

Galois module structure of Milnor K -theory mod p^s in characteristic p

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ABSTRACT. Let E be a cyclic extension of p th-power degree of a field F of characteristic p . For all $m, s \in \mathbb{N}$, we determine $K_m E/p^s K_m E$ as a $(\mathbb{Z}/p^s \mathbb{Z})[\text{Gal}(E/F)]$ -module. We also provide examples of extensions for which all of the possible nonzero summands in the decomposition are indeed nonzero.

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Let F be a field of characteristic p . Let $K_m F$ denote the m th Milnor K -group of F and $k_m F = K_m F/pK_m F$ (see, for instance, [Mi] and [Ma, Chapter 14]). If E/F is a Galois extension of fields, let $G = \text{Gal}(E/F)$ denote the associated Galois group. In [BLMS] the structure of $k_m E$ as an $\mathbb{F}_p G$ -module was determined when G is cyclic of p th-power order. In this paper we determine the Galois module structure of $K_m E$ modulo p^s for $s \in \mathbb{N}$ and these same G . We also provide examples of extensions for which the possible free summands in the decomposition are all nonzero. These examples together with the results in [BLMS] show that the dimensions over \mathbb{F}_p of indecomposable $\mathbb{F}_p[\text{Gal}(E/F)]$ -modules occurring as direct summands

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of $k_m E$ are all powers of p and that all dimensions p^i , $i = 0, 1, \dots, n$, indeed occur in suitable examples.

Since the classification problem of $(\mathbb{Z}/p^s\mathbb{Z})G$ -modules for cyclic G is non-trivial and has not been completely solved — see, for instance, [T] for results and references — it is a pleasant surprise that the $(\mathbb{Z}/p^s\mathbb{Z})G$ -modules $K_m E/p^s K_m E$ have a simple description. The main ingredients we use to obtain this description are the lack of p -torsion in $K_m E$, due to Izhboldin [I], together with the result [BLMS] for the case $s = 1$ — which also depends on Izhboldin's result.

Suppose that E/F is cyclic of degree p^n , and for $i = 0, \dots, n$, let E_i/F be the subextension of degree p^i of E/F and $G_i := \text{Gal}(E_i/F)$. Set $R_s := \mathbb{Z}/p^s\mathbb{Z}$ and let \mathbb{Z}_p be the ring of p -adic integers. We write $\iota_{F,E}: K_m F \rightarrow K_m E$ and $N_{E/F}: K_m E \rightarrow K_m F$ for the natural inclusion and norm maps, and we use the same notation for the induced maps between $K_m F/p^s K_m F$ and $K_m E/p^s K_m E$.

Theorem. *Let $s \in \mathbb{N}$. There exists an isomorphism of $R_s G$ -modules*

$$K_m E/p^s K_m E \simeq \bigoplus_{i=0}^n Y_i$$

where:

- Y_n is a free $R_s G$ -module of rank $\dim_{\mathbb{F}_p} N_{E/F} k_m E$.
- Y_i , $0 \leq i < n$, is a free $R_s G_i$ -module of rank

$$\dim_{\mathbb{F}_p} N_{E_i/F} k_m E_i / N_{E_{i+1}/F} k_m E_{i+1}.$$

- For each $0 \leq i \leq n$, $Y_i \subseteq \iota_{E_i,E}(K_m E_i/p^s K_m E_i)$.

Passing to the projective limit with respect to s , we obtain

$$\widehat{K_m E} := \varprojlim_s K_m E/p^s K_m E \simeq \bigoplus_{i=0}^n \hat{Y}_i,$$

where each \hat{Y}_i is a free $\mathbb{Z}_p G_i$ -module of rank

$$\dim_{\mathbb{F}_p} N_{E_i/F} k_m E_i / N_{E_{i+1}/F} k_m E_{i+1}.$$

Remark. Observe that Y_0 is a trivial $R_s G$ -module. Moreover, the statements about the ranks of the Y_i are immediate consequences of the fact that $K_m E/p^s K_m E$ is a direct sum of free $R_s G_i$ -modules, $i = 0, 1, \dots, n$ — which is the main point of the theorem.

