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The isolated ideal of a correspondence associated with a topological quiver

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ABSTRACT. For a general C*-correspondence $\mathcal{E}$ a canonical saturated invariant ideal, on which the correspondence is not supported, is identified. The quotient correspondence is formed and the Cuntz–Pimsner C*-algebra of it is identified both as a relative Cuntz–Pimsner algebra for $\mathcal{E}$, and as a quotient of the Cuntz–Pimsner algebra for $\mathcal{E}$. For the C*-correspondence arising from a topological quiver this process amounts to restricting the base space of vertices to the closed subspace supporting the space of edges.

CONTENTS

Introduction

Associated with a correspondence $\mathcal{E}$ over a C*-algebra A is the Cuntz–Pimsner C*-algebra $\mathcal{O}_{\mathcal{E}}$. This is a universal C*-algebra for representations of the correspondence $\mathcal{E}$ subject to relations determined by an ideal of A. The algebra contains an isomorphic copy of the algebra A even though the actual correspondence $\mathcal{E}$ may only involve a part of A. In the following we form the Cuntz–Pimsner C*-algebra associated with the correspondence restricted to the part of A on which $\mathcal{E}$ lives, and show it is a relative Cuntz–Pimsner algebra for the correspondence $\mathcal{E}$. Since this relative Cuntz–Pimsner C*-algebra is a quotient of the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ by the part of A independent of $\mathcal{E}$, it is an algebra meriting consideration. Certainly the algebra $\mathcal{O}_{\mathcal{E}}$ is not simple if $\mathcal{E}$ is not based on all of A. If the correspondence $\mathcal{E}$ lives on all of A this relative Cuntz–Pimsner algebra is just the usual C*-algebra $\mathcal{O}_{\mathcal{E}}$. In the discrete case, namely when the correspondence $\mathcal{E}$ is associated with

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a directed graph, this relative Cuntz–Pimsner C*-algebra just ignores the abelian C*-algebra summand generated by the isolated points of the graph.

The organization of the article is as follows. In Section 1, after some preliminary definitions and concepts mainly following the literature, we introduce the ideal N in A of isolated points for a correspondence $\mathcal{E}$ over A, intrinsically described as the annihilating ideal of $\mathcal{E}$, and show that restricting this correspondence yields a new correspondence whose Cuntz–Pimsner algebra is a quotient of the Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$. This quotient algebra is seen to be a relative Cuntz–Pimsner algebra of the correspondence $\mathcal{E}$. The Cuntz–Pimsner algebra of the correspondence can never be simple if there are isolated points and, in a sense, there is no dynamical content for the part of the correspondence over this isolated ideal.

In the second section we apply this to the correspondence associated with a topological quiver ([MT]), or what could also be called a directed topological graph. The new correspondence is then associated with an altered topological quiver, namely one restricted over the topological space of nonisolated vertices. In [MT] two conditions, (L) and (K), are introduced on topological quivers as analogues of these conditions on graphs. The restricted topological quiver is then shown to satisfy condition (L), or condition (K), if and only if the original topological quiver satisfies the same condition. Since these two conditions reflect representation theoretic aspects of the correspondence this illustrates that restricting a correspondence to ignore the isolated points does not affect these crucial properties. As is the case in [MT], the topological constraints of the topological quiver context results in proofs that can be intricate.

Notation. If D is a subset of a topological space Y then the closure of D is denoted by $\mathrm{Cl}_Y D$, or if there is no ambiguity by $\overline{D}$, while $\partial_Y D = \overline{D} \cap (\overline{Y \setminus D})$ is the boundary of D. The interior of D is $\mathrm{Int}_Y D$. The algebra of continuous functions on Y is $C(Y)$, while if Y is locally compact Hausdorff $C_c(Y)$ is the algebra of continuous functions with compact support. Its closure in the uniform sup norm is the algebra of continuous functions that vanish at infinity, $C_0(Y)$. The supports of a function f or of a measure λ are denoted $\mathrm{supp}(f)$ and $\mathrm{supp}(\lambda)$ respectively. If $f : Y \to Z$ is a continuous map of topological spaces then $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ denote the domain and range respectively of f, and the dual map $f^{\sharp} : C(Z) \to C(Y)$ is given by $f^{\sharp}(h) = h \circ f$. By an ideal of a C*-algebra A we shall mean a closed two sided ideal, and if B is a subset of a C*-algebra A then $\mathcal{I}(B)$ denotes the ideal of A generated by B. For an ideal J of A, $J^{\perp}$ denotes the ideal $\{a \in A \mid ab = 0, (b \in J)\}$.

1. The isolated ideal

For results and conventions on C*-modules we follow Lance [L]; so if A is a C*-algebra a Hilbert A-module $\mathcal{E}$ is a Banach space which is a right A-module with an A-valued inner product $\langle , \rangle_A$, denoted $\langle , \rangle$ if the context is clear. The norm on $\mathcal{E}$ is given by $\|x\|^2 = \|\langle x, x\rangle\|$, $(x \in \mathcal{E})$; $\mathcal{L}(\mathcal{E})$ denotes the C*-algebra of adjointable operators on $\mathcal{E}$ while $\mathcal{K}(\mathcal{E})$, in analogy with the case when A is the complex numbers, is the closed two-sided ideal of compact operators $\overline{\mathrm{span}}\{\theta^{\mathcal{E}}_{x,y} | x, y \in \mathcal{E}\}$ where $\theta^{\mathcal{E}}_{x,y}(z) = x\langle y, z\rangle$, $(z \in \mathcal{E})$. If $\mathcal{E}$ is a Hilbert A-module the linear span of $\{\langle x, y\rangle \,| x, y \in \mathcal{E}\}$, denoted $\langle \mathcal{E},\mathcal{E}\rangle$, has closure a two-sided ideal of A. Note that $\mathcal{E}\langle \mathcal{E},\mathcal{E}\rangle$ is dense in $\mathcal{E}$ ([L]). The Hilbert module $\mathcal{E}$ is called full if $\langle \mathcal{E},\mathcal{E}\rangle$ is dense in

A. If A is a C*-algebra then A_A refers to the Hilbert module A over itself, where $\langle a,b\rangle = a^*b$ for $a,b\in A$.

Definition 1.1. If A is a C*-algebra then a *C*-correspondence* $\mathcal{E}$ over A is a right Hilbert A-module $\mathcal{E}$ together with a left action of A on $\mathcal{E}$ defined by a $*$-homomorphism $\phi_A : A \to \mathcal{L}(\mathcal{E})$, $a\cdot x = \phi(a)x$, for $a\in A, x\in\mathcal{E}$, where just ϕ is used if the context is clear. The correspondence is called *faithful* if ϕ is injective, and *nondegenerate* (or *essential* [MT]) if span$\{\phi(a)x \mid a\in A, x\in\mathcal{E}\}$ is dense in $\mathcal{E}$.

In the literature a correspondence $\mathcal{E}$ over A is also commonly referred to as a Hilbert bimodule over A, although this may also refer to a particular type of correspondence. The identity correspondence A over A is A viewed as a Hilbert module over itself with the left action given by left multiplication.

If $\mathcal{E}$ is a C*-correspondence over A, and B is a C*-algebra, we say (T,π) is a representation of $\mathcal{E}$ in B, written $(T,\pi):\mathcal{E}\to B$, if $T:\mathcal{E}\to B$ is a linear map, $\pi : A\to B$ a $*$-homomorphism with

(1) $T^*(x)T(y) = \pi(\langle x,y\rangle)$,
(2) $T(\phi(a)x) = \pi(a)T(x)$,
(3) $T(x\cdot a) = T(x)\pi(a)$,

for all $x,y\in\mathcal{E}$, $a\in A$.

The C*-subalgebra of B generated by $T(\mathcal{E})\cup\pi(A)$ is denoted $C^*(T,\pi)$. Note that the first condition ensures that T is an isometry not only if π is injective as is usually noted in the literature, but also if only the restriction of π to the ideal $\overline{\langle\mathcal{E},\mathcal{E}\rangle}$ of A is injective, since $\|T(x)\|^2 = \|\pi\langle x,x\rangle\| = \|\langle x,x\rangle\| = \|x\|^2$, $(x\in\mathcal{E})$. If $\rho : B\to C$ is a $*$-homomorphism of C*-algebras then $(\rho\circ T,\rho\circ\pi)$ is a representation of $\mathcal{E}$ in C, denoted $\rho\circ(T,\pi)$. If $\mathcal{E},\mathcal{F}$ are correspondences over A,B respectively then a morphism from $\mathcal{E}$ to $\mathcal{F}$ is a pair (T,Π) with Π a $*$-homomorphism from A to B, T: $\mathcal{E}\to\mathcal{F}$ a linear map with $\langle Tx,Ty\rangle_B = \Pi(\langle x,y\rangle_A)$ and $\phi_{\mathcal{F}}(\Pi(a))T(x) = T(\phi_{\mathcal{E}}(a)x)$ for $x,y\in\mathcal{E}$, $a\in A$. Thus a representation (T,π) of $\mathcal{E}$ is a morphism from $\mathcal{E}$ to the identity correspondence of B over B. A morphism (T,Π) from $\mathcal{E}$ to $\mathcal{F}$ yields a $*$-homomorphism $\Psi_T:\mathcal{K}(\mathcal{E})\to\mathcal{K}(\mathcal{F})$ by $\Psi_T(\theta_{x,y}) = \theta_{T(x),T(y)}$ for $x,y\in\mathcal{E}$ ([KPW]), so using the identification of $\mathcal{K}(B)$ with B, a representation (T,π) of $\mathcal{E}$ in a C*-algebra B yields a $*$-homomorphism $\Psi_T:\mathcal{K}(\mathcal{E})\to B$ given by $\theta_{x,y}\to T(x)T^*(y)$. The argument of Lemma 2.2 [KPW] showing that Ψ_T is injective if π is injective also serves to show that Ψ_T is injective if only the restriction of π to the ideal $\overline{\langle\mathcal{E},\mathcal{E}\rangle}$ of A is injective.

