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Lifting Möbius Groups

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Abstract. We construct explicit examples of subgroups of $\mathrm{PSL}(2,\mathbb{C})$, $\mathrm{PSU}(2)$ and $\mathrm{PSL}(2,\mathbb{R})$ with no elements of order 2 which cannot be lifted up to $\mathrm{SL}(2,\mathbb{C})$.

Contents

1. Introduction

Given a surjective homomorphism of groups $\pi : G \to H$, we say that the subgroup $K \leq H$ has a lift (to G via π) if there exists $\overline{K} \leq G$ with π an isomorphism from $\overline{K}$ to K. In other words, $\pi^{-1}(K)$ is a split extension (equivalently a semidirect product) of $\pi^{-1}(K) \cap \ker \pi$. Of course in general lifts are not unique and may not even exist but one case where they always do is when K is a free group. The universal property of free groups means that for any π, G and H, a function $f : S \to G$ sending each element of a free generating set S for K to any element above it can be extended to a homomorphism of K, and this will be injective.

Throughout this paper we consider the case of $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ with $\ker \pi = \{\pm I\}$ which is an important and well studied example, fundamental to the theory of Kleinian groups and hyperbolic geometry because $\mathrm{PSL}(2,\mathbb{C})$ is the group of all Möbius transformations and $\mathrm{SL}(2,\mathbb{C})$ consists of matrix representatives of those transformations. The first thing that can be said is that if K is a Möbius group, namely a subgroup of $\mathrm{PSL}(2,\mathbb{C})$, with elements of order 2 then no lifts of K can exist; this is most easily seen by noting that the only element in $\mathrm{SL}(2,\mathbb{C})$ of order 2 is $-I$. As for cases where lifts exist, it is shown in [2] using covering space theory that a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with no elements of order 2 can be lifted. Partial results of this nature have a long history; a thorough discussion is given in [7] where it is shown that, especially in the case of the existence of a lift

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of a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ without elements of order 2, the same results have often been rediscovered and republished.

However, surely there is a case left over which presents us with a question that is crying out to be asked, namely: if Γ is an indiscrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with no elements of order 2 then does Γ possess a lift? Although it is easy to find examples of specific indiscrete Möbius groups with lifts (merely take a indiscrete free group, such as a cyclic group generated by an elliptic transformation of infinite order), in this paper we show that the answer to our question is no in general. Moreover we show how to create explicit examples of Möbius groups with no lifts and no elements of order 2, or even with no elliptic elements, and we also find subgroups of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{PSU}(2)$ (which is isomorphic to $\mathrm{SO}(3)$) with no lifts and no elements of order 2, and these can be torsion free.

The proof proceeds by considering a class of subgroups of $\mathrm{PSL}(2,\mathbb{C})$ which are generated by four elements. It is easy to see that no member of this class possesses a lift, but this is established without mention of elements of order 2. The hope is then that something in the class has no elements of order 2, but in order to prove this we first have to show that no word in the four generators can be of order 2 for all these subgroups, and then we use transcendental numbers, and the idea of algebraic independence, to construct examples of specific groups in this class without elements of order 2.

In the last section we review the difference between lifts of subgroups and lifts of representations, which puts the results in Section 2 in a wider setting.

2. Results

The fact that given $\pm A, \pm B \in \mathrm{PSL}(2,\mathbb{C})$ we are able to regard the commutator $ABA^{-1}B^{-1}$ as a well-defined element of $\mathrm{SL}(2,\mathbb{C})$ can be useful for seeing that a Möbius group G has no lifts. For instance in [8] (Example 3) it is pointed out that if we take any subgroup $\Gamma = \langle A, B, C, D\rangle$ of $\mathrm{SL}(2,\mathbb{C})$ where the relation $ABA^{-1}B^{-1}CDC^{-1}D^{-1} = -I$ holds then the projection $\pi\Gamma$ (which will be a homomorphic image of the closed surface group of genus 2) cannot be lifted, because whatever matrices $\pm A, \pm B$ we choose to lie above πA and πB, the commutator of $\pm A$ and $\pm B$ will remain the same. This is also true for C and D, so $\pi(ABA^{-1}B^{-1})$ and $\pi(DCD^{-1}C^{-1})$ are equal but any matrices lying above them cannot be. It is not a priori evident that there must be elements of order 2 in $\pi\Gamma$, so we would hope that examples of this type could be used to construct a Möbius group with no elements of order 2.

