On a Teichmüller functor between the categories of complex tori and the Effros–Shen algebras

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ABSTRACT. A covariant functor from the category of the complex tori to the category of the Effros–Shen algebras is constructed. The functor maps isomorphic complex tori to the stably isomorphic Effros–Shen algebras. Our construction is based on the Teichmüller theory of the Riemann surfaces.

A. Complex tori. Let ω_1 and ω_2 be a pair of nonzero complex numbers linearly independent over $\mathbb{R}$. Consider a lattice $\Lambda := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in the complex plane $\mathbb{C}$ and the quotient space $\mathbb{C}/\Lambda$. The space $\mathbb{C}/\Lambda$ is known as a *complex torus.* It is easy to see that the conformal transformation of the complex plane $z \to \pm\frac{\omega_2}{\omega_1}z$ brings the complex torus to a normal form $\mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. The transformation law in a lattice implies that the complex tori $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ are conformally equivalent (isomorphic) whenever $\tau' \equiv \tau \bmod \text{SL}_2(\mathbb{Z})$, i.e., $\tau' = \frac{a+b\tau}{c+d\tau}$, where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

B. The Effros–Shen algebras. Let $\theta > 0$ be an irrational number given by the regular continued fraction $[a_0, a_1, a_2, \ldots]$, where $a_0 \in \mathbb{N} \cup 0$ and $a_i \in \mathbb{N}$ for $i \geq 1$. By an *Effros–Shen algebra* [1], one understands the *AF*-algebra $\mathbb{A}_\theta$ given by the Bratteli diagram in Figure 1. Here a_i indicates the number of edges in the upper row of the graph. Recall that the Effros–Shen algebras $\mathbb{A}_\theta, \mathbb{A}_{\theta'}$ are said to be stably isomorphic (Morita equivalent), if $\mathbb{A}_\theta \otimes \mathcal{K} \cong \mathbb{A}_{\theta'} \otimes \mathcal{K}$, where $\mathcal{K}$ is a C^*-algebra of compact operators. A remarkable result, proved in [1], says that $\mathbb{A}_\theta$ and $\mathbb{A}_{\theta'}$ are stably isomorphic if and only if $\theta' \equiv \theta \bmod \text{SL}_2(\mathbb{Z})$.

C. Motivation and background. Comparing the category of complex tori with that of the Effros–Shen algebras, one cannot fail to observe that for the

generic objects in the respective categories, the corresponding morphisms (modulo the inner automorphisms) are isomorphic as groups. Assuming that the observation is not a simple coincidence, one can ask the following question.

Main Problem. *Let $\mathcal{A}$ be the category of complex tori described in Section A, and let $\mathcal{B}$ be the category of Effros–Shen algebras described in Section B. Construct a functor (if any) $F : \mathcal{A} \to \mathcal{B}$, which maps isomorphic complex tori to stably isomorphic Effros–Shen algebras.*

The question attracted attention of both the algebraic geometers and the operator algebraists. Manin [6] and Soibelman [16], [17] were apparently the first to study such a functor. (Note that the authors usually consider a category of noncommutative tori [15], i.e., the universal C^*-algebras generated by the two unitary operators U and V, which satisfy the commutation relation $VU = e^{2\pi i \theta} UV$. However, it is well-known that the two objects are closely related [7].) The topic was pursued by Polishchuk [10]–[13] and Polishchuk–Schwarz [14], using the methods of homological algebra and algebraic geometry. The works of Kontsevich [3] and Soibelman–Vologodsky [18] develop the ideas of a homological mirror symmetry and the deformation quantization of the elliptic curves. Finally, Mahanta [4], Mahanta–van Suijlekom [5], Plazas [8], [9] and Taylor [20], [21] elaborated the ideas of Polishchuk–Schwarz and Manin, respectively.

D. Measured foliations. A measured foliation, $\mathcal{F}$, on a surface X is a partition of X into singular points $x_1, \ldots, x_n$ of order $k_1, \ldots, k_n$ and the regular leaves (1-dimensional submanifolds). On each open cover U_i of $X - \{x_1, \ldots, x_n\}$ there exists a nonvanishing real-valued closed 1-form ϕ_i such that:

(i) $\phi_i = \pm \phi_j$ on $U_i \cap U_j$.
(ii) At each x_i there exists a local chart $(u, v) : V \to \mathbb{R}^2$ such that for $z = u + iv$, we have $\phi_i = \text{Im}\big(z^{\frac{k_i}{2}} dz\big)$ on $V \cap U_i$ for some branch of $z^{\frac{k_i}{2}}$.

