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Entropy of shifts on topological graph C^*-algebras

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Abstract. We give entropy estimates for two canonical noncommutative shifts on C^*-algebras associated to some topological graphs

$$E = (E^0, E^1, s, r),$$

defined using a basis of the corresponding Hilbert bimodule $H(E)$. We compare their entropies with the growth entropies associated directly to the topological graph. We illustrate with some examples of topological graphs considered by Katsura, where the vertex and the edge spaces are unions of unit circles and more detailed computations can be done.

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1. Introduction

The topological entropy for automorphisms of nuclear C^*-algebras was introduced by Voiculescu in [Vo] and extended by Brown to exact C^*-algebras in [Br]. It was computed for many examples of automorphisms, endomorphisms, and completely positive (cp) maps by several authors, often using the fact that the restriction to a commutative C^*-algebra is a map for which

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the entropy is known. For a comprehensive treatment of various techniques of computation of this entropy, we refer to [NS].

Recall that the map

$$\Phi : \mathcal{O}_n \to \mathcal{O}_n, \quad \Phi(c) = \sum_{i=1}^{n} S_i c S_i^*,$$

where $\mathcal{O}_n$ is the Cuntz algebra with generators $S_1, \ldots, S_n$, is an endomorphism and has topological entropy $\log n$, see [Ch]. The map Φ leaves invariant both the AF-core $\mathcal{F}_n \cong UHF(n^\infty)$, which is generated by monomials $S_\alpha S_\beta^*$ with $|\alpha| = |\beta|$, and the abelian algebra $\mathcal{D}_n$, which is generated by monomials $S_\alpha S_\alpha^*$. Recall that for a word $\alpha = \alpha_1 \cdots \alpha_k \in \{1, 2, \ldots, n\}^k$, $k \in \mathbb{N}$, we define $S_\alpha := S_{\alpha_1} \cdots S_{\alpha_k}$ and $|\alpha| = k$. It is known that $\mathcal{D}_n$ is isomorphic to $C(X)$, where $X = \{1, 2, \ldots, n\}^{\mathbb{N}}$, and Φ is called a noncommutative shift, since $\Phi\,|_{\mathcal{D}_n}$ is conjugate to the map $\widetilde{\sigma} : C(X) \to C(X)$, $\widetilde{\sigma}(f) = f \circ \sigma$, where σ is the unilateral shift $\sigma : X \to X$, $\sigma(x_0 x_1 x_2 \ldots) = x_1 x_2 \ldots$.

In the case of the Cuntz–Krieger algebra $\mathcal{O}_\Lambda$, where Λ is an incidence matrix, the corresponding map Φ is no longer multiplicative, but it is a unital completely positive map. Boca and Goldstein (see [BG]) proved that the topological entropy of Φ is $\log \rho(\Lambda)$, where $\rho(\Lambda)$ is the spectral radius of Λ. This coincides with the classical topological entropy of the underlying Markov shift (X_Λ, σ). Similar results are obtained for graph C^*-algebras by Jeong and Park, see [JP1, JP2, JP3], and for higher-rank graph C^*-algebras by Skalski and Zacharias, see [SZ]. The Boca-Goldstein technique was also used by Kerr and Pinzari to analyze the noncommutative pressure and the variational principle in Cuntz–Krieger type C^*-algebras, see [KP].

Let H be a full Hilbert bimodule over a C^*-algebra A with a basis $\xi_1, \ldots, \xi_n$, in the sense that for all $\xi \in H$,

$$\xi = \sum_{i=1}^{n} \xi_i \langle \xi_i, \xi \rangle.$$

The Cuntz–Pimsner algebra $\mathcal{O}_H$ is generated by A and $S_i = S_{\xi_i}, 1 \le i \le n$ with relations

$$\sum_{i=1}^{n} S_i S_i^* = 1, \quad S_i^* S_j = \langle \xi_i, \xi_j \rangle, \quad a \cdot S_j = \sum_{i=1}^{n} S_i \langle \xi_i, a \cdot \xi_j \rangle,\ a \in A,$$

see [KPW]. We consider the unital completely positive map (ucp map)

$$\Phi : \mathcal{O}_H \to \mathcal{O}_H, \quad \Phi(c) = \sum_{i=1}^{n} S_i c S_i^*,$$

and call it a noncommutative shift. Notice that Φ leaves invariant the core $\mathcal{F}_H$, generated by monomials $S_\alpha a S_\beta^*$ with $a \in A$ and $|\alpha| = |\beta|$, and also leaves invariant the subalgebra $\mathcal{C}_H$ generated by monomials $S_\alpha a S_\alpha^*$. If $H = \mathbb{C}^n$, we recover the canonical endomorphism considered by Choda, since $\mathcal{O}_H = \mathcal{O}_n$, the Cuntz algebra generated by n isometries $S_1, \dots, S_n$. It is our goal to study the map Φ and its entropy for more general finitely generated Hilbert bimodules.

In Section 2 we recall the definition of the C^*-algebras $C^*(E)$ and $\mathcal{F}_E$ associated to a topological graph E. In Section 3 we give more details about the structure of these C^*-algebras, in the presence of a basis $\{\xi_i\}_{1\leq i\leq n}$ of the Hilbert bimodule. We also define the ucp map Φ associated with such a basis. In the context of topological graph C^*-algebras there is another candidate for the notion of noncommutative shift, denoted by Ψ. This is defined only on the core algebra $\mathcal{F}_E$, using the embeddings $\mathcal{K}(H^{\otimes k}) \to \mathcal{K}(H^{\otimes k+1})$ for some particular topological graphs. For these graphs, we can associate an étale groupoid as in [De2], and the map Ψ restricted to the diagonal $\mathcal{D}_E = C(E^\infty)$ coincides with the shift map on the unit space of the groupoid identified with the space of infinite paths E^∞. In Section 4, we define the loop and block entropies for topological graphs E and compute them for several examples. In Section 5 we study the entropy of the noncommutative shifts Φ and Ψ, and the relationship with these growth entropies. The main result computes the entropy of Φ in terms of the spectral radius of an incidence matrix. The entropy of Ψ is bounded below by the loop entropy. Compared to the situation of discrete graphs or higher rank graphs, new phenomena occur, since $\Phi\mid_{\mathcal{F}_E}$ and Ψ are different and they may have different entropies. We illustrate this with several examples in Section 6. We also recover earlier computations from [De1].

Our results are similar to results of Pinzari, Watatani and Yonetani in [PWY]. In the final stages of writing this paper, we also learned that Yamashita obtained similar entropy computations for a particular class of toplogical graphs, called circle correspondences [Y].

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2. Topological graphs and their C^*-algebras

Topological graphs were studied in [De2] under the name continuous graphs, and generalized by several authors. We use the terminology and many facts from papers of Katsura, see [Ka1]. Let $E = (E^0, E^1, s, r)$ be a topological graph. Recall that E^0, E^1 are locally compact spaces, and $s, r : E^1 \to E^0$ are continuous maps such that s is a local homeomorphism. We think of points in E^0 as vertices, and of elements $e \in E^1$ as edges from $s(e)$ to $r(e)$. Several examples of topological graphs will be considered in Section 4.

