Preliminary Examination

Complex Analysis

June 1997

1. Suppose that f is an entire function and $f(\mathbf{C}) \cap \{w : \text{Re } w = 0\} = \emptyset$. Prove that f is constant.

2. Find a conformal mapping of the open unit disk onto the domain

$$\Omega = \left\{ w : |w + \frac{1}{2}| > \frac{1}{2} \right\} \bigcap \{w : |w| < 1\} \ .$$

3. Suppose that f is a holomorphic function in an open disk D, f is continuous in $\overline{D}$ and $|f(z)|$ is constant for $z \in \partial D$. Prove that f is a rational function.

4. Determine all polynomials P such that $I(r) = \displaystyle\int_{|z|=r} \frac{1}{P(z)} dz$ has the property that $I(r) \neq 0$ for all $r > 0$ for which $I(r)$ is well-defined.

5. Give an example of a function f which is holomorphic in $\mathbf{C} \backslash \{z_0\}$ for some $z_0 \neq 0$, has an essential singularity at z_0 and is continuous in $\{z : |z| \leq |z_0|\}$. Show that the function given actually has these properties.

6. Suppose that the function f is holomorphic in $\{z : |z| < R\}$, and for each r $(0 < r < R)$ let $L(r)$ denote the length of the curve $w = f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Show that $L(r) \geq 2\pi r |f'(0)|$ and determine all functions for which equality holds.

7. Suppose that the function f is holomorphic in $\{z : |z| < R\}$ for some $R > 0$. Prove that $f(z) = \displaystyle\frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\theta} + z}{re^{i\theta} - z} \text{Re}\{f(re^{i\theta})\} d\theta + i \text{ Im } f(0)$ for $|z| < r < R$.

8. Suppose that f is an entire function, and for $r > 0$ let $M_f(r) = \sup\{|f(z)| : |z| \leq r\}$. Assume that $0 < \alpha < 1$ and let

$$L(\alpha) = \lim_{r \to \infty} \frac{M_f(\alpha r)}{M_f(r)} \ .$$

   (a) Determine $L(\alpha)$ in the case f is a polynomial.

   (b) Show that $L(\alpha) = 0$ if f is not a polynomial.