For $\mathcal{E}$ a C*-correspondence over A use $J(\mathcal{E})$ to denote the ideal $\phi^{-1}(\mathcal{K}(\mathcal{E}))$ of A and $J_{\mathcal{E}}$ to denote the ideal $J(\mathcal{E})\cap(\ker\phi)^{\perp}$.

Definition 1.2. For $\mathcal{E}$ a C*-correspondence over A, K an ideal in $J(\mathcal{E})$, a representation (T,π) of $\mathcal{E}$ in a C*-algebra B is *coisometric* on K if $\Psi_T(\phi(a)) = \pi(a)$ for all $a\in K$.

Given a C*-correspondence $\mathcal{E}$ over A and K an ideal in $J(\mathcal{E})$ there is a representation $(T_{\mathcal{E}},\pi_{\mathcal{E}})$ of $\mathcal{E}$ which is coisometric on K and universal among all such representations ([FMR]), in the sense that if (T,π) is a representation of $\mathcal{E}$ in a C*-algebra B which is coisometric on K then there is a $*$-homomorphism $\rho : C^*(T_{\mathcal{E}},\pi_{\mathcal{E}})\to B$ with $(T,\pi) = \rho\circ(T_{\mathcal{E}},\pi_{\mathcal{E}})$. The C*-algebra $C^*(T_{\mathcal{E}},\pi_{\mathcal{E}})$ is called the relative Cuntz–Pimsner algebra of $\mathcal{E}$ determined by K and denoted $\mathcal{O}(K,\mathcal{E})$. When $K=0$ the

C*-algebra $\mathcal{O}(K,\mathcal{E})$ is denoted $\mathcal{T}(\mathcal{E})$ and called the universal Toeplitz C*-algebra for $\mathcal{E}$.

For C*-correspondences Pimsner ([P]) originally introduced the (augmented) C*-algebra $\mathcal{O}_{\mathcal{E}}$, where ϕ was injective, as $\mathcal{O}(J(\mathcal{E}),\mathcal{E})$. The algebra $\mathcal{O}(J(\mathcal{E}),\mathcal{E})$ was then used as the Cuntz–Pimsner algebra of a correspondence with general ϕ (Remark 2.14 of [MT]). The coisometric condition on the smaller ideal $J_{\mathcal{E}}$ first arose in Theorem 1.1 of (the preprint of) [B1], where the graph C*-algebra $C^*(E)$ for a general directed graph E — with any (countable) number of edges, sources, sinks, and isolated vertices — was obtained as a relative Cuntz–Pimsner C*-algebra using $J_{\mathcal{E}}$. In [K1] the ideal $J_{\mathcal{E}}$ was viewed as the maximal ideal on which ϕ is an injection into $\mathcal{K}(\mathcal{E})$, and the C*-algebra $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})$ was investigated as the appropriate analogue of the Cuntz–Pimsner algebra for general C*-correspondences $\mathcal{E}$. In this paper we investigate a relative Cuntz–Pimsner algebra of a general C*-correspondence, where the ideal used to define a universal C*-algebra for coisometric representations is in general larger than the ideal $J_{\mathcal{E}}$; thus, since $\pi(a) = \Psi_T(\phi(a)) = \Psi_T(0) = 0$ for those a in this larger ideal of coisometry satisfying $\phi(a) = 0$, the representation of $\mathcal{E}$ into this universal C*-algebra will not be injective in general. What this amounts to, for a general C*-correspondence $\mathcal{E}$ over A, is to view the parts of A that $\mathcal{E}$ is not supported on as superfluous to the Cuntz–Pimsner algebra of the correspondence.

Definition 1.3 ([MT]). For $\mathcal{E}$ a C*-correspondence over A, an ideal I in A is *$\mathcal{E}$-invariant* if $\phi(I)\mathcal{E} \subseteq \mathcal{E}I$. Such an invariant ideal is called *$\mathcal{E}$-saturated* if

$$\{a \in J_{\mathcal{E}} \mid \phi(a)\mathcal{E} \subseteq \mathcal{E}I\} \subseteq I.$$

Proposition 1.4. *Let $N \subseteq A$ and $\mathcal{E}$ be a C*-correspondence over A.*

(1) *N satisfies $N\mathcal{E} = 0$ if and only if $N \subseteq \ker\phi$. If N is an ideal then it is $\mathcal{E}$-invariant.*
(2) *$\mathcal{E}N = 0$ if and only if $N \subseteq \langle\mathcal{E},\mathcal{E}\rangle^{\perp}$. In this case if N is an invariant ideal then it is $\mathcal{E}$-saturated.*

The ideal $N = \ker(\phi) \cap \langle\mathcal{E},\mathcal{E}\rangle^{\perp}$ is $\mathcal{E}$-invariant and $\mathcal{E}$-saturated.

Proof. Part (1) is clear. If $\mathcal{E}N = 0$ and $n \in N$ then $fn = 0, (f \in \mathcal{E})$, so $0 = \langle e, fn\rangle = \langle e, f\rangle\, n$ for all $e, f \in \mathcal{E}$, and thus $n \in \langle\mathcal{E},\mathcal{E}\rangle^{\perp}$. Conversely, let $N \subseteq \langle\mathcal{E},\mathcal{E}\rangle^{\perp}$. Then $\mathcal{E}\,\langle\mathcal{E},\mathcal{E}\rangle\, N = 0$, and since $\mathcal{E}\,\langle\mathcal{E},\mathcal{E}\rangle$ is dense in $\mathcal{E}$ it follows that $\mathcal{E}N = 0$. If $\mathcal{E}N = 0$ then $\{a \in J_{\mathcal{E}} \mid \phi(a)\mathcal{E} \subseteq \mathcal{E}N\} \subseteq J_{\mathcal{E}} \cap \ker(\phi) \subseteq \ker(\phi)^{\perp} \cap \ker(\phi) = 0$ which is contained in N, so N is $\mathcal{E}$-saturated. □

Corollary 1.5. *Let $\mathcal{E}$ be a C*-correspondence over A. The maximal ideal N of A satisfying $N\mathcal{E} = \mathcal{E}N = 0$ is $\ker(\phi) \cap \langle\mathcal{E},\mathcal{E}\rangle^{\perp}$, and N is $\mathcal{E}$-invariant and $\mathcal{E}$-saturated.*

For nontrivial $\mathcal{E}$, so for $\mathcal{E} \neq 0$, the nonzero ideal $\overline{\langle\mathcal{E},\mathcal{E}\rangle}$ is contained in $N^{\perp}$, so N is never an essential ideal in A. It is worth noticing that in many examples $N = 0$; indeed the point of view taken is that if $N \neq 0$ then we should consider a new correspondence where $N = 0$, cf. Proposition 1.9. If α is an automorphism of a C*-algebra A the correspondence associated with this ([P]) is given by the left action A on the Hilbert module A_A where $\phi(a)b = \alpha(a)b$ for $a, b \in A$. Clearly $\ker\phi = 0$ so $N = 0$. If α is an endomorphism of A the correspondence is $\mathcal{E} = \overline{\alpha(A)A}$, where the closure is taken in A, with the same inner product as before and with $\phi(a)b = \alpha(a)b$ for $a \in A$, $b \in \mathcal{E}$ ([MS1]). Here $\ker\phi = \ker\alpha$, a closed α-invariant ideal of A. Since it is usually the case that one only considers injective endomorphisms, $N = 0$ here also. The correspondence giving rise to the Cuntz algebra also has $N = 0$, as will soon be clear.

Let $q_N : A \to A/N$ (or just q) denote the quotient map. With the $\mathcal{E}$-invariant ideal N of A, and the fact, noted above, that $\mathcal{E}N = 0$, we may form the new C*-correspondence $\mathcal{E}/\mathcal{E}N = \mathcal{E}$ over the C*-algebra A/N where $\phi_{A/N} : A/N \to \mathcal{L}(\mathcal{E})$ given by $\phi_{A/N} \circ q = \phi$ is clearly well-defined since $N \subset \ker \phi$. The right action of A/N on $\mathcal{E}$ is given by $x \cdot q(a) = xa$, and $\langle x, y\rangle_{A/N} = q(\langle x, y\rangle_A)$ for $x, y \in \mathcal{E}$ and $a \in A$.

Definition 1.6. For $\mathcal{E}$ a C*-correspondence over A let N (or $N(\mathcal{E})$ if required) denote the ideal $\ker(\phi_A) \cap \langle \mathcal{E},\mathcal{E}\rangle^{\perp}$ of A and $\mathcal{E}_N$ denote the C*-correspondence over A/N.

The pair (I, q) is a morphism of the correspondence $\mathcal{E}$ over A to the correspondence $\mathcal{E}_N$ over A/N, where I is the identity map on $\mathcal{E}$.

We have $q_N^{-1}\phi_{A/N}^{-1}(\mathcal{K}(\mathcal{E})) = \phi^{-1}(\mathcal{K}(\mathcal{E}))$ and $q_N^{-1}(\ker \phi_{A/N}) = \ker \phi$ so, since q_N is a surjection, $\ker \phi_{A/N} = q_N(\ker \phi)$ and $q_N(J(\mathcal{E})) = J(\mathcal{E}_N)$. In general if I is an ideal of a C*-algebra A and $q : A \to B$ is a surjective $*$-homomorphism then $q(I)$ is an ideal of B and $q(I^{\perp}) \subseteq q(I)^{\perp}$. With this observation it is clear that $q_N(J_{\mathcal{E}}) \subseteq [\ker \phi_{A/N}]^{\perp} \cap \phi_{A/N}^{-1}(\mathcal{K}(\mathcal{E})) = J_{\mathcal{E}_N}$.