1. Proof of the Theorem

We prove the result by induction on s . The case $s = 1$ is [BLMS, Theorem 2]. Assume therefore that $s > 1$ and the result holds for $s - 1$:

$$K_m E/p^{s-1} K_m E = \bigoplus \tilde{Y}_i$$

with each \tilde{Y}_i a free $R_{s-1} G_i$ -module in the image of $\iota_{E_i,E}$, $0 \leq i \leq n$.

For each i with $0 \leq i \leq n$, let $B_{s-1,i} \subseteq {}_i E_{i,E} K_m E_i / p^{s-1} K_m E_i$ be an $R_{s-1} G_i$ -base for the free $R_{s-1} G_i$ -module \tilde{Y}_i . By induction the cardinality of $B_{s-1,i}$ is

$$|B_{s-1,i}| = \begin{cases} \dim_{\mathbb{F}_p} N_{E_i/F} k_m E_i / N_{E_{i+1}/F} k_m E_{i+1}, & i < n \\ \dim_{\mathbb{F}_p} N_{E/F} k_m E, & i = n. \end{cases}$$

Set

$$B_{s-1} := \cup_{0 \leq i \leq n} B_{s-1,i} \subseteq K_m E / p^{s-1} K_m E.$$

For each i , let $\mathcal{B}_i \subseteq \iota_{E_i,E}(K_m E_i)$ be a set of representatives for the elements of $B_{s-1,i}$, and let $B_{s,i} \subseteq K_m E / p^s K_m E$ be chosen to make the following first diagram commutative. The second diagram simply recalls where our $\mathcal{B}_i, B_{s,i}$ and $B_{s-1,i}$ are located.

$$\begin{array}{ccc} \mathcal{B}_i & \xrightarrow{\text{mod } p^s} & B_{s,i} \\ \text{mod } p^{s-1} \downarrow & \swarrow \text{mod } p^{s-1} & \\ B_{s-1,i} & & \end{array} \qquad \begin{array}{ccc} K_m E & \longrightarrow & K_m E / p^s K_m E \\ \downarrow & \swarrow & \\ K_m E / p^{s-1} K_m E & & \end{array}$$

Hence for each i we have bijections

$$\mathcal{B}_i \leftrightarrow B_{s,i} \leftrightarrow B_{s-1,i}$$

and $|B_{s,i}| = |B_{s-1,i}|$.

First we observe that every nonzero ideal V of $R_s G_i$ contains

$$p^{s-1}(\tau - 1)^{p^i - 1},$$

where τ is any fixed generator of G_i . Indeed consider $0 \neq \beta \in V$. By multiplying by an appropriate power of p , we may assume $0 \neq \beta \in p^{s-1} R_s G_i$. Let us write

$$\beta = \sum_{j=k}^{p^i - 1} c_j (\tau - 1)^j,$$

where each $c_j \in p^{s-1} R_s$, $j = k, \dots, p^i - 1$, and $c_k \notin p^s R_s = \{0\}$, say $c_k = p^{s-1} \tilde{c}_k$ with $\tilde{c}_k \notin p R_s$. Using the fact that $p^{s-1}(\tau - 1)^{p^i} = 0$ in $R_s G_i$ we see that we can multiply β by $(\tau - 1)^{p^i - k - 1}$ to obtain

$$0 \neq \tilde{c}_k p^{s-1} (\tau - 1)^{p^i - 1} \in V.$$

Since $\tilde{c}_k \in U(R_s)$, the units of R_s , we see that $p^{s-1}(\tau - 1)^{p^i - 1} \in V$ as asserted.

Set Y_i to be the $R_s G$ -submodule of $K_m E / p^s K_m E$ generated by $B_{s,i}$. It is clear that $Y_i \subseteq \iota_{E_i,E}(K_m E_i / p^s K_m E_i)$ and hence Y_i is an $R_s G_i$ -module.

