New York Journal of Mathematics
New York J. Math. **15** (2009) 19–35.

Theta function and Bergman metric on Abelian varieties

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Abstract. In this note, we find explicit balanced embeddings for a principally polarized Abelian variety. As a consequence, we are able to give a very simple proof of the fact (cf. Donaldson, 2001) that balanced metrics converge to the flat metric.

Contents

1. Introduction

In [D], Donaldson proved the following beautiful result: Let $(X, \mathcal{O}_X(1))$ be a n-dimensional projective manifold, polarized by an ample line bundle $\mathcal{O}_X(1)$. Suppose that $(X, \mathcal{O}_X(1))$ has no continuous automorphisms and there exists a constant scalar curvature Kähler (cscK) metric in the Kähler class $c_1(\mathcal{O}_X(1))$. Then the projective embedding $X \hookrightarrow \mathbb{P}H^0(X, \mathcal{O}_X(k))$

Received September, 2007.
Mathematics Subject Classification. 11G10, 32Q20.
Key words and phrases. Theta function, Abelian variety, Bergman metric.
The work is partially supported by RGC grant CUHK403106 from the Hong Kong government.

ISSN 1076-9803/09

induced by $\mathcal{O}_X(k)$ can always be balanced for k sufficiently large, that is, for each large k there exists a basis $\left\{z_i^{(k)}\right\}$ of $H^0\left(X, \mathcal{O}_X(k)\right)$ such that

$$\int_X \left(\frac{z_i^{(k)} \overline{z}_j^{(k)}}{\left|z^{(k)}\right|^2} - \frac{\delta_{ij}}{N_k+1} \right) \frac{\omega_{\mathrm{FS}}^n}{n!} = 0$$

where $N_k + 1 = \dim H^0(X, \mathcal{O}_X(1))$ and ω_{FS} is the Fubini–Study metric on $\mathbb{P}^{N_k}$. Moreover, if we let ω_k denote ω_{FS}/k induced from the balanced embedding, then

$$|\omega_k - \omega_\infty|_{C^r(\omega_\infty)} = O(1/k), \tag{1}$$

where ω_∞ is the cscK metric in the class $c_1(\mathcal{O}_X(1))$ and $|\cdot|_{C^r(\omega_\infty)}$ is the C^r-norm with respect to the metric ω_∞. Since the condition of balanced embedding is equivalent to the Chow stability in the geometric invariant theory (GIT), Donaldson's result gives an affirmative answer to one direction of the conjecture raised by Yau many years ago, that is the existence of a cscK metric should be related to the stability of the polarized pair $(X, \mathcal{O}_X(1))$ in the GIT sense.

As Donaldson pointed out, although Mumford proved that any smooth Riemann surface is always Chow stable with respect to the canonical polarization [Mum2], his theorem does not give a new proof of the uniformization theorem. This is because we have to assume a priori that hyperbolic metric exists on the Riemann surface in order to apply Donaldson's convergence result. So to get the a priori convergence of the balanced metrics for Riemann surfaces remains a challenge for the moment. This is the motivation of the current work. More precisely, in this note we prove the following:

Theorem 1. *Let $(A_\Omega, \mathcal{O}_{A_\Omega}(1))$ be a principally polarized Abelian variety of dimension g with period matrix $\Omega = [I, Z]$. For sufficiently large $l \in \mathbb{N}$, the projective embedding induced by a basis*

$$\left\{ \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \right\}_{a,b \in (\mathbb{Z}[1/l]/\mathbb{Z})^g}$$

for $H^0(A_\Omega, \mathcal{O}_{A_\Omega}(l^2))$ (see Section 4 for the definition) is balanced. And if we normalize the balanced metrics, each of them converges to the flat metric on A_Ω in C^r for any $r > 0$.

It was well-known that the balanced embeddings of $\mathbb{P}^N$ are always isometric. There is no need to prove the convergence (1). The Abelian varieties are the first example that we do need convergence. Convergence is proven by an elementary method without using any asymptotic analysis of the Bergman kernel as in [T], [R], [Z]. By the Abel–Jacobi theorem, every Riemann surface can be embedded into its Jacobian which is principally polarized. This work shall be regarded as the first step toward the proof of the a priori C^∞-convergence of the balanced metrics for high genus Riemann surfaces.

Acknowledgement. The first author would like to thank professor Conan Leung for his interest.

2. Abelian varieties and Theta functions

In this section, we collect some basic facts about principally polarized Abelian varieties and properties of Theta functions, which will be used in the later sections. Good references for the materials presented here are [GH] and [Mum1].

2.1. $\mathcal{O}_{A_\Omega}(1)$ and Theta function ϑ. Let $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{\lambda_1, \cdots, \lambda_{2g}\}$ be a lattice in $\mathbb{C}^g$ with its period matrix given by

$$\Omega := [\lambda_1, \cdots, \lambda_{2g}] = [I, Z]$$

where Z satisfies $Z^t = Z$ and $\operatorname{Im} Z > 0$. Then $A^g_\Omega = \mathbb{C}^g/\Lambda$ is the principally polarized Abelian variety defined by Ω and the canonical factors $\{e_\alpha, e_{g+\alpha}\}$ for the principal polarization $\mathcal{O}_{A_\Omega}(1) \to A^g_\Omega$ are given by

$$e_a(z) \equiv 1 \text{ and } e_{g+\alpha}(z) = \exp \pi i\left(-Z_{\alpha\alpha} - 2z_\alpha\right).$$

Notice that the canonical factors $\{e_\alpha, e_{g+\alpha}\}$ satisfy

$$\begin{aligned}
&e_{g+\beta}\left(z + \lambda_{g+\alpha}\right) e_{g+\alpha}(z) \\
&= \exp \pi i\left(-Z_{\beta\beta} - 2z_\beta - 2Z_{\beta\alpha} - Z_{\alpha\alpha} - 2z_\alpha\right) \\
&= e_{g+\alpha}\left(z + \lambda_{g+\beta}\right) e_{g+\beta}(z).
\end{aligned}$$