Proposition 1.7. (1) *If I is an ideal in A then I is $\mathcal{E}$-invariant if and only if $q_N(I)$ is $\mathcal{E}_N$-invariant.*

(2) *If H is an ideal in A/N and H is $\mathcal{E}_N$-saturated then $q_N^{-1}(H)$ is $\mathcal{E}$-saturated.*

Proof. By definition $q(I)$ is $\mathcal{E}_N$-invariant if and only if $\phi_N(q(I))\mathcal{E} \subseteq \mathcal{E}q(I)$. Since $\phi(I)\mathcal{E} = \phi_N(q(I))\mathcal{E}$ and $\mathcal{E}q(I) = \mathcal{E}I$ the first part follows.

Given $a \in J_{\mathcal{E}}$ with $\phi(a)\mathcal{E} \subseteq \mathcal{E}q^{-1}(H)$ we need to show that $a \in q^{-1}(H)$. However $q(a) \in q(J_{\mathcal{E}}) \subseteq J_{\mathcal{E}_N}$ by the preceding comment and $\phi_N(q(a))\mathcal{E} = \phi(a)\mathcal{E} \subseteq \mathcal{E}q^{-1}(H) = \mathcal{E}H$, and since H is $\mathcal{E}_N$ -saturated we have $q(a) \in H$. Thus $a \in q^{-1}(H)$. $\square$

Definition 1.8. For $\mathcal{E}$ a C*-correspondence over A and $\mathcal{E}_N$ the correspondence over A/N with $N = N(\mathcal{E})$ let $J(N)$ denote the ideal $q_N^{-1}(J_{\mathcal{E}_N})$ of A. Let $(T_{\mathcal{E}}, \pi_{\mathcal{E}})$ be the universal representation of $\mathcal{E}$ coisometric on $J(N)$.

The comments preceding the previous proposition show $J_{\mathcal{E}} \subseteq J(N) \subseteq J(\mathcal{E})$. The later inclusion is crucial as it allows us to define the C*-algebra $\mathcal{O}(J(N), \mathcal{E})$, the relative Cuntz–Pimsner algebra generated by the images of $T_{\mathcal{E}}$ and $\pi_{\mathcal{E}}$. If $N = 0$ then $\mathcal{E}_N = \mathcal{E}$ and $J(N) = J_{\mathcal{E}} \cap (\ker \phi)^{\perp} = J_{\mathcal{E}}$, so $\mathcal{O}(J(N), \mathcal{E})$ is the usual C*-algebra $\mathcal{O}_{\mathcal{E}} = \mathcal{O}(J_{\mathcal{E}}, \mathcal{E})$ characterized in [K2].

Proposition 1.9. *If $\mathcal{E}$ is a C*-correspondence over A and $\mathcal{E}_N$ the correspondence over A/N then the ideal $N(\mathcal{E}_N)$ of A/N is zero and so*

$$\mathcal{O}(J(N(\mathcal{E}_N)), \mathcal{E}_N) = \mathcal{O}(J_{\mathcal{E}_N}, \mathcal{E}_N).$$

Proof. The ideal $N(\mathcal{E}_N) = \ker(\phi_{A/N}) \cap \langle \mathcal{E},\mathcal{E}\rangle_{A/N}^{\perp}$ where $N = \ker(\phi_A) \cap \langle \mathcal{E},\mathcal{E}\rangle^{\perp}$. Since $\ker \phi_{A/N} = q_N(\ker \phi)$ it is enough to show that if $a \in \ker(\phi)$ with $q_N(a) \in$

$(\langle \mathcal{E},\mathcal{E}\rangle_{A/N})^\perp$ then $a \in \langle \mathcal{E},\mathcal{E}\rangle^\perp$. Now $(\langle \mathcal{E},\mathcal{E}\rangle_{A/N})^\perp = (q_N \langle \mathcal{E},\mathcal{E}\rangle)^\perp$, so for such elements a we have $q_N(a\langle \mathcal{E},\mathcal{E}\rangle) = 0$. Thus $a\langle \mathcal{E},\mathcal{E}\rangle \subseteq \ker q_N = N \subseteq \langle \mathcal{E},\mathcal{E}\rangle^\perp$, so $a(\langle \mathcal{E},\mathcal{E}\rangle \langle \mathcal{E},\mathcal{E}\rangle) = 0$ from which it follows that $a(\langle \mathcal{E},\mathcal{E}\rangle) = 0$, i.e., $a \in \langle \mathcal{E},\mathcal{E}\rangle^\perp$. □

For K an ideal of A contained in $J(\mathcal{E})$ and (T,π) a universal covariant representation of $\mathcal{E}$ in $\mathcal{O}(K,\mathcal{E})$ coisometric on K there is a $*$-homomorphism δ $(= \delta_{(T,\pi)})$: $J(\mathcal{E}) \to \mathcal{O}(K,\mathcal{E})$ defined by $\delta(a) = \pi(a) - \Psi_T(\phi(a))$, $(a \in J(\mathcal{E}))$ ([MT]). Since $J_\mathcal{E} \subseteq J(N)$, the representation $(T_\mathcal{E},\pi_\mathcal{E})$ of $\mathcal{E}$ is also coisometric on $J_\mathcal{E}$ and the universal property yields a surjective $*$-homomorphism $\tau : \mathcal{O}(J_\mathcal{E},\mathcal{E}) \to \mathcal{O}(J(N),\mathcal{E})$ satisfying $\tau \circ (T,\pi) = (T_\mathcal{E},\pi_\mathcal{E})$ where (T,π) is a universal representation of $\mathcal{E}$ in $\mathcal{O}(J_\mathcal{E},\mathcal{E})$ coisometric on $J_\mathcal{E}$. The kernel of τ is the ideal in $\mathcal{O}(J_\mathcal{E},\mathcal{E})$ generated by $\delta(J(N))$ where $\delta = \delta_{(T,\pi)}$ (Lemma 8.21 of [MT]).

The observations that $N \subseteq q^{-1}(J_{\mathcal{E}_N}) = J(N)$, that the equality $\pi_\mathcal{E} = \Psi_{T_\mathcal{E}} \circ \phi$ holds on $J(N)$, and that $N \subseteq \ker(\phi)$, together imply that $N \subseteq \ker \pi_\mathcal{E}$ and so $\pi(N) \subseteq \ker \tau$. Since π is injective on A it follows that $\ker \tau \neq 0$ and $\mathcal{O}(J_\mathcal{E},\mathcal{E})$ can never be simple if $N \neq 0$.

Theorem 1.10. *If $\mathcal{E}$ is a C*-correspondence over A and $\mathcal{E}_N$ the correspondence over A/N then the relative Cuntz–Pimsner C*-algebra $\mathcal{O}(J(N),\mathcal{E})$, which is a quotient of the Cuntz–Pimsner C*-algebra $\mathcal{O}(J_\mathcal{E},\mathcal{E})$ of $\mathcal{E}$, is isomorphic to the Cuntz–Pimsner C*-algebra $\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ of $\mathcal{E}_N$.*

Proof. If (T,π) is a representation of the C*-correspondence $\mathcal{E}_N$ over A/N in a C*-algebra B then $(T,\pi\circ q)$ is a representation of the C*-correspondence $\mathcal{E}$ over A in B. Furthermore if (T,π) is coisometric on an ideal J with $J_{\mathcal{E}_N} \subseteq J \subseteq \phi^{-1}_{\mathcal{E}_N}(\mathcal{K}(\mathcal{E}_N))$ then $(T,\pi\circ q)$ is coisometric on $q^{-1}(J)$. Applying this to the universal representation (T,π) of $\mathcal{E}_N$ coisometric on $J_{\mathcal{E}_N}$ and using the universal representation $(T_\mathcal{E},\pi_\mathcal{E})$ of $\mathcal{E}$ coisometric on $J(N)$ yields a $*$-homomorphism $\rho : \mathcal{O}(J(N),\mathcal{E}) \to \mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ with $\rho \circ T_\mathcal{E} = T$ and $\rho \circ \pi_\mathcal{E} = \pi \circ q$.

Since $N \subseteq \ker \pi_\mathcal{E}$ there is a well-defined map $\pi_0 : A/N \to \mathcal{O}(J(N),\mathcal{E})$ with $\pi_\mathcal{E} = \pi_0 \circ q$. It is straightforward to check that $(T_\mathcal{E},\pi_0)$ is a representation of the C*-correspondence $\mathcal{E}_N$ coisometric on $J_{\mathcal{E}_N}$, so there is a $*$-homomorphism σ : $\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N) \to \mathcal{O}(J(N),\mathcal{E})$ with $\sigma \circ T = T_\mathcal{E}$ and $\sigma \circ \pi = \pi_0$. By checking that $\rho\circ\sigma$ is the identity on the images of T and π, and similarly that $\sigma\circ\rho$ is the identity on the images of $T_\mathcal{E}$ and $\pi_\mathcal{E}$, we see that $\rho = \sigma^{-1}$. □

Note that [K2] has some general conditions under which a relative Cuntz–Pimsner C*-algebra is itself the Cuntz–Pimsner C*-algebra for another correspondence, however it is not clear how to apply this here. At the very least one would need to apply several results from [K2] and also prove, for example, that the '$\mathcal{O}$-pair $(N, J(N))$' is the same as the '$\mathcal{O}$-pair $\omega(\pi,t)$' where (π,t) is the universal representation of $\mathcal{O}(J(N),\mathcal{E})$.

