Complex Prelim, September 1998

Do any six problems

1. Suppose f is analytic in $|z| < 1$ and
$$f(\frac{1}{n^2}) = \frac{1}{n}$$
for all $n > 2$. Show that f is identically zero in $|z| < 1$.

2. Suppose f is entire and $|f(z)| \leq \log(1 + |z|)$ for all z. Show that f is identically zero.

3. Show that the function
$$u(x, y) = \arctan(y/x)$$
is harmonic in the (open) right half plane. Find its harmonic conjugate there.

4. Suppose $\{f_n\}$ is a sequence of analytic functions in $|z| < 1$. If
$$\lim_{n\to\infty} \int_{|z|<1} |f_n(z)|\, dA(z) = 0,$$
where dA is area measure on $|z| < 1$, show that
$$\lim_{n\to\infty} f_n(z) = 0$$
for all $|z| < 1$.

5. Evaluate the integeral
$$I = \int_C \frac{\sin \zeta}{\zeta(\pi - 6\zeta)^2}\, d\zeta,$$
where C is the positively oriented unit circle.

6. Suppose f is entire and
$$|f(ne^{i\theta})| \leq \exp(n \cos \theta)$$
for all $n \geq 1$ and $\theta \in [0, 2\pi]$. Show that $f(z) = ce^z$ for some constant c with $|c| \leq 1$.

7. Find a conformal map from the unit disk $|z| < 1$ onto the region $|argz| < \pi/3$.