Let $\{x_1, \cdots, x_g, y_1, \cdots, y_g\}$ be the coordinates of the basis dual to

$$\{\lambda_1, \cdots, \lambda_{2g}\} \subset \mathbb{C}^g.$$

Then the first Chern class of $\mathcal{O}_{A_\Omega}(1)$ is given by

$$\begin{aligned}
\omega &:= \sum_{\alpha=1}^{g} dx_\alpha \wedge dy_\alpha \\
&= \sum_{\alpha,\beta} (\operatorname{Im} Z)^{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta \in \wedge^{1,1}(A_\Omega)
\end{aligned}$$

where

$$\begin{aligned}
z_\alpha &= x_\alpha + Z_{\alpha\beta} y_\beta \\
\bar{z}_\alpha &= x_\alpha + \overline{Z}_{\alpha\beta} y_\beta
\end{aligned}$$

and $(\operatorname{Im} Z)^{\alpha\beta}$ is the inverse of $\operatorname{Im} Z$. The unique global holomorphic section (up to scalar multiplication) of $\mathcal{O}_{A_\Omega}(1)$ is then given by

$$\vartheta(z, \Omega) := \sum_{m \in \mathbb{Z}^g} \exp \pi i\left(m^t Z m + 2m^t z\right).$$

In particular, ϑ satisfies

$$\vartheta(z+m,\Omega)=\vartheta(z,\Omega)$$
$$\vartheta(z+Zm,\Omega)=\exp-\pi i\left(m^tZm+2m^tz\right)\vartheta(z,\Omega)$$

for any $m\in\mathbb{Z}^g$.

2.2. Metric on $\mathcal{O}_{A_\Omega}(1)$. Let h be any Hermitian metric on $\mathcal{O}_{A_\Omega}(1)$, then it must satisfy

$$h(z)\left|\vartheta(z)\right|^2=h(z+Zm)\left|\vartheta(z+Zm)\right|^2 \text{ and } h(z+m)=h(z)$$

for all $m\in\mathbb{Z}^g$. Since $\vartheta(z+Zm,\Omega)=\exp-\pi i\left(m^tZm+2m^tz\right)\vartheta(z,\Omega)$, this implies that

$$\begin{aligned}h(z)&=h(z+Zm)\left|\exp-\pi i\left(m^tZm+2m^tz\right)\right|^2\\&=h(z+Zm)\exp-\pi i\left(m^tZm-m^t\overline{Z}m+2m^t(z-\overline{z})\right)\\&=h(z+Zm)\exp 2\pi\left(m^t\mathrm{Im}\,Zm+2m^ty\right).\end{aligned}$$

Now if in addition we require the curvature form of h to be

$$\omega=dz^t\left(\mathrm{Im}\,Z\right)^{-1}d\overline{z},$$

that is, $\partial\overline{\partial}\log h=\omega$, then this will force h to take the following form:

$$\begin{aligned}h(z)&=\exp-2\pi\left(y^t\left(\mathrm{Im}\,Z\right)^{-1}y+\frac{c^ty}{2}\right)\\&=\exp\frac{\pi}{2}\left((z-\overline{z})^t\left(\mathrm{Im}\,Z\right)^{-1}(z-\overline{z})+c^ti(z-\overline{z})\right)\end{aligned}$$

for some $c\in\mathbb{R}^g$. On the other hand, the identity

$$\begin{aligned}h(z+Zm)=\exp\frac{\pi}{2}\Big(&(z-\overline{z}+2i\mathrm{Im}\,Zm)^t\left(\mathrm{Im}\,Z\right)^{-1}(z-\overline{z}+2i\mathrm{Im}\,Zm)\\&+c^ti(z-\overline{z}+2i\mathrm{Im}\,Zm)\Big)\\=&\,h(z)\exp\left(-2\pi m^t\mathrm{Im}\,Zm+2\pi m^ti(z-\overline{z})-\pi c^t\mathrm{Im}\,Zm\right)\\=&\,h(z)\exp-2\pi\left(m^t\mathrm{Im}\,Zm+2m^ty\right)\end{aligned}$$

implies

$$c^t\mathrm{Im}\,Zm=0 \text{ for all } m\in\mathbb{Z}^g,$$

i.e., $c=0$. Hence we obtain:

Proposition 2 (cf. [GH]). *The only metric on $\mathcal{O}_{A_\Omega}(1)$ with curvature ω, up to a scalar multiple, is given by*

$$h(z):=\exp-2\pi y^t\left(\mathrm{Im}\,Z\right)^{-1}y.$$

Moreover, we have

$$h(z+Za)=h(z)\exp 2\pi\left(-a^t\mathrm{Im}\,Za-2a^ty\right)$$

for all $a\in\mathbb{R}^g$.

3. Finite Heisenberg groups

In this section, we recall the construction of irreducible representations of finite Heisenberg groups presented in [Mum]. They will supply the key ingredient of finding a balanced embedding $A_\Omega \subset \mathbb{P}^{l^{2g}-1}$.