The pair (U_i, ϕ_i) is called an atlas for the measured foliation $\mathcal{F}$. Finally, a measure μ is assigned to each segment $(t_0, t) \in U_i$, which is transverse to the leaves of $\mathcal{F}$, via the integral $\mu(t_0, t) = \int_{t_0}^{t} \phi_i$. The measure is invariant along the leaves of $\mathcal{F}$, hence the name. Note that in the case $X = T^2$ is a two-dimensional torus (our main concern), every measured foliation is given by a family of the parallel lines of a slope $\theta > 0$: see Figure 2.

E. The Hubbard–Masur homeomorphism. Let $T(g)$ be the Teichmüller space of the topological surface X of genus $g \geq 1$, i.e., the space of complex structures on X. Consider the vector bundle $p : Q \to T(g)$ over $T(g)$, whose fiber above a point $S \in T_g$ is the vector space $H^0(S, \Omega^{\otimes 2})$. Given a nonzero $q \in Q$ above S, we can consider the horizontal measured foliation $\mathcal{F}_q \in \Phi_X$ of q, where Φ_X denotes the space of equivalence classes of measured foliations

on X. If $\{0\}$ is the zero section of Q, the above construction defines a map $Q - \{0\} \longrightarrow \Phi_X$. For any $\mathcal{F} \in \Phi_X$, let $E_\mathcal{F} \subset Q - \{0\}$ be the fiber above $\mathcal{F}$. In other words, $E_\mathcal{F}$ is a subspace of the holomorphic quadratic forms, whose horizontal trajectory structure coincides with the measured foliation $\mathcal{F}$. Note that, if $\mathcal{F}$ is a measured foliation with simple zeroes (a generic case), then $E_\mathcal{F} \cong \mathbb{R}^n - 0$, while $T(g) \cong \mathbb{R}^n$, where $n = 6g - 6$ if $g \geq 2$ and $n = 2$ if $g = 1$.

Theorem (Hubbard–Masur [2]). *The restriction of p to $E_\mathcal{F}$ defines a homeomorphism (an embedding) $h_\mathcal{F} : E_\mathcal{F} \to T(g)$.*

F. The Teichmüller space and measured foliations. The Hubbard–Masur result implies that the measured foliations parametrize the space $T(g) - \{\mathrm{pt}\}$, where $\mathrm{pt} = h_\mathcal{F}(0)$. Indeed, denote by $\mathcal{F}'$ a vertical trajectory structure of q. Since $\mathcal{F}$ and $\mathcal{F}'$ define q, and $\mathcal{F} = \mathrm{Const}$ for all $q \in E_\mathcal{F}$, one gets a homeomorphism between $T(g) - \{\mathrm{pt}\}$ and Φ_X, where $\Phi_X \cong \mathbb{R}^n - 0$ is the space of equivalence classes of the measured foliations $\mathcal{F}'$ on X. Note that the above parametrization depends on a foliation $\mathcal{F}$. However, there exists a unique canonical homeomorphism $h = h_\mathcal{F}$ as follows. Let $\mathrm{Sp}(S)$ be the length spectrum of the Riemann surface S and $\mathrm{Sp}(\mathcal{F}')$ be the set positive reals $\inf \mu(\gamma_i)$, where γ_i runs over all simple closed curves transverse to the foliation $\mathcal{F}'$. A canonical homeomorphism $h = h_\mathcal{F} : \Phi_X \to T(g) - \{\mathrm{pt}\}$ is defined by the formula $\mathrm{Sp}(\mathcal{F}') = \mathrm{Sp}(h_\mathcal{F}(\mathcal{F}'))$ for $\forall \mathcal{F}' \in \Phi_X$. Thus, the following corollary is true.

