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On certain weighted moving averages and their differentiation analogues

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Abstract. Let (X, Σ, μ, T) be a measure-preserving dynamical system, and $\{I_n\}$ a sequence of intervals of nonnegative integers moving to infinity with increasing cardinality. Rosenblatt and Wierdl constructed optimal weights w_n for the averages of the form

$$\frac{1}{w_n} \sum_{k \in I_n} f \circ T^k$$

to converge a.e. in L_1. In this paper, we provide modified versions of those weights to address the question of optimality for more general weighted averages and their differentiation analogues.

Contents

1. Introduction 19
2. Weighted moving averages and optimal weights 21
3. Weighted differentiation averages and optimal weights 29
Acknowledgments 36
References 37

1. Introduction

Let (X, Σ, μ, T) be a measure-preserving dynamical system. Pointwise convergence of averages of the form $\frac{1}{\#I_n} \sum_{k \in I_n} f \circ T^k$ (called *moving averages*) depends on existence of maximal inequalities, where I_n's are intervals of integers and $\#I_n$ denotes the cardinality of I_n. Bellow et al. [4] provided a geometric criterion, the so-called *cone condition*, which controls boundedness of the associated maximal operators. Whenever this criterion fails, it is natural to investigate the rate of divergence. For the diverging moving averages $\frac{1}{n} \sum_{n^2 < k \le n^2+n} f \circ T^k$ (studied in [6]), multiplying by the weights $1/n$ remedies a.e. convergence. Rosenblatt and Wierdl

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[8] constructed appropriate weights, so-called *correct factors*, which impose maximal inequalities to the weighted averages in L_1. The correct factors are optimal, that is smallest possible, whenever the intervals I_n have increasing lengths.

Let A and B be sets of integers. We define the difference set

$$A - B = \{a - b : a \in A, b \in B\}.$$

For a sequence of nonempty finite sets of nonnegative integers $\{U_n\}_{n\in\mathbb{N}}$, the correct factors are defined by

$$Q_n = \# \bigcup_{i=1}^{n} (U_n - U_i).$$

Consider the set of indices

$$\mathcal{U}_n = \{i : \#U_i \leq \#U_n\}.$$

We define the *modified correct factors* by

$$\widetilde{Q}_n = \# \bigcup_{i\in\mathcal{U}_n} (U_n - U_i).$$

In this paper we show that the modified correct factors not only preserve the good qualities of the correct factors, that is the weighted operators satisfy maximal inequalities, but they further extend them in the sense that optimality holds for *all* moving averages. For example, consider the diverging moving averages over the intervals $I_{2n-1} = [n^2, n^2+n),\ I_{2n} = [n^3, n^3+n^2)$. In this case, the modified correct factors are optimal while the correct factors are not. Another significant advantage of the modified correct factors is that they can be used as a geometric criterion for the a.e. convergence of moving averages. This is presented in the following theorem.

Theorem 1.1. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of bounded intervals of nonnegative integers given by $I_n = [v_n, v_n + \#I_n)$ where $v_n, \#I_n$ tend to infinity. Then, for every measure-preserving system (X, Σ, μ, T), there is a constant C so that for every $f \in L_1(X)$ and $\lambda > 0$,*

$$\mu \left\{ x \in X : \sup_n \left| \frac{1}{\#I_n} \sum_{k\in I_n} f(T^k(x)) \right| > \lambda \right\} \leq C \frac{\|f\|_1}{\lambda}$$

if and only if

$$\sup_n \frac{\widetilde{Q}_n}{\#I_n} < +\infty.$$

Consider the moving averages over the intervals $I_{2n-1} = [0, (n+1)!),\ I_{2n} = [(n+1)!, (n+1)! + n!)$. Notice that $\sup_n \frac{Q_n}{\#I_n} = +\infty$, when $\sup_n \frac{\widetilde{Q}_n}{\#I_n} < +\infty$. By the latter and Theorem 1.1 we conclude that the associated moving averages converge pointwise (see also Remark 2.6 in [8]).

The interplay between ergodic theory and real variable harmonic analysis was established by Calderón's Transfer Principle in [5]. Consequently, parallel results appeared for example in [4] and [7], and in [2] and [3]. We modify the correct factors for differentiation operators analogous to moving averages, and we show that they display similar behavior. Furthermore, we investigate the question of optimality of these weights, which was not addressed in [8].

In Section 2, we provide in detail the properties of the modified correct factors and the proof of Theorem 1.1. Many examples and propositions are included to illustrate the differences between the new weights and the ones introduced in [8]. In Section 3, we discuss the differentiation analogue of the modified correct factor and analyze the similarities and differences in behavior relative to its ergodic version. Sufficient conditions are provided for the modified correct factors to be optimal.

2. Weighted moving averages and optimal weights

Throughout this section we stay in the ergodic framework of a measure-preserving system (X, Σ, μ, T). Let f be a μ-almost everywhere finite Σ-measurable function. Let I be a nonempty, finite set of nonnegative integers, and n, w be positive integers. We make use of the following types of operators

$$M_{(I,w)}f = \frac{1}{w}\sum_{n\in I} f\circ T^n,$$

$$M_I f = M_{(I,\#I)}f = \frac{1}{\#I}\sum_{n\in I} f\circ T^n.$$

Let Ω be an infinite collection of lattice points in $\mathbb{Z}^2$ with positive second coordinate. We denote by Ω_α the lattice points in the union of solid cones with aperture $\alpha > 0$ and vertex in Ω, and for any integer height $\lambda > 0$, we let

$$\Omega_\alpha(\lambda) = \{k \in \mathbb{Z} : (k, \lambda) \in \Omega_\alpha\}.$$

The maximal function associated with Ω is given by

$$M^*_\Omega f(x) = \sup_{(k,n)\in\Omega} \frac{1}{n}\sum_{j=0}^{n-1} |f(T^{k+j}x)|.$$

Definition 2.1. We say that M^*_Ω is *weak type* (p, p) for $p < \infty$ if

$$\mu\big\{x \in X : M^*_\Omega f(x) > \lambda\big\} \le \left(C\frac{\|f\|_p}{\lambda}\right)^p.$$

We say that M^*_Ω is *strong type* (p, p) for $1 < p \le \infty$ if it bounded from L_p to itself.

The following geometric criterion on the approach regions which characterizes the existence of maximal inequalities for ergodic moving averages is essential in understanding the statement of Theorem 1.1.

