compression of a cylinder with reference configuration given by:

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2 \, , \; 0 < x_3 < L\right\}$$

One specifies the compression ratio λ in the x_3 direction and look for axisymmetric (about the x_3 axis) deformations of the cylinder. (See Figure 2.) In addition we have the nonlinear Neumann type boundary condition of zero traction on the lateral boundary, and zero shear on the top and bottom of the cylinder. We solve the PDE (4.1a) subject to these boundary

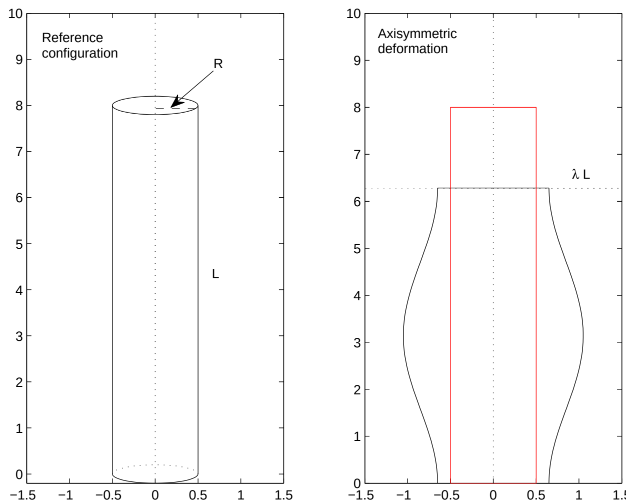

FIGURE 2. Axial compression of a cylindrical structure showing a cross section of an axisymmetric deformation.

conditions with Ω as above. We assume the material of the cylinder is homogenous which implies that the stored energy function W does not depend on $\mathbf{x}$. The trivial solution $\mathbf{f}_\lambda$ is now given by a uniform compression of the cylinder along the x_3 direction.

As posed, the results from Theorem 6 are not applicable to this problem because the boundary $\partial\Omega$ is not smooth. One can eliminate the corners in Ω by extending it periodically in the x_3 direction, but the resulting extended problem is not necessarily equivalent to the original one. However, by exploiting some hidden symmetries of the stress tensor, one can show [10, Proposition (2.3)] that solutions over the extended domain are equivalent to solutions of the original problem. For the stored energy function

$$W(\mathbf{F}) = \frac{1}{2}\,\mathbf{F}\cdot\mathbf{F} + \frac{1}{m}(\det \mathbf{F})^{-m}, \tag{4.4}$$

where $m > 0$, one can check that all the hypotheses of Theorem 6 hold and we get the existence of global bifurcating branches. Moreover, in this problem the CC fails exactly for one value of λ, and there is an infinite sequence of bifurcation points that converges to the value where the CC fails.

If we do not restrict to axisymmetric deformations, then one can show [19, Theorem 5.1] that the CC for this problem fails for a full interval of values of λ.

Example 8. In this example we consider the compression of a two dimensional rectangular slab. The presentation is based on the results in [18] and

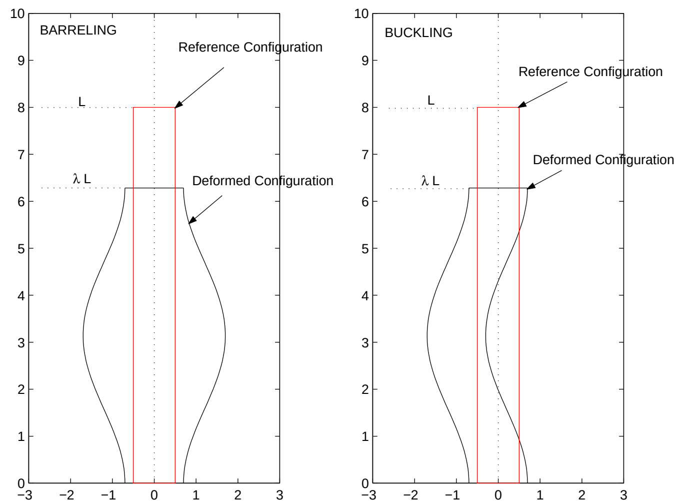

FIGURE 3. Compressions of either barreling or buckling type of a rectangular slab.

the reference configuration is now given by:

$$\Omega = \{(x, y) \,:\, -R < x < R \,,\; 0 < y < L\} \,.$$

One specifies the compression ratio λ in the y direction. We have the nonlinear Neumann type boundary condition of zero traction on the lateral boundary

$$\{(x, y) \,:\, x = \pm R,\; 0 \leq y \leq L\} \,,$$

and zero shear on the top and bottom of the slab:

$$\{(x, y) \,:\, -R \leq x \leq R,\; y = 0, L\} \,.$$

Contrary to the previous example, the deformations can break symmetry with respect to the compression axis. Thus these deformations can be of either barreling or buckling type. (See Figure 3.)