Corollary *There exists a canonical homeomorphism $h : \Phi_X \to T(g) - \{\mathrm{pt}\}$.*

G. A parametrization of $\mathbb{H} - \{\mathrm{pt}\}$ by the measured foliations. In the case $X = T^2$, the picture simplifies. First, notice that $T(1) \cong \mathbb{H}$. Since $q \neq 0$ there are no singular points and each $q \in H^0(S, \Omega^{\otimes 2})$ has the form $q = \omega^2$, where ω is a nowhere zero holomorphic differential on the complex torus S. (Note that ω is just a constant times dz, and hence its vertical trajectory structure is just a family of the parallel lines of a slope θ, see e.g., Strebel [19], pp. 54–55.) Therefore, Φ_{T^2} consists of the equivalence classes of the nonsingular measured foliations on the two-dimensional torus. It is well-known (the Denjoy theory), that every such foliation is measure equivalent to the foliation of a slope θ and a transverse measure $\mu > 0$, which is invariant along the leaves of the foliation (Figure 2). Thus, one obtains a canonical bijection $h : \Phi_{T^2} \to \mathbb{H} - \{\mathrm{pt}\}$.

H. Lattices. Let $\mathbb{C}$ be the complex plane. A *lattice* is a triple $(\Lambda, \mathbb{C}, j)$, where $\Lambda \cong \mathbb{Z}^2$ and $j : \Lambda \to \mathbb{C}$ is an injective homomorphism with discrete image. A *morphism* of lattices $(\Lambda, \mathbb{C}, j) \to (\Lambda', \mathbb{C}, j')$ is the identity $j \circ \psi = \varphi \circ j'$ where φ is a group homomorphism and ψ is a $\mathbb{C}$-linear map. It is not hard to see that any isomorphism class of a lattice contains a representative given by $j : \mathbb{Z}^2 \to \mathbb{C}$ such that $j(1,0) = 1$, $j(0,1) = \tau \in \mathbb{H}$. The category of lattices, $\mathcal{L}$, consists of $\mathrm{Ob}(\mathcal{L})$, which are lattices $(\Lambda, \mathbb{C}, j)$

and morphisms $H(L, L')$ between $L, L' \in \mathrm{Ob}(\mathcal{L})$ which coincide with the morphisms of lattices specified above. For any $L, L', L'' \in \mathrm{Ob}(\mathcal{L})$ and any morphisms $\varphi' : L \to L'$, $\varphi'' : L' \to L''$ a morphism $\phi : L \to L''$ is the *composite* of φ' and φ'', which we write as $\phi = \varphi'' \varphi'$. The *identity* morphism, 1_L, is a morphism $H(L, L)$. Note that the lattices are bijective with the complex tori via the formula $(\Lambda, \mathbb{C}, j) \mapsto \mathbb{C}/j(\Lambda)$. Therefore, $\mathcal{L} \cong \mathcal{A}$.

I. Pseudo-lattices. Let $\mathbb{R}$ be the real line. A *pseudo-lattice* (of rank 2) is a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^2$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of the pseudo-lattices $(\Lambda, \mathbb{R}, j) \to (\Lambda', \mathbb{R}, j')$ is the identity $j \circ \psi = \varphi \circ j'$, where φ is a group homomorphism and ψ is an inclusion map (i.e., $j'(\Lambda') \subseteq j(\Lambda)$). Any isomorphism class of a pseudo-lattice contains a representative given by $j : \mathbb{Z}^2 \to \mathbb{R}$, such that $j(1, 0) = \lambda_1$, $j(0, 1) = \lambda_2$, where λ_1, λ_2 are positive reals. The pseudo-lattices make up a category, which we denote by $\mathcal{PL}$.

Lemma 1. *The pseudo-lattices are bijective with the measured foliations on the torus via the formula $(\Lambda, \mathbb{R}, j) \mapsto \mathcal{F}_{\lambda_2/\lambda_1}^{\lambda_1}$, where $\mathcal{F}_{\lambda_2/\lambda_1}^{\lambda_1}$ is a foliation of the slope $\theta = \lambda_2/\lambda_1$ and measure $\mu = \lambda_1$.*