Theorem 2.2 ([4])**.** (i) *Assume that there exist constants $A < +\infty$ and $\alpha > 0$ such that $\#\Omega_\alpha(\lambda) \le A\lambda$ for all integers $\lambda > 0$; then M^*_Ω is weak type $(1, 1)$ and strong type (p, p) for $1 < p \le \infty$.*

(ii) *If M^*_Ω is weak type (p, p) for some $p > 0$ then for every $\alpha > 0$ there exists $A_\alpha < +\infty$ such that for all integers $\lambda > 0$ we have $\#\Omega_\alpha(\lambda) \le A_\alpha\lambda$.*

Remark 2.3. For $I_n = [v_n, v_n + l_n)$ with v_n, $l_n \to \infty$ we set $\Omega = \{(v_n, l_n)\}_{n\in\mathbb{N}}$. Then Theorem 2.2 characterizes the pointwise convergence of the moving averages M_{I_n} in all L_p spaces, $p \ge 1$. This is a consequence of the Banach Principle, which asserts that the set of functions where a.e. convergence holds is a closed set, and the fact that $\{f \in L_p(X) : f \circ T = f\} \oplus \mathrm{cl}_{\|\cdot\|_p}\{f - f \circ T : f \in L_\infty\}$ is a dense subset of that set.

Question 2.4. Suppose that $\sup_k M_{I_k}|f|$ is not weak type $(1,1)$. What are the optimal (i.e., smallest) weights w_k so that $\sup_k M_{(I_k,w_k)}|f|$ becomes weak type $(1,1)$?

We use the modified correct factors as weights. The next two theorems show the sufficiency of the weights $\widetilde{Q}_n$ to impose pointwise convergence, and their optimality. Their proofs are variations of the corresponding theorems in [8], and therefore omitted. Interested readers may refer to [1] for details.

Theorem 2.5. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of finite sets of positive integers. For every measure-preserving system (X,Σ,μ,T), $f\in L_1(X)$ and $\lambda>0$ we have*

$$\mu\left\{\sup_n |M_{(I_n,\widetilde{Q}_n)}f|>\lambda\right\}\leq \frac{\|f\|_1}{\lambda}.$$

Let $\alpha_n=\#\mathfrak{I}_n$. It is easy to see that whenever I_n are bounded intervals one has the estimate

$$\widetilde{Q}_n\leq\sum_{i\in\mathfrak{I}_n}(\#I_n+\#I_i-1)\leq 2\alpha_n\cdot\#I_n. \tag{2.1}$$

As a consequence of Equation (2.1) and Theorem 2.5 we have the next corollary.

Corollary 2.6. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of bounded intervals of nonnegative integers. For any measure-preserving system (X,Σ,μ,T), $f\in L_1(X)$ and $\lambda>0$, we have*

$$\mu\left\{\sup_n |M_{(I_n,\alpha_n\cdot\#I_n)}f|>\lambda\right\}\leq \frac{2}{\lambda}\|f\|_1.$$

Thus, for $f\in L_1(X)$,

$$M_{(I_n,\alpha_n\cdot\#I_n)}f(x)\to 0 \text{ a.e.}$$

For sequences $\{I_n\}_{n\in\mathbb{N}}$ of intervals with lengths tending to infinity, $\widetilde{Q}_n$ are optimal weights.

Theorem 2.7. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of bounded intervals of nonnegative integers with $\{\#I_n\}$ tending to infinity. Let $c(n)$ be a sequence of positive numbers satisfying*

$$\lim_{n\to\infty}c(n)=+\infty.$$

For every nonatomic ergodic probability measure-preserving system (X,Σ,μ,T) there is a residual set in L_1 of functions so that

$$\sup_n |M_{(I_n,\widetilde{Q}_n/c(n))}f(x)|=+\infty$$

for almost every $x\in X$.

The largeness of $\widetilde{Q}_n$ in terms of the size of the intervals is a measure of whether these weights are trivial or not. An upper bound is given in (2.1).

Remark 2.8. If

$$\sum_n\frac{\#I_n}{w_n}<+\infty,$$

then w_n are trivial sufficient factors for $M_{(I_n,w_n)}$ to satisfy weak type (1,1) maximal inequalities. It is enough to notice that

$$\mu\left\{\sup_n |M_{(I_n,w_n)}f| > \lambda\right\} \leq \sum_n \mu\{|M_{(I_n,w_n)}f| > \lambda\}$$
$$\leq \sum_n \frac{\#I_n}{w_n} \cdot \frac{\|f\|_1}{\lambda}$$
$$\leq C\frac{\|f\|_1}{\lambda}.$$

As a consequence of Theorem 2.7 we conclude that $\widetilde{Q}_n$ are nontrivial sufficient weights.

Corollary 2.9. *If $\{I_n\}_{n\in\mathbb{N}}$ is a sequence of bounded intervals of nonnegative integers with $\{\#I_n\}$ tending to infinity, then*

$$\sum_n \frac{\#I_n}{\widetilde{Q}_n} = +\infty.$$

A new characterization of pointwise convergence for moving averages is given in terms of the relative size of the length of the intervals to the corresponding correct factor. It indicates the connection between the size of the cross sections involved in Theorem 2.2 and the correct factors.

Proof of Theorem 1.1. Suppose that $\widetilde{Q}_n \leq C \cdot \#I_n$ for some constant C and every $n \in \mathbb{N}$. Then for every $f \in L_1$,

$$|M_{I_n}f(x)| \leq M_{I_n}|f|(x) \leq CM_{(I_n,\widetilde{Q}_n)}|f|(x),$$

which implies that for every $\lambda > 0$

$$\mu\left\{\sup_n |M_{I_n}f| > \lambda C\right\} \leq \mu\left\{\sup_n M_{(I_n,\widetilde{Q}_n)}|f| > \lambda\right\},$$

and Theorem 2.5 finishes the proof in this direction.

Conversely, assume that M_{I_n} satisfy weak type (1,1) maximal inequalities. By Theorem 2.2, there exist constants $A < \infty$ and $\alpha > 0$ such that $\#\Omega_\alpha(s) \leq As$ for every positive integer height s. Without loss of generality we may assume that $2\alpha \geq \pi/2$.

Fix n_0. We prove the existence of a constant C, independent of n_0, for which $\widetilde{Q}_{n_0} \leq Cl_{n_0}$.

