**Preliminary Exam**

**Complex Analysis**

**August 1999**

1. Find an analytic function $f(z)$ whose real part is $(z = x + iy)$.

$$\text{Re } f(z) = xy - 10 \ .$$

   Does such a function exist? Justify your answer.

2. Find the general form of an entire function $f(z)$ satisfying

$$|f(z)| \leq A + B|z|^{3/2}, \text{ where } A \text{ and } B \text{ are constants .}$$

3. Find the general form of a function $f(z)$ which is analytic inside the ellipse $D$ $(z = x + iy)$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \ ,$$

   continuous in $\overline{D}$, and

$$\text{Im } f(z) = -5 \quad (z \in \partial\mathbf{D})$$

4. Find a conformal mapping from $\mathbf{C} \backslash \{[0, +\infty)\}$ to the unit disk.

5. 1. Prove that for any polynomial $p$ and any $a \in \Delta$

$$p(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{p(e^{i\theta})}{1 - e^{-i\theta}a} d\theta$$

   2. Deduce from 5.1 that

$$|p(a)| \leq \left[ \frac{1}{(1 - |a|^2)} \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta \right]^{1/2}$$

6. Let $f$ be analytic in the unit disk and map the unit disk into itself given $f(1/2) = 0$. Prove that $|f'(1/2)| \leq \dfrac{4}{3}$.

7. Let

$$f(z) = \frac{1}{z} \cdot \frac{1-2z}{z-2} \cdot \ldots \cdot \frac{1-10z}{z-10}$$

Find $\displaystyle\int_{|z|=100} f(z)dz$.

8. Let $f(z) \not\equiv 0$ be a meromorphic function in $\mathbf{C}$ such that

$$|f(z)| = 1 \quad (|z| = 1)$$

and

$$f\left(\frac{1}{2}\right) = 0 \ .$$

Can $f$ be an entire function?