Let $\mathbb{V}$ be the complex vector space of entire functions on $\mathbb{C}^g$. For $a, b \in \mathbb{R}^g$ and $f \in \mathbb{V}$, we introduce two operators on $\mathbb{V}$:

$$\begin{aligned} \mathfrak{S}_b f(z) &:= f(z+b), \\ \mathfrak{T}_\alpha f(z) &:= \exp \pi i(a^t Za + 2a^t z) f(z+Za). \end{aligned}$$

It is clear that they obey the following rules

$$\begin{aligned} \mathfrak{S}_{b_1} \circ \mathfrak{S}_{b_2} &= \mathfrak{S}_{b_1+b_2} \\ \mathfrak{T}_{a_1} \circ \mathfrak{T}_{a_2} &= \mathfrak{T}_{a_1+a_2}. \end{aligned}$$

for $a_i, b_i \in \mathbb{R}^g$. On the other hand,

$$\begin{aligned} \mathfrak{S}_b \left(\mathfrak{T}_a f\right)(z) &= \left(\mathfrak{T}_a f\right)(z+b) \\ &= \exp \pi i \left(a^t Za + 2a^t (z+b)\right) f(z+b+Za) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_a \left(\mathfrak{S}_b f\right)(z) &= \exp \pi i \left(a^t Za + 2a^t z\right) \left(\mathfrak{S}_b f\right)(z+Za) \\ &= \exp \pi i \left(a^t Za + 2a^t z\right) f(z+b+Za), \end{aligned}$$

implies the following commutative relation

$$\mathfrak{S}_b \circ \mathfrak{T}_a = \exp 2\pi i a^t b \mathfrak{T}_a \circ \mathfrak{S}_b.$$

Recall that the $2g+1$ dimensional *Heisenberg group* $\mathcal{G}$ is $U(1) \times \mathbb{R}^g \times \mathbb{R}^g$ with the multiplication defined by

$$(\lambda, a, b)\left(\lambda', a', b'\right) = \left(\lambda\lambda' \exp 2\pi i b^t a', a+a', b+b'\right).$$

It has a natural action on $\mathbb{V}$ via

$$\begin{aligned} &((\lambda, a, b) \cdot f)(z) \\ &= \lambda \left(\mathfrak{T}_a \circ \mathfrak{S}_b f\right)(z) \\ &= \lambda \exp \pi i \left(a^t Za + 2a^t z\right) f(z+Za+b) \end{aligned}$$

since

$$\begin{aligned} &(\lambda, a, b)\left(\left(\lambda', a', b'\right) \cdot f\right)(z) \\ &= \lambda\lambda' \left(\mathfrak{T}_a \circ \mathfrak{S}_b \circ \mathfrak{T}_{a'} \circ \mathfrak{S}_{b'} f\right)(z) \\ &= \lambda\lambda' \exp 2\pi i b^t a' \left(\mathfrak{T}_{a+a'} \circ \mathfrak{S}_{b+b'} f\right)(z). \end{aligned}$$

To make this representation unitary, we introduce a norm on $\mathbb{V}$

$$\begin{aligned}\|f\|^2 &:= \int_{\mathbb{C}^g} |f|^2 h(z)\,\omega^g \\ &= \int_{\mathbb{C}^g} |f|^2 \exp\left(-2\pi y^t (\operatorname{Im} Z)^{-1} y\right)\omega^g.\end{aligned}$$

Proposition 3. *For $\forall a, b \in \mathbb{R}^g$, we have*

$$\begin{aligned}h(z)\,|\mathfrak{S}_b f|^2 &= \left(h\,|f|^2\right)(z+b), \\ h(z)\,|\mathfrak{T}_a f|^2 &= h(z+Za)\,|f(z+Za)|^2.\end{aligned}$$

Proof. For $a, b \in \mathbb{R}^g$, we have

$$h(z)\,|\mathfrak{S}_b f|^2 = h(z)\,|f(z+b)|^2 = h(z+b)\,|f(z+b)|^2$$

and

$$\begin{aligned}h(z)\,|\mathfrak{T}_a f(z)|^2 &= h(z)\left|\exp \pi i\left(a^t Z a + 2a^t z\right) f(z+Za)\right|^2 \\ &= h(z)\exp 2\pi\left(-a^t \operatorname{Im} Z a - 2a^t y\right)|f(z+Za)|^2 \\ &= h(z+Za)\,|f(z+Za)|^2.\end{aligned}$$ □

Corollary 4. *For any $a, b \in \mathbb{R}^g, \mathfrak{S}_b$ and $\mathfrak{T}_a$ are unitary operators on $(\mathbb{V}, \|\cdot\|)$. In particular, the action of $\mathcal{G}$ on $(\mathbb{V}, \|\cdot\|)$ is unitary.*

Proof. The only thing we need to check is that

$$\begin{aligned}\|\mathfrak{T}_a f\|^2 &= \int_{\mathbb{C}^g} |\mathfrak{T}_a f(z)|^2 h(z)\,\omega^g \\ &= \int_{\mathbb{C}^g} |f(z+Za)|^2 h(z+Za)\,\omega^g \\ &= \int_{\mathbb{C}^g} |f(z)|^2 h(z)\,\omega^g \\ &= \|f\|^2.\end{aligned}$$ □

Remark 5. The $\mathcal{G}$ action on $\mathbb{V}$ is the classical Stone–Von Neumann representation.

Let us introduce a discrete subgroup of $\mathcal{G}$

$$\Gamma := \left\{(1, a, b) \in \mathcal{G} | a, b \in \mathbb{Z}^g\right\}.$$

Now $\vartheta(z, \Omega)$, up to scalars, can be characterized as the unique Γ-invariant entire function on $\mathbb{C}^g$. For a fixed $l \in \mathbb{N}$, let

$$l\Gamma := \{(1, a, b) \in \mathcal{G} \mid a, b \in (l\mathbb{Z})^g\} \subset \Gamma$$

and let $\mathbb{V}_l$ be the set of entire functions $f(z)$ on $\mathbb{C}^g$, invariant under the action of $l\Gamma$. That is, $f \in \mathbb{V}_l$ if and only if for any $a, b \in (l\mathbb{Z})^g$, we have

$$f(z) = \mathfrak{S}_b f(z) := f(z+b),$$
$$f(z) = \mathfrak{T}_\alpha f(z) := \exp \pi i \left(a^t Z a + 2a^t z\right) f(z + Za).$$

To get a good basis of $\mathbb{V}_l$, we need the following:

Proposition 6. *$f \in \mathbb{V}_l$ if and only if*

$$f(z) = \sum_{n \in (\mathbb{Z}[1/l])^g} c_n \exp \pi i \left(n^t Z n + 2n^t z\right)$$

such that $c_n = c_m$ if $n - m \in (l\mathbb{Z})^g$. In particular, $\dim \mathbb{V}_l = l^{2g}$.