Theorem 1.11. *Let $\mathcal{E}$ be a C*-correspondence over A and K an ideal of A contained in $J(\mathcal{E})$. If (T_K,π_K) is a universal covariant representation of $\mathcal{E}$ coisometric on K then $\pi_K(N)$ is an ideal in the relative Cuntz–Pimsner C*-algebra $\mathcal{O}(K,\mathcal{E})$ and the quotient C*-algebra $\mathcal{O}(K,\mathcal{E})/\pi_K(N)$ is isomorphic to $\mathcal{O}(q(K),\mathcal{E}_N)$. In particular the Toeplitz C*-algebra $\mathcal{T}(\mathcal{E}_N)$ is isomorphic to the quotient of $\mathcal{T}(\mathcal{E})$ by the ideal N.*

Proof. Since N is an ideal in A, $\pi_K(N)$ is an ideal in $\pi_K(A)$. Also $\phi(N)\mathcal{E} = \mathcal{E}N = 0$ implies that $\pi_K(N)T_K(\mathcal{E}) = T_K(\mathcal{E})\pi_K(N) = 0$ and so $\pi_K(N)$ is an ideal in $C^*(T_K, \pi_K)$.

Let $q_\mathcal{E} : \mathcal{O}(K, \mathcal{E}) \to \mathcal{O}(K, \mathcal{E})/\pi_K(N)$ denote the canonical quotient map and define $T_0 = q_\mathcal{E} \circ T_K$ and $\pi_0 : A/N \to \mathcal{O}(K, \mathcal{E})/\pi_K(N)$ the well-defined map sending $q(a)$ to $q_\mathcal{E}\pi_K(a)$, $(a \in A)$. One can check that (T_0, π_0) is a covariant representation of the correspondence $\mathcal{E}_N$ in the C*-algebra $\mathcal{O}(K, \mathcal{E})/\pi(N)$ and that $q_\mathcal{E} \circ \delta_{(T_K,\pi_K)} = \delta_{(T_0,\pi_0)} \circ q$. Since (T_K, π_K) is coisometric on K we have $\delta_{(T_K,\pi_K)}(K) = 0$, so $\delta_{(T_0,\pi_0)}q(K) = 0$ and therefore (T_0, π_0) is coisometric on $q(K)$.

For (T_u, π_u) a universal representation of $\mathcal{E}_N$ coisometric on $q(K)$ the universal property yields a surjective $*$-homomorphism $\rho : \mathcal{O}(q(K), \mathcal{E}_N) \to \mathcal{O}(K, \mathcal{E})/\pi(N)$ with $\rho \circ (T_u, \pi_u) = (T_0, \pi_0)$. Since $(T_u, \pi_u \circ q)$ is a covariant representation of $\mathcal{E}$ coisometric on K, the universal property yields a $*$-homomorphism

$$\sigma : \mathcal{O}(K, \mathcal{E}) \to \mathcal{O}(q(K), \mathcal{E}_N)$$

with $(T_u, \pi_u \circ q) = \sigma \circ (T_K, \pi_K)$. Thus σ contains $\pi(N)$ in its kernel and therefore defines a $*$-homomorphism $\widetilde{\sigma} : \mathcal{O}(K, \mathcal{E})/\pi(N) \to \mathcal{O}(q(K), \mathcal{E}_N)$ satisfying

$$(T_u, \pi_u \circ q) = \widetilde{\sigma} \circ q_\mathcal{E} \circ (T_K, \pi_K).$$

We see that $\widetilde{\sigma} \circ \rho$ is the identity map on $\mathcal{O}(q(K), \mathcal{E}_N)$ by checking that

$$\widetilde{\sigma} \circ \rho(T_u(x)) = \widetilde{\sigma}(T_0(x)) = \widetilde{\sigma}(q_\mathcal{E} \circ T_K(x)) = \sigma(T_K(x)) = T_u(x)$$

and that

$$\widetilde{\sigma} \circ \rho(\pi_u(q(a))) = \widetilde{\sigma}(\pi_0(q(a))) = \widetilde{\sigma}(q_\mathcal{E}\pi_K(a)) = \sigma(\pi_K(a)) = \pi_u(q(a)),$$

$(x \in \mathcal{E}, a \in A)$. This implies that ρ is injective, and so an isomorphism. Thus (T_0, π_0) is a universal representation of $\mathcal{E}_N$ coisometric on $q(K)$.

When $K = 0$, π_K is injective and the last statement follows. □

Note that if π_K is injective then π_0 is injective.

To see that the map ρ in the above proof is an isomorphism one could conceivably have applied the relative gauge invariant uniqueness theorem of [K2] since it is clear that there is a gauge action on $\mathcal{O}(K, \mathcal{E})/\pi_K(N)$. However, it is not straightforward to verify the second condition of this result in our context, so a self contained approach was used.

Recall the surjective $*$-homomorphism $\tau : \mathcal{O}(J_\mathcal{E}, \mathcal{E}) \to \mathcal{O}(J(N), \mathcal{E})$ described after Proposition 1.9.

Corollary 1.12. *If $\mathcal{E}$ is a C*-correspondence over A and (T, π) a universal representation of $\mathcal{E}$ coisometric on $J_\mathcal{E}$ then $\pi(N)$ is an ideal in $\mathcal{O}(J_\mathcal{E}, \mathcal{E})$ contained in* $\ker \tau$. *The quotient C*-algebra $\mathcal{O}(J_\mathcal{E}, \mathcal{E})/\pi(N)$ is isomorphic to $\mathcal{O}(q(J_\mathcal{E}), \mathcal{E}_N)$. Furthermore, $J_\mathcal{E} = 0$ if and only if $q(J_\mathcal{E}) = 0$.*

Proof. N is an ideal of $J(N)$ and $\delta = \delta_{(T,\pi)}$ is a $*$-homomorphism so $\delta(N)$ is an ideal in $\delta(J(N))$. Now $N \subseteq \ker \phi$ so $\delta = \pi$ on N, and $\mathcal{I}(\pi(N)) = \mathcal{I}(\delta(N))$ is an ideal in $\mathcal{I}(\delta(J(N))) = \ker \tau$. However $\pi(N) = \mathcal{I}(\pi(N))$.

Apply the previous theorem with $K = J_\mathcal{E}$, so $(T, \pi) = (T_K, \pi_K)$. Here (T_0, π_0) is now a universal representation of $\mathcal{E}_N$ coisometric on $q(J_\mathcal{E})$. Note that π_0 is injective since π is injective.

Since $N \subseteq \ker\phi$ we have $J_{\mathcal{E}} \subseteq (\ker\phi)^{\perp} \subseteq N^{\perp}$, so q is injective on $J_{\mathcal{E}}$. It follows that $q(J_{\mathcal{E}}) = 0$ implies $J_{\mathcal{E}} = 0$. □

It is worth pointing out that the map T_0 in the corollary is an isometry. To see this first note that the map q restricted to the ideal $\overline{\langle\mathcal{E},\mathcal{E}\rangle}$ is injective so an isometry, since $\overline{\langle\mathcal{E},\mathcal{E}\rangle} \cap N \subseteq \overline{\langle\mathcal{E},\mathcal{E}\rangle} \cap \overline{\langle\mathcal{E},\mathcal{E}\rangle}^{\perp} = 0$. For (T,π) a covariant representation of a correspondence we had noted above that T is an isometry as long as $\pi|_{\langle\mathcal{E},\mathcal{E}\rangle}$ is injective. Since $q(\langle\mathcal{E},\mathcal{E}\rangle) = \langle\mathcal{E},\mathcal{E}\rangle_{A/N}$ it follows that $T_0 = q_{\mathcal{E}} \circ T$ must also be an isometry.

The C*-algebra $\mathcal{O}(J(N),\mathcal{E})$, which is isomorphic to the quotient of $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})$ by $\ker\tau = \mathcal{I}(\delta(J(N)))$, is therefore isomorphic to the quotient of $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(N)$ by the ideal generated by $\delta(J(N))/\pi(N)$ where $\delta = \delta_{(T,\pi)}$, so by the ideal $q_{\mathcal{E}}\mathcal{I}(\delta(J(N))) = \mathcal{I}(\delta_{(T_0,\pi_0)}q(J(N))) = \mathcal{I}(\delta_{(T_0,\pi_0)}(J_{\mathcal{E}_N}))$. Since (T,π) is coisometric on $J_{\mathcal{E}}$, we have $\delta(J_{\mathcal{E}}) = 0$, so $q(J_{\mathcal{E}}) \subseteq \ker\delta_{(T_0,\pi_0)}$ and $\delta_{(T_0,\pi_0)}(J_{\mathcal{E}_N}) = \delta_{(T_0,\pi_0)}(J_{\mathcal{E}_N}\backslash q(J_{\mathcal{E}}))$. Note that there is a well-defined map $\widetilde{\delta}_{(T_0,\pi_0)}$ from the quotient space $\phi_N^{-1}(\mathcal{K})/q(J_{\mathcal{E}})$ to $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(\mathcal{N})$ with $\widetilde{\delta}_{(T_0,\pi_0)}([a]) = \delta_{(T_0,\pi_0)}(a)$, $a \in \phi_N^{-1}(\mathcal{K})$.

Corollary 1.13. *$\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ is isomorphic to the quotient of $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(N)$ by the ideal generated by $\delta_{(T_0,\pi_0)}(J_{\mathcal{E}_N}\backslash q(J_{\mathcal{E}})) = \widetilde{\delta}_{(T_0,\pi_0)}(J_{\mathcal{E}_N}/q(J_{\mathcal{E}}))$. In particular $\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ is isomorphic to $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(N)$ if $q(J_{\mathcal{E}}) = J_{\mathcal{E}_N}$.*

Proof. The C*-algebra $\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ is isomorphic to $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(N)$ if and only if $\delta_{(T_0,\pi_0)}(J_{\mathcal{E}_N}\backslash q(J_{\mathcal{E}})) = 0$. □

One condition that ensures $q(J_{\mathcal{E}}) = J_{\mathcal{E}_N}$ is if $A = N \oplus M$ as a direct sum of C*-algebras. In this case the ideal $M = N^{\perp}$, and since $N \subseteq \ker\phi$ we have $J_{\mathcal{E}} \subseteq (\ker\phi)^{\perp} \subseteq N^{\perp} = M$. For any ideal J of A we have $J = (N\cap J) \oplus (M \cap J)$; for if $a \in J$ with $a = n+m$, $n \in N$, $m \in M$ then for e_λ an approximate unit of N, $e_\lambda a = e_\lambda n + e_\lambda m = e_\lambda n \to n$. However, $e_\lambda a \in J$ so $n \in J$, and $m \in J$ also. Identifying M with A/N, the $*$-homomorphism ϕ_N becomes the restriction of ϕ to M, and so $\phi_N^{-1}(\mathcal{K}) = M \cap \phi^{-1}(\mathcal{K})$, $(\ker\phi_N)^{\perp} = M \cap (\ker\phi)^{\perp}$, and $J_{\mathcal{E}_N} = \phi_N^{-1}(\mathcal{K}) \cap (\ker\phi_N)^{\perp} = M \cap J_{\mathcal{E}} = J_{\mathcal{E}}$. Thus $\mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ is isomorphic to $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})/\pi(N)$ if $A = N \oplus M$.

