Erratum to "On the Chung-Diaconis-Graham random process"

Martin Hildebrand
Department of Mathematics and Statistics
University at Albany
State University of New York
Albany, NY 12222

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Abstract

This note corrects a flawed statement in the paper "On the Chung-Diaconis-Graham random process".

1 The Flaws and Corrections

On p. 352 of [1], the claim that

$$\lim_{t\to\infty} \frac{G\left(\frac{2^{t-1}-2^{t-j-1}}{p}\right) G\left(\frac{2^{t-1}-2^{j-1}}{p}\right) G\left(\frac{2^{t-2}-2^{j-2}}{p}\right)}{G\left(\frac{2^{t-1}}{p}\right) G\left(\frac{2^{t-2}}{p}\right) G\left(\frac{2^{j-2}}{p}\right)} = \frac{G(1/2)}{G(0)} < 1$$

can be shown has flaws. If j is constant, the limit is $G(0.5*(1-2^{-j}))/G(0)$. If $b=0$, then $G(1/2)/G(0)=1$.

First we shall consider the case where $b \neq 0$. Let $M = \sup_{x\in[1/4,1/2]} G(x)$. Then, we can show the fact that $M<1$. To see this fact, we will let $H(x) = |ae^{2\pi ix}+b|$ in the case $a\neq 0$. $H(x)$ is decreasing on $[0,1/2]$, and $a+b-H(1/4)>0$. For $x\in[1/4,1/2]$, we have $G(x) \leq H(x)+c = 1-(a+b-H(x)) \leq 1-(a+b-H(1/4))$. If $a=0$, then $c>0$ and a similar argument using $|ce^{-2\pi ix}+b|$ applies. Since $G(0)=1$, we conclude that

$$\limsup_{t\to\infty} \frac{G\left(\frac{2^{t-1}-2^{t-j-1}}{p}\right) G\left(\frac{2^{t-1}-2^{j-1}}{p}\right) G\left(\frac{2^{t-2}-2^{j-2}}{p}\right)}{G\left(\frac{2^{t-1}}{p}\right) G\left(\frac{2^{t-2}}{p}\right) G\left(\frac{2^{j-2}}{p}\right)} \leq M/G(0) < 1,$$

and the rest of the argument is unaffected.

If $b=0$, Case 2 of Theorem 1 can be proved by considering the following random processes on the integers mod p:

1. $X_0 = 0$ and $X_{n+1} = 2X_n + b_n \pmod p$ where $P(b_n = 1) = a$ and $P(b_n = -1) = 1 - a$

2. $Y_0 = 0$ and $Y_{n+1} = 2Y_n + d_n \pmod p$ where $P(d_n = 2) = a$ and $P(d_n = 0) = 1 - a$

3. $Z_0 = 0$ and $Z_{n+1} = 2Z_n + e_n \pmod p$ where $P(e_n = 1) = a$ and $P(e_n = 0) = 1 - a$

If $P_n(s) = Pr(X_n = s)$, $Q_n(s) = Pr(Y_n = s)$, and $R_n(s) = Pr(Z_n = s)$, then $\|P_n - U\| = \|Q_n - U\| = \|R_n - U\|$. To see this, we can let $d_n = 2e_n$ and $b_n = d_n - 1$ so that $Y_n = 2Z_n$ and $X_n = Y_n - \sum_{j=0}^{n-1} 2^j$ for $n \geq 1$. To conclude the case where $b = 0$, use the argument with $b \neq 0$ on the third random process (provided that $a \neq 1/2$ so that we are in Case 2).

2 Acknowledgment

The author thanks Ravi Montenegro for pointing out a flaw in the original argument.

References

[1] Hildebrand, M. On the Chung-Diaconis-Graham random process. *Elect. Comm. in Probab.* **11** (2006), 347-356.