Proof. For any $f \in \mathbb{V}_l$, by the invariance of f under the action of $\mathfrak{S}_l$, it has expansion

$$f(z) = \sum_{n \in (\mathbb{Z}[1/l])^g} c_n \exp 2\pi i n^t z.$$

On the other hand, for $m \in (l\mathbb{Z})^g$, $\mathfrak{T}_m f = f$ implies that

$$\begin{aligned}
\mathfrak{T}_m f(z) &= \exp \pi i \left(m^t Z m + 2m^t z\right) f(z + Zm) \\
&= \exp \pi i \left(m^t Z m + 2m^t z\right) \sum_{n \in (\mathbb{Z}[1/l])^g} c_n \exp 2\pi i n^t (z + Zm) \\
&= \sum_{n \in (\mathbb{Z}[1/l])^g} c_n \exp \pi i (m + 2n)^t Z m \exp 2\pi i (n+m)^t z \\
&= f(z) \\
&= \sum_{n \in (\mathbb{Z}[1/l])^g} c_{n+m} \exp 2\pi i (n+m)^t z
\end{aligned}$$

hence we have

$$\begin{aligned}
c_{n+m} &= c_n \exp \pi i (m + 2n)^t Z m \\
&= c_n \exp \left(\pi i m^t Z m + 2n^t Z m\right) \\
&= c_n \exp \left(\pi i (n+m)^t Z (n+m) - n^t Z n\right)
\end{aligned}$$

which means

$$c_{n+m} \exp \left(-\pi i (n+m)^t Z (n+m)\right) = c_n \exp \left(-\pi i n^t Z n\right).$$

So if we define $c'_n := c_n \exp \left(-\pi i n^t Z n\right)$ then

$$\begin{aligned}
f(z) &= \sum_{n \in (\mathbb{Z}[1/l])^g} c_n \exp 2\pi i n^t z \\
&= \sum_{n \in (\mathbb{Z}[1/l])^g} c'_n \exp \pi i \left(n^t Z n + 2n^t z\right)
\end{aligned}$$

and $c'_n = c'_m$ for $n - m \in (l\mathbb{Z})^g$. □

Let $\mu_m \subset U(1)$ denote the subgroup of m^{th} roots of 1. For $l \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{G}_l &:= \left\{(\lambda, a, b) \in \mathcal{G} \mid \lambda \in \mu_{l^2}; a, b \in (\mathbb{Z}[1/l])^g\right\}/l\Gamma \\ &= \mu_{l^2} \times \left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g \times \left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g\end{aligned}$$

with the multiplication induced from $\mathcal{G}$. Now for $a, b \in (\mathbb{Z}[1/l])^g$, the elements $\mathfrak{S}_b, \mathfrak{T}_a \in \mathcal{G}$ commute with $l\Gamma$, (since for $a' \in (\mathbb{Z}[1/l])^g$ and $b \in l\Gamma$ we have $\exp 2\pi i b^t a' = 1$) hence they act on $\mathbb{V}_l$. This descends to an action of $\mathcal{G}_l$ on $\mathbb{V}_l$, and the generators $\mathfrak{S}_b, \mathfrak{T}_a, \forall a, b \in (\mathbb{Z}[1/l])^g$ act on $\mathbb{V}_l$ as follows:

$$\begin{aligned}&\mathfrak{S}_b\left(\sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp \pi i \left(n^t Z n + 2n^t z\right)\right) \\ &= \sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp \pi i \left(n^t Z n + 2n^t (z+b)\right) \\ &= \sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp 2\pi i n^t b \exp \pi i \left(n^t Z n + 2n^t z\right)\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{T}_a\left(\sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp \pi i \left(n^t Z n + 2n^t z\right)\right) \\ &= \exp \pi i \left(a^t Z a + 2a^t z\right) \sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp \pi i \left(n^t Z n + 2n^t (z + Za)\right) \\ &= \sum_{n\in(\mathbb{Z}[1/l])^g} c_n \exp \pi i \left((n+a)^t Z (n+a) + 2(n+a)^t z\right) \\ &= \sum_{n\in(\mathbb{Z}[1/l])^g} c_{n-a} \exp \pi i \left(n^t Z n + 2n^t z\right).\end{aligned}$$

This motivates us to introduce a basis for $\mathbb{V}_l$:

$$s_c(z, \Omega) := \sum_{n\in c+(l\mathbb{Z})^g} \exp \pi i \left(n^t Z n + 2n^t z\right) \text{ for } c \in \left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g.$$

A direct calculation implies the following lemma from which we obtain the irreducibility of the $\mathcal{G}_l$ action on $\mathbb{V}_l$.

Lemma 7. *For any $a, b \in \mathbb{Z}[1/l]$, we have*

$$\begin{aligned}\mathfrak{S}_b s_c &= \exp\left(2b^t c\pi i\right) s_c, \\ \mathfrak{T}_a s_c &= s_{c+a}.\end{aligned}$$

4. Main theorem

Now we are ready to introduce the basis $\left\{\vartheta\left[\begin{array}{c} a \\ b \end{array}\right]\right\}$ of $H^0\left(A_\Omega, \mathcal{O}_{A_\Omega}\left(l^2\right)\right)$ whose induced projective embedding is balanced.