Corollary 1.14. *If $A = N \oplus M$ as a direct sum of C*-algebras then*

$$\mathcal{O}(J_{\mathcal{E}},\mathcal{E}) \cong N \oplus \mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N).$$

Proof. Let (T,π) denote a universal covariant representation of $\mathcal{E}$ coisometric on $J_{\mathcal{E}}$. Note again that π is injective on A. Theorem 1.11 applied with $K = J_{\mathcal{E}}$ noted that the covariant representation (T_0,π_0) of $\mathcal{E}_N$ is coisometric on $q(J_{\mathcal{E}})$ which is equal to $J_{\mathcal{E}_N}$ under our hypothesis. One can check that $(T_0, \pi|_N \oplus \pi_0)$ is then a covariant representation of $\mathcal{E}$ in the C*-algebra $\pi(N) \oplus \mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ which is coisometric on $J_{\mathcal{E}}$. Since this representation admits a gauge action, and since $\pi|_N \oplus \pi_0$ is injective on $N \oplus M = A$, the surjection of $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})$ to $\pi(N) \oplus \mathcal{O}(J_{\mathcal{E}_N},\mathcal{E}_N)$ given by the universal property is, by the gauge invariant uniqueness theorem ([K3]), actually an injection. □

2. Topological quivers

In [MT] the authors show that a certain topological condition, called Condition (K), on a topological quiver implies that the Cuntz–Pimsner C*-algebra of the topological quiver has only gauge invariant ideals ([MT, Theorem 9.10]). Furthermore, the Cuntz–Pimsner C*-algebra enjoys the Cuntz–Krieger uniqueness property if the topological quiver satisfies Condition (L) ([MT, Theorem 6.16]). We show that the restricted topological quiver satisfies Condition (K), or Condition (L), if and only if the original topological quiver does.

Following [MT], $G = (X, E, r, s, \lambda)$ is a topological quiver when X, E are a pair of second countable locally compact Hausdorff spaces, $r : E \to X$ and $s : E \to X$ a pair of continuous maps (the range and source maps) with r open, and λ a family $\{\lambda_x \mid x \in X\}$ of Radon measures on E with

(1) $\mathrm{supp}(\lambda_x) = r^{-1}(x)$, $(x \in X)$,
(2) $x \to \int_E f(\alpha) d\lambda_x(\alpha) \in C_c(X)$ for $f \in C_c(X)$.

A topological quiver G defines a C*-correspondence $\mathcal{E}$ (or $\mathcal{E}(G)$) over the C*-algebra $A = C_0(X)$ as follows: for $f, g \in C_c(E)$ an A-valued inner product given by

$$\langle f, g\rangle (x) = \int_{r^{-1}(x)} \overline{f(\alpha)} g(\alpha) d\lambda_x(\alpha), (x \in X),$$

defines a norm on $C_c(E)$ with completion $\mathcal{E}$, a Hilbert module over $C_0(X)$. The left action coming from a $*$-homomorphism $\phi : A \to \mathcal{L}(\mathcal{E})$ and the right action of A on an element h of $C_c(E)$ are given by

$$h \cdot g = h(r^{\sharp}(g))$$
$$\phi(g)h = (s^{\sharp}(g))h,$$

for $g \in A$.

We briefly include some comments regarding ideals in an abelian C*-algebra. If D is a closed subset of a compact space Y then D is compact in the subspace topology and the dual $i^{\sharp}$ of the inclusion $i : D \to Y$ is a surjective $*$-homomorphism $i^{\sharp} : C(Y) \to C(D)$ with kernel the ideal $I_D = \{f \in C(Y) |\ \ f|_D = 0\}$ determined by D. We obtain the exact sequence

$$0 \to C_0(Y \setminus D) \to C(Y) \xrightarrow{i^{\sharp}} C(D) \to 0.$$

For the situation that D is closed in a locally compact space Y, thus locally compact in the subspace topology, then $Y \setminus D$ is open in Y and also in its one point compactification $Y_{\{\infty\}}$. Thus $Y_{\{\infty\}} \setminus D = D \cup \{\infty\}$ is closed, so compact in $Y_{\{\infty\}}$ and may be identified with the one point compactification of the locally compact space D, since $Y \setminus D = Y_{\{\infty\}} \setminus D_{\{\infty\}}$. Now $0 \to C_0(D) \to C(D_{\{\infty\}}) \to \mathbb{C} \to 0$ is exact for any locally compact space D, so applying this to the spaces $Y \setminus D, Y$, and D, and arranging these exact sequences in an array we obtain via the 5-lemma the exact sequence

$$0 \to C_0(Y \setminus D) \to C_0(Y) \xrightarrow{i^{\sharp}} C_0(D) \to 0.$$

In [MT], X_{sink} denotes the open set of X with $\ker\phi \cong C_0(X_{\text{sink}})$ and it is shown that $X_{\text{sink}} = X \diagdown \overline{s(E)}$, or equivalently that $\overline{s(E)} = \{x \in X \mid f(x) = 0, (f \in \ker\phi)\}$, so $\ker\phi = \left\{f \in C_0(X) \mid f|_{\overline{s(E)}} \equiv 0\right\}$. Recall that $\overline{\langle \mathcal{E},\mathcal{E}\rangle}$ is an ideal of A.

Proposition 2.1. *The ideal* $\overline{\langle \mathcal{E},\mathcal{E}\rangle}$ *of* A *is* $\{f \in A \mid f \equiv 0$ *on* $X \diagdown r(E)\}$.

Proof. Since r is an open map, $X \diagdown r(E)$ is closed. If $x \in X$ with $r^{-1}(x) = \phi$ then $\langle f, g\rangle (x) = 0$ for $f, g \in C_c(E)$, a dense subspace of $\mathcal{E}$. Thus

$$X \diagdown r(E) \subseteq \left\{x \in X \mid f(x) = 0, (f \in \overline{\langle \mathcal{E},\mathcal{E}\rangle})\right\}.$$

To show the reverse inclusion it is enough to show that given $x \in r(E)$ there is $h \in C_c(E)$ with $\langle h, h\rangle (x) \neq 0$. Since $\operatorname{supp}(\lambda_x) = r^{-1}(x) \neq \phi$ there is a positive $g \in C(r^{-1}(x))$ with $\lambda_x(|g|^2) \gneqq 0$. If K is the compact support of g in $r^{-1}(x)$, then K must also be compact in E, so by Urysohn's Lemma there is an $f \in C_c(E)$ with $f|_K \equiv 1$. By Tietze's Extension Theorem there is a continuous function $l : \operatorname{supp}(f) \to \mathbb{R}$ with $l|_K = g$. Setting h to be the element of $C_c(E)$ which is $l \cdot f$ on $\operatorname{supp}(f)$ and 0 on $E \diagdown \operatorname{supp}(f)$ we have

$$\begin{aligned}
\langle h, h\rangle (x) &= \int_{r^{-1}(x)} |l|^2 (\alpha) d\lambda_x(\alpha) \\
&\geq \int_{r^{-1}(x)\cap K} |l|^2 (\alpha) d\lambda_x(\alpha) \\
&= \int_{r^{-1}(x)\cap K} |g|^2 (\alpha) d\lambda_x(\alpha) \\
&= \int_{r^{-1}(x)} |g|^2 (\alpha) d\lambda_x(\alpha) \gneqq 0.
\end{aligned}$$

□

In general for $U \subseteq X$ an open set and $I = \left\{f \in C_0(X) \mid f|_{X \diagdown U} \equiv 0\right\}$ the ideal $C_0(U)$ of $C_0(X)$ determined by the closed set $X \diagdown U$ we have that the ideal $I^\perp$ is determined by the closed set $\overline{U}$. Thus the ideal $[\overline{\langle \mathcal{E},\mathcal{E}\rangle}]^\perp$of A is determined by the closed set $\overline{r(E)}$. The next proposition follows.

Proposition 2.2. *For* (X, E, r, s, λ) *a topological quiver and* $\mathcal{E}$ *the associated correspondence over* $A = C_0(X)$, *the ideal* $N = \ker(\phi) \cap [\overline{\langle \mathcal{E},\mathcal{E}\rangle}]^\perp$ *of* A *is determined by the closed set* $\overline{s(E) \cup r(E)}$; *namely*

$$N = \left\{f \in C_0(X) \mid f|_{\overline{s(E)\cup r(E)}} \equiv 0\right\}.$$

Recall the terminology from [MT], where X_{fin} and $X_{\text{reg}} = X_{\text{fin}} \diagdown \overline{X_{\text{sink}}}$ are open sets so that the ideals $C_0(X_{\text{fin}})$ and $C_0(X_{\text{reg}})$ of A are equal to $J(\mathcal{E}) = \phi^{-1}(\mathcal{K}(\mathcal{E}))$ and $J_{\mathcal{E}} = \phi^{-1}(\mathcal{K}(\mathcal{E})) \cap (\ker\phi)^\perp$ respectively.

