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On ε-Pisot numbers

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Abstract. An algebraic integer whose other conjugates over the field of the rationals $\mathbb{Q}$ are of modulus less than ε, where $0 < \varepsilon \leq 1$, is called an ε-Pisot number. A Salem number is a real algebraic integer greater than 1 all of whose other conjugates over $\mathbb{Q}$ belong to the closed unit disc, with at least one of them of modulus 1. Let K be a number field generated over $\mathbb{Q}$ by a Salem number. We prove that there is a finite subset, say F_ε, of the integers of K such that each Salem number generating K over $\mathbb{Q}$ can be written as a sum of an element of F_ε and an ε-Pisot number. We also show some analytic properties of the set of ε-Pisot numbers.

Contents

1. Introduction

Pisot numbers were discovered more than a century ago during research in the uniform distribution of real sequences by A. Thue and then by G. H. Hardy (see, e.g., [1]). A Pisot (or Pisot–Vijayaraghavan) number is a real algebraic integer greater than 1, all of whose other conjugates lie inside the open unit disc. In this manuscript, when we speak about a conjugate, the minimal polynomial or the degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$. It was Pisot's thesis that provided a link to harmonic analysis as described in some papers of R. Salem who introduced a related class of algebraic numbers, namely Salem numbers. A real algebraic integer greater than 1 is called a Salem number if all its other conjugates are of

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modulus at most 1 and at least one conjugate lies on the unit circle. A Salem number is always of even degree. It has exactly one real conjugate in the open unit disc, namely its reciprocal, and all remaining conjugates are pairwise complex conjugates on the unit circle. Pisot and Salem numbers are quite rich in arithmetical properties which explain their appearance in various questions of harmonic analysis, automata theory, dynamical systems, ergodic theory, etc. (see, e.g., [5, 6, 9, 10, 13, 15]), and their role has always been important in the development of such theories.

Many results are known about the set S of Pisot numbers. For example, S is closed in the real line $\mathbb{R}$ [12], and the positive root, say θ_1, of the polynomial x^3-x-1 is the minimal element of S [14]. There is an algorithm due essentially to J. Dufresnoy and C. Pisot [7], but developed by D. W. Boyd [2, 3, 4], to determine the structure of the set S in a finite interval. This algorithm has been firstly used by J. Dufresnoy and C. Pisot to find all Pisot numbers less than $1.6183\dots$ and to show that the positive root, say θ_∞, of the polynomial x^2-x-1 is the minimal element of the derived set of S (see [1, Theorem 7.2.1]). By a construction due to R. Salem, every Pisot number is a limit of a sequence of elements of the set T of Salem numbers. The questions whether the set $S\cup T$ is closed or whether $\inf T>1$ are still unanswered.

Let $0<\varepsilon\leq 1$ be given. An algebraic integer is called an ε-Pisot number if all its other conjugates have modulus less than ε [8]. It is clear that the rational integers are the ε-Pisot numbers of degree 1. It is easy to check that an ε-Pisot number, say α, of degree ≥ 2 has modulus greater than 1 and so $\alpha\in\mathbb{R}$ and $|\alpha|\in S$. Let S_ε be set of ε-Pisot numbers. Then, $S_\varepsilon=-S_\varepsilon$, $S_1=S\cup(-S)\cup\{-1,0,1\}$ and $S_1=\bigcup\limits_{0<\varepsilon<1} S_\varepsilon$. The next result gives some elementary analytic properties of the set S_ε.

Theorem 1. *For each $\varepsilon<1$ the set S_ε is discrete but not uniformly, and*

$$\min(S_\varepsilon\cap\ ]1,\infty[)=\begin{cases}2 & \text{if } \varepsilon\leq 1/\theta_\infty\\ \theta_\infty=1.6180\dots & \text{if } 1/\theta_\infty<\varepsilon\leq 1/\sqrt{\theta_4}\\ \theta_4=1.4655\dots & \text{if } 1/\sqrt{\theta_4}<\varepsilon\leq 1/\sqrt{\theta_1}\\ \theta_1=1.3247\dots & \text{if } 1/\sqrt{\theta_1}<\varepsilon,\end{cases}$$

where θ_4 is the fourth smallest Pisot number (the minimal polynomial of θ_4 is x^3-x^2-1).