Choose an integer constant C_1 such that $(C_1 - 1)\tan\alpha \geq 1$. Then

$$C_1 \geq 1 + \frac{1}{\tan\alpha} > 1.$$

Therefore, at height $s = C_1 l_{n_0}$ we have

$$\#\Omega_\alpha(s) \leq AC_1 l_{n_0}. \tag{2.2}$$

Notice that

$$\Omega_\alpha(s) = \bigcup_{\{n:\, l_n < s\}} J_n^s$$

where J_n^s are the intervals of nonnegative integers given by

$$J_n^s = v_n + \tan\alpha(-(s-l_n),(s-l_n)).$$

We reflect about the origin and appropriately translate J_n^s to obtain U_n^s, given by

$$U_n^s = v_{n_0} - v_n + \tan\alpha(-(s-l_n),(s-l_n)).$$

Then, by Equation (2.2),

$$\# \bigcup_{\{n:\, l_n<s\}} U_n^s = \# \bigcup_{\{n:\, l_n<s\}} J_n^s \\ \le AC_1 l_{n_0}. \tag{2.3}$$

For all n with $l_n \le l_{n_0} < s$ we have

$$\begin{aligned} I_{n_0} - I_n &= v_{n_0} - v_n + (-l_n, l_{n_0}) \\ &\subseteq v_{n_0} - v_n + (-l_{n_0}, l_{n_0}) \\ &\subseteq v_{n_0} - v_n + \tan\alpha(-l_{n_0}(C_1-1), l_{n_0}(C_1-1)) \\ &\subseteq v_{n_0} - v_n + \tan\alpha(-(C_1 l_{n_0} - l_n), C_1 l_{n_0} - l_n)) \\ &= U_n^s. \end{aligned}$$

Therefore, using also Equation (2.3),

$$\begin{aligned} \widetilde{Q}_{n_0} &= \# \bigcup_{n\in\mathcal{A}_{n_0}} (I_{n_0} - I_n) \\ &\le \# \bigcup_{n\in\mathcal{A}_{n_0}} U_n^s \\ &\le \# \bigcup_{\{n:\, l_n<s\}} U_n^s \\ &\le C l_{n_0}. \end{aligned}$$

□

Definition 2.10. Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence of linear operators from L_1 to L_1. We say that the *strong sweeping out property* holds for the operators M_n if for every $\varepsilon > 0$ there is a set $A \in \Sigma$ such that:

(a) $\mu(A) < \varepsilon$,
(b) $\limsup_{n\to\infty} M_n\chi_A(x) = 1$ a.e., and
(c) $\liminf_{n\to\infty} M_n\chi_A(x) = 0$ a.e.

One may think of strong sweeping out property as the extreme of divergence.

Example 2.11. Let $I_k = [v_k, v_k + l_k)$ with $v_k = 2^k$ and

$$l_k = \begin{cases} 2s, \text{ if } k = 2s-1 \\ s, \text{ if } k = 2s. \end{cases}$$

Then

$$\begin{aligned} \widetilde{Q}_{2s-1} &= \# \left(\bigcup_{i=1}^{2s-1} (I_{2s-1} - I_i) \cup \bigcup_{j=s}^{2s} (I_{2s-1} - I_{2j}) \right) \\ &\ge s^2. \end{aligned}$$

It follows that

$$\sup_s \frac{\widetilde{Q}_{2s-1}}{\#I_{2s-1}} = +\infty,$$

and therefore a weak type $(1,1)$ maximal inequality is not possible for M_{I_n}. In particular, the corresponding moving averages satisfy the strong sweeping out property (see Corollary 5 in [4]).

In the previous example, the strong sweeping out property is not a coincidence, as is pointed out below.

Theorem 2.12. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of finite intervals of nonnegative integers with lengths tending to infinity. If*

$$\sup_n \frac{\widetilde{Q}_n}{\#I_n} = +\infty$$

then in every nonatomic ergodic probability measure-preserving system (X, Σ, μ, T) the operators M_{I_n} satisfy the strong sweeping out property.

Proof. Similar to the proof of Theorem 2.5b in [8]. □

Our next task is to compare the two versions of correct factors. We assume nondecreasing lengths for the intervals I_n and we investigate how much larger is the modified correct factor.

First, consider intervals with not strictly increasing lengths, for which the correct factors are not necessarily optimal.

Proposition 2.13. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint bounded intervals of nonnegative integers with lengths satisfying:*

(1) $\#I_i < \#I_{i+1}$ *for every* $1 \leq i < k_1$ *and every* $m_n \leq i < k_{n+1}$ *for* $n \in \mathbb{N}$,
(2) $\#I_i = \#I_{k_n}$ *for every* $k_n \leq i \leq m_n$ *for* $n \in \mathbb{N}$,

where k_n and m_n are two increasing sequences of positive integers with the property $\sup_n(m_n - k_n) = +\infty$. Then $\sup_n \frac{\widetilde{Q}_n}{\#I_n} = +\infty$. Moreover, if

$$\sup_n \frac{(m_n - k_n + 1)\cdot \#I_{k_n}}{\sum_{i=1}^{n-1}(m_i - k_i + 1)\cdot \#I_{k_i}} < +\infty$$

then

$$\sup_n \frac{\widetilde{Q}_n}{Q_n} < +\infty.$$

Proof. Let $I_n = [v_n, v_n + l_n)$. Fix $n = k_s + j$ for some s and $0 \leq j \leq m_s - k_s$. Let

$$A_i = \bigcup_{r=0}^{m_i - k_i} (I_{k_s+j} - I_{k_i+r}).$$

By the definition of Q_n and $\widetilde{Q}_n$,

$$\begin{aligned} Q_n &= \#\underbrace{\bigcup_{i=1}^{n}(I_n - I_i)}_{B} \\ &= \bigcup_{i=1}^{k_1-1}(I_n - I_i) \cup \bigcup_{i=1}^{s-1}\bigcup_{r=1}^{k_{i+1}-m_i-1}(I_n - I_{m_i+r}) \cup \bigcup_{i=1}^{s-1} A_i \cup \bigcup_{p=0}^{j}(I_n - I_{k_s+p}), \\ \widetilde{Q}_n &= \#B \cup \bigcup_{t=j+1}^{m_s-k_s}(I_{k_s+j} - I_{k_s+t}). \end{aligned}$$

Since I_n are disjoint, we obtain the following bounds for $\widetilde{Q}_n$ and $\cup A_i$:

$$Q_n + (m_s - k_s - j)l_{k_s} \leq \widetilde{Q}_n \leq Q_n + 2(m_s - k_s + 1)l_{k_s} \tag{2.4}$$

$$\#\bigcup_{i=1}^{s-1} A_i \geq \sum_{i=1}^{s-1}(m_i - k_i + 1)l_{k_i}. \tag{2.5}$$

From the left-hand side of (2.4) we conclude that $\sup_n \frac{\widetilde{Q}_n}{\#I_n} = +\infty$.