Definition 2.1. The C^*-algebra of a topological graph E, denoted $C^*(E)$ is defined to be the Cuntz–Pimsner algebra $\mathcal{O}_H$, where the Hilbert bimodule $H = H(E)$ over the C^*-algebra $A = C_0(E^0)$ is obtained as the completion of $C_c(E^1)$ using the inner product

$$\langle \xi, \eta \rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e), \quad \xi, \eta \in C_c(E^1)$$

and the multiplications

$$(\xi \cdot f)(e) = \xi(e)f(s(e)), \quad (f \cdot \xi)(e) = f(r(e))\xi(e).$$

The core algebra $\mathcal{F}_E$ is the fixed point algebra under the gauge action, see below.

Recall that a Hilbert bimodule over a C^*-algebra A (sometimes called a C^*-correspondence from A to A) is a Hilbert A-module H with a left action of A given by a homomorphism $\varphi : A \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ denotes the C^*-algebra of all adjointable operators on H. A Hilbert bimodule is called full if the inner products generate A. For $n \geq 0$ we denote by $H^{\otimes n}$ the Hilbert bimodule obtained by taking the tensor product of n copies of H, balanced over A (for $n = 0, H^{\otimes 0} = A$). For $n = 2$, the inner product is given by

$$\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \eta, \varphi(\langle \xi, \xi' \rangle)\eta' \rangle,$$

and it is inductively defined for general n.

A *Toeplitz representation* of a Hilbert bimodule H over A in a C^*-algebra C is a pair (τ, π) with $\tau : H \to C$ a linear map and $\pi : A \to C$ a $*$-homomorphism, such that

$$\tau(\xi a) = \tau(\xi)\pi(a), \quad \tau(\xi)^*\tau(\eta) = \pi(\langle \xi, \eta \rangle), \quad \tau(\varphi(a)\xi) = \pi(a)\tau(\xi).$$

The C^*-algebra generated by the images of π and τ in C is denoted by $C^*(\tau, \pi)$. The corresponding universal C^*-algebra is called the Toeplitz algebra of H, denoted by $\mathcal{T}_H$. If H is full, then $\mathcal{T}_H$ is generated by elements $\tau^n(\xi)\tau^m(\eta)^*, m, n \geq 0$, where $\tau^0 = \pi$ and for $n \geq 1, \tau^n(\xi_1 \otimes \cdots \otimes \xi_n) = \tau(\xi_1) \cdots \tau(\xi_n)$ is the extension of τ to $H^{\otimes n}$. Note that A is isomorphic to a subalgebra of $\mathcal{T}_H$.

The rank one operators $\theta_{\xi,\eta}$ given by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle$ generate an essential ideal of $\mathcal{L}(H)$, denoted $\mathcal{K}(H)$. It is known that $\mathcal{K}(H) \cong H \otimes H^*$, where H^* is the dual of H, regarded as a left A-module. The rank one operator $\theta_{\xi,\eta} \in \mathcal{K}(H)$ is identified with $\xi \otimes \eta^* \in H \otimes H^*$. The map τ defines a homomorphism $\psi : \mathcal{K}(H) \to C$ such that $\psi(\xi \otimes \eta^*) = \tau(\xi)\tau(\eta)^*$. A representation (τ, π) is *Cuntz–Pimsner covariant* if $\pi(a) = \psi(\varphi(a))$ for all a in the ideal

$$I_H = \varphi^{-1}(\mathcal{K}(H)) \cap (\ker \varphi)^\perp.$$

The Cuntz–Pimsner algebra $\mathcal{O}_H$ is universal with respect to the covariant representations, and it is a quotient of $\mathcal{T}_H$. The universal properties allow us to define a gauge action of $\mathbb{T}$ on $\mathcal{T}_H$ and $\mathcal{O}_H$ such that

$$z \cdot (\tau^n(\xi)\tau^m(\eta)^*) = z^{n-m}\tau^n(\xi)\tau^m(\eta)^*, \quad z \in \mathbb{T}.$$

The *core* C^*-algebra $\mathcal{F}_H$ is the fixed point algebra $\mathcal{O}_H^{\mathbb{T}}$, and it is generated by the union of the algebras $\mathcal{K}(H^{\otimes k})$, $k \geq 0$. For more details about the algebras $\mathcal{F}_H, \mathcal{T}_H$, and $\mathcal{O}_H$ we refer to the papers of Pimsner ([Pi]) and Katsura ([Ka1]).

Remark 2.2. Notice that, since Hilbert bimodules over commutative C^*-algebras are associated to continuous fields of Hilbert spaces, it follows that for $H = H(E)$, where $E = (E^0, E^1, s, r)$ is a topological graph with s surjective, $\mathcal{K}(H)$ is a continuous trace C^*-algebra over E^0. The elements of $\mathcal{K}(H)$ can be thought as compact operator valued continuous functions on E^0 which vanish at infinity, and the elements of $\mathcal{L}(H)$ can be considered as bounded operator valued continuous functions on E^0.

3. Finitely generated bimodules and noncommutative shifts

Definition 3.1. The full Hilbert bimodule H over A is finitely generated if it has a basis $\{\xi_i\}_{1\leq i\leq n}$, in the sense that for all $\xi \in H$,

$$\xi = \sum_{i=1}^{n} \xi_i \langle \xi_i, \xi \rangle.$$

It is easy to check that, in this case, $H^{\otimes k}$ has basis $\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}$, where $i_1, \dots, i_k \in \{1, 2, \dots, n\}$. Denote by S_i the image of ξ_i in the Cuntz–Pimsner algebra $\mathcal{O}_H$. Then $\mathcal{O}_H$ is generated by A and $S_i, 1 \leq i \leq n$ with relations

$$\sum_{i=1}^{n} S_i S_i^* = 1, \quad S_i^* S_j = \langle \xi_i, \xi_j \rangle, \quad a \cdot S_j = \sum_{i=1}^{n} S_i \langle \xi_i, a \cdot \xi_j \rangle, \ a \in A,$$

see [KPW]. For a word $\alpha = \alpha_1 \cdots \alpha_k \in \{1, 2, \dots, n\}^k$ of length $|\alpha| = k$, let $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$.

Lemma 3.2. *Assume that the full Hilbert bimodule H over A has a basis $\{\xi_i\}_{1\leq i\leq n}$ as above. Then $\mathcal{K}(H) = \mathcal{L}(H)$, and the Cuntz–Pimsner algebra $\mathcal{O}_H$ is the closed linear span of monomials $S_\alpha a S_\beta^*$, where α, β are arbitrary words and $a \in A$. The core algebra $\mathcal{F}_H$ is the closed linear span of monomials $S_\alpha a S_\beta^*$ with $|\alpha| = |\beta|$.*

Proof. Since $\sum_{i=1}^{n} S_i S_i^* = 1$, it is clear that $\mathcal{K}(H)$ is unital, and therefore $\mathcal{K}(H) = \mathcal{L}(H)$. It suffices to show that the product of two monomials as above is a sum of monomials of the same form. Consider the product $S_\alpha a S_\beta^* S_\gamma b S_\delta^*$ for some words $\alpha, \beta, \gamma, \delta$ and $a, b \in A$. Using the relations $S_i^* S_j = \langle \xi_i, \xi_j \rangle \in A$ and $a \cdot S_j = \sum_{i=1}^{n} S_i \langle \xi_i, a \cdot \xi_j \rangle$ repeatedly, for $|\beta| = |\gamma|$ we get that $S_\beta^* S_\gamma$ belongs to A; for $|\beta| > |\gamma|$ we get $S_\beta^* S_\gamma = S_{\beta'}^* c$ with $|\beta'| = |\beta| - |\gamma|$ and $c \in A$; and for $|\beta| < |\gamma|$ we get $S_\beta^* S_\gamma = d S_{\gamma'}$ with $|\gamma'| = |\gamma| - |\beta|$ and $d \in A$. In the first case we are done. In the second and the third case, we use again the relation $a \cdot S_j = \sum_{i=1}^n S_i \langle \xi_i, a \cdot \xi_j \rangle$ repeatedly to write

$$S_{\beta'}^* c = (c^* S_{\beta'})^* = \left(\sum_j S_{\mu^j} c_j \right)^* = \sum_j c_j^* S_{\mu^j}^*$$

and respectively

$$dS_{\gamma'} = \sum_i S_{\nu^i} d_i,$$

where μ^j, ν^i are words, and $c_j, d_i \in A$. We get

$$S_\alpha a S_\beta^* S_\gamma b S_\delta^* = \sum_j S_\alpha a c_j^* S_{\mu^j}^* b S_\delta^*$$

and

$$S_\alpha a S_\beta^* S_\gamma b S_\delta^* = \sum_i S_\alpha a S_{\nu^i} d_i b S_\delta^*.$$