Let

$$X = \{(A,B,C,D) \in \mathrm{SL}(2,\mathbb{C})^4 : ABA^{-1}B^{-1}CDC^{-1}D^{-1} = -I\}$$

so that, from above, the image under π of any group generated by an element of X cannot possibly have a lift. For each reduced word w in the free group on four elements we wish to show that there exists $(A,B,C,D) \in X$ with $w(\pi A, \pi B, \pi C, \pi D)$ not an element of order 2. We might think that this would be straightforward by evaluating $w(A,B,C,D)$ using particular matrices such as the identity (for instance if our relation above had I rather than $-I$ on the right-hand side then putting all four matrices equal to I would do it immediately), but the problem is that if one of A,B,C,D is set equal to the identity then the defining relation of X requires that we need two elements of $\mathrm{SL}(2,\mathbb{C})$ with commutator $-I$. There is only one pair of matrices with this property (up to simultaneous conjugation in $\mathrm{SL}(2,\mathbb{C})$), generating a quaternionic group, and they project down to two rotations of order 2 whose product is also of order 2. It can be assumed that the axes of these three Möbius transformations pass through the centre of the Riemann sphere and are mutually perpendicular. We now exploit this information while proceeding carefully in order to extract a contradiction from the assumption that there is a word w with $w(\pi A, \pi B, \pi C, \pi D)$ of order 2 for all $(A, B, C, D) \in X$.

Lemma 1. *For all words w in the free group on four elements, there exists an element $(A, B, C, D) \in X$ with $w(\pi A, \pi B, \pi C, \pi D)$ not of order* 2.

Proof. We proceed by considering the parity of the exponent sums a, b, c, d of A, B, C, D in the word w which is claimed always to be of order two. We first take matrices A, B, C, D with corresponding Möbius transformations $\alpha, \beta, \gamma, \delta$ such that A, B generate a quaternionic group, with γ an element of infinite order and two fixed points equal to those of $\alpha\beta$, and $D = I$. Then we have a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with presentation

$$\langle \alpha, \beta, \gamma, \delta : \alpha^2 = \beta^2 = \delta = \mathrm{id},\ \alpha\beta = \beta\alpha,\ \alpha\gamma\alpha = \beta\gamma\beta = \gamma^{-1} \rangle$$

and we now use the relations to put $w(\alpha, \beta, \gamma, \delta)$ into one of the normal forms γ^k, $\alpha\gamma^k$, $\beta\gamma^k$, $\alpha\beta\gamma^k$ for $k \in \mathbb{Z}$, noting that this does not change the parity of a, b or c. But γ^k is not of order 2, and nor is $\alpha\beta\gamma^k$ unless $k = 0$. Thus if c is odd then exactly one of a and b is odd, and if c is even then a and b are not both even. Repeating the argument with C and D generating a quaternionic group, taking one of A, B whose exponent sum has odd parity (there is at least one) to be of infinite order with the same fixed points as $\gamma\delta$ and the other the identity, we conclude that one of c, d is odd (say c) and one is even. But going back to the first case, we now have one of a, b is odd (say a) and one is even.

Finally we take the case where A and B again generate a quaternionic group, with C chosen such that γ is an element of infinite order sharing the same fixed points as α, and $D = I$. We now get the group of Möbius transformations

$$\langle \alpha, \beta, \gamma, \delta : \alpha^2 = \beta^2 = \delta = \mathrm{id},\ \alpha\beta = \beta\alpha,\ \alpha\gamma\alpha = \gamma,\ \beta\gamma\beta = \gamma^{-1} \rangle$$

again with normal forms γ^k, $\alpha\gamma^k$, $\beta\gamma^k$, $\alpha\beta\gamma^k$, and as the parity of a, b, c is preserved throughout with a, c odd and b even, $w(\alpha, \beta, \gamma, \delta)$ must be $\alpha\gamma^k$. This has infinite order unless $k = 0$, which is not the case as c is odd. $\square$

It is the case that a Möbius transformation α has order two if and only if the trace of either of the matrices above α is zero. If we knew that X was connected then a Baire category argument along with Lemma 1 would establish our result but it would not be constructive, unlike what follows.