We consider a stored energy function of the form

$$\hat{W}(\nabla \mathbf{f}, \nabla^2 \mathbf{f}) = W(\nabla \mathbf{f}) + \varepsilon \nabla^2 \mathbf{f} \vdots \nabla^2 \mathbf{f},$$

where $W(\mathbf{F})$ is given by (4.4), and $\varepsilon > 0$ is a perturbation parameter. The boundary value problem (4.1) has to be modified for this $\hat{W}$. For simplicity of exposition, we do not discuss the modified boundary conditions (cf. [18]). The PDE is now given by:

$$\varepsilon \Delta^2 \mathbf{u} - \operatorname{div} \mathbf{S}_\lambda(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x})) = \mathbf{0}, \quad \text{in } \Omega, \tag{4.5}$$

with $\mathbf{S}_\lambda$ the stress tensor corresponding to W, and $\mathbf{f}_\lambda$ a uniform compression along the y axis. As posed, Theorem 6 is not applicable to this problem because of the corners in $\partial\Omega$. Once again we extend Ω periodically in the y

direction so as to eliminate the corners, and some hidden symmetries of the stress tensor are used to check that solutions over the extended domain are equivalent to solutions of the original problem.

Because of the structure of the differential operator in (4.5) and the corresponding boundary conditions, the linearized problem is always strongly elliptic and satisfies the CC. Thus we can apply now Theorem 6 (first part, local bifurcation only) to get the existence of local branches of nontrivial solutions, provided (H1)–(H3) hold. However, even for the function (4.4) above, one can only check numerically conditions (H1) and (H3). This example suggests that in general, if the CC always holds, then the number of bifurcation points must be finite. In this problem, the number of bifurcation points monotonically increases as $\varepsilon \longrightarrow 0$. For the corresponding problem with $\varepsilon = 0$, it has been shown [25] that the CC fails at exactly one value λ_*, and the number of bifurcation points is infinite and accumulating at λ_*.

5. Violations of the CC

What happens when the CC fails? We have observed in several problems in elasticity, like the one in Example 7 for the axial compression of a cylinder (see also [23], [25], [26], [20]), a recurrent relation between violation of the CC and the existence of sequences of possible bifurcation points accumulating at a point where the CC fails. (See Figure 4.) This observation motivates the following two conjectures:

(a) If λ_c is an accumulation point of the set:

$$\{\lambda : G_{\mathbf{u}}(\lambda, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0} \text{ has nontrivial solutions}\}, \tag{5.1}$$

then the linear operator $G_{\mathbf{u}}(\lambda_c, \mathbf{0})$ fails to satisfy the complementing condition.

(b) If λ_c is a boundary point and a member of the set

$$\{\lambda : G_{\mathbf{u}}(\lambda, \mathbf{0}) \text{ does not satisfy the CC}\},$$

then there exists a sequence (λ_n) in the set (5.1) such that $\lim_n \lambda_n = \lambda_c$.

Both conjectures have been shown to be true for the problem of the axial axisymmetric compressions of a cylinder discussed in Example 7 for more general stored energy functions than (4.4). A function $W(\mathbf{F})$ is *isotropic* (see eg. [3] or [4]) if:

$$W(\mathbf{F}) = \Phi(I_1, I_2, I_3),$$

where (I_1, I_2, I_3) are the principal invariants of $\mathbf{F}^t\mathbf{F}$ and Φ is a symmetric function of its arguments. For axisymmetric deformations of the cylinder, the vector $\boldsymbol{\xi}$ in (3.14) is restricted to $(0, 0, \pm 1)$. Moreover since the W above is independent of $\mathbf{x}$ and because of the axial symmetry of the cylinder, the function (3.15) depends only on λ. We say that λ^* *is a simple value for which the CC fails* if λ^* is a simple root of (3.15).