In [Mum], Mumford introduces the following:

Definition 8. For $a, b \in \mathbb{Q}^g$,

$$\begin{aligned}
\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z, \Omega) &:= \left(\mathfrak{S}_b \circ \mathfrak{T}_a\right) \vartheta(z, \Omega) \\
&= \exp \pi i\left(a^t Z a+2 a^t(z+b)\right) \vartheta(z+Z a+b, \Omega) \\
&= \sum_{m \in \mathbb{Z}^g} \exp \pi i\left((m+a)^t Z(m+a)+2(m+a)^t(z+b)\right).
\end{aligned}$$

They satisfy the following properties:

Proposition 9 ([Mum])**.** *For $a, a', b, b' \in(\mathbb{Z}[1/l])^g$ and $p, q \in \mathbb{Z}^g$, we have:*

(1) $\vartheta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](z, \Omega)=\vartheta(z, \Omega)$.

(2) $\mathfrak{S}_{b'} \vartheta\left[\begin{array}{c} a \\ b \end{array}\right]=\vartheta\left[\begin{array}{c} a \\ b+b' \end{array}\right]$.

(3) $\mathfrak{T}_{a'} \vartheta\left[\begin{array}{c} a \\ b \end{array}\right]=\exp \left(-2 \pi i b^t a'\right) \vartheta\left[\begin{array}{c} a'+a \\ b \end{array}\right]$.

(4) $\vartheta\left[\begin{array}{c} a+p \\ b+q \end{array}\right]=\exp \left(2 \pi i a^t q\right) \vartheta\left[\begin{array}{c} a \\ b \end{array}\right]$.

(5)
$$h(z)\left|\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](z, \Omega)\right|^2=h|\vartheta|^2(z+Z a+b).$$

Notice that

$$\begin{aligned}
\vartheta\left[\begin{array}{c} 0 \\ 0 \end{array}\right] &= \vartheta(z) \\
&= \sum_{m \in \mathbb{Z}^g} \exp \pi i\left(m^t Z m+2 m^t z\right) \\
&= \sum_{p \in(\mathbb{Z} / l \mathbb{Z})^g} \sum_{n \in p+(l \mathbb{Z})^g} \exp \pi i\left(n^t Z n+2 n^t z\right) \\
&= \sum_{p \in(\mathbb{Z} / l \mathbb{Z})^g} s_p
\end{aligned}$$

implies that

$$\begin{aligned}\vartheta\left[\begin{array}{c}a\\b\end{array}\right] &= \mathfrak{S}_b\mathfrak{T}_a\vartheta\\ &= \sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\left(\mathfrak{S}_b\circ\mathfrak{T}_a s_p\right)(z)\\ &= \sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\exp\left(2\pi i b^t\left(p+a\right)\right)s_{p+a}.\end{aligned}$$

For a fixed k, denote $\varepsilon_k := (0,\cdots,\overset{k^{th}}{1},\cdots 0)\in\mathbb{Z}^g$, let

$$\widetilde{\vartheta}_a := \left\{\vartheta\left[\begin{array}{c}a\\q/l\end{array}\right]\right\}_{q\in(\mathbb{Z}/l\mathbb{Z})^g}\quad\text{and } \widetilde{s}_a := \left\{s_{p+a}\right\}_{p\in(\mathbb{Z}/l\mathbb{Z})^g}$$

be l^g-dimensional vectors, then we have

$$\widetilde{\vartheta}_a = \mathfrak{U}_a\widetilde{s}_a$$

with

$$\mathfrak{U}_a = \left\{\exp 2\pi i q^t\left(p+a\right)/l\right\}_{q,p\in(\mathbb{Z}/l\mathbb{Z})^g}.$$

Lemma 10. *For any* $r\in(\mathbb{Z}/l\mathbb{Z})^g$

$$\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\exp 2\pi i r^t p/l = 0.$$

Proof. For simplicity let us first assume that r is primitive, that is r is not a multiple of some element in $(\mathbb{Z}/l\mathbb{Z})^g$, then we have the following exact sequence of Abelian groups

$$0 \longrightarrow \ker r \longrightarrow (\mathbb{Z}/l\mathbb{Z})^g \xrightarrow{r\cdot} \mathbb{Z}/l\mathbb{Z} \longrightarrow 0$$

and $\ker r\cong(\mathbb{Z}/l\mathbb{Z})^{g-1}$. For any $p\in(\mathbb{Z}/l\mathbb{Z})^g$, we have a decomposition of $p=p^\top+p^\perp$ with $p^\top\in\ker r$ and $p^\perp\in\ker r^\perp$. Thus we obtain

$$\begin{aligned}\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\exp 2\pi i r^t p/l &= \sum_{q\in\ker r}\sum_{p\in\ker r^\perp}\exp 2\pi i r^t\left(q+p\right)/l\\ &= \sum_{q\in\ker r}\sum_{p\in\ker r^\perp}\exp 2\pi i r^t p/l\\ &= l^{g-1}\sum_{p\in\ker r^\perp}\exp 2\pi i r^t p/l\\ &= 0.\end{aligned}$$

where we have used the fact that

$$\sum_{k=0}^{l-1}\exp 2\pi i a k/l = 0 \text{ for any } 0\le a\le l-1.$$

for the last identity.

If r is not primitive, then it is a multiple of a primitive vector $r_0 \in (\mathbb{Z}/l\mathbb{Z})^g$. We may replace r by r_0 and use the same argument. The details are left to the readers. □

Corollary 11. *$\mathfrak{U}_a/l^{g/2}$ is a $l^g \times l^g$-unitary matrix, i.e., $\mathfrak{U}_a/l^{g/2} \in U(l^g)$.*

Proof. Since $\mathfrak{U}_a = \operatorname{diag}\left[\exp 2\pi i q^t a/l\right] \mathfrak{U}'_a$ with $\mathfrak{U}'_a := \left\{\exp 2\pi i q^t p/l\right\}$, all we need to show is that $\mathfrak{U}'_a$ is unitary, which follows from the above lemma. □

In conclusion, we have:

Proposition 12.