Proof. Define a pairing by the formula $(\gamma, \mathrm{Re}\,\omega) \mapsto \int_\gamma \mathrm{Re}\,\omega$, where $\gamma \in H_1(T^2, \mathbb{Z})$ and $\omega \in H^0(S; \Omega)$. The trajectories of the closed differential $\phi := \mathrm{Re}\,\omega$ define a measured foliation on T^2. Thus, in view of the pairing, the linear spaces Φ_{T^2} and $\mathrm{Hom}(H_1(T^2, \mathbb{Z}); \mathbb{R})$ are isomorphic. Notice that the latter space coincides with the space of the pseudo-lattices. To obtain an explicit bijection formula, let us evaluate the integral:

$$(1) \qquad \int_{\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2} \phi = \mathbb{Z} \int_{\gamma_1} \phi + \mathbb{Z} \int_{\gamma_2} \phi = \mathbb{Z} \int_0^1 \mu dx + \mathbb{Z} \int_0^1 \mu dy,$$

where $\{\gamma_1, \gamma_2\}$ is a basis in $H_1(T^2, \mathbb{Z})$. Since $\frac{dy}{dx} = \theta$, one gets:

$$\begin{cases} \int_0^1 \mu dx = \mu = \lambda_1 \\ \int_0^1 \mu dy = \int_0^1 \mu \theta dx = \mu \theta = \lambda_2. \end{cases}$$

Thus, $\mu = \lambda_1$ and $\theta = \frac{\lambda_2}{\lambda_1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

It follows from Lemma 1 and the canonical bijection $h : \Phi_{T^2} \to \mathbb{H} - \{\mathrm{pt}\}$, that $\mathcal{L} \cong \mathcal{PL}$ are the equivalent categories.

J. Projective pseudo-lattices. Finally, a *projective pseudo-lattice* (of rank 2) is a triple $(\Lambda, \mathbb{R}, j)$, where $\Lambda \cong \mathbb{Z}^2$ and $j : \Lambda \to \mathbb{R}$ is a homomorphism. A morphism of the projective pseudo-lattices $(\Lambda, \mathbb{C}, j) \to (\Lambda', \mathbb{R}, j')$ is the identity $j \circ \psi = \varphi \circ j'$, where φ is a group homomorphism and ψ is an $\mathbb{R}$-linear map. (Notice, that unlike the case of the pseudo-lattices, ψ is a scaling map as opposed to an inclusion map. Thus, the two pseudo-lattices can be projectively equivalent, while being distinct in the category $\mathcal{PL}$.) It is not hard to see that any isomorphism class of a projective pseudo-lattice contains

a representative given by $j : \mathbb{Z}^2 \to \mathbb{R}$ such that $j(1,0) = 1$, $j(0,1) = \theta$, where θ is a positive real. Note that the projective pseudo-lattices are bijective with the Effros–Shen algebras, via the formula $(\Lambda, \mathbb{R}, j) \mapsto \mathbb{A}_\theta$. The projective pseudo-lattices make up a category, which we shall denote by $\mathcal{PPL}$.

Lemma 2. $\mathcal{PPL} \cong \mathcal{B}$.

Proof. An isomorphism $\varphi : \Lambda \to \Lambda'$ acts by the formula $1 \mapsto a + b\theta$, $\theta \mapsto c + d\theta$, where $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. Therefore,

$$\theta' = \frac{c + d\theta}{a + b\theta} = \theta \bmod \mathrm{SL}_2(\mathbb{Z}).$$

Thus, the isomorphic projective pseudo-lattices map to the stably isomorphic Effros–Shen algebras. $\square$

K. The map F and main results. To finish the construction of a map $F : \mathcal{A} \to \mathcal{B}$, consider a composition of the following morphisms:

$$(2) \qquad \mathcal{A} \xrightarrow{\sim} \mathcal{L} \xrightarrow{\sim} \mathcal{PL} \xrightarrow{F} \mathcal{PPL} \xrightarrow{\sim} \mathcal{B},$$

where all the arrows, but F, have been defined. To define F, let $PL \in \mathcal{PL}$ be a pseudo-lattice, such that $PL = PL(\lambda_1, \lambda_2)$, where $\lambda_1 = j(1,0)$, $\lambda_2 = j(0,1)$ are positive reals. Let $PPL \in \mathcal{PPL}$ be a projective pseudo-lattice, such that $PPL = PPL(\theta)$, where $j(1,0) = 1$ and $j(0,1) = \theta$ is a positive real. Then $F : \mathcal{PL} \to \mathcal{PPL}$ is given by the formula $PL(\lambda_1, \lambda_2) \longmapsto PPL\left(\frac{\lambda_2}{\lambda_1}\right)$. It is easy to see, that $\mathrm{Ker}\, F \cong (0, \infty)$ and F is not an injective map. Since all the arrows, but F, in the formula (2) are the isomorphisms between the categories, one gets a map $F : \mathcal{A} \to \mathcal{B}$.