Recall that a subset X of $\mathbb{R}$ is uniformly discrete if the usual distance between two distinct elements of X is greater than a positive constant depending only on X; a uniformly discrete set is a discrete set, that is a set with no finite limit point. The proof of Theorem 1 appears in Section 3. In the next section we will be concerned by elements of S_ε which belong to certain real number fields. Recall also that a subset, say K, of the complex field $\mathbb{C}$ is called a number field if it is an extension of $\mathbb{Q}$ by an algebraic number, that is there is an algebraic number, say α, such that $K = \mathbb{Q}(\alpha)$, where $\mathbb{Q}(\alpha) = \{a_0 + a_1\alpha + \cdots + a_{d-1}\alpha^{d-1},\ a_k \in \mathbb{Q}\ \forall k \in \{0, \ldots, d-1\}\}$ and d is the degree of α; in this case we say that K is generated by α, and the degree of K is d. A real number field is a number field which is contained in $\mathbb{R}$. Using some results of Y. Meyer on harmonious sets [10], we shall prove the following:

Theorem 2. *Let K be a number field generated by a Salem number. Then, for each $0 < \varepsilon \leq 1$, there is a finite subset $F = F(K, \varepsilon)$ of the integers of K such that each Salem number generating K can be written as a sum of an element of F and an element of S_ε generating K.*

2. ε-Pisot numbers in a number field

For a real number field K and an $\varepsilon \in]0, 1]$, let

$$P = P(K, \varepsilon) := \{\alpha \in S_\varepsilon \cap K,\ K = \mathbb{Q}(\alpha)\}.$$

Then, a corollary of a result of C. Pisot (see [1, Theorem 8.1.4]), asserts that P is not finite. In [8], A. H. Fan and J. Schmeling have proved that the set P is relatively dense in $\mathbb{R}$, that is there is a positive constant ρ depending only on P, such that each interval of length ρ contains an element of P. Using some results of Y. Meyer on harmonious sets, the author pointed out that P is a model set in $\mathbb{R}$ [16]. We shall recall the definition of a real model set in the proof of Theorem 2. A model set is a Meyer set, and a subset X of $\mathbb{R}$ is called a Meyer set if X is relatively dense and the set

$$X - X = \{x - x',\ x \in X,\ x' \in X\}$$

is uniformly discrete [11]. From the proof of Theorem 1 it is easy to see that the set $S_\varepsilon \cap K$ is not uniformly discrete, and so $S_\varepsilon \cap K$ is not a Meyer set.

By analogy, let

$$T_K := \{\tau \in T \cap K,\ K = \mathbb{Q}(\tau)\}.$$

It is clear that $T_K = \emptyset$ for "almost all" real number fields K with even degree (for instance, if $K = \mathbb{Q}(\alpha)$, where the algebraic number α has more than two real conjugates, then $T_K = \emptyset$). Furthermore, if $K = \mathbb{Q}(\tau)$, for some Salem number τ, then the following "known" result shows that Salem numbers generating K are rare compared with the elements of the set P.

Proposition 3. *Let K be a number field generated by a Salem number. Then, $T_K = \{\tau_1^n,\ \ n \in \mathbb{N}\}$, where $\tau_1 = \min T_K$ and $\mathbb{N}$ is the set of positive rational integers.*

Proof. Let τ be a an element of the set T_K and let $\sigma_1, \sigma_2, \ldots, \sigma_d$ be the distinct embeddings of K in $\mathbb{C}$, where σ_1 is the identity of K and $\sigma_2(K) \subset \mathbb{R}$. Then, $\sigma_2(\tau) = 1/\tau$ and $|\sigma_j(\tau)| = 1$ for $j \in \{3, 4, \ldots, d\}$. Let n be the greatest positive rational integer such that $\tau_1^n \leq \tau$. Then, $\tau_1^n \leq \tau < \tau_1^{n+1}$ and so