Combining (2.4) and (2.5),

$$\frac{\widetilde{Q}_n}{Q_n} \leq 1 + 2\frac{(m_s - k_s + 1)l_{k_s}}{\sum_{i=1}^{s-1}(m_i - k_i + 1)l_{k_i}} < C. \qquad \square$$

Remark 2.14. The converse of the previous proposition does not hold. For example, the intervals $\{I_n\}_{n\in\mathbb{N}}$ given by $I_{n+i} = [n! + i(n-1)!, n! + (i+1)(n-1)!)$ for every $0 \leq i \leq n-1$, satisfy

$$\sup_n \frac{(m_n - k_n + 1)\cdot \#I_{k_n}}{\sum_{i=1}^{n-1}(m_i - k_i + 1)\cdot \#I_{k_i}} = +\infty,$$

and simultaneously

$$\frac{\widetilde{Q}_n}{Q_n} < +\infty.$$

The fact that the A_i's defined in the proof of the previous proposition are all disjoint plays a crucial role.

Next we consider intervals with increasing lengths for which the correct factors are optimal.

Lemma 2.15. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint bounded intervals of nonnegative integers with lengths increasing to infinity. Then Q_n is an increasing sequence of positive integers.*

Proof. Let $I_n = [v_n, v_n + l_n)$. Fix $n \in \mathbb{N}$, and let $k \in \{1, \cdots, n-1\}$. We denote

$$D_{n,k} = I_n - I_k.$$

We relate the sets $D_{n,k} \cup D_{n,k+1}$ and $D_{n+1,k} \cup D_{n+1,k+1}$.

There are two possibilities for $D_{n,k} \cap D_{n,k+1}$:

- $D_{n,k} \cap D_{n,k+1} \neq \emptyset$:
  this is equivalent to $v_{k+1} - v_k - l_k < l_n$ which implies that
  $$v_{k+1} - v_k - l_k < l_{n+1} \text{ or } D_{n+1,k} \cap D_{n+1,k+1} \neq \emptyset.$$
  Then
  $$D_{n,k} \cup D_{n,k+1} = v_n + (-v_{k+1} - l_{k+1}, -v_k + l_n),$$
  and
  $$D_{n+1,k} \cup D_{n+1,k+1} = v_{n+1} + (-v_{k+1} - l_{k+1}, -v_k + l_{n+1}).$$
- $D_{n,k} \cap D_{n,k+1} = \emptyset$:
  this is equivalent to $v_{k+1} - v_k \geq l_k + l_n$ and
  $$D_{n,k} \cup D_{n,k+1} = v_n + (-v_k - l_k, -v_k + l_n) \cup (-v_{k+1} - l_{k+1}, -v_{k+1} + l_n).$$
  Then we have two possible cases for $v_{k+1} - v_k - l_k - l_{n+1}$:
  - if $v_{k+1} - v_k - l_k < l_{n+1}$ then $D_{n+1,k} \cap D_{n+1,k+1} \neq \emptyset$, which implies
  $$D_{n+1,k} \cup D_{n+1,k+1} = v_{n+1} + (-v_{k+1} - l_{k+1}, -v_k + l_{n+1}).$$
  - if $v_{k+1} - v_k - l_k \geq l_{n+1}$ then $D_{n+1,k} \cap D_{n+1,k+1} = \emptyset$, which implies
  $$D_{n+1,k} \cup D_{n+1,k+1} = v_{n+1} + (-v_k - l_k, -v_k + l_{n+1}) \cup (-v_{k+1} - l_{k+1}, -v_{k+1} + l_{n+1}).$$

Hence,
$$v_{n+1} - v_n + D_{n,k} \cup D_{n,k+1} \subset D_{n+1,k} \cup D_{n+1,k+1},$$
which produces
$$Q_n = \# \bigcup_{k=1}^{n} D_{n,k} = \# \left(v_{n+1} - v_n + \bigcup_{k=1}^{n} D_{n,k} \right) \leq \# \bigcup_{k=1}^{n+1} D_{n+1,k} = Q_{n+1}. \qquad \square$$

Proposition 2.16. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of finite intervals of nonnegative integers with lengths increasing to infinity. Assume that:*

(1) *$\#I_i < \#I_{i+1}$ for every $1 \leq i < k_1$ and every $m_n \leq i < k_{n+1}$ for $n \in \mathbb{N}$,*
(2) *$I_i = [v_{k_n} + (i - k_n)l_{k_n}, v_{k_n} + (i - k_n + 1)l_{k_n})$ for every $k_n \leq i \leq m_n$ for $n \in \mathbb{N}$,*

where k_n and m_n are two increasing sequences of positive integers with the property $\sup_n(m_n - k_n) = +\infty$. Then
$$\sup_n \frac{\widetilde{Q}_n}{Q_n} < +\infty \text{ if and only if } \sup_n \frac{(m_n - k_n)l_{k_n}}{Q_n} < +\infty.$$

Proof. Notice that for $m_n \leq j < k_{n+1}$ we have $\widetilde{Q}_j = Q_j$.