$$\begin{bmatrix} \widetilde{\vartheta}_0 \\ \widetilde{\vartheta}_{\varepsilon_1} \\ \vdots \\ \widetilde{\vartheta}_{(1-1/l)\sum_{i=1}^g \varepsilon_i} \end{bmatrix} = \begin{bmatrix} \mathfrak{U}_0 & 0 & & 0 \\ 0 & \mathfrak{U}_{\varepsilon_1} & & \\ & & \ddots & \\ 0 & & & \mathfrak{U}_{(1-1/l)\sum_{i=1}^g \varepsilon_g} \end{bmatrix} \begin{bmatrix} \widetilde{s}_0 \\ \widetilde{s}_{\varepsilon_1} \\ \vdots \\ \widetilde{s}_{(1-1/l)\sum_{i=1}^g \varepsilon_g} \end{bmatrix}$$

where $\mathfrak{U}_a \in U(l^g)$ for each $a \in \left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g$. In particular, we have

$$\sum_{a,b \in \left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g} \left|\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\Omega)\right|^2 = l^g \sum_{p \in (\mathbb{Z}/l\mathbb{Z})^g} \sum_{a \in \left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)} |s_{p+a}(z)|^2 .$$

The next lemma gives a geometric interpretation of the space $\mathbb{V}_l$.

Lemma 13. *Let*

$$\begin{aligned} l^* : A_\Omega &\longrightarrow A_\Omega \\ z &\longmapsto lz \end{aligned}$$

be the rescaling map. Then $\deg l^*\mathcal{O}_{A_\Omega}(1) = l^2$ and

$$\mathbb{V}_l = H^0\left(l^*\mathcal{O}_{A_\Omega}(1)\right) = H^0\left(\mathcal{O}_{A_\Omega}\left(l^2\right)\right).$$

Proof. Let

$$\begin{aligned} l^* : \mathbb{C}/\Lambda &\longrightarrow \mathbb{C}/\Lambda \\ z &\longmapsto lz \end{aligned}$$

be the rescaling map and let $\left\{e'_\alpha, e'_{g+\alpha}\right\}$ be the canonical factor associated with $l^*\mathcal{O}_{A_\Omega}(1)$. Then we have

$$e'_\alpha(z) = 1 = e^{l^2}_\alpha(z) \text{ and } e'_{g+\alpha}(z) = \exp \pi i\left(-Z_{\alpha a} l^2 - 2z_\alpha l^2\right) = e^{l^2}_{g+\alpha}(z).$$

In fact, $\mathcal{O}_{A_\Omega}(1)$ is a symmetric line bundle, which means

$$l^*\mathcal{O}_{A_\Omega}(1) = \mathcal{O}_{A_\Omega}\left(l^2\right).$$

On the other hand, for $f \in \mathbb{V}_l$ we have

$$f(l(z+m)) = f(lz+lm) = f(lz) \text{ for } m \in \mathbb{Z}^g$$

and

$$\begin{aligned} f\left(l\left(z+\lambda_{g+\alpha}\right)\right) &= \exp \pi i\left(-Z_{\alpha\alpha} l^2 - 2z_\alpha l^2\right) f\left(lz\right) \\ &= e'_{g+\alpha}\left(z\right) f\left(lz\right) \\ &= e^{l^2}_{g+\alpha}\left(z\right) f\left(lz\right). \end{aligned}$$

Thus $f \in H^0\left(l^*\mathcal{O}_{A_\Omega}(1)\right) = H^0\left(\mathcal{O}_{A_\Omega}\left(l^2\right)\right)$. A dimension count then implies

$$\begin{aligned} \varphi : \mathbb{V}_l &\longrightarrow H^0\left(A_\Omega, \mathcal{O}\left(l^2\right)\right) \\ f\left(z\right) &\longmapsto f\left(lz\right) \end{aligned}$$

is an isomorphism. □

Now we introduce an inner product on $\mathbb{V}_l$.

Definition 14. For $f\left(z\right) \in \mathbb{V}_l$ we define

$$\left|f\left(z\right)\right|_h^2 := \int_{\mathbb{D}_{l\Lambda}} \left|f\left(z\right)\right|^2 h\left(z\right) \omega^n$$

where $\mathbb{D}_\Lambda$ is the fundamental domain associated to Λ. In particular,

$$\left|\vartheta_{0,0}\left(z\right)\right|_h^2 := \int_{\mathbb{D}_{l\Lambda}} \left|\vartheta\left(z\right)\right|^2 h\left(z\right) \omega^g = l^{2g} \int_{\mathbb{D}_\Lambda} \left|\vartheta\left(z\right)\right|^2 h\left(z\right) \omega^g.$$

It possesses the following properties.

Lemma 15. *For $a, b \in \left(\mathbb{Z}\left[1/l\right]\right)^g$, we have:*

(1)

$$\begin{aligned} h\left(z\right)\left|\left(\mathfrak{S}_b f\right)\left(z\right)\right|^2 &= \left(h\left|f\right|^2\right)\left(z+b\right), \\ h\left(z\right)\left|\left(\mathfrak{T}_a f\right)\left(z\right)\right|^2 &= \left(h\left|f\right|^2\right)\left(z+Za\right). \end{aligned}$$

(2) *The actions $\{\mathfrak{S}_b, \mathfrak{T}_a\}$ are unitary with respect to the L^2-metric induced from h.*

(3) $\left\{\frac{s_c}{l^{g/2}|\vartheta|_h}\right\}_{c\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$ *forms an orthonormal basis of $\left(\mathbb{V}_l, |\cdot|_h\right)$.*

Proof. Part (1) is a special case of Proposition 3.