Definition 2.3. Let X_{source} be the open set $X \diagdown \overline{r(E)}$, so $C_0(X_{\text{source}})$ is isomorphic to the ideal

$$\left\{f \in C_0(X) \mid f|_{\overline{r(E)}} \equiv 0\right\} = [\overline{\langle \mathcal{E},\mathcal{E}\rangle}]^\perp.$$

Define $X_{\text{isol}} = X_{\text{source}} \cap X_{\text{sink}} = X \diagdown \overline{s(E) \cup r(E)}$. and set D (or D_G) $= \overline{s(E) \cup r(E)}$.

Since $\ker\phi = C_0(X_{\rm sink})$ and $X_{\rm sink} = X \setminus \overline{s(E)}$, $[\ker\phi]^\perp = C_0(\mathrm{Int}(\overline{s(E)}))$. From the definitions we have $C_0(X_{\rm reg}) = J_{\mathcal{E}} \subseteq [\ker\phi]^\perp$, so it follows that $X_{\rm reg} \subseteq \mathrm{Int}(\overline{s(E)})$. In particular $X_{\rm reg}$ is contained in the closed set $D = \overline{s(E) \cup r(E)}$.

If $\mathcal{E}$ is the C*-correspondence over $A = C_0(X)$ associated with the topological quiver $G = (X, E, r, s, \lambda)$ and $N = \ker\phi \cap [\overline{\langle \mathcal{E}, \mathcal{E}\rangle}]^\perp$ is the ideal $C_0(X_{\rm isol})$ of A, form the correspondence $\mathcal{E}_N$ over $A/N \cong C_0(X)/C_0(X_{\rm isol}) = C_0(D)$ as in Section 1. The correspondence $\mathcal{E}_N$ over A/N is the same as the correspondence associated with the restricted topological quiver $G^N = (D, E, r, s, \lambda)$ where we continue to use r and s to denote the appropriate maps, now viewed with ranges in D. The natural quotient map $q : A \to A/N$ is the map $i^\sharp : C_0(X) \to C_0(D)$ where $i : D \to X$ is the inclusion of the closed subset D in X . We have $\phi_N \circ q = \phi$ where $\phi : C_0(X) \to \mathcal{L}(\mathcal{E})$ and $\phi_N : C_0(D) \to \mathcal{L}(\mathcal{E})$ define the left actions of A and $C_0(D)$ respectively on $\mathcal{E}$. This implies that $q(\ker\phi) = \ker\phi_N$. Similarly $q(f) \in \phi_N^{-1}(\mathcal{K}(\mathcal{E}))$ if and only if $f \in \phi^{-1}(\mathcal{K}(\mathcal{E}))$, so $q(\phi^{-1}(\mathcal{K}(\mathcal{E}))) = \phi_N^{-1}(\mathcal{K}(\mathcal{E}))$. For any ideal $I = \left\{ f \in C_0(X) \mid f|_{X\setminus U} \equiv 0 \right\}$ of $C_0(X)$ where $U \subseteq X$ is open, so $I = C_0(U)$, the ideal $q(I) = \left\{ g \in C_0(D) \mid f|_D \equiv g,\ f|_{D\cap(X\setminus U)} \equiv 0 \right\} = C_0(D\cap U)$ since $D \setminus D\cap(X\setminus U) = D\cap U$. It follows that $D_{\rm fin} = X_{\rm fin}\cap D$ and $D_{\rm sink} = X_{\rm sink}\cap D$.

Recall that the ideal $J_{\mathcal{E}} = \phi^{-1}(\mathcal{K}(\mathcal{E})) \cap [\ker\phi]^\perp$ of $A = C_0(X)$ is contained in the ideal $J(N) = q^{-1}(J_{\mathcal{E}_N})$. In the present situation $J_{\mathcal{E}} = C_0(X_{\rm reg})$ and $J_{\mathcal{E}_N} = C_0(D_{\rm reg})$. Thus the inclusion $q(J_{\mathcal{E}}) \subseteq J_{\mathcal{E}_N}$ is equivalent to $i^\sharp(C_0(X_{\rm reg})) \subseteq C_0(D_{\rm reg})$. Since $i^\sharp(C_0(X_{\rm reg})) = C_0(D\cap X_{\rm reg})$, the above inclusion of ideals yields $D \cap X_{\rm reg} \subseteq D_{\rm reg}$. However we have already seen that $X_{\rm reg} \subseteq D$, so we have that $X_{\rm reg} \subseteq D_{\rm reg}$, where both are open subsets of D.

Apply the results of the first section to $A = C_0(X)$ and $N = C_0(X\backslash D)$, and view the restriction of the map $q = i^\sharp : A \to A/N$ to domain $J_{\mathcal{E}} = C_0(X_{\rm reg})$ and codomain $J_{\mathcal{E}_N} = C_0(D_{\rm reg})$ as the natural inclusion of $C_0(X_{\rm reg})$ in $C_0(D_{\rm reg})$. Then

$$J_{\mathcal{E}_N}/q(J_{\mathcal{E}}) = C_0(D_{\rm reg})/C_0(X_{\rm reg}) \cong C_0(D_{\rm reg}\backslash X_{\rm reg})$$

and the C*-algebra $\mathcal{O}(C_0(D_{\rm reg}), \mathcal{E}_N)$ is isomorphic to the quotient of

$$\mathcal{O}(C_0(X_{\rm reg}), \mathcal{E})/C_0(X\backslash D)$$

by the ideal generated by $\widetilde{\delta}_{(T_0,\pi_0)}(C_0(D_{\rm reg}\backslash X_{\rm reg}))$, where $\mathcal{E}$ is the correspondence over A associated with a topological quiver, or in fact any correspondence over A.

We briefly recall some terminology from [MT].

Definition 2.4. If $G = (X, E, r, s, \lambda)$ is a topological quiver, a path (of length n) in G is a finite sequence $\alpha := \alpha_1 ... \alpha_n$ with $\alpha_k \in E$ and $r(\alpha_k) = s(\alpha_{k+1})$ for $1 \le k \lneqq n$. Denote the paths of length n by E^n , viewed as a subspace of ΠE and extend r, s to E^n by $r^n : E^n \to X$, $s^n : E^n \to X$ where $r^n(\alpha) = r(\alpha_n)$ and $s^n(\alpha) = s(\alpha_1)$. A path $\alpha \in E^n$ is a loop if $s(\alpha_1) = r(\alpha_n)$ with base point $s^n(\alpha)$. A loop $\alpha \in E^n$ has an exit if there is a $\beta \in E$ and a $k \in \{1, ..., n\}$ with $s(\beta) = s(\alpha_k)$ and $\beta \ne \alpha_k$. We say G satisfies condition (L) if the set of base points of loops in G with no exits has empty interior. Let $B = \{x \in X \mid x \text{ is a base point of a loop in } G \text{ with no exits}\}$ and $B_N = \left\{x \in D \mid x \text{ is a base point of a loop in } G^N \text{ with no exits}\right\}$.

Theorem 2.5. *The topological quiver $G = (X, E, r, s, \lambda)$ satisfies condition* (L) *if and only if the restricted topological quiver $G^N = (D, E, r, s, \lambda)$ satisfies condition* (L).

Proof. We show that $\mathrm{Int}_X B = \mathrm{Int}_D B$ using that D is closed in X and that r is an open map. In general $\mathrm{Cl}_D B = D \cap \mathrm{Cl}_X B \subseteq \mathrm{Cl}_X B$ for any $B \subseteq D$, so $\partial_D B = \mathrm{Cl}_D B \cap \mathrm{Cl}_D(D\setminus B) \subseteq \mathrm{Cl}_X B \cap \mathrm{Cl}_X(D\setminus B) \subseteq \mathrm{Cl}_X B \cap \mathrm{Cl}_X(X\setminus B) = \partial_X B$. Now D is closed in X and since $B \subseteq D$, $\mathrm{Cl}_X B$ is contained in D and $\mathrm{Cl}_D B = \mathrm{Cl}_X B$. Thus $\mathrm{Int}_X B = \mathrm{Cl}_X B \setminus \partial_X B = \mathrm{Cl}_D B \setminus \partial_X B \subseteq \mathrm{Cl}_D B \setminus \partial_D B = \mathrm{Int}_D B$.

To see the other inclusion first note that base points of loops lie in $r(E)$, so $B \subseteq r(E)$. Also $r(E)$ is open in X, so open in D. Thus $B \subseteq \mathrm{Int}_X D$. If $z \in \mathrm{Int}_D B$ then there is a neighbourhood N_z of z in X with $(N_z \cap D) \subseteq B \subseteq \mathrm{Int}_X D$. Thus $N_z \cap \mathrm{Int}_X D \subseteq B \cap \mathrm{Int}_X D \subseteq \mathrm{Int}_X D$. Since $z \in B \subseteq \mathrm{Int}_X D$, the set $N_z \cap \mathrm{Int}_X D$ is an open subset of B containing z and so $z \in \mathrm{Int}_X B$. □

In order to discuss condition (K) on topological quivers we first recall some material from [MT]. If $G = (X, E, r, s, \lambda)$ is a topological quiver and $U \subseteq X$ is open, U is called hereditary (for G) if $r^{-1}(s(u)) \subseteq U$. A hereditary (open) set U is called saturated (for G) if $\{v \in X_{\mathrm{reg}} \mid r(s^{-1}(v)) \subseteq U\} \subseteq U$. If $\mathcal{E}$ is the C*-correspondence over $A = C_0(X)$ associated with the topological quiver G, then U is hereditary if and only if the ideal $I = C_0(U)$ is $\mathcal{E}$-invariant. In this case, the new C*-correspondence $\mathcal{E} = \mathcal{E}/\mathcal{E}I$ over the C*-algebra A/I is the correspondence associated with the topological quiver $G_U = (X\setminus U, E\setminus r^{-1}(U), r_U, s_U, \lambda^U)$, where r_U, s_U, and λ^U are just the respective restrictions of r, s, to $E\setminus r^{-1}(U)$ and λ to $X\setminus U$. Note also ([MT]) that U is saturated for G is equivalent to $C_0(U)$ being an $\mathcal{E}$-saturated ideal of A. The topological quiver G is said to satisfy condition (K) if the topological quiver G_U satisfies condition (L) for all open, hereditary, saturated sets U in X.