Fix $k_n \leq j < m_n$. A basic calculation shows that
$$\bigcup_{i=j+1}^{m_n} (I_j - I_i) = (-(m_n - j + 1)l_{k_n}, 0).$$

Hence,
$$\begin{aligned}\widetilde{Q}_j &= Q_j + \#(-(m_n - j + 1)l_{k_n}, -l_{k_n}) \\ &= Q_j + (m_n - j)l_{k_n}\end{aligned}$$

which gives

$$\max_{k_n \le j < m_n} \frac{\widetilde{Q}_j}{Q_j} = 1 + \max_{k_n \le j < m_n} \frac{(m_n - j) l_{k_n}}{Q_j}.$$

Since

$$\max_{k_n \le j < m_n} (m_n - j) = m_n - k_n$$

and, by Lemma 2.15,

$$\inf_{k_n \le j < m_n} Q_j = Q_{k_n},$$

we obtain

$$\max_{k_n \le j < m_n} \frac{\widetilde{Q}_j}{Q_j} = 1 + \frac{(m_n - k_n) l_{k_n}}{Q_{k_n}},$$

and finally

$$\sup_n \frac{\widetilde{Q}_j}{Q_j} = 1 + \sup_n \frac{(m_n - k_n) l_{k_n}}{Q_{k_n}}.$$ □

Remarks 2.17. Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of finite intervals of nonnegative integers with lengths increasing to infinity (not necessarily strictly). Q_n are optimal weights. We examine how much $\widetilde{Q}_n$ differ from Q_n. Notice that $\lim_n \frac{\widetilde{Q}_n}{Q_n} = +\infty$ is not possible since Q_n and $\widetilde{Q}_n$ agree for infinitely many indices n.

(1) If the lengths of $\{I_n\}$ strictly increase to infinity, then $Q_n = \widetilde{Q}_n$ for every n.
(2) If the lengths of $\{I_n\}$ increase to infinity and the number of intervals with the same length is uniformly bounded, then Q_n and $\widetilde{Q}_n$ are equivalent up to a constant.
(3) If the lengths of $\{I_n\}$ increase to infinity and the number of intervals with the same length is unbounded, then it is possible to have Q_n and $\widetilde{Q}_n$ to be equivalent up to a constant or not. Consider the intervals given for each n by $I_{n+i} = [2^n + in, 2^n + (i+1)n)$ for every $0 \le i \le n-1$. Then we have $\sup_n \frac{\widetilde{Q}_n}{Q_n} < +\infty$, by Proposition 2.13. To the other end, consider the sequence of intervals $\{I_n\}$ given for each n by $I_{n+i} = [n! + in, n! + (i+1)n)$ for every $0 \le i \le (n-1)! - 1$. Then, by Proposition 2.16 we have $\sup_n \frac{\widetilde{Q}_n}{Q_n} = +\infty$. Notice that in the last example consecutive A_i's are not disjoint, which is in contrast to the example in Remark 2.14.

Remarks 2.18. (1) Consider a sequence of finite intervals of nonnegative integers $\{I_n\}_{n\in\mathbb{N}}$ with lengths tending not monotonically to infinity. Depending both on the number of intervals that spoil the increasing property, and the number of those with the same length, the modified correct factor can be much smaller than the original one. There is a unique permutation π such that $\{I_{\pi(n)}\}_{n\in\mathbb{N}}$ has increasing lengths and is "moving" to infinity. For intervals of the same cardinality, the smaller the left endpoint of I_n, the smaller the index after π.
(2) The question of optimal weights for general moving averages with the number of intervals at each step not uniformly bounded remains open. It may involve a generalized cone condition which is still missing.

3. Weighted differentiation averages and optimal weights

We consider the space $(\mathbb{R}^n, \mathcal{L}, m)$ where $\mathcal{L}$ is the σ-algebra of the Lebesgue measurable sets and m is the Lebesgue measure. Let I be a set of positive measure in $\mathbb{R}$ and Q be a positive real number. We make use of the following operators

$$N_{(I,w)}f(\cdot) = \frac{1}{w}\int_I f(t+\cdot)\,dt,$$

$$N_I f(\cdot) = N_{(I,m(I))}f(\cdot) = \frac{1}{m(I)}\int_I f(t+\cdot)\,dt.$$

Let Ω be an infinite collection of points in $\mathbb{R}^{n+1}_+$ and Ω_α be the union of solid cones with aperture $\alpha > 0$ and vertex in Ω. For any positive height λ, $\Omega_\alpha(\lambda)$ is defined by

$$\Omega_\alpha(\lambda) = \{x \in \mathbb{R}^n : (x,\lambda) \in \Omega_\alpha\}.$$

The maximal operator associated with Ω_α is denoted by

$$N^*_{\Omega_\alpha} f(\cdot) = \sup_{(x,y)\in\Omega_\alpha} \frac{1}{m(B(x,y))}\int_{B(x,y)} |f(t+\cdot)|\,dt.$$

Theorem 3.1 ([7]). (i) *Assume that there exist constants $A < +\infty$ and $\alpha > 0$ such that $m(\Omega_\alpha(\lambda)) \le A\lambda$ for every $\lambda > 0$; then $N^*_{\Omega_\alpha}$ is weak type $(1,1)$ and strong type (p,p) for $1 < p \le \infty$.*

(ii) *If $N^*_{\Omega_\alpha}$ is weak type (p,p) for some $p > 0$ then for every $\alpha > 0$ there exists $A_\alpha < +\infty$ such that for all $\lambda > 0$ we have $m(\Omega_\alpha(\lambda)) \le A_\alpha\lambda$.*

Remark 3.2. Let $I_n = [v_n, v_n + l_n)$ be on the real line with $v_n, l_n \to 0$. For $\Omega = \{(v_n, l_n)\}_{n\in\mathbb{N}}$ the previous theorem characterizes the pointwise behavior of the operators N_{I_n}.

Question 3.3. Suppose that $\sup_k N_{I_k}|f|$ is not weak type $(1,1)$. What are the optimal weights w_k so that $\sup_k N_{(I_k,w_k)}|f|$ becomes weak type $(1,1)$?

Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of sets of real numbers with nonzero Lebesgue measure. Let

$$\mathcal{I}_n = \{i : m(I_i) \le m(I_n)\}.$$

We define $\widetilde{Q}_n$ by

$$\widetilde{Q}_n = m\left(\bigcup_{i\in\mathcal{I}_n} (I_n - I_i)\right).$$

Theorem 3.4. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of sets of real numbers of finite, nonzero Lebesgue measure. There exists constant $C > 0$ so that for every $f \in L_1(\mathbb{R})$ and $\lambda > 0$,*

$$m\left\{\sup_n |N_{(I_n,\widetilde{Q}_n)} f| > \lambda\right\} \le \frac{C}{\lambda}\|f\|_1.$$

Remarks 3.5. (1) If

$$\sum_n \frac{m(I_n)}{w_n} < +\infty$$

then w_n are trivial sufficient factors for $N_{(I_n,w_n)}$ to satisfy weak type (1,1) maximal inequalities (see also Remark 2.8).