Theorem 2. *There exist subgroups of* $\mathrm{PSL}(2,\mathbb{C})$ *which contain no elements of order two but which have no lifts to* $\mathrm{SL}(2,\mathbb{C})$.

Proof. Note that in the proof of Lemma 1 we are free to take a conjugate of any (A, B, C, D) which is used and that throughout the proof we never needed πA to have just one fixed point; either we can assume it has two or $A = I$, so that by normalising we can assume that A fixes 0 and infinity. Therefore we define X_0 to be the subset of X where A is a diagonal matrix. Letting

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad C = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \quad D = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$$

where $\lambda \neq 0$ and $ad - bc, \alpha_i\delta_i - \beta_i\gamma_i$ are equal to 1 for $i = 1$ and 2, we would like to find what relations hold between the various entries to ensure that (A, B, C, D) is in X. Let us set

$$CDC^{-1}D^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix};$$

we will not need to know the exact expressions for $\alpha, \beta, \gamma, \delta$, just that they are polynomials in $\alpha_i, \beta_i, \gamma_i, \delta_i$ for $i = 1, 2$. By replacing ad with $1 + bc$ and solving for λ^2, then c, a and d, we find that setting

$$(1) \qquad a = -\frac{\beta(1+\alpha)}{b(\alpha+\delta+2)}, \quad c = -\frac{(\alpha+1)(\delta+1)}{b(\alpha+\delta+2)}, \quad d = \frac{b(\alpha\delta-1)}{\beta(1+\alpha)}, \quad \lambda^2 = -\frac{1+\delta}{1+\alpha}$$

for $\alpha+\delta \neq -2$; $b, \beta \neq 0$ and $\alpha, \delta \neq -1$, we obtain matrices such that $(A, B, C, D) \in X$ which can be easily checked. We define $Y \subseteq X_0$ to be all (A, B, C, D) of this form with β_i nonzero for $i = 1, 2$ so that we can replace γ_i with $(\alpha_i\delta_i - 1)/\beta_i$. If $(A, B, C, D) \in Y$ then it is the case that all entries in A, B, C, D, and hence all entries in all elements of $\Gamma = \langle A, B, C, D\rangle$, are rational functions of the eight variables $\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2, b, \lambda$ subject only to the above inequalities and $\lambda^2 = -(1+\delta)/(1+\alpha)$. We then take the first seven variables to be complex numbers that are algebraically independent (over $\mathbb{Q}$), that is

$$p(\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2, b) \neq 0$$

for any polynomial $p \in \mathbb{Q}[t_1, \dots, t_7]$ that is not identically zero. As a result we are guaranteed that none of the problem equalities will hold.

Now let us suppose that there is a w with the rational function $\operatorname{tr} w(A, B, C, D) = 0$ when these particular seven values are inserted (and either of the two possible values for λ). We must then have a polynomial p in our eight variables equal to zero. Writing this as $p(\lambda) = p_0(\lambda^2) + \lambda p_1(\lambda^2)$, we have $p_0^2 - \lambda^2 p_1^2 = 0$ and as we can replace the λ^2 in terms of our seven algebraically independent variables, this must be identically zero. This means that for all $(A, B, C, D) \in Y$, either $\operatorname{tr} w(A, B, C, D) = 0$ or $\operatorname{tr} w(-A, B, C, D) = 0$ where $-A$ occurs by replacing λ with $-\lambda$, but $\operatorname{tr} w(-A, B, C, D) = \pm w(A, B, C, D)$ so the second case implies the first anyway. We would now like to apply Lemma 1 to Y in order to show that no word w can have trace identically zero throughout all of Y but a little care is needed because it could be that for some of the Möbius transformations used in the proof of the Lemma the problem equalities will be satisfied and so the matrix entries will not be defined. We get round this by noting that we can certainly apply the proof of Lemma 1 to X_0 instead of X without change. Although in the course of this proof we will take points $p = (A, B, C, D)$ which lie in $X_0 - Y$, we can always find a sequence p_n in Y with $p_n \to p$ by ensuring that none of the problem equalities hold for p_n, so that the corresponding matrices will be of the form in (1). Thus we can now apply Lemma 1 to the set Y, and if we have to use $(A, B, C, D) \in X_0 - Y$ with $\operatorname{tr} w(A, B, C, D) = t \neq 0$, then taking (A_n, B_n, C_n, D_n) in Y tending to (A, B, C, D), we will have $\operatorname{tr} w(A_n, B_n, C_n, D_n) \to t$ so that for some point in Y the trace of w is not zero. This is a contradiction and so inserting algebraically independent numbers into the matrices whose form is given in (1) gives us explicit Möbius groups with no lifts but no elements of order 2. $\square$