We can use again the above relations to write $S_{\mu^j}^* b$ as $\sum_k b_{jk} S_{\mu^{jk}}^*$ with $b_{jk} \in A$ and $a S_{\nu^i} = \sum_l S_{\nu^{il}} a_{il}$ with $a_{il} \in A$ to get the desired result. For the core algebra we can use the same method, keeping track of the lengths of the involved words. □

Corollary 3.3. *If in addition $\langle \xi_i, \xi_j \rangle = 0$ for all $1 \le i \ne j \le n$, then S_i are partial isometries. Set $Q_i := S_i^* S_i$ and $P_i := S_i S_i^*$ for $i = 1, \dots, n$. Suppose that $Q_i = \sum_{j=1}^n \Lambda(i,j) P_j$ for some incidence matrix Λ. Then the nonzero elements $S_\mu P_i S_\nu^*$ for $|\mu| = |\nu| = k$ and $i = 1, \dots, n$ form a system of matrix units generating a finite-dimensional C^*-algebra F_k. If, moreover, A is commutative and unital with $A \subset \mathcal{K}(H)$, then there is an isomorphism $F_k \otimes A \cong \mathcal{K}(H^{\otimes k}) \cong \mathcal{L}(H^{\otimes k})$, and the Cuntz–Pimsner algebra $\mathcal{O}_H$ contains a copy of the Cuntz–Krieger algebra $\mathcal{O}_\Lambda$. The core $\mathcal{F}_H$ is isomorphic to the inductive limit $\varinjlim \mathcal{K}(H^{\otimes k}) \cong \varinjlim F_k \otimes A$, where the embeddings are determined by the left action of A, and it contains a copy of the stationary AF-algebra determined by Λ. There is a commutative subalgebra $\mathcal{C}_H \subset \mathcal{F}_H$, generated by monomials $S_\alpha a S_\alpha^*$ with $a \in A$, which contains o copy of $C(X_\Lambda)$, where $X_\Lambda = \{(x_k) \in \{1, 2, \dots, n\}^{\mathbb{N}} : \Lambda(x_k, x_{k+1}) = 1\}$.*

Proof. Since $\langle \xi_i, \xi_j \rangle = 0$ for $i \ne j$, we get $\xi_i = \xi_i \langle \xi_i, \xi_i \rangle$, which implies $S_i = S_i S_i^* S_i$ for all i. It follows that each S_i is a partial isometry, and therefore P_i, Q_i are projections. Under our assumptions, it is clear that S_i generate a copy of the Cuntz–Krieger algebra $\mathcal{O}_\Lambda$. The isomorphism $F_k \otimes A \cong \mathcal{K}(H^{\otimes k})$ is given by the map $S_\mu P_i S_\nu^* \otimes a \mapsto S_\mu a P_i S_\nu^*$. The rest follows from the previous lemma. □

Definition 3.4. The canonical noncommutative shift associated to a basis $\{\xi_i\}_{1 \le i \le n}$ of the Hilbert bimodule H over A is the ucp map

$$\Phi : \mathcal{O}_H \to \mathcal{O}_H, \ \Phi(c) = \sum_{i=1}^n S_i c S_i^*.$$

Remark 3.5. Notice that Φ leaves invariant the core algebra $\mathcal{F}_H$, generated by monomials $S_\alpha a S_\beta^*$ with $a \in A$ and $|\alpha| = |\beta|$, and also leaves invariant the subalgebra $\mathcal{C}_H$ generated by monomials $S_\alpha a S_\alpha^*$. In the hypotheses of the above corollary, the map Φ also leaves invariant the subalgebra $C(X_\Lambda)$, and its restriction is conjugate to the map induced by the Markov shift $\sigma_\Lambda : X_\Lambda \to X_\Lambda, \sigma_\Lambda(x_0x_1x_2\cdots) = x_1x_2\cdots$.

In our examples of topological graphs, E^0 and E^1 are compact and s and r are surjective local homeomorphisms. In that case, it follows that the left action of $A = C(E^0)$ on $H(E)$ is by compact operators. Moreover, the Hilbert module $H = H(E)$ will have a basis $\xi_1, \xi_2, \ldots, \xi_n$ satisfying

$$\langle \xi_i, \xi_j \rangle = \delta_{ij} Q_i,$$

where $Q_i \in A$ are projections. It follows from the corollary that the C^*-algebra $C^*(E) = \mathcal{O}_H$ is generated by $A = C(E^0)$ and n partial isometries $S_1, S_2, \ldots, S_n$ with orthogonal ranges satisfying some commutation relations determined by the range map r.

For these topological graphs, there is also a groupoid approach for $C^*(E)$. For $k \geq 2$, let's define E^k to be the space of paths of length k in the topological graph and $E^\infty = \varprojlim E^k$ to be the space of infinite paths. For $k \geq 0$, we define the equivalence relation R_k on E^k by

$$R_k = \{(e, f) \in E^k \times E^k : s(e) = s(f)\},$$

with the induced product topology. Note that R_0 is just the diagonal in $E^0 \times E^0$. The set

$$\Gamma = \Gamma(E) = \{(x, p-q, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \sigma^p(x) = \sigma^q(y)\},$$

with natural operations $(x,k,y)(y,l,z) = (x,k+l,z), (x,k,y)^{-1} = (y,-k,x)$ and appropriate topology, becomes an amenable étale groupoid, and its C^*-algebra is isomorphic to $C^*(E)$, see [De2].

Definition 3.6. The map $\varphi : C(E^0) \to C^*(R_1)$ induces embeddings $\Psi_k : C^*(R_k) \to C^*(R_{k+1})$ for each $k \geq 0$, which define another noncommutative shift $\Psi : \mathcal{F}_E \to \mathcal{F}_E$. This is a $*$-homomorphism.

Remark 3.7. Recall that we have isomorphisms $C^*(R_k) \cong \mathcal{K}(H^{\otimes k})$. There is a natural diagonal isomorphic to $C(E^\infty)$ inside $\varinjlim C^*(R_k) \cong \mathcal{F}_E$, and the restriction $\Psi|_{C(E^\infty)}$ is conjugate to the map defined by the shift

$$\sigma : E^\infty \to E^\infty,$$
$$\sigma(e_1e_2e_3\cdots) = e_2e_3\cdots,$$

which is a surjective local homeomorphism.