(2) Addressing the question of optimality for the weights $\widetilde{Q}_n$, in the sense of Theorem 2.7, it is natural to first investigate the convergence of $\sum_n \frac{m(I_n)}{\widetilde{Q}_n}$ (see also Corollary 2.9).

Let $I_n = [v_n, v_n + l_n)$ and let d_n denote the distance between I_n and I_{n+1}. Suppose that l_n, v_n decrease strictly to zero and d_n decrease to zero but not strictly. Fix n and let $k \geq n$.

$$\begin{aligned} m((I_n - I_k) \cap (I_n - I_{k+1})) \neq 0 \Leftrightarrow& \\ v_n - v_k + l_n > v_n - v_{k+1} - l_{k+1} \Leftrightarrow& \\ l_n > v_k - (v_{k+1} + l_{k+1}) \Leftrightarrow& \\ l_n > d_k.& \end{aligned}$$

Let $k(n)$ be the smallest integer $i \geq n$ so that $d_i < l_n$. It follows immediately that from $k(n)$ onwards the consecutive $I_n - I_k$ have nontrivial intersection. Therefore

$$\begin{aligned} \widetilde{Q}_n = Q_n &= \sum_{i=n}^{k(n)-1} (l_i + l_n) + (v_n + l_n) - (v_n - v_{k(n)} - l_{k(n)}) \\ &\geq (k(n) - n) l_n. \end{aligned} \tag{3.1}$$

As a consequence of (3.1) we have the following lemma.

Lemma 3.6. *For a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ as given above:*

(i) *If $\sup_n (k(n) - n) = +\infty$ then $\sup_n \frac{Q_n}{l_n} = +\infty$.*
(ii) *If $\sum_n \frac{1}{k(n)-n} < +\infty$ then $\sum_n \frac{l_n}{Q_n} < +\infty$.*

Proposition 3.7. *There exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ for which*

$$\sum_n \frac{m(I_n)}{\widetilde{Q}_n} < +\infty.$$

Proof. Choose $l_n = \frac{1}{2^n} \searrow 0$. Fix n_0, to be determined later. Let

$$w(n) = (n_0 + n) + (n_0 + n)^2.$$

Let $\{d_n\}$ be given by

$$d_n = \begin{cases} l_{n_0} & \text{if } n_0 \leq n < n_0 + n_0^2 \\ l_{n_0+s+1} & \text{if } w(s) \leq n < w(s+1) \text{ for some } s \geq 0. \end{cases}$$

Then d_n decreases to zero but not strictly. Such a choice of l_n and d_n makes $\{I_n\}$ well-defined, since

$$\sum_{n=n_0}^{\infty} l_n < +\infty,$$

and

$$\begin{aligned}\sum_{n=n_0}^{\infty} d_n &= n_0^2 l_{n_0} + \sum_{s=0}^{\infty}[w(s+1)-w(s)]l_{n_0+s+1}\\ &= \frac{n_0^2}{2^{n_0}} + 2\sum_{s=0}^{\infty}\frac{s+n_0+1}{2^{s+n_0+1}}\\ &= \frac{n_0^2}{2^{n_0}} + 2\sum_{s=n_0+1}^{\infty}\frac{s}{2^s}\end{aligned}$$

which is finite, by the integral test.

The integer n_0 is chosen so that the intervals move to the origin.

Next, we compute v_n: $w(s) \le n < w(s+1)$ for some $s \ge 0$, so

$$\begin{aligned}v_n &= \sum_{i=n}^{\infty} d_i + \sum_{i=n+1}^{\infty} l_i\\ &= [w(s+1)-n]l_{n_0+s+1} + 2\sum_{i=n_0+s+1}^{\infty}\frac{i}{2^i} + \sum_{i=n+1}^{\infty} l_i\\ &= \frac{(s+1)^2+(2n_0+1)(s+1)+(n_0^2+n_0)-n}{2^{n_0+s+1}} + 2\sum_{i=n_0+s+1}^{\infty}\frac{i}{2^i} + \sum_{i=n+1}^{\infty}\frac{i}{2^i}\\ &= (\mathrm{I})_n + (\mathrm{II})_n + (\mathrm{III})_n < +\infty.\end{aligned}$$

As $n \to \infty$ all three terms in the last equation tend to zero.

Moreover, $v_n \searrow 0$: If $n+1$ is in the same block then $(\mathrm{I})_n > (\mathrm{I})_{n+1}$, $(\mathrm{II})_n = (\mathrm{II})_{n+1}$ and $(\mathrm{III})_n > (\mathrm{III})_{n+1}$. If $n+1$ is in the next block, namely $n = w(s+1)-1$, then

$$\begin{aligned}v_n &= \frac{1}{2^{n_0+s+1}} + 2\sum_{k=n_0+s+1}^{\infty}\frac{k}{2^k} + \sum_{i=w(s+1)}^{\infty}\frac{1}{2^i},\\ v_{n+1} &= \frac{s+n_0+2}{2^{n_0+s+1}} + 2\sum_{k=n_0+s+2}^{\infty}\frac{k}{2^k} + \sum_{i=w(s+1)+1}^{\infty}\frac{1}{2^i},\end{aligned}$$

and $v_n > v_{n+1}$ holds, since

$$\frac{s+n_0+2}{2^{n_0+s+1}} < \frac{1}{2^{n_0+s+1}} + 2\frac{n_0+s+1}{2^{n_0+s+1}}.$$

Fix n. For $n \le i \le n+n^2-1$ we have $d_i \ge l_n$, when $d_{n+n^2} = l_{n+1} < l_n$. Therefore $k(n) = n+n^2$, and Lemma 3.6 finishes the proof. □

The previous proposition already indicates the existence of delicate differences between moving averages and their differentiation analogues. There exists a characterization of weak boundedness of the maximal operator of N_{I_n} in terms of the relative size of $\widetilde{Q}_n$ to $m(I_n)$. This characterization is analogous to the one for ergodic moving averages.