Clearly a union or finite intersection of open hereditary subsets of X is open and hereditary. Since $N = C_0(X\setminus D)$ is an invariant saturated ideal in $\mathcal{E}(G)$, $X\setminus D$ is an open hereditary subset of X. Therefore, if U is an open hereditary set for $G = (X, E, r, s, \lambda)$ then so is $U \cup (X\setminus D)$. In this case, if U is also saturated for G then, since $r(s^{-1}(v)) \subseteq U$ if and only if $r(s^{-1}(v)) \subseteq U \cup (X\setminus D)$, and $X_{\mathrm{reg}} \cap (X\setminus D) = \phi$, we see that $U \cup (X\setminus D)$ is also saturated.

Lemma 2.6. *Let $V \subseteq X$ be open and $V_D = D \cap V$. Then:*

(1) *V_D is hereditary in G^N if and only if V is hereditary in G.*
(2) *If V_D is saturated in G^N then V is saturated in G.*

Proof. The first part follows from Proposition 1.4. Suppose V is saturated in G^N. To show V is saturated in G we need to show that if $v \in X$ with $r(s^{-1}(v)) \subseteq V$ then $v \in V$. However $X_{\mathrm{reg}} \subseteq D_{\mathrm{reg}}$, so v is in D_{reg}, and $r(s^{-1}(v)) = r(s^{-1}(v)) \cap D \subseteq V \cap D = V_D$. Since V_D is saturated in G^N we have $v \in V_D \subseteq V$. □

Lemma 2.7. *If $z \in D_{\mathrm{reg}} \setminus X_{\mathrm{reg}}$, then:*

(1) *$z \in \partial_X \overline{s(E)} \setminus \overline{[\overline{r(E)} \cap X_{\mathrm{sink}}]}$.*
(2) *There is a neighbourhood N_z of z such that $N_z \cap \mathrm{Int}_X \overline{s(E)} \subseteq X_{\mathrm{reg}}$ and $N_z \cap \overline{r(E)} = \phi$.*

Proof. Since D is closed we see that $\mathrm{Cl}_D(A) = \mathrm{Cl}_X(A)$ for any subset A of D. If $z \in D_{\mathrm{reg}} = D_{\mathrm{fin}} \setminus \overline{D_{\mathrm{sink}}}$ then $z \in D_{\mathrm{fin}} = X_{\mathrm{fin}} \cap D \subseteq X_{\mathrm{fin}}$. Since $z \notin X_{\mathrm{reg}} = X_{\mathrm{fin}} \setminus \overline{X_{\mathrm{sink}}}$ then z must be in $\overline{X_{\mathrm{sink}}}$. However $z \notin X_{\mathrm{sink}}$ since $z \notin D_{\mathrm{sink}} = X_{\mathrm{sink}} \cap D$. Thus $z \in \partial_X(X_{\mathrm{sink}}) = \partial_X(\overline{s(E)})$. Since $D_{\mathrm{sink}} = X_{\mathrm{sink}} \cap D = X_{\mathrm{sink}} \cap \overline{r(E)}$ and $z \notin \overline{D_{\mathrm{sink}}}$, statement (1) follows.

We noted that $z \in X_{\text{fin}}$ which is open, so there is a neighbourhood N of z with $N \subseteq X_{\text{fin}}$. Since $z \in \partial_X(s(E))$ we have $N \cap \operatorname{Int}_X \overline{s(E)}$ is a nonempty open subset in X_{fin}. Since $\operatorname{Int}_X \overline{s(E)} \cap \overline{X_{\text{sink}}} = \phi$ this open subset is in $X_{\text{reg}} = X_{\text{fin}} \setminus \overline{X_{\text{sink}}}$. To show that N can be chosen disjoint from $\overline{r(E)}$ it is enough to see that $z \notin \overline{r(E)}$, or since $z \in \partial_X \overline{s(E)}$, that $z \notin \overline{r(E)} \cap \partial_X \overline{s(E)} = \overline{r(E)} \cap \partial_X(X_{\text{sink}})$. However, the latter is a subset of $[X_{\text{sink}} \cap \overline{r(E)}]$ since any neighbourhood of a point in $\overline{r(E)} \cap \partial_X(X_{\text{sink}})$ meets both X_{sink} and $r(E)$. By (1) we are done. $\square$

Lemma 2.8. *Let $V \subseteq X$ be open, $V_D = V \cap D$, and*

$$B_V = \{z \in D_{\text{reg}} \setminus X_{\text{reg}} \mid r(s^{-1}(z)) \subseteq V\}.$$

For $z \in B_V$ there is a neighbourhood M_z of z with $r(s^{-1}(M_z)) \subseteq V$, $M_z \cap \overline{r(E)} = \phi$.

Proof. Let M_k, $k \in \mathbb{N}$ be a decreasing countable neighbourhood base of z. We may choose $M_k \cap \overline{r(E)} = \phi$ by Lemma 2.7. By Proposition 3.15 of [MT], since $z \in D_{\text{fin}}$, there is a neighbourhood W of Z with both $\overline{W \cap D}$ and $s^{-1}(\overline{W \cap D})$ compact in D, and $r|_{s^{-1}(W \cap D)}$ a local homeomorphism. Suppose $rs^{-1}(M_k) = rs^{-1}(M_k \cap D) \nsubseteq V$ for all k. Then for each k there is an $\alpha_k \in s^{-1}(M_k)$ with $r(\alpha_k) \notin V$. There is k_0 such that $W \cap D \supseteq M_k \cap D$ for $k \geq k_0$, so $\alpha_k \in s^{-1}(\overline{W \cap D})$ for $k \geq k_0$. By compactness there is $\alpha \in s^{-1}(\overline{W \cap D})$ an accumulation point of α_k, so $\alpha_{k_j} \to \alpha$. By continuity $s(\alpha_{k_j}) \to s(\alpha)$. Since $s(\alpha_{k_j}) \in M_{k_j}$ for all j, $s(\alpha_{k_j}) \to z$ also, and so $z = s(\alpha)$ i.e., $\alpha \in s^{-1}(z)$. Since $z \in B_V$, $r(\alpha) \in V$. Now by the choice of α_k, $r(\alpha_{k_j}) \in X \setminus V$, so $r(\alpha) = \lim r(\alpha_{k_j}) \in X \setminus V$, a contradiction. $\square$

Definition 2.9. Let $G = (X, E, r, s, \lambda)$ be a topological quiver and

$$D = \overline{r(E) \cup s(E)}.$$

For V an open hereditary subset of X define $\widetilde{V} = V \cup \{\cup\{M_z \mid z \in B_V\}\}$ where M_z is chosen as in the previous lemma.

Lemma 2.10. *If V is hereditary then $\widetilde{V}$ is also; if V is saturated then $\widetilde{V}$ is also.*

Proof. First suppose V is hereditary. If $\alpha \in s^{-1}(\widetilde{V})$ then $\alpha \in s^{-1}(V)$ or $\alpha \in s^{-1}(M_z)$ for some $z \in B_V$. In the case that $\alpha \in s^{-1}(V)$ we have $r(\alpha) \in V \subseteq \widetilde{V}$ since V is hereditary. For $\alpha \in s^{-1}(M_z)$ with $z \in B_V$ then $r(\alpha) \in rs^{-1}(M_z) \subseteq V \subseteq \widetilde{V}$. Thus $rs^{-1}(\widetilde{V}) \subseteq \widetilde{V}$. To show the remaining claim it is enough, by Lemma 2.6, to show that $\widetilde{V} \cap D$ is saturated in $G^N = (D, E, r, s, \lambda)$. Suppose $v \in D_{\text{reg}}$ and $r(s^{-1}(v)) \subseteq \widetilde{V} \cap D$. We need to show that $v \in \widetilde{V} \cap D$. Since the sets M_z added to V are all disjoint from $\overline{r(E)}$ the condition $r(s^{-1}(v)) \subseteq \widetilde{V} \cap D$ implies $r(s^{-1}(v)) \subseteq V$. We consider two cases; $v \notin X_{\text{reg}}$ and $v \in X_{\text{reg}}$. In the first case $v \in D_{\text{reg}} \setminus X_{\text{reg}}$. Since $r(s^{-1}(v)) \subseteq V$ we have $v \in B_V$, and so $v \in \widetilde{V} \cap D$. On the other hand if $v \in X_{\text{reg}}$, then since $r(s^{-1}(v)) \subseteq V$ and V is saturated, we have $v \in V$; and since $v \in D$ we have $v \in \widetilde{V} \cap D$. $\square$

Theorem 2.11. *The topological quiver $G = (X, E, r, s, \lambda)$ satisfies condition* (K) *if and only if the restricted topological quiver $G^N = (D, E, r, s, \lambda)$ satisfies condition* (K).

Proof. First assume G satisfies condition (K). Let $V \subseteq D$ be an arbitrary open, hereditary, saturated set. If $W \subseteq X$ open with $W \cap D = V$, then $W \cup (X \setminus D) = \widetilde{V} \subseteq X$ is open with $\widetilde{V} \cap D = V$. We have

$$G_{\widetilde{V}} = (X \setminus \widetilde{V}, E \setminus r^{-1}(\widetilde{V}), r_{\widetilde{V}}, s_{\widetilde{V}}, \lambda^{\widetilde{V}}) = (D \setminus V, E \setminus r^{-1}(V), r_V, s_V, \lambda^V) = (G^N)_V.$$

Since $\widetilde{V}$ is saturated and hereditary by Lemma 2.6, $G_{\widetilde{V}}$, and therefore also $(G^N)_V$ satisfies condition (L). Since V was arbitrary, G^N satisfies condition (K).