We now have:

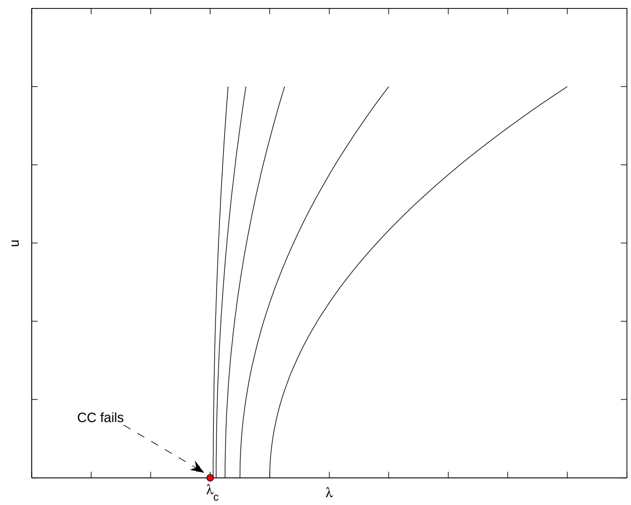

FIGURE 4. Sequence of bifurcating branches accumulating at a point where the CC fails.

Theorem 9 ([19]). *Let the stored energy function W be isotropic and consider the BVP of Example 7. Let (λ_n) be a sequence of possible bifurcation points. If λ^* is an accumulation point of (λ_n), then the complementing condition fails for $\lambda = \lambda^*$. On the other hand, if λ^* is a simple value for which the CC fails for the linearized problem, then there exists a sequence (λ_n) of possible bifurcation points that converge to λ^*.*

Since the eigenfunctions corresponding to each of these possible bifurcation points have more nodes (i.e., are more "oscillatory") as the mode number increases, we get that locally the corresponding nonlinear equilibrium solutions develop high spatial oscillations near the boundary. This is consistent with an interpretation within the static theory of elasticity of violation of the complementing condition as a material *wrinkling* type instability (see [19]). It remains as an open problem to show whether or not this result is true for more general boundary value problems.

6. Closing remarks

The complementing condition plays a central role in the applicability of a generalized degree theory suitable for problems in nonlinear elasticity that are non reducible to ode's. For fully two or three dimensional problems, the verification of the hypotheses in Theorem 6 can be very difficult, in particular the one dimensional kernel (H1) and the crossing condition (H3). Although in principle the complementing condition is simpler to check as it reduces to an algebraic equation, computing this algebraic equation (cf.

(3.14)) and showing whether it has solutions or not, is not necessarily an easy task.

We have seen that failure of the complementing condition is tantamount (at least for deformations of cylindrical structures) to the existence of possible bifurcation points accumulating at the point where the complementing condition fails for the linearized problem. If true for more general boundary value problems, such a statement would provide a very powerful bifurcation theorem as failure of the complementing condition would imply the existence of a sequence of possible bifurcation points accumulating at the point where the complementing condition fails, and in general checking whether the complementing condition fails or not could be a much simpler problem than characterizing the values of λ in Theorem 6 for which the linearized problem $G_{\mathbf{u}}(\lambda, \mathbf{0}) \cdot \mathbf{v} = \mathbf{0}$ has nontrivial solutions $\mathbf{v}$.

Appendix A. The CC as an algebraic condition

We employ the notation introduced in Section 2. The operator (2.1) is *elliptic* if

$$\sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x})\boldsymbol{\eta}^{\mathbf{s}} \neq 0, \tag{A.1}$$

for any nonzero $\boldsymbol{\eta} \in \mathbb{R}^n$ and $\mathbf{x} \in \overline{\Omega}$. Recall that if $\mathbf{s} = (s_1, \ldots, s_n)$, then $\boldsymbol{\eta}^{\mathbf{s}} = \eta_1^{s_1} \cdots \eta_n^{s_n}$.

Corresponding to the operators (2.8), we define the associated polynomials:

$$P(r) = \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0)(\boldsymbol{\xi} + r\mathbf{n})^{\mathbf{s}},$$
$$P_j(r) = \sum_{|\mathbf{s}|=m_j} b_{j\,\mathbf{s}}(\mathbf{x}_0)(\boldsymbol{\xi} + r\mathbf{n})^{\mathbf{s}}, \quad 1 \leq j \leq m,$$

where as before $\boldsymbol{\xi}$ is a nonzero vector with $\boldsymbol{\xi} \cdot \mathbf{n} = 0$, and $\mathbf{n} = \mathbf{n}(\mathbf{x}_0)$ is the unit outer normal to $\partial\Omega$ at $\mathbf{x}_0$. If $n > 2$, the ellipticity condition (A.1) implies (see [1]) that P has exactly[2] m roots $r_1, \ldots, r_m$ (which we assume to be distinct), with negative imaginary part. When $n = 2$, this condition has to be explicitly assumed.