$$1 \leq \tau' := \tau/\tau_1^n < \tau_1. \tag{1}$$

It is clear that τ' is an algebraic integer, since a Salem number is a unit (recall that the other conjugates of a Salem number are its reciprocal and complex numbers of modulus 1) and powers of a unit are units. Moreover, the conjugates of τ' are among the numbers $\sigma_1(\tau') = \tau'$, $\sigma_2(\tau/\tau_1^n) = \tau_1^n/\tau = 1/\tau'$, $\sigma_3(\tau)/\sigma_3(\tau_1)^n, \dots, \sigma_d(\tau)/\sigma_d(\tau_1)^n$. Suppose that $\tau \neq \tau_1^n$. It follows by the relation (1) that τ' has one real conjugate greater than 1, namely τ', one real conjugate less than 1, namely $1/\tau'$, and the remaining conjugates have modulus 1, as $|\sigma_j(\tau)/\sigma_j(\tau_1)^n| = |\sigma_j(\tau)| / |\sigma_j(\tau_1)^n| = 1/1$ for $j \in \{3, 4, \dots, d\}$. Consequently, each conjugate of τ' is repeated only one time by the action of the embeddings of K in $\mathbb{C}$, and so the conjugates of τ' are exactly the numbers $\sigma_j(\tau')$, where $j \in \{1, 2, \dots, d\}$; thus $\tau' \in T_K$, and this last relation together with (1) yields a contradiction. □

Proof of Theorem 2. For the following definition of a model set in $\mathbb{R}$ we shall refer to [10]. For the general definition of a model set in a locally compact abelian group which uses the notion of a cut and project scheme see the excellent expository paper of R. V. Moody [11]. Assume that for some rational integer $n \geq 2$ there exist a bounded subset Ω of the Euclidean space $\mathbb{R}^{n-1}$ with nonempty interior, and n linear forms $l_1, l_2, \dots, l_n$ on the $\mathbb{R}$-vector space $\mathbb{R}^n$ satisfying the following three conditions:

(C1) The forms $l_1, l_2, \dots, l_n$ are $\mathbb{R}$-linearly independent.
(C2) The coefficients of one of these forms, say l_1, are $\mathbb{Z}$-linearly independent, where $\mathbb{Z}$ is the ring of the rational integers.
(C3) If l is a nonzero linear form on $\mathbb{R}^n$ with rational integer coefficients, then the vectors $l, l_2, \dots, l_n$ are also $\mathbb{R}$-linearly independent.

Then, a subset Λ of $\mathbb{R}$ of the form

$$\Lambda = \{l_1(v),\ v \in \mathbb{Z}^n,\ (l_2(v), \dots, l_n(v)) \in \Omega\}$$

is called a model set (or a cut and project set) in $\mathbb{R}$. The set Ω is the window of the model set Λ [11]. To be more precise, we also say that the model set Λ is defined by the window Ω and the linear forms $l_1, l_2, \dots, l_n$. The scheme of the proof of Theorem 2 is very simple: we shall exhibit two real model sets defined by the same linear forms on the space $\mathbb{R}^d$, where d is the degree of the field K, and then we use the following immediate corollary of Proposition 7.9 of [11].

Lemma 4. *Let Λ_1 and Λ_2 be two real model sets defined by the same linear forms. Then there is a real finite set, say F, such that $\Lambda_1 \subset \Lambda_2 + F$.*

In fact the two model sets, say also Λ_1 and Λ_2, we shall use are contained in the ring $\mathbb{Z}_K$ of the integers of K. Let $\{\omega_1, \omega_2, \dots, \omega_d\}$ be a base of the $\mathbb{Z}$-module $\mathbb{Z}_K$, and let $\tau \in T_K$. Recall that d is even and $d \geq 4$. Set $s := (d-2)/2$. Let $\sigma_1, \sigma_2, \dots, \sigma_d$ be the distinct embeddings of K in $\mathbb{C}$, where σ_1 is the identity of K, $\sigma_2(K) \subset \mathbb{R}$ and $\sigma_{j+s}(\tau)$ is the complex conjugate $\overline{\sigma_j(\tau)}$ of $\sigma_j(\tau)$ for $j \in \{3, \dots, 2+s\}$. Now, consider the linear forms $l_1, \dots, l_d$ defined on the space $\mathbb{R}^d$ by the relations