4. Growth entropies of topological graphs

Recall that a *path* of length k in a topological graph $E = (E^0, E^1, s, r)$ is a concatenation $e_1 e_2 \cdots e_k$ of edges in E^1 such that $s(e_i) = r(e_{i+1})$ for all i. The maps s and r extend naturally to E^k. A vertex $v \in E^0$ is viewed as a path of length 0. For each vertex $v \in E^0$ and $k \geq 1$ let

$$E_s^k(v) := \{e \in E^k \mid s(e) = v\}, \quad E_r^k(v) := \{e \in E^k \mid r(e) = v\}$$

and

$$E_\ell^k(v) := \{e \in E^k \mid s(e) = r(e) = v\},$$

the set of loops of length k based at v. We will be mostly interested in locally finite topological graphs in the sense that each vertex emits and receives a finite number of edges. In that case, the sets $E_s^k(v), E_r^k(v)$ and $E_\ell^k(v)$ are finite for each k and v. If both maps s and r are onto, then $E_s^k(v), E_r^k(v)$ are nonempty for all k and v, but $E_\ell^k(v)$ may be empty. Define

$$E_\ell^k := \bigcup_{v \in E^0} E_\ell^k(v).$$

Definition 4.1. Assume E is a locally finite topological graph. The loop entropy $h_\ell(E)$ and the block entropy $h_b(E)$ of E are defined by

$$h_\ell(E) = \limsup_{k \to \infty} \frac{1}{k} \log |E_\ell^k|, \quad h_b(E) = \sup_{v \in E^0} \limsup_{k \to \infty} \frac{1}{k} \log |E_r^k(v)|,$$

where $|L|$ denotes the cardinality of L.

Remark 4.2. For a topological graph E where r is also a local homeomorphism, we can interchange s and r to get a new topological graph, denoted E^t and called the transposed graph. We have $h_\ell(E) = h_\ell(E^t)$ but $h_b(E) \neq h_b(E^t)$ in general. It is known that the loop entropy and the block entropy make sense for discrete graphs, and that these entropies can be computed in terms of the incidence matrix. For example, if the irreducible finite graph E is given by a matrix Λ_E, then it is known that $h_\ell(E) = h_b(E) = \log \rho(\Lambda_E)$, where $\rho(\Lambda_E)$ denotes the spectral radius. Since $\rho(\Lambda_E) = \rho(\Lambda_E^t)$, for finite graphs we have $h_b(E) = h_b(E^t)$. This equality fails for infinite graphs, as was shown by Jeong and Park in Example 3.3 of [JP2]. For topological graphs, see the next examples.

Example 4.3. Let X be a compact metric space, and $\sigma : X \to X$ a continuous surjective map. Define the topological graph $E = E(X, \sigma) = (E^0, E^1, s, r)$, where $E^0 = E^1 = X,\ s = \text{id}$ and $r = \sigma$. For $k \geq 2$, the space

$$E^k = \{(x_1, \ldots, x_k) \in X \times \cdots \times X \mid x_j = \sigma(x_{j+1}), j = 1, \ldots, k-1\}$$

is homeomorphic to X. For a fixed $v \in E^0 = X$ we can identify $E_s^k(v)$ with $\{v\}$, $E_r^k(v)$ with $\sigma^{-k}(v)$ and E_ℓ^k with $\{v \in X \mid \sigma^k(v) = v\}$. Hence

$$h_\ell(E) = \limsup_{k\to\infty} \frac{1}{k} \log |\{x \in X \mid \sigma^k(x) = x\}|,$$
$$h_b(E) = \sup_{v\in X} \limsup_{k\to\infty} \frac{1}{k} \log |\sigma^{-k}(v)|.$$

Recall that for an expansive map $\sigma : X \to X$ of a metric space X,

$$h_{\text{top}}(\sigma) \geq \limsup_{k\to\infty} \frac{1}{k} \log |\{x \in X \mid \sigma^k(x) = x\}|,$$

see Theorem 8.16 in [Wa], which gives the inequality

$$h_{\text{top}}(\sigma) \geq h_\ell(E(X, \sigma)).$$

Corollary 4.4. *A topological graph E with E^1 a compact metric space determines a local homeomorphism*

$$\sigma = \sigma_E : E^\infty \to E^\infty,$$
$$\sigma(e_1e_2e_3\dots) = e_2e_3\dots,$$

where

$$E^\infty = \left\{e_1e_2\dots \in (E^1)^{\mathbb{N}} \mid s(e_j) = r(e_{j+1})\right\}$$

is the space of infinite paths. Then the set of periodic points of σ coincides with the union of loops $\bigcup_{k\geq 1} E_\ell^k$. If σ is expansive, it follows that

$$h_{\text{top}}(\sigma) \geq h_\ell(E).$$

Example 4.5. Consider the topological graphs $E_{p,q}$ from Example A.6 of [Ka3]. Recall that the vertex and edge spaces are copies of the unit circle $\mathbb{T}$, the source and range maps are $s(z) = z^p$ and $r(z) = z^q$, where p, q are integers with $p \geq 1$ and $q \neq 0$. Let's assume that $\gcd(p, q) = 1$. Then it is easy to see that

$$|E_s^k(v)| = p^k, \quad |E_r^k(v)| = |q|^k, \quad |E_\ell^k| = \left|p^k - |q|^k\right|.$$

It follows that for this topological graph, $h_b(E) = \log |q|$, $h_b(E^t) = \log p$, and $h_\ell(E) = \log \max\{p, |q|\}$. The space of infinite paths $E_{p,q}^\infty$ is homeomorphic to

$$\Sigma(p, q) = \left\{(z_0, z_1, z_2, \dots) \in \mathbb{T}^{\mathbb{N}} \mid z_k^p = z_{k+1}^q\right\},$$

which is a 1-dimensional solenoid. In particular, the shift

$$\sigma : \Sigma(p, q) \to \Sigma(p, q),$$
$$\sigma(z_0, z_1, z_2, \dots) = (z_1, z_2, \dots)$$

has entropy $\geq \log \max\{p, |q|\}$. For a more general result about the entropy of automorphisms of solenoids, see [Kw] and [LW].

Example 4.6. Consider the topological graph $E = (E^0, E^1, s, r)$ where $E^0 = \mathbb{T}$, $E^1 = \mathbb{T} \times \{1, 2\}$, $s(z, 1) = z^2$, $r(z, 1) = z$, $s(z, 2) = z$, $r(z, 2) = z^3$. The space of infinite paths is the generalized solenoid

$$E^\infty = \big\{(z_k, a_k)_{k\geq 0} \in (\mathbb{T} \times \{1, 2\})^{\mathbb{N}} \mid \\ a_k = 1 \Rightarrow\ z_{k+1}^2 = z_k,\ a_k = 2 \Rightarrow\ z_{k+1} = z_k^3\big\}.$$

Counting all the possible words $(z_1, a_1)(z_2, a_2) \cdots (z_m, a_m)$ for a fixed $v = r(z_1, a_1) \in \mathbb{T}$, we get $|E_r^m(v)| = 4^m$. It follows that $h_b(E) = \log 4$. Counting all the possible words $(z_1, a_1)(z_2, a_2) \cdots (z_m, a_m)$ for a fixed $v = s(z_m, a_m) \in \mathbb{T}$, we get $|E_s^m(v)| = 3^m$, hence $h_b(E^t) = \log 3$. Counting all the loops $(z_1, a_1)(z_2, a_2) \cdots (z_m, a_m)$, we get

$$|E_\ell^m| = \sum_{k=0}^{m} \binom{m}{k} |2^{m-k} - 3^k|.$$

It follows that $h_\ell(E) = \log 4$, since

$$\begin{aligned} 4^m - 3^m &= \left|\sum_{k=0}^{m} \binom{m}{k} (2^{m-k} - 3^k)\right| \\ &\leq \sum_{k=0}^{m} \binom{m}{k} |2^{m-k} - 3^k| \\ &\leq \sum_{k=0}^{m} \binom{m}{k} 3^k = 4^m. \end{aligned}$$