For part (2), we have

$$\begin{aligned} \left\|\mathfrak{S}_b f\left(z\right)\right\|_h^2 &= \int_{\mathbb{D}_{l\Lambda}} \left|f\left(z+b\right)\right|^2 h\left(z+b\right) \omega^g = \int_{\mathbb{D}_{l\Lambda}} \left|f\left(z\right)\right|^2 h\left(z\right) \omega^g \\ \left\|\mathfrak{T}_a f\left(z\right)\right\|_h^2 &= \int_{\mathbb{D}_{l\Lambda}} \left|f\left(z+Za\right)\right|^2 h\left(z+Za\right) \omega^g = \int_{\mathbb{D}_{l\Lambda}} \left|f\left(z\right)\right|^2 h\left(z\right) \omega^g. \end{aligned}$$

For part (3), since $\{\mathfrak{S}_c\}_{c\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$ forms a family of commuting operators, the eigenvectors $\{s_c\}_{c\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$ are mutually orthogonal. On the other hand Lemma 7 implies that for all $a, c \in \left(\mathbb{Z}\left[1/l\right]\right)^g$

$$\left|s_{c+a}\right|_h^2 = \left|\mathfrak{T}_a s_c\right|_h^2 = \left|s_c\right|_h^2.$$

If we set $c=-a$ then we have

$$|s_c|_h^2 = |s_0|_h^2 .$$

This implies

$$|\vartheta|_h^2 = \sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g} |s_p|_h^2 = l^g |s_0|_h^2 . \qquad \square$$

Now let

$$\begin{aligned} \Phi : A_\Omega &\longrightarrow \mathbb{P}^{l^{2g}-1} \\ z &\longmapsto [s_c(lz)]_{c\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g} \end{aligned}$$

be the projective embedding induced by $\{s_c\}_{c\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$. Then $\Phi^*\omega_{\mathrm{FS}}$ is invariant under the action of $\mathfrak{S}_b$ and $\mathfrak{T}_a$ for $a,b\in\left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g$ since both $\mathfrak{S}_b$ and $\mathfrak{T}_a$ are unitary. The Fubini–Study metric is given by

$$\|f(lz)\|_{\mathrm{FS}}^2 := \int_{A_\Omega} \frac{|f(lz)|^2}{\sum_c |s_c(lz)|^2} \Phi^*\omega_{\mathrm{FS}}.$$

This together with the identities

$$\begin{aligned} h(z)\left|(\mathfrak{S}_b f)(z)\right|^2 &= \left(h|f|^2\right)(z+b) \\ h(z)\left|(\mathfrak{T}_a f)(z)\right|^2 &= \left(h|f|^2\right)(z+Za) \end{aligned}$$

and

$$\begin{aligned} |\mathfrak{S}_b s_c| &= |s_c| \\ \sum_c |\mathfrak{T}_a s_c|^2 &= \sum_c |s_c|^2 \end{aligned}$$

would imply

$$\begin{aligned} \|(\mathfrak{S}_b f)(lz)\|_{\mathrm{FS}}^2 &= \int_{A_\Omega} \frac{|(\mathfrak{S}_b f)(lz)|^2}{\sum_c |s_c(lz)|^2} \Phi^*\omega_{\mathrm{FS}} \\ &= \int_{A_\Omega} \frac{|(\mathfrak{S}_{lb} f)(lz)|^2 h(lz)}{\sum_c |(\mathfrak{S}_b s_c)(lz)|^2 h(lz)} \Phi^*\omega_{\mathrm{FS}} \\ &= \int_{A_\Omega} \frac{\left(h|f|^2\right)(lz+b)}{\left(\sum_c h|s_c|^2\right)(lz+b)} (\mathfrak{S}_b\circ\Phi)^*\omega_{\mathrm{FS}} \\ &= \int_{A_\Omega} \left\{\frac{|f|^2}{\sum_c |s_c|^2}\Phi^*\omega_{\mathrm{FS}}\right\}(lz+b) \\ &= \|f(lz)\|_{\mathrm{FS}}^2 , \end{aligned}$$

and

$$\begin{aligned}\|(\mathfrak{T}_a f)(lz)\|^2_{\mathrm{FS}} &= \int_{A_\Omega} \frac{|(\mathfrak{T}_a f)(lz)|^2}{\sum_c |s_c(lz)|^2} \Phi^* \omega_{\mathrm{FS}} \\ &= \int_{A_\Omega} \frac{|(\mathfrak{T}_a f)(lz)|^2 h(lz)}{\sum_c |(\mathfrak{T}_a s_c)(lz)|^2 h(lz)} (\mathfrak{T}_a \circ \Phi)^* \omega_{\mathrm{FS}} \\ &= \int_{A_\Omega} \left\{ \frac{|f|^2}{\sum_c |s_c|^2} \Phi^* \omega_{\mathrm{FS}} \right\} (lz + Za) \\ &= \|f(lz)\|^2_{\mathrm{FS}}.\end{aligned}$$

Hence we obtain the following:

Proposition 16. *For $a, b \in (\mathbb{Z}[1/l])^g$:*

(1) $\mathfrak{S}_b, \mathfrak{T}_a$ *are unitary operators acting on* $(\mathbb{V}_l, \|\cdot\|_{\mathrm{FS}})$.
(2) *The action* $\{s_c\}_{c \in \left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$ *is an orthonormal basis for* $(\mathbb{V}_l, \|\cdot\|_{\mathrm{FS}})$.

Recall from [D] that a projective embedding of $\varphi : A_\Omega \to \mathbb{P}^{l^{2g}-1}$ is called *balanced* if we have

$$\int_{A_\Omega} \left(\frac{z_i \bar{z}_j}{|z|^2} - \frac{\delta_{ij}}{l^{2g}} \right) \varphi^* \omega^g_{\mathrm{FS}} = 0.$$

Corollary 17. *The embedding defined by*

$$\begin{aligned}\Theta : A_\Omega &\longrightarrow \mathbb{P}^{l^{2g}-1} \\ z &\longmapsto \left[\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) \right]_{a,b \in \left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}\end{aligned}$$

is balanced.