$$l_j(x_1, \dots, x_d) = \sum_{k=1}^{d} x_k \sigma_j(\omega_k)$$

when $j \in \{1, 2\}$, and

$$l_j(x_1, \dots, x_d) = \sum_{k=1}^{d} x_k(\sigma_j(\omega_k) + \sigma_{j+s}(\omega_k))/2$$

and

$$l_{j+s}(x_1, \dots, x_d) = \sum_{k=1}^{d} x_k(\sigma_j(\omega_k) - \sigma_{j+s}(\omega_k))/2i,$$

where $i^2 = -1$, when $j \in \{3, \dots, 2+s\}$. As the determinant of the forms $l_1, \dots, l_d$ is not zero and is a multiple of the discriminant of the field K, the vectors $l_1, \dots, l_d$ are $\mathbb{R}$-linearly independent; thus the condition (C1) is true. A similar computation shows that (C3) is satisfied. Moreover, the coefficients, namely $\omega_1, \dots, \omega_d$, of l_1 are $\mathbb{Z}$-linearly independent, because $\{\omega_1, \dots, \omega_d\}$ is a $\mathbb{Z}$-base of $\mathbb{Z}_K$. Consequently, for this choice of the linear forms (and so of the space $\mathbb{R}^d$) there is a model set

$$\Lambda = \left\{ \sum_{k=1}^{d} p_k \omega_k, (p_1, \dots, p_d) \in \mathbb{Z}^d, (l_2(p_1, \dots, p_d), \dots, l_d(p_1, \dots, p_d)) \in \Omega \right\}$$

corresponding to each bounded set Ω of $\mathbb{R}^{d-1}$ with nonempty interior. Now, set Λ_1 the model set defined by the window

$$[-1, 1] \times \bigcap_{j=1}^{s} \{(y_1, y_2, \dots, y_{2s}) \in \mathbb{R}^{2s},\ y_j^2 + y_{j+s}^2 \leq 1\}.$$

Then, a simple computation shows that $\eta \in \Lambda_1$ if and only if $\eta \in \mathbb{Z}_K$ and $|\sigma_j(\eta)| \leq 1$ for each $j \in \{2, 3, \dots, 2+s\}$. Hence,

$$T_K \subset \Lambda_1. \tag{2}$$

For a given $\varepsilon \in]0, 1]$, set Λ_2 to be the model set defined by the window

$$\left( ]-\varepsilon, \varepsilon[\times \bigcap_{j=1}^{s} \{(y_1, y_2, \dots, y_{2s}) \in \mathbb{R}^{2s},\ y_j^2 + y_{j+s}^2 < \varepsilon^2\} \right) - \{(0, 0, \dots, 0)\}.$$

Then, $\beta \in \Lambda_2$ if and only if $\beta \in \mathbb{Z}_K$ and

$$|\sigma_j(\beta)| < \varepsilon \quad \text{for each } j \in \{2, 3, \dots, 2+s\}.$$

Thus $\Lambda_2 \subset S_\varepsilon$. Let $\beta \in \Lambda_2$. It follows by the relation $\prod_{j=1}^{d} |\sigma_j(\beta)| \in \mathbb{N}$, that $|\beta| > 1$ and each conjugate of β is repeated only one time by the action of the embeddings of K in $\mathbb{C}$. Consequently, β is of degree d and

$$\Lambda_2 \subset P. \tag{3}$$

Finally, Lemma 4 together with relations (2) and (3) yield

$$T_K \subset \Lambda_1 \subset \Lambda_2 + F \subset P + F,$$

where F is a real finite set, and the result follows immediately by considering the subset of F whose elements are effectively of the form $\eta - \beta$, where $\eta \in T_K$ and $\beta \in P$. □