Example 4.7. More generally, for a discrete graph $E = (E^0, E^1, s, r)$ with no sinks, Katsura considers in [Ka3] the topological graph $E \times_{p,q} \mathbb{T}$ with vertex space $E^0 \times \mathbb{T}$, edge space $E^1 \times \mathbb{T}$, source map $s(e, z) = (s(e), z^{p(e)})$ and range map $r(e, z) = (r(e), z^{q(e)})$, where $p : E^1 \to \{1, 2, 3, \dots\}$ and $q : E^1 \to \mathbb{Z}$ are two maps. The infinite path space is the generalized solenoid

$$(E \times_{p,q} \mathbb{T})^\infty \\ = \Big\{(e_1, e_2, \dots; z_1, z_2, \dots) \in E^\infty \times \mathbb{T}^\infty \mid z_k^{p(e_k)} = z_{k+1}^{q(e_{k+1})} \text{ for } k = 1, 2, \dots\Big\}.$$

Define the matrices P, Q, where

$$P(v, w) = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} p(e), \quad Q(v, w) = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} q(e)$$

for $v, w \in E^0$.

We conjecture that, for E finite, under certain conditions on the maps p and q, we have

$$\begin{aligned} h_b(E \times_{p,q} \mathbb{T}) &= \log \rho(Q), \\ h_b((E \times_{p,q} \mathbb{T})^t) &= \log \rho(P), \\ h_\ell(E \times_{p,q} \mathbb{T}) &= \log \max\{\rho(P), \rho(Q)\}, \end{aligned}$$

but we were unable to prove it at this time.

5. Entropy of noncommutative shifts

We recall the definition and a few useful facts concerning the topological entropy of completely positive maps. The reader may consult [Br] or [NS] for an extensive treatment. Let A be an exact C^*-algebra, $\pi : A \to \mathcal{L}(\mathcal{H})$ a faithful *-representation on a Hilbert space $\mathcal{H}$, and $\omega \subset A$ a finite subset. For $\delta > 0$ we put

$$\mathrm{CPA}(\pi, A) =$$
$$\{(\phi, \psi, B) \mid \phi : A \to B, \psi : B \to \mathcal{L}(\mathcal{H}) \text{ contractive cp maps}, \dim(B) < \infty\},$$
$$\mathrm{rcp}(\pi, \omega, \delta)$$
$$= \inf\{\mathrm{rank}(B) \mid (\phi, \psi, B) \in \mathrm{CPA}(\pi, A), ||\psi \circ \phi(a) - \pi(a)|| < \delta \ \ \forall\, a \in \omega\},$$

where $\mathrm{rank}(B)$ denotes the dimension of a maximal abelian subalgebra of B. Since the completely positive rank $\mathrm{rcp}(\pi, \omega, \delta)$ is independent of the choice of π, we may write $\mathrm{rcp}(\omega, \delta)$ instead of $\mathrm{rcp}(\pi, \omega, \delta)$.

Definition 5.1. Let $A \subset \mathcal{L}(\mathcal{H})$ be a C^*-algebra, $\Phi : A \to A$ a cp map, $\omega \subset A$ finite and $\delta > 0$. Put

$$\begin{aligned} \mathrm{ht}(\Phi, \omega, \delta) &= \limsup_{k \to \infty} \frac{1}{k} \log \left(\mathrm{rcp} \left(\bigcup_{i=0}^{k-1} \Phi^i(\omega), \delta \right) \right) \\ \mathrm{ht}(\Phi, \omega) &= \sup_\delta \mathrm{ht}(\Phi, \omega, \delta) \\ \mathrm{ht}(\Phi) &= \sup_\omega \mathrm{ht}(\Phi, \omega). \end{aligned}$$

The number $\mathrm{ht}(\Phi) \in [0, \infty]$ is called the topological entropy of Φ.

Remark 5.2. We collect here some useful facts:

(1) If $A_0 \subset A$ is a Φ-invariant C^*-subalgebra of A, then
$$\mathrm{ht}(\Phi \mid_{A_0}) \le \mathrm{ht}(\Phi).$$
(2) If $\{\omega_k\}$ is an increasing sequence of finite subsets of A such that the linear span of $\displaystyle\bigcup_{k,m \ge 0} \Phi^m(\omega_k)$ is dense in A, then $\mathrm{ht}(\Phi) = \sup_k \mathrm{ht}(\Phi, \omega_k)$.
(3) If $\sigma : X \to X$ is a continuous map on a compact metric space X, then $\mathrm{ht}(\widetilde{\sigma}) = h_{\mathrm{top}}(\sigma)$, where $\widetilde{\sigma} : C(X) \to C(X)$, $\widetilde{\sigma}(f) = f \circ \sigma$, and h_{top} is the classical topological entropy defined in Chapter 7 of [Wa].

For the remainder of this paper, we consider a topological graph $E = (E^0, E^1, s, r)$ such that E^0 and E^1 are compact metric spaces, and s, r are surjective local homeomorphisms. In this case, it follows that the left action of $A = C(E^0)$ on $H = H(E)$ is by compact operators. Moreover, we assume that the Hilbert module $H = H(E)$ has a basis $\xi_1, \xi_2, \dots, \xi_n$ satisfying

$$\langle \xi_i, \xi_j \rangle = \delta_{ij} Q_i,$$

where $Q_i \in A = C(E^0)$ are projections. Suppose that $Q_i = \sum_{j=1}^n \Lambda(i,j) P_j$ for some incidence matrix Λ, where $P_i = S_i S_i^*$. Recall that the C^*-algebra $C^*(E) = \mathcal{O}_H$ is generated by A and n partial isometries $S_1, S_2, \dots, S_n$ with orthogonal ranges satisfying some commutation relations determined by the range map r. It contains a copy of the Cuntz–Krieger algebra $\mathcal{O}_\Lambda$. Denote by $G = G(\Lambda)$ the finite graph defined by the incidence matrix Λ. Notice that S_μ for $\mu \in \{1, 2, \dots, n\}^k$ is nonzero precisely when μ is a path of length k in this finite graph. We are interested in determining the entropy of the noncommutative shifts Φ and Ψ defined in Section 3 for such a topological graph E.

Lemma 5.3. *Consider a topological graph E as above. For each $m \geq 1$ there is a $*$-homomorphism $\chi_m : \mathcal{O}_H \to M_{w(m)} \otimes \mathcal{O}_H$ given by*

$$\chi_m(x) = \sum_{|\mu|=|\nu|=m} S_\mu S_\nu^* \otimes S_\mu^* x S_\nu,$$

where $w(m)$ denotes the number of paths of length m in the associated graph G.