Note that all these groups with no lifts and no elements of order 2 must be indiscrete by the result in [2], even though a priori we have no reason to suppose this. Moreover there certainly exist discrete groups in X but their projections must contain elements of order 2.

It might be asked how one knows the existence of and actually finds $z_1, \ldots, z_n \in \mathbb{C}$ which are algebraically independent. We can do this by using a famous theorem of Lindemann [1] from 1882, that if $x_1, \ldots, x_n$ are distinct algebraic numbers and $y_1, \ldots, y_n$ are any algebraic numbers that are not all 0 then

$$\sum_{i=1}^{n} y_i e^{x_i} \neq 0.$$

In particular, $\{x_i\}$ being algebraic numbers linearly independent over $\mathbb{Q}$ implies that e^{x_i} are algebraically independent over $\mathbb{Q}$. Also n-tuples of $\mathbb{C}$ that are algebraically independent are dense in $\mathbb{C}^n$. This is a consequence of the following parallel of the Steinitz exchange theorem (which can be found in [3] or most introductory textbooks on Galois theory). If $L : K$ is a field extension, C a subset of L which is algebraically independent over K, and A a subset of L such that $L : K(A)$ is algebraic then the cardinality of C is no bigger than that of A and there exists a set D with the same cardinality of A such that $C \subseteq D \subseteq A \cup C$ and $L : K(D)$ is algebraic. Therefore if $C = \{z_1, \ldots, z_n\}$ is a given algebraically independent set over $K = \mathbb{Q}$ and we let $L = \mathbb{Q}(z_1, \ldots, z_n, \alpha_1, \ldots, \alpha_n)$ for any nonzero algebraic numbers α_i and $A = \{\alpha_1 z_1, \ldots, \alpha_n z_n\}$, then if A is not algebraically independent we could remove elements until it was while still keeping $L : K(A)$ algebraic, which contradicts the above result.

Corollary 3. *There exist subgroups of* $\mathrm{PSL}(2, \mathbb{R})$ *with no lift to* $\mathrm{SL}(2, \mathbb{R})$ *and no elements of order* 2*.*

Proof. Merely take seven algebraically independent real numbers and form the above matrices; one has only to choose entries for C and D so that $\lambda^2 = -(1 + \delta)/(1 + \alpha)$ is positive, and this property can be preserved by moving to six algebraically independent values that are suitably close. $\square$

As a consequence of the famous inequality of Jørgensen [5], he shows in [6] that if every elliptic element of a non-elementary subgroup of $\mathrm{PSL}(2, \mathbb{R})$ is of finite order then the subgroup is discrete. However this does not hold for non-elementary subgroups of $\mathrm{PSL}(2, \mathbb{C})$, as was shown by Greenberg in 1962 in [4] where an indiscrete group which is free on a countably infinite number of generators is constructed with only the identity having a real trace. In our case, as all such subgroups of $\mathrm{PSL}(2, \mathbb{R})$ that we create in Corollary 3 are indiscrete and non-elementary, they must contain an elliptic element of infinite order even if it is not immediately apparent which one.