Using the Binomial Theorem to expand each of the terms in $(\boldsymbol{\xi} + r\mathbf{n})^{\mathbf{s}}$, we get that

$$P(r) = \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} r^{|\boldsymbol{\beta}|}, \tag{A.2a}$$

$$P_j(r) = \sum_{|\mathbf{s}|=m_j} b_{j\,\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} r^{|\boldsymbol{\beta}|}, \tag{A.2b}$$

[2]The polynomial P is of degree $2m$. The other m roots have positive imaginary part.

where $1 \leq j \leq m$. Here $\boldsymbol{\beta} \leq \mathbf{s}$ means that $\beta_k \leq s_k$ for $1 \leq k \leq n$, and

$$\binom{\mathbf{s}}{\boldsymbol{\beta}} = \binom{s_1}{\beta_1} \cdots \binom{s_n}{\beta_n}.$$

The following result shows that the CC is equivalent to a certain algebraic condition on the polynomials $\{P_j(r)\}$ and $P(r)$.

Proposition 10. *The BVP (2.5) satisfy the complementing condition for the pair $(\mathbf{x}_0, \mathbf{n})$ if and only if the polynomials $\{P_j(r)\}$ are linearly independent modulo the polynomial $(r - r_1) \cdots (r - r_m)$.*

Proof. Upon recalling the identity

$$\mathrm{D}^{\mathbf{s}}(uv) = \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} \mathrm{D}^{\boldsymbol{\beta}} u \mathrm{D}^{\mathbf{s}-\boldsymbol{\beta}} v,$$

we get that for the function (2.9), the BVP (2.7) with $\alpha = 0$ is equivalent to the following initial value problem for w:

$$\text{(A.3a)} \qquad \sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} i^{|\mathbf{s}|-|\boldsymbol{\beta}|} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} w^{|\boldsymbol{\beta}|}(s) = 0,$$

$$\text{(A.3b)} \qquad \sum_{|\mathbf{s}|=m_j} b_{j\,\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} i^{|\mathbf{s}|-|\boldsymbol{\beta}|} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} w^{|\boldsymbol{\beta}|}(0) = 0,$$

where $1 \leq j \leq m$ and $s = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}$. If we look for solutions of the form $w(s) = \mathrm{e}^{\rho s}$, then (A.3a) reduces to:

$$\sum_{|\mathbf{s}|=2m} a_{\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} i^{|\mathbf{s}|-|\boldsymbol{\beta}|} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} \rho^{|\boldsymbol{\beta}|} = 0.$$

Since for $|\mathbf{s}| = 2m$ we have that $i^{|\mathbf{s}|-|\boldsymbol{\beta}|} = (-1)^m (-i)^{|\boldsymbol{\beta}|}$, it follows from (A.2a) that the above equation is equivalent to:

$$(-1)^m P(-i\rho) = 0.$$

Hence we have that $\rho_k = i r_k$, $1 \leq k \leq m$, are roots of this equation. Since r_k has negative imaginary part, ρ_k has positive real part. It follows that

$$w(s) = \sum_{k=1}^{m} c_k \mathrm{e}^{i r_k s},$$

is the general solution of (A.3a) satisfying that $w(s) \longrightarrow 0$ as $s \longrightarrow -\infty$. If now we substitute this expression for w into (A.3b) we get that:

$$\sum_{|\mathbf{s}|=m_j} b_{j\,\mathbf{s}}(\mathbf{x}_0) \sum_{\boldsymbol{\beta} \leq \mathbf{s}} \binom{\mathbf{s}}{\boldsymbol{\beta}} \mathbf{n}^{\boldsymbol{\beta}} \boldsymbol{\xi}^{\mathbf{s}-\boldsymbol{\beta}} i^{|\mathbf{s}|-|\boldsymbol{\beta}|} \sum_{k=1}^{m} c_k i^{|\boldsymbol{\beta}|} r_k^{|\boldsymbol{\beta}|} = 0,$$

which after simplification is equivalent to:

$$\sum_{k=1}^{m} c_k P_j(r_k) = 0, \quad 1 \leq j \leq m.$$

Thus the only solution of the BVP (2.7) with $\alpha = 0$ of the form (2.9), is the one with $w \equiv 0$ if and only if the matrix

$$\begin{pmatrix} P_1(r_1) & \cdots & P_1(r_m) \\ \vdots & \ddots & \vdots \\ P_m(r_1) & \cdots & P_m(r_m) \end{pmatrix}$$

is nonsingular. The requirement that the rows of this matrix be linearly independent, is equivalent to that of the polynomials $\{P_j(r)\}$ be linearly independent modulo the polynomial $(r - r_1)\cdots(r - r_m)$. □

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