Proof.

$$\begin{aligned}
\chi_m(x)\chi_m(y) &= \left(\sum_{\mu,\nu} S_\mu S_\nu^* \otimes S_\mu^* x S_\nu \right) \left(\sum_{\mu',\nu'} S_{\mu'} S_{\nu'}^* \otimes S_{\mu'}^* y S_{\nu'} \right) \\
&= \sum_{\mu,\nu,\mu',\nu'} S_\mu S_\nu^* S_{\mu'} S_{\nu'}^* \otimes S_\mu^* x S_\nu S_{\mu'}^* y S_{\nu'} \\
&= \sum_{\mu,\nu'} S_\mu S_{\nu'}^* \otimes S_\mu^* x \left(\sum_\nu S_\nu S_\nu^* \right) y S_{\nu'} \\
&= \chi_m(xy),
\end{aligned}$$

since $\sum_{|\nu|=m} S_\nu S_\nu^* = 1$. □

Theorem 5.4. *Consider $E = (E^0, E^1, s, r)$ a topological graph with E^0 and E^1 compact metric spaces such that s and r are surjective local homeomorphisms. Assume that $A = C(E^0)$ is finitely generated, in the sense that there are $a_1, a_2, \dots, a_q \in A$ such that polynomials in a_j and a_j^* are dense in A.*

Assume also that there are elements $\xi_1, \xi_2, \ldots, \xi_n$ in $H = C(E^1)$ such that

$$\xi = \sum_{i=1}^{n} \xi_i \langle \xi_i, \xi \rangle \quad \text{for all } \xi \in H, \text{ and}$$
$$\langle \xi_i, \xi_j \rangle = \delta_{ij} Q_i \quad \text{for some projections } Q_i \in A,\ i = 1, \ldots, n.$$

Let S_i be the image of ξ_i in $\mathcal{O}_H$, and let $P_i = S_i S_i^*$ for $i = 1, \ldots, n$. Assume $Q_i = \sum_{j=1}^{n} \Lambda(i,j) P_j$, for $\Lambda = \Lambda_E$, the incidence matrix of the associated finite graph. Define the canonical ucp map

$$\Phi : C^*(E) \to C^*(E),\ \ \Phi(c) = \sum_{i=1}^{n} S_i c S_i^*.$$

If $C^*(E)$ is simple, then $\mathrm{ht}(\Phi) = \log \rho(\Lambda_E)$, where $\rho(\Lambda_E)$ denotes the spectral radius of the matrix Λ_E.

Proof. Since $C^*(E)$ contains a copy of $\mathcal{O}_\Lambda$ such that $\Phi(\mathcal{O}_\Lambda) \subset \mathcal{O}_\Lambda$, it follows that $\mathrm{ht}(\Phi) \geq \log \rho(\Lambda_E)$, (see also Section 6 of [PWY]). For the other inequality, we use the maps χ_m from the previous lemma. Since $C^*(E)$ is simple, it follows that $\chi_m : C^*(E) \to \chi_m(C^*(E)) \subset M_{w(m)} \otimes C^*(E)$ is a $*$-isomorphism. Note that for $l \geq 1$ we have

$$\Phi^l(c) = \sum_{|\gamma| = l} S_\gamma c S_\gamma^*,\ \ c \in C^*(E).$$

For $k \geq 1$, $|\beta| \leq |\alpha| \leq k_0$, $m \geq k + k_0$, $l \leq k - 1$ and $a = a_1^{p_1} \cdots a_q^{p_q}$, $0 \leq p_j \leq k_0$, $j = 1, \ldots, q$, one has as in [BG] (see also [SZ])

$$(\chi_m \circ \Phi^l)(S_\alpha a S_\beta^*) = \sum_{j=1}^{h(m,k)} y_j \otimes c_j,$$

where $y_j \in M_{w(m)}$ are partial isometries, and $c_j \in C^*(E)$.

For any $k \geq 1$ consider

$$\omega(k) = \{S_\alpha a S_\beta^* : |\beta| \leq |\alpha| \leq k,\ \ a = a_1^{p_1} \cdots a_q^{p_q},\ 0 \leq p_j \leq k,\ j = 1, \ldots, q\}.$$

Notice that $\omega_k = \omega(k) \cup \omega(k)^*$ is an increasing sequence of finite subsets of $C^*(E)$ such that the span of their union is dense in $C^*(E)$. For a fixed $k_0 \geq 1$ and $\delta > 0$ we have

$$\limsup_{k \to \infty} \frac{1}{k} \log \mathrm{rcp} \left(\bigcup_{i=0}^{k-1} \Phi^i(\omega(k_0)), \delta \right) \leq \log \rho(\Lambda_E).$$

Indeed, fix $k \geq 1$, and let $m = k + k_0$. Since $C^*(E)$ is nuclear, there exists $(\phi_0, \psi_0, M_{m_0}) \in \mathrm{CPA}(\mathrm{id}, C^*(E))$ such that

$$\|\psi_0(\phi_0(c_j)) - c_j\| < \frac{\delta}{h(m,k)},\ j = 1, \ldots, h.$$

Consider $B = M_{w(m)} \otimes M_{m_0}$ and let $\mathcal{H}$ be a Hilbert space on which $C^*(E)$ acts faithfully. The $*$-isomorphism $\chi_m^{-1} : \chi_m(C^*(E)) \to C^*(E)$ extends to a cp map $\psi_m : M_{w(m)} \otimes C^*(E) \to \mathcal{L}(\mathcal{H})$ with $\|\psi_m\| = 1$. Consider the cp maps $\phi = (\mathrm{id} \otimes \phi_0) \circ \chi_m : C^*(E) \to B$ and $\psi = \psi_m \circ (\mathrm{id} \otimes \psi_0) : B \to \mathcal{L}(\mathcal{H})$. For $c = S_\alpha a S_\beta^* \in \omega(k_0)$ we get

$$\begin{aligned}\|\psi(\phi(\Phi^l(c))) - \Phi^l(c)\| &= \|\psi_m(\mathrm{id} \otimes \psi_0\phi_0)((\chi_m \circ \Phi^l)(c)) - \Phi^l(c)\| \\ &= \|\psi_m(\mathrm{id} \otimes \psi_0\phi_0)((\chi_m \circ \Phi^l)(c)) - \psi_m((\chi_m \circ \Phi^l)(c))\| \\ &\leq \|(\mathrm{id} \otimes \psi_0\phi_0)((\chi_m \circ \Phi^l)(c)) - (\chi_m \circ \Phi^l)(c)\| \\ &= \left\| \sum_{j=1}^{h(m,k)} y_j \otimes (\psi_0\phi_0(c_j) - c_j) \right\| < \delta.\end{aligned}$$

Hence

$$\mathrm{rcp}\left(\bigcup_{i=0}^{k-1} \Phi^i(\omega(k_0)), \delta \right) \leq m_0 w(m) = m_0 w(k + k_0),$$

which gives

$$\limsup_{k\to\infty} \frac{1}{k} \log \left(\bigcup_{i=0}^{k-1} \Phi^i(\omega(k_0)), \delta \right) \leq \limsup_{k} \frac{1}{k} \log w(k) = \log \rho(\Lambda_E).$$

We conclude that $\mathrm{ht}(\Phi) = \log \rho(\Lambda_E)$. □

Proposition 5.5. *Consider $E = (E^0, E^1, s, r)$ a topological graph as in the previous theorem. Assume that the shift $\sigma_E : E^\infty \to E^\infty$ is expansive. Then the noncommutative shift $\Psi : \mathcal{F}_E \to \mathcal{F}_E$ defined by the maps $\Psi_k : C^*(R_k) \to C^*(R_{k+1})$ has entropy $\mathrm{ht}(\Psi) \geq h_\ell(E)$.*