Each element $b \in B_{s,i}$ generates in $K_m E / p^s K_m E$ a free $R_s G_i$ -module M_b , as follows. Suppose that M_b is not a free $R_s G_i$ -module. Then the annihilator of b in $R_s G_i$ is a nonzero ideal of $R_s G_i$. Let $\hat{b} \in \mathcal{B}_i$ and $\tilde{b} \in B_{s-1,i}$ correspond to b under the bijection above. Let also σ be a generator of G and $\bar{\sigma}$ its

image in G_i . Since every nonzero ideal of $R_s G_i$ contains $p^{s-1}(\bar{\sigma} - 1)^{p^i - 1}$, for some $\hat{c} \in K_m E$ we have

$$p^{s-1}(\bar{\sigma} - 1)^{p^i - 1} \hat{b} = p^s \hat{c}.$$

Since $K_m E$ has no p -torsion [I, Theorem A], we obtain

$$p^{s-2}(\bar{\sigma} - 1)^{p^i - 1} \hat{b} = p^{s-1} \hat{c}.$$

Then in $K_m E / p^{s-1} K_m E$

$$p^{s-2}(\bar{\sigma} - 1)^{p^i - 1} \tilde{b} = 0,$$

contradicting the fact that \tilde{b} lies in the $R_{s-1} G_i$ -base $B_{s-1,i}$ for \tilde{Y}_i . (Alternatively, we could use [T, Theorem 5.1] to show that M_b is a free $R_s G_i$ -module.)

Now set $B_s := \cup_{0 \leq i \leq n} B_{s,i} \subseteq K_m E / p^s K_m E$. Suppose we have a relation

$$\sum_{b \in B_s} r(b) b = 0,$$

where for $b \in B_{s,i}$ we have $r(b) \in R_s G_i$, and all but a finite number of $r(b)$ are zero. We write $r(b) \in R_s G_{i(b)}$. Let $\tilde{b} \in B_{s-1}$ correspond to b under the natural bijection, and similarly let $\tilde{r}(b) \in R_{s-1} G$ be the image of $r(b) \in R_s G$.

Working mod p^{s-1} we have $\sum \tilde{r}(b) \tilde{b} = 0$. Since each \tilde{b} lies in B_{s-1} , we deduce that $r(b) \in p^{s-1} R_s G_{i(b)}$. Write $r(b) = ps(b)$ for elements $s(b) \in R_s G_{i(b)}$. We rewrite the original relation as

$$\sum ps(b) b = 0.$$

Just as before we divide by p to obtain in $K_m E / p^{s-1} K_m E$

$$\sum \tilde{s}(b) \tilde{b} = 0.$$

Again since each $\tilde{b} \in B_{s-1}$, we deduce that $s(b) \in p^{s-1} R_s G_{i(b)}$. But then $r(b) = ps(b) = 0 \in R_s G_{i(b)}$ for each $r(b)$, as desired.

Hence for each i in $0 \leq i \leq n$ we have that Y_i is a direct sum of free $R_s G_i$ -modules M_b for $b \in B_{s,i}$, and moreover that $\sum Y_i = \oplus Y_i$. By Nakayama's Lemma, since \mathcal{B} generates $K_m E / p^{s-1} K_m E$ it also generates $K_m E / p^s K_m E$, and hence $\oplus Y_i = K_m E / p^s K_m E$. (More explicitly, choose $\alpha \in K_m E / p^s K_m E$ and $\hat{\alpha} \in K_m E$ a lift of α . Since B_{s-1} spans $K_m E / p^{s-1} K_m E$ we have

$$\hat{\alpha} = \sum_{b \in \mathcal{B}} f_b b + p^{s-1} \hat{\gamma}$$

for some $\hat{\gamma} \in K_m E$, where each $f_b \in R_s G$ and all but finitely many $f_b = 0$. We also have $\hat{\gamma} = \sum g_b b + p^{s-1} \hat{\delta}$ for some $\hat{\delta} \in K_m E$, where again each $g_b \in R_s G$ and all but finitely many $g_b = 0$. Therefore

$$\alpha = \sum_{b \in \mathcal{B}} (f_b + p^{s-1} g_b) b$$

as elements of $K_m E / p^s K_m E$. □

2. Examples of E/F with $Y_i \neq \{0\}$ for all i

Let p be an arbitrary prime number, and let q be an arbitrary prime number or 0. We show that for each $n, m \in \mathbb{N}$ there exists a cyclic field extension E/F of degree p^n and characteristic q such that for each $i, 0 \leq i < n$,

$$\dim_{\mathbb{F}_p} N_{E_i/F} k_m E_i / N_{E_{i+1}/F} k_m E_{i+1} \neq 0$$

and

$$\dim_{\mathbb{F}_p} N_{E/F} k_m E \neq 0.$$

Recall that we index E_i such that $F \subseteq E_i \subseteq E$ and $[E_i : F] = p^i$.