Proof. It follows from the fact that

$$\left\{ \frac{l^{g/2} s_c}{\|\vartheta\|_{\mathrm{FS}}} \right\}_{c \in \left(\frac{\mathbb{Z}[1/l]}{l\mathbb{Z}}\right)^g}$$

forms an orthonormal basis for $(\mathbb{V}_l, \|\cdot\|_{\mathrm{FS}})$, and the map Φ and Θ differ by an unitary transformation. □

Remark 18. Geometrically, the finite Heisenberg group $\mathcal{G}_l$ acts on the embedding leaving $\operatorname{Im}\Phi$ invariant. And $\{\mathfrak{S}_b, \mathfrak{T}_a\}$ acts on $\Phi(A_\Omega)$ via translations under the group law. Although we can not get a homomorphism from A_Ω to $U(l^2)$, we do have $\mathcal{G}_l \subset U(l^2)$.

Finally, we are ready to state our main result.

Theorem 19.

$$\sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}\left|\vartheta\left[\begin{array}{c}a\\b\end{array}\right](z,\Omega)\right|^2=l^g\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)}\left|s_{p+a}(z)\right|^2.$$

$$\lim_{l\to\infty}l^{-g}\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g}\sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}\left|s_{p+a}(z)\right|^2h(z)=\int_{A_\Omega}\left|\vartheta(z)\right|^2h(z)\,dx\wedge dy.$$

To prove this we need the following elementary lemma.

Lemma 20. *Let f be a smooth real-valued doubly periodic function on $\mathbb{C}^g$ with period matrix $\Omega=(I,Z)\in M_{g\times 2g}(\mathbb{C})$. Let $\mathbb{D}_\Omega\subset\mathbb{C}^g$ be a fundamental domain. Then*

$$\frac{\det\operatorname{Im}Z}{l^{2g}}\sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}f(lz+Za+b)$$

$$=\int_{\mathbb{D}_\Omega}f(z)\frac{\omega^g_{\mathbb{C}^g}}{\det\operatorname{Im}Z}+O\left(l^{-3}\right)\ \textit{for } l\to\infty$$

where $\omega_{\mathbb{C}^g}:=\sum_i dz_i\wedge d\bar{z}_i$ is the standard Kähler form on $\mathbb{C}^g$.

Proof. By performing affine transformations, we may assume that $Z=iI$. First, we notice that f being smooth implies that

$$\lim_{l\to\infty}\frac{1}{l^{2g}}\sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}f(z+ai+b)=\int_{\mathbb{D}_\Omega}f\omega^g_{\mathbb{C}^g}$$

for any fixed $z\in\mathbb{D}_\Omega$. Moreover we have that

$$\left|\frac{1}{l^{2g}}\sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}f(z+ai+b)-\int_{\mathbb{D}_\Omega}f\omega^g_{\mathbb{C}^g}\right|\le Cl^{-3}$$

with C being a constant depending only on the sup-norm of the second derivative of f but not on z. In particular, this implies

$$\left|\frac{1}{l^{2g}}\sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g}f(lz+ai+b)-\int_{\mathbb{D}_\Omega}f\omega^g_{\mathbb{C}^g}\right|\le Cl^{-3}.\qquad\square$$

Proof of Theorem 19. By Proposition 3, we have

$$\sum_{a,b}\left|\vartheta\left[\begin{array}{c}a\\b\end{array}\right](lz)\right|^2h(lz)$$

$$=\sum_{a,b}\left|\vartheta(lz+a\tau+b)\right|^2h(lz+a\tau+b)$$

hence

$$\begin{aligned} & l^g \sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g} \sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g} |s_{p+a}(lz)|^2 h(lz) \\ & = \sum_{a,b\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)^g} \left|\vartheta\left[\begin{array}{c} a \\ b \end{array}\right](lz,\Omega)\right|^2 h(lz) \\ & = \sum_{a,b} |\vartheta(lz+Za+b)|^2 h(lz+Za+b). \end{aligned}$$

By Lemma 20, we have

$$\begin{aligned} & \lim_{l\to\infty} \frac{\mathrm{Vol}(A_\Omega)}{l^{2g}} \sum_{a,b} |\vartheta(lz+a\tau+b)|^2 h(lz+a\tau+b) \\ & = \int_{A_\Omega} |\vartheta(z)|^2 h(z)\,\omega^g. \end{aligned}$$

Hence

$$\begin{aligned} & \lim_{l\to\infty} l^{-g} \sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g} \sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)} |s_{p+a}(lz)|^2 h(lz) \\ & \qquad = \frac{1}{\mathrm{Vol}(A_\Omega)} \int_{A_\Omega} |\vartheta(z)|^2 h(z)\, dx\wedge dy. \qquad \square \end{aligned}$$

Corollary 21. *The balanced metric $\Theta_l^*\omega_{\mathrm{FS}}/l^2$ converges to ω_0, the flat metric on A_Ω, as a C^∞ function as $l\to\infty$.*

Proof. Let h_{FS} denote the Fubini–Study metric on $\mathcal{O}_{A_\Omega}(l^2)$ induced via the embedding Θ. Then

$$h_{\mathrm{FS}} = \frac{h^{l^2}}{\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g} \sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)} |s_{p+a}(lz)|^2 h(lz)}.$$

On the other hand

$$\begin{aligned} \frac{\Theta_l^*\omega_{\mathrm{FS}}}{l^2} &= \frac{\partial\bar\partial \log h_{\mathrm{FS}}}{l^2} \\ &= \partial\bar\partial\log h - \frac{\partial\bar\partial\log\left(\sum_{p\in(\mathbb{Z}/l\mathbb{Z})^g} \sum_{a\in\left(\frac{\mathbb{Z}[1/l]}{\mathbb{Z}}\right)} |s_{p+a}(lz)|^2 h(lz)\right)}{l^2} \\ &= \omega_0 - \frac{\partial\bar\partial\log\left(l^g\|\vartheta\|_h + O\left(l^{g-3}\right)\right)}{l^2} \to \omega_0. \end{aligned}$$

$\square$

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This paper is available via http://nyjm.albany.edu/j/2009/15-2.html.