Corollary 4. *There exist subgroups of*

$$\mathrm{PSU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, a\bar{a} + b\bar{b} = 1 \right\} / \pm I$$

which contain no elements of order two but which have no lifts to $\mathrm{SU}(2)$*.*

Proof. We require seven algebraically independent complex numbers $\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2, b$ such that all four matrices created will lie in SU(2). To see this can be done, choose x_i, y_i, u_i for $i = 1, 2$ to be six algebraically independent reals such that $x_i^2 + y_i^2 + u_i^2 < 1$, let v_i be either square root of $1 - x_i^2 - y_i^2 - u_i^2$ and define $\alpha_i = x_i + iy_i, \beta_i = u_i + iv_i$, $\delta_i = x_i - iy_i$. This implies that C, D are now in SU(2) and thus so is $-DCD^{-1}C^{-1}$. Note that $(\alpha+1)(\delta+1)/(\alpha+\delta+2) = \mu$ is real and between 0 and 1, because $|\alpha| < 1$ and $\delta = \overline{\alpha}$. We now choose $b_1 \in \mathbb{R}$ so that $b_1^2 < \mu$ and with x_i, y_i, u_i, b_1 algebraically independent. Setting b_2 to be a square root of $\mu - b_1^2$ and $b = b_1 + ib_2$, we can now solve for a, c, d and λ to form the matrices A and B which we can check by a straightforward calculation will lie in SU(2). To see that our seven complex numbers are again algebraically independent, we form the extension field k by adjoining all of them to the rationals, whence we will have $x_i, iy_i \in k$, as well as u_i (because $\beta_i, \overline{\beta_i}$ are in k) and the entries of $CDC^{-1}D^{-1}$. This will imply that $|b|^2$ $(= -bc)$, thus $\bar{b}$ and then b_1, lie in k too.

Thus k contains $l = Q(x_i, iy_i, u_i, b_1)$ and as $\mathbb{Q}(x_i, iy_i, u_i, b_1, i) = \mathbb{Q}(x_i, y_i, u_i, b_1, i)$ is an algebraic extension of both l and the field $\mathbb{Q}(x_i, y_i, u_i, b_1)$, we have algebraic independence. □

We finish by displaying a stronger property of our examples already constructed, which follows by the same techniques.

Theorem 5. *There exist subgroups of* PSL(2, ℝ) (*respectively* PSU(2)) *with no lifts to* SL(2, ℝ) (*respectively* SU(2)) *which are torsion free. There also exist subgroups of* PSL(2, ℂ) *with no lifts to* SL(2, ℂ) *which contain no elliptic elements.*

Proof. For SL(2, ℝ) and SU(2), take any of the examples that we have constructed above. Suppose in such a group we have a matrix of finite order, then it has a trace that is an algebraic number $\alpha \in (-2, 2)$. Thus we have a word w on four letters A, B, C, D with

$$\operatorname{tr} w(A, B, C, D) = r(\alpha_1, \beta_1, \delta_1, \alpha_2, \beta_2, \delta_2, b, \lambda) = \alpha$$

where r is a rational function in these variables, as before. Thus taking an irreducible $f \in \mathbb{Z}[X]$ with $f(\alpha) = 0$, we have $f(r(\lambda)) = 0$ (suppressing the other variables temporarily). We can write $f(r(\lambda)) = \gamma_0(\lambda^2) + \lambda\gamma_1(\lambda^2)$ where γ_0 and γ_1 are also rational functions, and note that

$$r(-\lambda) = \operatorname{tr} w(-A, B, C, D) = \pm \operatorname{tr} w(A, B, C, D).$$

Then $f(r(\lambda))f(r(-\lambda))$ is (on substituting for λ^2) a rational function of seven algebraically independent numbers, thus is identically zero. This means that for all values of our variables either $\operatorname{tr} w(A, B, C, D)$ or $\operatorname{tr} w(-A, B, C, D)$ is a root of f. But making A, B generate a quaternionic group and $C = D = I$, we obtain a group $\langle A, B, C, D\rangle$ whose only traces are equal to 0 or ± 2. Thus these are the only possible values for α, but we already know that our groups have no elements with trace 0.