2.1. The case $m = 1$.

Fix p, q , and n as above and set $m = 1$. We construct a field extension E/F as above together with elements

$$\begin{aligned} x_i &\in N_{E_i/F}(E_i^\times) \setminus N_{E_{i+1}/F}(E_{i+1}^\times) F^{\times p}, \quad 0 \leq i < n, \\ x_n &\in N_{E/F}(E^\times) \setminus F^{\times p}. \end{aligned}$$

Let A/B be a field extension of degree p^n such that the characteristic of B is q . Index the subfields A_i of A/B such that $[A_i : B] = p^i$, and denote by $\iota_{B, A_i} : K_1 B \hookrightarrow K_1 A_i$ the natural inclusion. Let σ be a generator of $\text{Gal}(A/B)$ and set $\sigma_i = \sigma|_{A_i}$, the restricted map. Finally, assume that there exist elements $x_0, x_1, \dots, x_n \in B^\times$ such that the following condition holds:

$$\begin{aligned} &[\iota_{B, A_j}(x_j)]^{p^{n-j-1}} \notin \langle [\iota_{B, A_j}(x_1)]^{p^{n-1}}, [\iota_{B, A_j}(x_2)]^{p^{n-2}}, \dots, [\iota_{B, A_j}(x_n)] \rangle \\ (*) \quad &\subseteq A_j^\times / N_{A/A_j}(A^\times), \quad 0 \leq j < n, \\ &x_n \notin B^{\times p}, \end{aligned}$$

where $[x]$ denotes the class of x and $\langle S \rangle$ the subgroup generated by a set S in the named factor group. At the end of this section we shall create an example where condition $(*)$ holds.

Now consider cyclic algebras

$$\mathcal{A}_j = (A/B, \sigma, x_j^{p^{n-j}}), \quad 1 \leq j \leq n.$$

Observe that

$$[\mathcal{A}_j] = [(A_j/B, \sigma_j, x_j)] \in \text{Br}(B), \quad 1 \leq j \leq n$$

([P, Chapter 15, Corollary b]), where $\text{Br}(B)$ denotes the Brauer group of B . Let F be the function field of the product of the Brauer–Severi varieties of $\mathcal{A}_1, \dots, \mathcal{A}_n$ (see [SV, page 735]; see also [J, Chapter 3] for basic properties of Brauer–Severi varieties).

Let $E = A \otimes_B F$. Since F is a regular extension of B , we see that E/F is a cyclic extension of degree p^n . We denote again as σ the generator of $\text{Gal}(E/F)$ which restricts to $\sigma \in \text{Gal}(A/B)$, and we write $E_k = A_k \otimes_B F$

for $k = 0, 1, \dots, n$. Now $[\mathcal{A}_j \otimes_B F] = 0 \in \text{Br}(F)$, $j = 1, \dots, n$, because F splits each \mathcal{A}_j . Hence

$$0 = [(E/F, \sigma, x_j^{p^{n-j}})] = [(E_j/F, \sigma_j, x_j)],$$

and so $x_j \in N_{E_j/F}(E_j^\times)$ as desired (see [P, Chapter 15, page 278]).

However, we claim that

$$\begin{aligned} x_j &\notin (N_{E_{j+1}/F}(E_{j+1}^\times))F^{\times p}, \quad 0 \leq j < n, \\ x_n &\notin F^{\times p}. \end{aligned}$$

Since $x_n \notin B^{\times p}$ by hypothesis and F/B is a regular extension, we have $x_n \notin F^{\times p}$. Assume then that $0 \leq j < n$ and, contrary to our statement,

$$x_j \in (N_{E_{j+1}/F}(E_{j+1}^\times))F^{\times p}.$$

Then we have $x_j f^p \in N_{E_{j+1}/F}(E_{j+1}^\times)$ for some $f \in F^\times$. Hence

$$[(E_{j+1}/F, \sigma_{j+1}, x_j f^p)] = 0 \in \text{Br}(F)$$

and so

$$\begin{aligned} [(E_{j+1}/F, \sigma_{j+1}, x_j)] &= -[(E_{j+1}/F, \sigma_{j+1}, f^p)] \\ &= -[(E_j/F, \sigma_j, f)]. \end{aligned}$$

(In the case $j = 0$, we use $(E_0/F, \sigma_0, f)$ to denote the zero element in $\text{Br}(F)$.) Consequently $(E_{j+1}/F, \sigma_{j+1}, x_j)$ is split by E_j (see [P, Chapter 15, Proposition b]).