Proof. The natural diagonal $C(E^\infty)$ of the groupoid $\Gamma(E)$ defined at the end of Section 3 is invariant under the map Ψ and its restriction $\Psi\,|_{C(E^\infty)}$ is conjugate to the map induced by shift $\sigma_E : E^\infty \to E^\infty$, hence $\mathrm{ht}(\Psi) \geq h_{\mathrm{top}}(\sigma)$. Now apply Corollary 4.4. □

Corollary 5.6. *The noncommutative shifts $\Phi, \Psi : \mathcal{F}_E \to \mathcal{F}_E$ may have different entropies.*

Proof. Since $\mathcal{F}_E$ is invariant under Φ, from Theorem 5.4 we get

$$\mathrm{ht}(\Phi\,|_{\mathcal{F}_E}) = \log \rho(\Lambda_E).$$

From the above proposition, $\mathrm{ht}(\Psi) \geq h_\ell(E)$. Examples 6.2 and 6.3 in the next section show that $\mathrm{ht}(\Phi)$ and $\mathrm{ht}(\Psi)$ indeed may be different. □

6. Examples

Example 6.1. Let X be a compact space, let $\sigma : X \to X$ a continuous surjective map, and let $E = E(X, \sigma) = (E^0, E^1, s, r)$ be the topological graph we defined earlier, where $E^0 = E^1 = X$, $s = \mathrm{id}$ and $r = \sigma$. We identify the Hilbert bimodule $H = H(E)$ and its tensor powers $H^{\otimes n}$ with $C(X)$. The endomorphism $\widetilde{\sigma} : C(X) \to C(X)$, $\widetilde{\sigma}(f) = f \circ \sigma$ coincides with the left action $\varphi_r : C(X) \to \mathcal{K}(H)$ defined by $r = \sigma$, after the identification of $\mathcal{K}(H)$ with $C(X)$. Consider (τ, π) a Toeplitz representation of H. Then the constant function 1 in $C(X)$ determines an element U in $C^*(\tau, \pi)$ satisfying $U^*U = 1$, $U\pi(f) = \tau(f)$ and $\pi(f)U = \tau(\widetilde{\sigma}(f))$ for $f \in C(X)$. The map $\psi : \mathcal{K}(H) \to C^*(\tau, \pi)$ can be expressed as $\psi(f) = U\pi(f)U^*$, hence $\psi(\varphi_r(f)) = U\pi(\widetilde{\sigma}(f))U^*$. The C^*-algebra $\mathcal{O}_E$ is the universal C^*-algebra generated by a copy of $C(X)$ and a unitary u satisfying $u^*fu = \widetilde{\sigma}(f)$. In particular, for σ a homeomorphism, $\mathcal{O}_E$ is isomorphic to the crossed product $C(X) \rtimes_\sigma \mathbb{Z}$. For details, see the results following Example 2 in [Ka1].

The Hilbert bimodule H has basis $\{\xi_1\}$, where ξ_1 is determined by the constant function 1. The canonical map $\Phi : \mathcal{O}_E \to \mathcal{O}_E$ is given by $\Phi(c) = ucu^*$. Note that $\Phi(\widetilde{\sigma}(f)) = f$, so Φ is a left inverse for $\widetilde{\sigma}$. If σ is a homeomorphism, then it is known that $\mathrm{ht}(\Phi) = h_{\mathrm{top}}(\sigma)$.

Example 6.2. Consider again the topological graphs $E_{p,q}$ of Katsura [Ka3], where p and q are integers with $p \geq 1$ and $q \neq 0$, $E^0 = E^1 = \mathbb{T}$, $s(z) = z^p$ and $r(z) = z^q$, as in Example 4.5. The Hilbert bimodule $H_{p,q} = H(E_{p,q}) = C(E^1)$ has a basis $\{\xi_1, \xi_2, \ldots, \xi_p\}$, where $\xi_k(z) = \frac{1}{\sqrt{p}} z^{k-1}, k = 1, \ldots, p$. We have

$$\langle \xi_j, \xi_k \rangle(z) = \frac{1}{p} \sum_{w^p = z} \overline{w^j} w^k = \delta_{jk} \cdot 1 \quad \text{and} \quad \sum_{j=1}^{p} \theta_{\xi_j, \xi_j} = 1.$$

Indeed, let $z = \exp(it)$. Then $w^p = z$ has solutions

$$w_m = \exp(i(t + 2m\pi)/p),$$

$m = 0, \ldots, p-1$. We have

$$\begin{aligned}\sum_{w^p=z} \overline{w^j} w^k &= \sum_{m=0}^{p-1} \exp(i(k-j)(t + 2m\pi)/p) \\ &= \exp(i(k-j)t/p) \sum_{m=0}^{p-1} (\exp(i(k-j)2\pi/p))^m \\ &= \delta_{kj} \cdot 1.\end{aligned}$$

A similar computation gives

$$\sum_{j=1}^{p} \xi_j \langle \xi_j, \eta \rangle(z) = \sum_{j=1}^{p} \xi_j(z) \sum_{w^p = z^p} \overline{\xi_j(w)} \eta(w) = \eta(z).$$

It follows that the C^*-algebra $\mathcal{O}_{E_{p,q}}$ is generated by the unitary $u \in C(E^0) = C(\mathbb{T})$, where $u(z) = z$ and p isometries $S_1, S_2, \ldots, S_p$ satisfying the relations

$$S_j^* S_k = \delta_{jk} \cdot 1, \quad \sum_{j=1}^{p} S_j S_j^* = 1, \quad uS_k = S_{k+q},$$

where $S_{k+q} = S_l u^m$ using the unique $l \in \{1, 2, \ldots, p\}$ with $k + q = l + pm$. The isometries $S_1, S_2, \ldots, S_p$ generate a copy of the Cuntz algebra $\mathcal{O}_p$, and the finite graph Λ associated to the topological graph $E_{p,q}$ has one vertex and p edges. From Theorem 5.4, we get

$$\mathrm{ht}(\Phi) = \log p.$$

We have $\mathcal{L}(H_{p,q}) = \mathcal{K}(H_{p,q}) \cong \mathbb{M}_p \otimes C(\mathbb{T})$. Let $q = pm + r$, where $r \in \{0, 1, \dots, p-1\}$. If $1 \leq r + k \leq p$, then $u \cdot \xi_k = \xi_{r+k} \cdot u^m$, and therefore the inclusion $\Psi_0 = \varphi : C(E^0) \to \mathcal{K}(H_{p,q})$ is given by

$$u \mapsto \sum_{k=1}^{p} S_{r+k} u^m S_k^*;$$

if $p + 1 \leq r + k \leq 2p - 1$, then $u \cdot \xi_k = \xi_{r+k-p} \cdot u^{m+1}$, and Ψ_0 it is given by

$$u \mapsto \sum_{k=1}^{p} S_{r+k-p} u^{m+1} S_k^*.$$

The C^*-algebra $\mathcal{F}_{E_{p,q}}$ is isomorphic to the inductive limit

$$C(\mathbb{T}) \xrightarrow{\Psi_0} \mathbb{M}_p \otimes C(\mathbb{T}) \xrightarrow{\Psi_1} \mathbb{M}_{p^2} \otimes C(\mathbb{T}) \xrightarrow{\Psi_2} \cdots ,$$

where the embeddings Ψ_k are obtained from Ψ_0 by tensoring with the identity. Recall from Example 4.5 that the space of infinite paths $E_{p,q}^{\infty}$ is homeomorphic to

$$\Sigma(p, q) = \left\{(z_0, z_1, z_2, \dots) \in \mathbb{T}^{\mathbb{N}} \mid z_k^p = z_{k+1}^q\right\},$$

which is a 1-dimensional solenoid if $\gcd(p, q) = 1$. The map

$$\Psi : \mathcal{F}_{E_{p,q}} \to \mathcal{F}_{E_{p,q}}$$

determined by the maps Ψ_k, when restricted to $C(E_{p,q}^{\infty})$, is conjugate with the shift σ, which has entropy $h_{\mathrm{top}}(\sigma) \geq \max\{\log p, \log |q|\}$ for $\gcd(p, q) = 1$. It follows that $\mathrm{ht}(\Psi) \geq \log \max\{p, |q|\}$.