To obtain an example in PSL(2, ℂ) with no elliptics, we take 14 algebraically independent reals which make up the real and imaginary parts of our seven complex numbers (which will then also be algebraically independent). If we had a word w with real trace in our group then splitting $\operatorname{tr} w(A, B, C, D)\operatorname{tr} w(-A, B, C, D) = u+iv$ into real and imaginary parts, we have that because v is zero for our particular chosen values, it must be the case that v is identically zero. But $\operatorname{tr}(w)$ is an analytic function of these variables and so must be real and constant. So using the same group as above, our traces can only be 0 and ± 2, with 0 already eliminated. $\square$

As a byproduct we obtain examples of non-elementary indiscrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ with no elliptic elements, which provide a finitely generated version of Greenberg's counterexample. We can also use algebraic independence to construct finitely generated free groups having no elliptic elements which are non-elementary and indiscrete. Such groups will possess lifts whereas our examples in Theorem 5 are non-free groups without lifts, thus providing an indirect proof of indiscreteness.

3. Lifts of representations

We finish by discussing the difference between lifting of subgroups and lifting of representations. If $\pi : \overline{G} \to G$ is a surjective homomorphism of groups and $\mathrm{Hom}(\Gamma, G)$ is the set of all representations from the group Γ to G then we say that the representation $\rho \in \mathrm{Hom}(\Gamma, G)$ has a lift if there exists $\bar{\rho} \in \mathrm{Hom}(\Gamma, \overline{G})$ with $\rho = \pi\bar{\rho}$. This does not imply in general that the subgroup $\rho(\Gamma)$ of G has a lift to $\overline{G}$ via π but we do obtain a lift $\bar{\rho}(\Gamma)$ if ρ is faithful, for then π is injective from $\bar{\rho}(\Gamma)$ to $\rho(\Gamma)$. However, taking our usual $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ and $\Gamma = \mathbb{Z}_{4k}$ to be a cyclic group, we can lift the representation $\rho : \mathbb{Z}_{4k} \to \mathbb{Z}_{2k} \leq \mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$ but $\rho(\Gamma)$ has elements of order 2. The "basic observation" equivalent to the fact that a subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with elements of order 2 does not lift is probably that a representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ does not lift if there exists $\gamma \in \Gamma$ of order 2 which is not in $\ker \rho$, for then $\pi\bar{\rho}(\gamma)$ being of order 2 would imply that $\bar{\rho}(\gamma)$ was of order 4.

Conversely though, if ρ is any representation in $\mathrm{Hom}(\Gamma, G)$ with the subgroup $\rho(\Gamma)$ having a lift L to $\overline{G}$ then we can obtain a lift $\bar{\rho}$ of ρ by setting $\bar{\rho} = \pi^{-1}\rho$ from Γ to $\overline{G}$, so that $\bar{\rho}(\Gamma) = L$.

In specialising to our case of $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$, we have that if $\bar{\rho}$ is any lift of the representation $\rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ then all lifts above ρ are of the form $\epsilon \cdot \bar{\rho}$ (multiplication) for $\epsilon : \Gamma \to \ker \pi$ any homomorphism. But assuming that $\rho(\Gamma)$ can be lifted to $L = \bar{\rho}(\Gamma)$, the subgroup $\epsilon \cdot \bar{\rho}(\Gamma)$ of $\overline{G}$ is a lift of $\rho(\Gamma)$ if and only if $\ker \rho \leq \ker \epsilon$, as only then will π be injective on $\epsilon \cdot \bar{\rho}(\Gamma)$. This process provides new lifts of the subgroup $\rho(\Gamma)$ and all such lifts can be obtained in this way.