Now suppose G^N satisfies condition (K). For $V \subseteq X$ open, hereditary, and saturated for G, we need to show G_V has condition (L); namely

$$\mathrm{Int}_{X \setminus V}\{v \text{ a base point for a loop in } G \text{ with no exit}\} = \phi.$$

By Lemma 2.10 $\widetilde{V}$ is open, hereditary, and saturated, so by Lemma 2.6 $\widetilde{V} \cap D$ is open and hereditary. The proof of Lemma 2.10 showed $\widetilde{V} \cap D$ is saturated for G^N. Since G^N satisfies condition (K), $G^N_{\widetilde{V} \cap D}$ satisfies condition (L). We note that $G^N_{\widetilde{V} \cap D} = G_{\widetilde{V} \cup (X \setminus D)}$, where, by the comments preceding Lemma 2.6, $\widetilde{V} \cup (X \setminus D)$ is open, hereditary, and saturated; so $G_{\widetilde{V} \cup (X \setminus D)}$ also satisfies condition (L). Thus

$$\mathrm{Int}_{X \setminus (\widetilde{V} \cup (X \setminus D))}\big\{v \text{ a base point for a loop in } G_{\widetilde{V} \cup (X \setminus D)} \text{ with no exit}\big\} = \phi.$$

However, since $X \setminus D$ and the sets M_z are disjoint from $\overline{r(E)}$, $r^{-1}(\widetilde{V} \cup (X \setminus D)) = r^{-1}(V)$ and $E \setminus r^{-1}(\widetilde{V} \cup (X \setminus D)) = E \setminus r^{-1}(V)$, so any edge under consideration in $G_{\widetilde{V} \cup (X \setminus D)}$ is also under consideration in G, and

$$\big\{v \text{ a base point for an exitless loop in } G_{\widetilde{V} \cup (X \setminus D)}\big\}$$
$$= \{v \text{ a base point for an exitless loop in } G_V\}.$$

In general the interior of a set decreases if we take the interior with respect to a larger set, so

$$\mathrm{Int}_{X \setminus V}\{v \text{ a base point for an exitless loop in } G_V\} = \phi. \qquad \square$$

The following example illustrates some of the concepts above. The quiver G we consider is basically a discrete topological graph where the base space has been enlarged to have a nondiscrete component. For $\mathcal{E}$ the C*-correspondence associated with G, the two C*-algebras $\mathcal{O}(J_{\mathcal{E}}, \mathcal{E})$ and $\mathcal{O}(J_{\mathcal{E}_N}, \mathcal{E}_N)$ are compared.

Let X be a nondiscrete second countable locally compact Hausdorff space with $p_0, p_1 \in X$, $X = \{p_0\} \cup B$, and $p_0 \notin B, p_1 \in B$. The edge space E is the singleton $\{e\}$ with r and s continuous maps from E to X defined by $r(e) = p_0$, $s(e) = p_1$. Note that the map r is open. Set $\lambda_x = 0$ for $x \in X \setminus \{p_0\}$, and λ_{p_0} a probability measure concentrated at $\{e\}$. Then $G = (X, E, r, s, \lambda)$ is a topological quiver, in fact a topological relation defined by a function ([B2]) since E may be viewed as the subset $\{(p_0, p_1)\}$ of $X \times X$. The correspondence $\mathcal{E} = \mathbb{C} \cdot \delta_e$ over $C_0(X)$ is described as follows: the map $\phi : C(X) \to \mathcal{L}(\mathcal{E})$ giving the left action is defined by multiplication by the scalar $f(p_1)$, while the right action of $C_0(X)$ on $\mathcal{E}$ is given by multiplication by the scalar $f(p_0)$, $f \in C_0(X)$. We have $\langle \delta_e, \delta_e \rangle = \delta_{p_0}$, so the element $\delta_e \otimes \delta_e$ of $\mathcal{K}(\mathcal{E})$ is the identity map of $\mathcal{E}$. Thus $\phi(f) = f(p_1)\delta_e \otimes \delta_e$ for $f \in C_0(X)$, $\phi^{-1}(\mathcal{K}(\mathcal{E})) = C_0(X)$, and $X_{\mathrm{fin}} = X$. Since $\ker \phi = \{f \in C_0(X) \mid f(p_1) = 0\}$ we have $X_{\mathrm{sink}} = X \setminus \{p_1\}$. Thus $\overline{X_{\mathrm{sink}}} = X$ and $X_{\mathrm{reg}} = \phi$; the ideal $J_{\mathcal{E}} = 0$.

The closed subset $D = \overline{r(E) \cup s(E)}$ of X is $\{p_0, p_1\}$, so the correspondence $\mathcal{E}_N$ over $C_0(D) = C(D)$ is the same as the correspondence associated with the discrete graph given by a single directed edge from the vertex p_1 to the vertex p_0. We also note that $D_{\text{sink}} = \{p_0\}$, $D_{\text{fin}} = D$, $D_{\text{reg}} = \{p_1\}$. Thus the C*-algebra $\mathcal{O}(J_{\mathcal{E}_N}, \mathcal{E}_N)$ is the universal C*-algebra generated by a partial isometry S with orthogonal initial and final ranges which sum up to the identity.

Since $J_{\mathcal{E}} = 0$ the coisometric condition on the universal representation vanishes and the Cuntz–Pimsner C*-algebra $\mathcal{O}(J_{\mathcal{E}}, \mathcal{E})$ is just $\mathcal{T}(\mathcal{E})$ the Toeplitz C*-algebra for $\mathcal{E}$. This is the universal C*-algebra generated by a partial isometry T $(= T(e))$ and the abelian C*-algebra $C_0(X)$ with the initial space T^*T equal to the projection $\delta_{p_0}$of $C_0(X)$ and $f \cdot T = f(p_1)T$ for $f \in C_0(X)$. We have $T^*TT = (\delta_{p_0}) \cdot T = \delta_{p_0}(p_1)T = 0$, so the final space of T is orthogonal to the initial space T^*T. The quotient C*-algebra $\mathcal{T}(\mathcal{E})/C_0(D)$ is the Toeplitz C*-algebra $\mathcal{T}(\mathcal{E}_N)$ for the correspondence $\mathcal{E}_N$, so for the discrete directed graph, while its quotient by the ideal generated by $\widetilde{\delta}_{(T_0,\pi_0)}(C_0(D_{\text{reg}}))$ is the C*-algebra $\mathcal{O}(J_{\mathcal{E}_N}, \mathcal{E}_N)$, namely the C*-algebra of this discrete directed graph. Since $C_o(D_{\text{reg}}) = C(\{p_1\}) = \mathbb{C}\delta_{p_1}$ and $\phi_N(\delta_{p_1}) = \delta_e \otimes \delta_e$ we see that $\widetilde{\delta}_{(T_0,\pi_0)}(\delta_{p_1}) = \delta_{p_1} - TT^*$.

A very special case of a topological quiver occurs within the context of a function $f : \text{dom}(f) \to X$ where $\text{dom}(f)$ is a subspace of a locally compact space X. If π_1 and π_2: $X \times X \to X$ denote the canonical continuous projections onto the first and second coordinates respectively, set $r = \pi_1|_{\text{graph}(f)}$ and $s = \pi_2|_{\text{graph}(f)}$, continuous maps of the subspace $\text{graph}(f)$ of $X \times X$ to X.

Proposition 2.12. *Let X be a locally compact Hausdorff space. The map r : $\text{graph}(f) \to X$ is open if and only if $f : \text{dom}(f) \to X$ is continuous and $\text{dom}(f)$ is open in X. In this case $\text{dom}(f)$ is homeomorphic with $\text{graph}(f)$, and $\text{graph}(f)$ is locally compact in $X \times X$.*

Proof. First note that for A, B open subsets of X then

$$\text{graph}(f) \cap (A \times B) = \left\{(x, f(x)) \mid x \in A \cap f^{-1}(B)\right\}, \quad \text{so}$$
$$r(\text{graph}(f) \cap (A \times B)) = A \cap f^{-1}(B) = A \cap f^{-1}(B) \cap \text{dom}(f).$$

If f is continuous then $f^{-1}(B)$ is open in $\text{dom}(f)$ and r is an open map to the subspace $\text{dom}(f)$ of X ; since r is continuous, one to one, and onto $\text{dom}(f)$, r is a homeomorphism of $\text{graph}(f)$ with the subspace $\text{dom}(f)$ of X. If, in addition, $\text{dom}(f)$ is open in X, then r is an open map to X . Since $\text{dom}(f)$ is open in X it is also locally compact. Thus $\text{graph}(f)$ must also be locally compact in $X \times X$, so is therefore the intersection of a closed and open set in $X \times X$. In fact one can show directly that $\text{graph}(f)$ is the intersection of the closure of $\text{graph}(f)$ with the open set $\text{dom}(f) \times X$.

Conversely, if r : $\text{graph}(f) \to X$ is open, then $r(\text{graph}(f)) = \text{dom}(f)$, so $\text{dom}(f)$ is open in X and therefore locally compact. If B is open in X, $f^{-1}(B) = r(\text{graph}(f) \cap (X \times B))$ which is open in X and a subset of $\text{dom}(f)$, so also open in $\text{dom}(f)$. Thus f is continuous. □

Thus (cf. [B2]), for $f : \text{dom}(f) \to X$ a continuous function with $\text{dom}(f)$ open in X we may form the topological quiver $(X, \text{graph}(f), r, s, \mu)$, or, using the homeomorphism $r : \text{graph}(f) \to \text{dom}(f)$, $G = (X, \text{dom}(f), i, f, \lambda)$ where i is the inclusion of $\mathrm{dom}(f)$ in X and λ is the normalized measure on X with support $\overline{\{f(x)\}}$. Here the restricted topological quiver $G^N = (D, E, r, s, \lambda)$ where $D = \overline{\mathrm{dom}(f) \cup \mathrm{ran}(f)}$.

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