Theorem 3.8. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals of real numbers given by $I_n = [v_n, v_n + l_n)$, where l_n, v_n tend to zero. Then, there exists a constant $C > 0$*

so that for every $f \in L_1(\mathbb{R})$ and $\lambda > 0$,

$$m\left\{\sup_n |N_{I_n} f| > \lambda\right\} \leq \frac{C}{\lambda}\|f\|_1$$

if and only if

$$\sup_n \frac{\widetilde{Q}_n}{l_n} < +\infty.$$

Proof. Similar to the proof of Theorem 1.1. □

Remark 3.9. The proof of the above theorem includes that at any height l_n there exists a height $C_1 l_n$ for which we have

$$\widetilde{Q}_n \leq m(\Omega_\alpha(C_1 l_n)).$$

Therefore,

$$\frac{l_n}{\widetilde{Q}_n} \geq \frac{1}{C_1} \cdot \frac{C_1 l_n}{m(\Omega_\alpha(C_1 l_n))} \geq \frac{1}{C_1} \cdot \inf_{u \geq l_n} \frac{u}{m(\Omega_\alpha(u))}.$$

Let $\inf_{u \geq l_n} \frac{u}{m(\Omega_\alpha(u))} = \eta(l_n)$. Choosing suitably the angle α allows us to have C_1 as close to one as desired. Hence, we may assume that C_1 is almost one, and then

$$\frac{l_n}{\widetilde{Q}_n} N_{I_n} f(x) \geq \eta(l_n) N_{I_n} f(x).$$

That relates the modified correct factor $\widetilde{Q}_n$ with the weights used by Nagel and Stein [7] in their Section 3.

The following proposition shows that whenever $\sum_n m(I_n)/\widetilde{Q}_n < +\infty$, it is always possible to have a better factor than $\widetilde{Q}_n$.

Proposition 3.10. *Suppose that $\sup_n \frac{\widetilde{Q}_n}{m(I_n)} = +\infty$ and $\sum_n \frac{m(I_n)}{\widetilde{Q}_n} < +\infty$. Let $F_n = \rho_n m(I_n)$ where $\rho_n \ll \widetilde{Q}_n$ and $\sum_n \frac{1}{\rho_n} < +\infty$. Then, there exists a constant C such that for every $\lambda > 0$ and $f \in L_1(\mathbb{R})$,*

$$m\left\{\sup_n |N_{(I_n,F_n)} f| > \lambda\right\} \leq \frac{C}{\lambda}\|f\|_1.$$

Proof. We may assume that $f \geq 0$, and $\lambda = 1$. We replace the sup with max by using the interval $1 \leq n \leq U$ where U is a large number.

Let

$$A = \{x : N_{(I_n,F_n)} f(x) > 1 \text{ for some } n, 1 \leq n \leq U\}.$$

We show that $m(A) \leq C\|f\|_1$.

Notice that

$$A = \bigcup_{n=1}^{U} A_n$$

where

$$\begin{aligned} A_n &= \{x : N_{(I_n,F_n)} f(x) > 1\} \\ &= \{x : N_{I_n} f(x) > \rho_n\}. \end{aligned}$$

By Chebyshev's inequality,

$$m(A_n) \le \frac{1}{\rho_n}\|N_{I_n} f\|_1 \le \frac{1}{\rho_n}\|f\|_1.$$

Then

$$m(A) \le \sum_{n=1}^{U} m(A_n) \le \|f\|_1 \sum_{n=1}^{U} \frac{1}{\rho_n} < \|f\|_1 \sum_{n=1}^{\infty} \frac{1}{\rho_n} = C\|f\|_1. \qquad \square$$

Proposition 3.7 shows the possibility of convergence of the series $\sum_n m(I_n)/\widetilde{Q}_n$, in which case $\widetilde{Q}_n$ are not the optimal weights. Now we show that divergence is also possible. We use the same notation as in Proposition 3.10.

Let

$$\varepsilon_n = \frac{l_n}{\sum_{k>n} l_k}.$$

Lemma 3.11. *If $\sum_n l_n < +\infty$ then $\sum_n \varepsilon_n = +\infty$.*

Proof. Let $\sigma_n = \sum_{k=n}^{\infty} l_k$. Then $\sigma_n \to 0$. Notice that

$$\varepsilon_n = (\sigma_n - \sigma_{n+1})/\sigma_{n+1}.$$

It follows that

$$\sum_n \varepsilon_n \ge \int_0^c \frac{1}{x}\,dx = +\infty. \qquad \square$$

Lemma 3.12. *Suppose $l_n, v_n \searrow 0$, $\varepsilon_n \to 0$, and for d_n one of the following conditions holds:*

(1) $\sum_{k\ge n} d_k \le C l_n$.
(2) $\sum_{k\ge n} d_k = R_n l_n$ *with* $\sup_n R_n = +\infty$ *and* $\sup_n R_n\varepsilon_n < +\infty$.
(3) $\sum_{k\ge n} d_k = R_n l_n$ *with* $\sup_n R_n = +\infty$, $\sup_n R_n\varepsilon_n = +\infty$ *and* $\sum_n \frac{1}{R_n} = \infty$.

Then

$$\sup_n \frac{Q_n}{l_n} = +\infty \text{ and } \sum_n \frac{l_n}{Q_n} = +\infty.$$

Proof. Suppose that condition (1) holds. Since $l_n, v_n \searrow 0$ there exists an upper bound for the weight Q_n,

$$Q_n \le v_n + 2l_n,$$

which gives an upper bound for Q_n/l_n,

$$\frac{Q_n}{l_n} \le 2 + \frac{v_n}{l_n}. \tag{3.2}$$

Decomposing v_n in terms of l_n and d_n we have

$$\begin{aligned} v_n &= \sum_{k>n} l_k + \sum_{k\ge n} d_k \\ &\le \frac{l_n}{\varepsilon_n} + C l_n \\ &\le C\frac{l_n}{\varepsilon_n} \end{aligned} \tag{3.3}$$

where we use the definition of ε_n, condition (1), and $\varepsilon_n \to 0$.

On the other hand,

$$\begin{aligned} v_n &\geq \sum_{k>n} l_k \\ &= \frac{l_n}{\varepsilon_n}, \end{aligned}$$

which implies

$$\frac{v_n}{l_n} \geq \frac{1}{\varepsilon_n} \to +\infty.$$

Moreover,

$$\sup_n \frac{v_n}{l_n} = +\infty$$

produces, by (3.2),

$$\sup_n \frac{Q_n}{l_n} = +\infty.$$

Finally, Equation (3.3) yields

$$\frac{Q_n}{l_n} \leq \frac{C}{\varepsilon_n} \Rightarrow$$
$$\sum_n \frac{l_n}{Q_n} \geq C \sum_n \varepsilon_n.$$

By Lemma 3.11, we conclude that

$$\sum_n \frac{l_n}{Q_n} = +\infty.$$

The cases when conditions (2) or (3) are satisfied follow similarly. □

Example 3.13. $I_n = [\frac{1}{n}, \frac{1}{n} + \frac{1}{n^2})$ satisfy the hypothesis of Lemma 3.12 with condition (1).