3. Proof of Theorem 1

Let α be an ε-Pisot number, where $\varepsilon < 1$, and let $\alpha_1 := \alpha$, $\alpha_2, \dots, \alpha_d$ be the conjugates of α. If $d = 1$, then $\alpha \in \mathbb{Z}$. By the relation $\left|\prod_{j=1}^{d} \alpha_j\right| \geq 1$, we have

$$|\alpha| > \varepsilon^{1-d} \tag{4}$$

when $d \geq 2$. It follows by (4) that for each finite interval I, there is a positive constant $D(\varepsilon, I)$ such that $d \leq D(\varepsilon, I)$ when $\alpha \in I$. Hence, the degree and the conjugates of the algebraic integer α are bounded when $\alpha \in S_\varepsilon \cap I$; thus α takes at most a finite number of values, the set $S_\varepsilon \cap I$ is finite and S_ε is discrete. Let $s_n = \sum_{j=1}^{d} \alpha_j^n$, where $n \in \mathbb{N}$. Then,

$$|s_n - \alpha^n| = \left|\sum_{j=2}^{d} \alpha_j^n\right| < (d-1)\varepsilon^n$$

and so S_ε is not uniformly discrete, since $s_n \in \mathbb{Z} \subset S_\varepsilon$, $\alpha^n \in S_{\varepsilon^n} \subset S_\varepsilon$ and $\lim_{n\to\infty} \varepsilon^n = 0$. To determine $\min(S_\varepsilon \cap\,]1, \infty[)$, notice first that

$$\theta_1 \leq \min(S_\varepsilon \cap\,]1, \infty[) \leq 2, \tag{5}$$

because $2 \in S_\varepsilon$ and $S_\varepsilon \cap\,]1, \infty[\, \subset S$. Let again $\alpha \in S_\varepsilon$ with degree $d \geq 2$. If $d = 2$ and $|\alpha| \leq 2$, then the minimal polynomial of α is of the form $x^2 - s_1 x \pm 1$, where $s_1 = \alpha_1 + \alpha_2 \in \{\pm 2, \pm 1\}$, and a simple computation shows that $\alpha = \pm\theta_\infty$. Now, suppose $d \geq 3$. If $\varepsilon \leq 1/\theta_\infty$, then the inequality (4) yields

$$|\alpha| > \theta_\infty^2 = \theta_\infty + 1 > 2,$$

and so by the relation (5) we have

$$\min(S_\varepsilon \cap\,]1, \infty[) = 2,$$

because $-1/\theta_\infty$ is a conjugate of θ_∞ and $\theta_\infty \notin S_\varepsilon$. Similarly as in the case above, a simple computation shows that $\alpha \in \{\pm\theta_1, \pm\theta_4\}$ when $d = 3$ and $|\alpha| \leq \theta_\infty$ (to calculate α we may also use the fact that $|\alpha|$ is of degree 3 and

$|\alpha|$ belongs to the known set $S\cap\,]1,\theta_\infty]$). Moreover, if $d \geq 4$ and $\varepsilon \leq 1/\sqrt{\theta_4}$, then (4) implies

$$|\alpha| > \theta_4^{\frac{3}{2}} = 1.7\ldots > \theta_\infty.$$

It follows by (5) that

$$\min(S_\varepsilon \cap\,]1,\infty[) = \theta_\infty$$

when $1/\theta_\infty < \varepsilon \leq 1/\sqrt{\theta_4}$, since the numbers θ_1 and θ_4 have two nonreal conjugates with modulus $1/\sqrt{\theta_1}$ and $1/\sqrt{\theta_4}$, respectively. Notice also by the relation (5) that

$$\min(S_\varepsilon \cap\,]1,\infty[) = \theta_1$$

for $1/\sqrt{\theta_1} < \varepsilon$. Finally, by the relation (4) we have

$$|\alpha| > \theta_1^{\frac{3}{2}} = 1.5\ldots > \theta_4$$

when $d \geq 4$ and $\varepsilon \leq 1/\sqrt{\theta_1}$; thus

$$\min(S_\varepsilon \cap\,]1,\infty[) = \theta_4$$

for $1/\sqrt{\theta_4} < \varepsilon \leq 1/\sqrt{\theta_1}$. □

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