Example 6.3. Consider the topological graph $E = (E^0, E^1, s, r)$ from Example 4.6, where $E^0 = \mathbb{T}$, $E^1 = \mathbb{T} \times \{1, 2\}$, $s(z, 1) = z^2$, $r(z, 1) = z$, $s(z, 2) = z$, $r(z, 2) = z^3$. The Hilbert bimodule $H = H(E) = C(E^1)$ has a basis $\{\xi_1, \xi_2, \xi_3\}$, where $\xi_k(z, 1) = \frac{1}{\sqrt{2}} z^{k-1}$, $\xi_k(z, 2) = 0$ for $k = 1, 2$ and $\xi_3(z, 1) = 0$, $\xi_3(z, 2) = 1$. We have $\langle \xi_j, \xi_k \rangle = \delta_{jk} \cdot 1$. The C^*-algebra $\mathcal{O}_E$ is generated by $C(E^0) = C(\mathbb{T})$ and three isometries S_1, S_2, S_3 satisfying the relations

$$\sum_{j=1}^{3} S_j S_j^* = 1, \quad S_j^* S_k = \delta_{jk} \cdot 1, \quad f \cdot S_j = \sum_{k=1}^{3} S_k \langle \xi_k, f \cdot \xi_j \rangle,$$

where $(f \cdot \xi_j)(z, 1) = f(z) \frac{1}{\sqrt{2}} z^{j-1}$, $j = 1, 2$, $(f \cdot \xi_3)(z, 2) = f(z^3)$ for $f \in C(\mathbb{T})$. It follows from Theorem 5.4 that the C^*-algebra $\mathcal{O}_E$ contains a copy of the Cuntz algebra $\mathcal{O}_3$ and $\mathrm{ht}(\Phi) = \log 3$. Let $u \in C(\mathbb{T})$ be the generator $u(z) = z$. We get the relations

$$uS_1 = S_2, \quad uS_2 = S_1 u, \quad uS_3 = S_3 u^3.$$

We have $\mathcal{L}(H) = \mathcal{K}(H) \cong \mathbb{M}_3 \otimes C(\mathbb{T})$. The inclusion $\Psi_0 : C(E^0) \to \mathcal{L}(H)$ is determined by the map

$$u \mapsto \begin{pmatrix} 0 & u & 0 \\ 1 & 0 & 0 \\ 0 & 0 & u^3 \end{pmatrix}.$$

Indeed, the element $u \in C(E^0)$ is sent to the compact operator

$$\xi_2 \otimes \xi_1^* + \xi_1 u \otimes \xi_2^* + \xi_3 u^3 \otimes \xi_3^*,$$

because

$$(u \cdot \xi_1)(z,1) = z\xi_1(z,1) = \frac{1}{\sqrt{2}} z = \xi_2(z,1),$$

$$(u \cdot \xi_2)(z,1) = z\xi_2(z,1) = \frac{1}{\sqrt{2}} z^2 = (\xi_1 \cdot u)(z,1),$$

$$(u \cdot \xi_3)(z,2) = z^3\xi_3(z,2) = z^3 = (\xi_3 \cdot u^3)(z,2).$$

The C^*-algebra $\mathcal{F}_E$ is isomorphic to the inductive limit

$$C(\mathbb{T}) \to \mathbb{M}_3 \otimes C(\mathbb{T}) \to \mathbb{M}_{3^2} \otimes C(\mathbb{T}) \to \cdots$$

The first inclusion is given by $u \mapsto S_2 S_1^* + S_1 u S_2^* + S_3 u^3 S_3^*$, and the other inclusions are obtained by iterating this formula. The restriction of $\Psi : \mathcal{F}_E \to \mathcal{F}_E$ to the diagonal $C(E^\infty)$ is conjugate to the map induced by the shift σ on the generalized solenoid

$$E^\infty = \Big\{ (z_k, a_k)_{k\geq 0} \in (\mathbb{T} \times \{1,2\})^{\mathbb{N}} \mid a_k = 1 \Rightarrow\ z_{k+1}^2 = z_k,\ a_k = 2 \Rightarrow\ z_{k+1} = z_k^3 \Big\},$$

and has entropy $\mathrm{ht}(\Psi) \geq \log 4$, see Example 4.6.

Example 6.4. Consider the topological graph from Example 4.7. In the case E^0 and E^1 are finite, the Hilbert bimodule $H(p,q) = H(E \times_{p,q} \mathbb{T})$ has a basis $\{\xi_{e,k} \mid e \in E^1, k = 1, \dots, p(e)\}$, where $\xi_{e,k}(e,z) = \frac{1}{\sqrt{p(e)}} z^{k-1}$ and $\xi_{e,k}(e',z) = 0$ for $e' \neq e$. For each $v \in E^0$, let $u_v \in \mathcal{O}_{E \times_{p,q} \mathbb{T}}$ be the image of the generating unitary of $C(\{v\} \times \mathbb{T}) \subset C(E^0 \times \mathbb{T})$. Let $S_{e,k} \in \mathcal{O}_{E \times_{p,q} \mathbb{T}}$ be the image of $\xi_{e,k}$. Then $\mathcal{O}_{E \times_{p,q} \mathbb{T}}$ is generated by the family $\{u_v\}_{v \in E^0}$ of partial unitaries with orthogonal ranges and the family $\{S_{e,k}\}_{e \in E^1, 1 \leq k \leq p(e)}$ of partial isometries with orthogonal ranges satisfying the relations

$$S_{e,k}^* S_{e,k} = u_{s(e)}^* u_{s(e)} \text{ for } e \in E^1 \text{ and } 1 \leq k \leq p(e),$$

$$u_{r(e)} S_{e,k} = S_{e,k+q(e)} \text{ for } e \in E^1 \text{ and } 1 \leq k \leq p(e),$$

$$u_v^* u_v = \sum_{r(e)=v} S_{e,k} S_{e,k}^* \text{ for } v \in E^0,$$

where $S_{e,k+q(e)} = S_{e,k'}u^l_{s(e)}$ for the unique $k' \in \{1, 2, \ldots, p(e)\}$ and $l \in \mathbb{Z}$ with $k + q(e) = k' + p(e)l$, see Appendix A in [Ka3]. Consider the matrices P, Q, where

$$P(v,w) = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} p(e), \quad Q(v,w) = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} q(e)$$

for $v, w \in E^0$. The partial isometries $S_{e,k}$ generate a copy of the Cuntz–Krieger algebra $\mathcal{O}_\Lambda$ inside $\mathcal{O}_{E\times_{p,q}\mathbb{T}}$, and by Theorem 5.4 we get

$$\mathrm{ht}(\Phi) = \log \rho(\Lambda).$$

If the formulas conjectured in Example 4.7 were true, this would imply that, under certain conditions on the maps p and q,

$$\mathrm{ht}(\Psi) \geq \log \max\{\rho(P), \rho(Q)\}.$$

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