Preliminary Examination in Complex Analysis

June 4, 1996

Notation: $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ is the open unit disk

1. Let f be analytic in a nonempty connected open set U. Let F be a nonconstant entire function. Show that if $F(f(z)) = 0$ for all z in a neighborhood of some $z_0 \in U$, then f is constant in U.

2. (a) Find all constants c_1 and c_2 so that the functions

$$f_1(z) = c_1 z \quad \text{and} \quad f_2(z) = \frac{c_2}{z}$$

define conformal self-maps of the annulus $\mathcal{A} = \{z \in \mathbf{C} : a < |z| < b\}$ ($0 < a < b$ are given constants).

(b) Prove that there are no other conformal self-maps of $\mathcal{A}$.

3. Evaluate

$$\int_\gamma \frac{1 - \cos z}{(e^z - 1)\sin z}\, dz$$

where the path γ is the circle $|z| = e$ traversed once counterclockwise.

4. For $n \in \mathbf{N}$ show that

$$\int_\Delta \left| \frac{1 - z^n}{1 - z} \right|^2 dxdy = \pi\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$

5. Let f be analytic in Δ, and let $f(\Delta) \subseteq \Delta$. Prove that if $f(0) = 0$ and $f(a) = a$ for some $a \neq 0$, then $f(z) = z$.

6. Let f be analytic in Δ. Show that

$$\sup_{z \in \Delta} (1 - |z|^2)\, |f'(z)| \leq \sup_{z \in \Delta} |f(z)|\ .$$

7. Let $f(z)$ be analytic in Δ. Suppose

$$\lim_{r\uparrow 1} \int_0^{2\pi} |f(re^{i\theta})| d\theta = 0 \ .$$

Show that $f \equiv 0$.

8. Prove that the zero set $\mathcal{S}$ of $e^z + z$:

$$\mathcal{S} = \{z \in \mathbf{C} : e^z + z = 0\}$$

is nonempty: $\mathcal{S} \neq \emptyset$.

Bonus. Prove that $\mathcal{S}$ is an infinite set.

9. Find $w = f(z)$ that maps Δ conformally onto the strip $|\text{Im } w| < \dfrac{\pi}{2}$ so that $f(0) = 0$ and $f'(0) > 0$.