The result in [2] that a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with no elements of order 2 has a lift is set in the general framework of Γ being an abstract group, G a connected topological group and $\overline{G}$ a covering space of G with a compatible group structure, in which case $\mathrm{Hom}(\Gamma, G)$ has the subspace topology induced by the topology on the relevant product of G. It is also shown under these circumstances that the representation ρ lifts to $\overline{G}$ if and only if every representation in the path component of ρ in $\mathrm{Hom}(\Gamma, G)$ lifts. For examples using $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$, if Γ has torsion then $\mathrm{Hom}(\Gamma, G)$ will not be connected if there exists a faithful representation of Γ in G because $\rho \mapsto |\mathrm{tr}(\rho(\gamma))|$ is continuous with discrete image (including 2 and another point) for $\gamma (\neq 1)$ of finite order. But if Γ is the free group F_k then $\mathrm{Hom}(F_k, \mathrm{PSL}(2,\mathbb{C})) = \mathrm{PSL}(2,\mathbb{C})^k$ which is (path) connected, so that every $\mathrm{PSL}(2,\mathbb{C})$-representation of a finitely generated free group lifts to $\mathrm{SL}(2,\mathbb{C})$ (which we effectively saw earlier by sending a free generating set $\{\gamma_i\}$ of F_k to matrices above $\rho(\gamma_i)$ and extending to a homomorphism) because the trivial homomorphism always lifts. However, clearly there are plenty of (unfaithful) $\rho \in \mathrm{PSL}(2,\mathbb{C})^k$ with the subgroup $\rho(\Gamma)$ not possessing a lift.

We can now see that the space X in Section 2 is really obtained by taking $Y = \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ for Γ the fundamental group of the closed surface of genus 2 with presentation

$$\langle \alpha, \beta, \gamma, \delta : \alpha\beta\alpha^{-1}\beta^{-1}\gamma\delta\gamma^{-1}\delta^{-1} = 1\rangle$$

and noting that the function $f : Y \to \pm I$ that sends ρ to $ABA^{-1}B^{-1}CDC^{-1}D^{-1}$ (where A, B, C, D are either of the matrices above $\rho(\alpha), \rho(\beta), \rho(\gamma), \rho(\delta)$) is well-defined because every generator has even exponent sum in the relation. Therefore we have been considering those ρ with $f(\rho) = -I$, so that none of these representations have a lift and hence no subgroup $\rho(\Gamma)$ can be lifted. We can try to generalise this argument for Γ a group with a given finite presentation

$$\langle \gamma_1, \dots, \gamma_k : r_1 = \cdots = r_l = I\rangle$$

by taking the (abelianised) $l \times k$ presentation matrix over $\mathbb{Z}_2$, and if the columns do not span $\mathbb{Z}_2^l$ (such as if $l > k$) then a suitable change of some of the appearances of I to $-I$ in the presentation for Γ provides us as before with representations into $\mathrm{PSL}(2,\mathbb{C})$ with no lifts to $\mathrm{SL}(2,\mathbb{C})$, but for this to work we require that there do actually exist matrices in $\mathrm{SL}(2,\mathbb{C})$ satisfying the new equations. Luckily in our case there are, so that we then had plenty of non-liftable representations of Γ available in our search for ones with no elements of order 2 in their image.

References

[1] A. Baker, *Transcendental Number Theory*, Cambridge University Press, London-New York, 1975, MR 54 #10163, Zbl 0297.10013.

[2] M. Culler, *Lifting representations to covering groups*, Adv. in Math. **59** (1986), 64–70, MR 87g:22009, Zbl 0582.57001.

[3] D. J. H. Garling, *A Course in Galois Theory*, Cambridge University Press, Cambridge-New York, 1986, MR 88d:12007, Zbl 0608.12025.

[4] L. Greenberg, *Discrete subgroups of the Lorentz group*, Math. Scand. **10** (1962), 85–107, MR 25 #5128, Zbl 0118.03902.

[5] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 739–749, MR 55 #658.

[6] T. Jørgensen, *A note on subgroups of* $\mathrm{SL}(2,\mathbb{C})$, Oxford Quart. J. Maths **28** (1977) 209–211, MR 56 #3186.

[7] I. Kra, *On Lifting Kleinian Groups to* $\mathrm{SL}(2,\mathbb{C})$, in 'Differential Geometry and Complex Analysis' (I. Chavel, H. M. Farkas, Eds.), Springer-Verlag, Berlin, 1985, 181–193, MR 86h:30078, Zbl 0571.30037.

[8] M. Seppälä and T. Sorvali, *Liftings of Möbius groups to matrix groups*, Math. Scand. **72** (1993), 223–234, MR 95b:20071, Zbl 0797.30034.

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