The next theorem answers the question of optimality of the correct factor for a large class of differentiation operators.

Theorem 3.14. *Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint intervals of real numbers denoted by $I_n = [v_n, v_n + l_n)$ with l_n, v_n tending to zero. Suppose that*

$$\inf_n \frac{1}{\widetilde{Q}_n} \sum_{k>n} l_k > 0.$$

Let $\{c_n\}$ be a sequence of real numbers such that $\lim_n c_n = +\infty$. Then, for every $C > 0$ there is $\lambda > 0$ and $f \in L_1^+(\mathbb{R})$ so that

$$m\left\{\sup_n N_{(I_n, \frac{\widetilde{Q}_n}{c_n})} f > \lambda\right\} > \frac{C}{\lambda}\|f\|_1.$$

Proof. Fix $C > 0$. Let

$$\alpha_n = \frac{c_n}{\widetilde{Q}_n} \sum_{k>n} l_k.$$

Then

$$\alpha_n \geq c_n \inf_n \frac{1}{\widetilde{Q}_n} \sum_{k>n} l_k$$

and since $\lim_n c_n = +\infty$ we conclude that $\lim_n \alpha_n = +\infty$. Consequently, there exists $n_0 \in \mathbb{N}$ so that for every $n \geq n_0$ we have $\alpha_n > C$.

Choose

$$\lambda \in \left(\frac{c_{n_0}}{\widetilde{Q}_{n_0}}, \frac{c_{n_0+1}}{\widetilde{Q}_{n_0+1}} \right).$$

For $\mu = \delta_0$, and for all $n > n_0$ and all $x \in U_n = -I_n$,

$$\int_{I_n} d\mu(x+t) = 1.$$

Let

$$k_{n_0} = \max\left\{ n \geq n_0 : \frac{c_n}{\widetilde{Q}_n} \leq \lambda \text{ and } \frac{c_m}{\widetilde{Q}_m} \text{ for every } m > n \right\}.$$

Then for all $n > k_{n_0}$ we have

$$\frac{c_n}{\widetilde{Q}_n} \int_{I_n} d\mu(x+t) > \lambda$$

or

$$\left\{ x \in \mathbb{R} : N_{(I_n, \frac{\tilde{Q}_n}{c_n})}\mu(x) > \lambda \right\} \supseteq U_n.$$

Moreover, since U_n are disjoint,

$$\begin{aligned} m\left\{ \sup_n N_{(I_n, \frac{\tilde{Q}_n}{c_n})}\mu(x) > \lambda \right\} &\geq m\left\{ \sup_{n > k_{n_0}} N_{(I_n, \frac{\tilde{Q}_n}{c_n})}\mu(x) > \lambda \right\} \\ &\geq m\left(\bigcup_{n > k_{n_0}} U_n \right) \\ &= \sum_{n > k_{n_0}} l_n \\ &= \alpha_{k_{n_0}} \cdot \frac{\widetilde{Q}_{k_{n_0}}}{c_{k_{n_0}}} \\ &> C \cdot \frac{1}{\lambda} \\ &= \frac{C}{\lambda} \|\mu\|_1. \end{aligned}$$

Since $\sum_n l_n < +\infty$, a significant portion of this sum occurs in the first N terms, where N is sufficiently large. Therefore, for fixed $C > 0$ there are $\lambda > 0$ and N sufficiently large positive integer so that

$$m\left\{ x \in \mathbb{R} : \sup_{n \leq N} N_{(I_n, \frac{\tilde{Q}_n}{c_n})}\mu(x) > \lambda \right\} > \frac{C}{\lambda} \|\mu\|_1.$$

Consider an integrable function ϕ on $\mathbb{R}$ such that $\int \phi = 1$, and for $t > 0$ define $\phi_t(x) = 1/t\phi(x/t)$. Then it can be shown that

$$\lim_{t \to 0} \phi_t = \mu.$$

It follows that for every $\varepsilon > 0$ there is $t_0 > 0$ such that for all $t \geq t_0$ and $n \leq N$,

$$\int_{I_n} \phi_t(x+u)\, du > \int_{I_n} d\mu(x+u) - \frac{\varepsilon}{C}$$

where

$$C = \max_{n \leq N} \frac{c_n}{\widetilde{Q}_n}.$$

Therefore,

$$\sup_{n \leq N} N_{(I_n, \frac{c_n}{\widetilde{Q}_n})} \mu(x) > \lambda$$

implies

$$\sup_{n \leq N} N_{(I_n, \frac{c_n}{\widetilde{Q}_n})} \phi_t(x) > \lambda - \varepsilon.$$

Hence, for every $C > 0$ there exist $\lambda > 0$, $N \in \mathbb{N}$ and $t > 0$ such that

$$m\left\{x \in \mathbb{R} : \sup_{n \leq N} N_{(I_n, \frac{\widetilde{Q}_n}{c_n})} \phi_t(x) > \lambda - \varepsilon\right\} > \frac{C}{\lambda} \|\phi_t\|_1.$$

Taking $\varepsilon \to 0$ finishes the proof. □

Example 3.15. Lemma 3.12 with conditions (1) or (2) implies that

$$Q_n \leq v_n + 2l_n \leq \frac{C}{\varepsilon_n} l_n,$$

therefore

$$\frac{1}{Q_n} \sum_{k>n} l_k \geq C \frac{\varepsilon_n}{l_n} \sum_{k>n} l_k = C > 0,$$

which gives

$$\inf_n \frac{1}{Q_n} \sum_{k>n} l_k > 0.$$

For such sequences of intervals, $\widetilde{Q}_n$ are indeed the optimal factors for a maximal inequality to hold for $N_{(I_n, \widetilde{Q}_n)}$.

Remark 3.16. Since it is not clear whether the condition in Theorem 3.14 is also necessary for the modified correct factor to be optimal, further investigation is required.

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