But then

$$[(E_{j+1}/E_j, \sigma_{j+1}^j, \iota_{F, E_j}(x_j))] = 0 \in \text{Br}(E_j)$$

(see [D, page 74]). Hence $[(E/E_j, \sigma^{p^j}, \iota_{F, E_j}(x_j^{p^{n-j-1}}))] = 0 \in \text{Br}(E_j)$. But $E_j = A_j \otimes_B F$ is the function field of the product of the Brauer–Severi varieties of $\mathcal{A}_k \otimes_B A_j$ for $k = 1, \dots, n$. Therefore (see [SV, Theorem 2.3])

$$[(A/A_j, \sigma^{p^j}, \iota_{B, A_j}(x_j^{p^{n-j-1}}))] \in \langle [\mathcal{A}_k \otimes_B A_j], k = 1, \dots, n \rangle \subseteq \text{Br}(A_j).$$

Consequently

$$\begin{aligned} [\iota_{B, A_j}(x_j^{p^{n-j-1}})] &\in \langle [\iota_{B, A_j}(x_1)]^{p^{n-1}}, \dots, [\iota_{B, A_j}(x_n)] \rangle \\ &\in A_j^\times / N_{A/A_j}(A^\times), \end{aligned}$$

a contradiction to condition (*).

Thus we have shown that a required extension E/F exists with elements x_0, x_1, \dots, x_n , provided that we can produce a field extension A/B and elements $x_0, x_1, \dots, x_n \in B^\times$ such that condition (*) is valid. Now we show that such an extension and elements exist.

Let $B := C(x_0, x_1, \dots, x_n)$, where C is a field of characteristic q and x_0, x_1, \dots, x_n are algebraically independent elements over C . Assume also that there exists a cyclic extension D/C of degree p^n with Galois group

$G = \langle \sigma \rangle$. Finally, let $A := D(x_0, x_1, \dots, x_n)$. Thus A/B is a cyclic extension of degree p^n .

We claim that condition $(*)$ holds. Clearly $x_n \notin B^{\times p}$. Contrary to our claim assume that

$$x_j^{p^{n-j-1}} = x_1^{c_1 p^{n-1}} \cdots x_n^{c_n} N_{A/A_j}(\gamma)$$

where $c_i \in \mathbb{Z}$, $\gamma \in A^\times$, and $0 \leq j < n$.

Consider the discrete valuation v_j on A such that $v_j(x_j) = 1$, $v_j(x_k) = 0$ if $j \neq k$, and $v_j(\alpha) = 0$ for $\alpha \in D$. Let $H_j = \text{Gal}(A/A_j)$. Then $|H_j| = p^{n-j}$ and $v_j(\sigma(\gamma)) = v_j(\gamma)$ for each $\sigma \in H_j$. Thus we see that

$$v_j(x_j^{p^{n-j-1}}) = p^{n-j-1},$$

while v_j of the right-hand side of the equation is a multiple of p^{n-j} . (Recall that $N_{A/A_j}(\gamma)$ is the product of p^{n-j} conjugates of γ .) Hence the valuations, one equal to p^{n-j-1} and one equal to a multiple of p^{n-j} , disagree, and we have a contradiction.

Thus we see that the equation above is impossible. Hence condition $(*)$ is valid and we have established the desired example of E/F in the case $m = 1$.