Theorem 1. *The map $F : \mathcal{A} \to \mathcal{B}$ is a covariant noninjective functor with $\mathrm{Ker}\, F \cong (0, \infty)$, which maps isomorphic complex tori to stably isomorphic Effros–Shen algebras.*

Proof. (i) Let us show that F maps isomorphic complex tori to stably isomorphic Effros–Shen algebras. Let $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ be a complex torus. Recall that the periods $\omega_1 = \int_{\gamma_1} \omega_E$ and $\omega_2 = \int_{\gamma_2} \omega_E$, where $\omega_E = dz$ is an invariant (Néron) differential on the complex torus and $\{\gamma_1, \gamma_2\}$ is a basis in $H_1(T^2, \mathbb{Z})$. The map F can be written as:

$$(3) \qquad \mathbb{C}/\Lambda_{(\int_{\gamma_2} \omega_E)/(\int_{\gamma_1} \omega_E)} \xmapsto{F} \mathbb{A}_{(\int_{\gamma_2} \phi)/(\int_{\gamma_1} \phi)},$$

where $\phi = \mathrm{Re}\,\omega$ is a closed differential defined earlier. Note that every isomorphism in the category $\mathcal{A}$ is induced by an orientation preserving automorphism, φ, of the torus T^2. The action of φ on the homology basis $\{\gamma_1, \gamma_2\}$ of T^2 is given by the formula

$$\begin{cases} \gamma_1' = a\gamma_1 + b\gamma_2 \\ \gamma_2' = c\gamma_1 + d\gamma_2, \end{cases} \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

The functor F acts by the formula:

$$(4) \qquad \tau = \frac{\int_{\gamma_2} \omega_E}{\int_{\gamma_1} \omega_E} \longmapsto \theta = \frac{\int_{\gamma_2} \phi}{\int_{\gamma_1} \phi}.$$

(a) From the left-hand side of (4), one obtains

$$\begin{cases} \omega_1' = \int_{\gamma_1'} \omega_E = \int_{a\gamma_1 + b\gamma_2} \omega_E = a \int_{\gamma_1} \omega_E + b \int_{\gamma_2} \omega_E = a\omega_1 + b\omega_2 \\ \omega_2' = \int_{\gamma_2'} \omega_E = \int_{c\gamma_1 + d\gamma_2} \omega_E = c \int_{\gamma_1} \omega_E + d \int_{\gamma_2} \omega_E = c\omega_1 + d\omega_2, \end{cases}$$

and therefore $\tau' = \frac{\int_{\gamma_2'} \omega_E}{\int_{\gamma_1'} \omega_E} = \frac{c+d\tau}{a+b\tau}$.

(b) From the right-hand side of (4), one obtains

$$\begin{cases} \lambda_1' = \int_{\gamma_1'} \phi = \int_{a\gamma_1 + b\gamma_2} \phi = a \int_{\gamma_1} \phi + b \int_{\gamma_2} \phi = a\lambda_1 + b\lambda_2 \\ \lambda_2' = \int_{\gamma_2'} \phi = \int_{c\gamma_1 + d\gamma_2} \phi = c \int_{\gamma_1} \phi + d \int_{\gamma_2} \phi = c\lambda_1 + d\lambda_2, \end{cases}$$

and therefore $\theta' = \frac{\int_{\gamma_2'} \phi}{\int_{\gamma_1'} \phi} = \frac{c+d\theta}{a+b\theta}$. Comparing (a) and (b), one concludes that F maps isomorphic complex tori to stably isomorphic Effros–Shen algebras.

(ii) Let us show that F is a covariant functor, i.e., F does not reverse the arrows. Indeed, it can be verified directly using the above formulas, that $F(\varphi_1 \varphi_2) = \varphi_1 \varphi_2 = F(\varphi_1)F(\varphi_2)$ for any pair of the isomorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(T^2)$. Theorem 1 is proved. $\qquad\square$
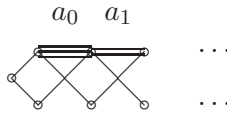
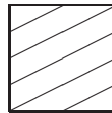FIGURE 1. The Effros–Shen algebra $\mathbb{A}_\theta$.

FIGURE 2. A measured foliation on the torus $\mathbb{R}^2/\mathbb{Z}^2$.

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