2.2. The case $m > 1$.

Fix m, n , and q as above and let L/K be a field extension satisfying the case $m = 1$ with the same n and q . Let $x_0, x_1, \dots, x_n \in K^\times$ such that

$$\begin{aligned} x_i &\in N_{L_i/K}(L_i^\times) \setminus N_{L_{i+1}/K}(L_{i+1}^\times)K^{\times p}, \quad 0 \leq i < n, \\ x_n &\in N_{L/K}(L^\times) \setminus K^{\times p}. \end{aligned}$$

Consider the field of the iterated power series $F := K((y_1)) \cdots ((y_{m-1}))$. Then $E := L \otimes_B F$ is a cyclic extension of degree p^n over F . For each $j \in \{0, 1, \dots, n\}$ consider the element

$$\alpha_j = \{x_j, y_1, \dots, y_{m-1}\} \in k_m F.$$

(If $m = 1$ then $\alpha_j = \{x_j\}$.) By our hypothesis and the projection formula for the norm map in K -theory, we have

$$\alpha_j \in N_{E_j/F} k_m E_j.$$

Now for $0 \leq j < n$ we shall prove by induction on $m \in \mathbb{N}$ that

$$\alpha_j \notin N_{E_{j+1}/F} k_m E_{j+1}.$$

If $m = 1$ then our statement is true by the choice of the field extension L/K and the elements x_j . Assume then that $m > 1$ and that our statement is true for $m - 1$.

Consider the complete discrete valuation v on F with uniformizer y_{m-1} and residue field $F_v = K((y_1)) \cdots ((y_{m-2}))$ if $m > 2$ and $F_v = K$ if $m = 2$.

For the sake of simplicity we denote by E' the field E_{j+1} , and denote the unique extension of v on E' again by v . Since we are considering an

unramified extension we assume that both valuations are normalized. Let $\partial : k_m F \rightarrow k_{m-1} F_v$ and $k_m E' \rightarrow k_{m-1} E'_v$ be the homomorphisms induced by residue maps in Milnor K -theory. Then applying [K, Lemma 3] we see that the following diagram is commutative:

$$\begin{array}{ccc}
 k_m E' & \xrightarrow{\partial} & k_{m-1} E'_v \\
 N_{E'/F} \downarrow & & \downarrow N_{E'_v/F_v} \\
 k_m F & \xrightarrow{\partial} & k_{m-1} F_v.
 \end{array}$$

If $\alpha_j \in N_{E'/F} k_m E'$, then $\partial \alpha_j \in N_{E'_v/F_v} k_{m-1} E'_v$.

But $\partial \alpha_j = \{x_j, y_1, \dots, y_{m-2}\}$ if $m > 2$ and $\partial \alpha_j = \{x_j\}$ if $m = 2$, a contradiction in either case. Therefore we have constructed a field extension E/F with the desired properties.

Remarks. (1) In [MSS1] we determined the $\mathbb{F}_p G$ -module structure of $k_1 E$ for all cyclic extensions E/F of degree p^n , where G is the Galois group. In particular, the decomposition does not depend upon the characteristic of the base field. The ranks of the free $\mathbb{F}_p G_i$ -summands appearing in that decomposition are determined by the images of the various $N_{E_i/F}(E_i^\times)/N_{E_{i+1}/F}(E_{i+1}^\times)$ in $E^\times/E^{\times p}$.

(2) When no primitive p th root of unity lies in F , we have that $F^\times/F^{\times p}$ embeds in $E^\times/E^{\times p}$. Therefore the construction given above for field extensions E/F applies in this case, and these free $\mathbb{F}_p G_i$ -summands do indeed occur for any characteristic. When a primitive p th root of unity is in F^\times , the kernel of the homomorphism $F^\times/F^{\times p} \rightarrow E^\times/E^{\times p}$ is generated by a class $[a] \in F^\times/F^{\times p}$, where $E_1 = F(\sqrt[p]{a})$. In this case it is enough to additionally require that $a \in N_{E/F}(E^\times)$ when $p = 2$ and $n = 1$ (since the condition is automatic otherwise), and the construction above of field extensions E/F applies again.

(3) If F contains a primitive p th-root of unity and $m = 1$, the decomposition contains at most one other indecomposable module, a cyclic $\mathbb{F}_p G$ -module of dimension $p^k + 1$ for $k \in \{-\infty, 0, 1, \dots, n - 1\}$ (where we set $p^{-\infty} = 0$). In [MSS2] we showed that all of these modules are realizable as well.

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