Ph.D. Preliminary Examination

Complex Analysis

August 26, 1994

1. Find an explicit conformal map from the region

$$\{z : |z| < 1\} - \{x \in \mathbf{R} : x \leq 0\}$$

onto the upper halfplane $\{\text{Im } z > 0\}$.

2. Find the explicit Laurent series of the function

$$f(z) = \frac{1}{z(z-3)}$$

on the annulus $\{z : 1 < |z-1| < 2\}$ centered at 1.

3. Let $D \subset \mathbf{C}$ be open and connected, and fix $z_0 \in D$; set $A(D, z_0) = \{|f'(z_0)| : f$ holomorphic on D and $|f(z)| < 1$ for $z \in D\}$. Prove that $A(D, z_0)$ is a compact subset of $\mathbf{R}$. What is $A(\mathbf{C}, z_0)$?

4. Let f be holomorphic in the connected region $\Omega \subset \mathbf{C}$, and assume that there exists a nonempty open set $U \subset \Omega$, such that $|f(z)| = 1$ for all $z \in U$. Show that f is constant in Ω.

5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on the closed unit disc. Prove that

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \ .$$

6. Suppose h is holomorphic in a neighborhood of $\{z : |z| \leq R\}$, and that $h(z) \neq 0$ for $|z| = R$.

(a) Use the Theorem of Residues to show that

$$\oint_{|z|=R} \frac{h'(z)}{h(z)} dz = 2\pi i \; Z_R(h) \; ,$$

where $Z_R(h)$ is the number of zeroes of h in $\{|z| < R\}$, counted with multiplicities.

(b) Use (a) to prove that if f and g satisfy the same hypotheses as h, and if

$$|f - g| < |f| \text{ on } \{|z| = R\} \; ,$$

then $Z_R(f) = Z_R(g)$.

7. Use the Theorem of Residues for appropriate contours to evaluate

$$\int_{-\infty}^{\infty} \frac{\sqrt{x+i}}{1+x^2} dx \; ,$$

where on $\{\text{Im } z > 0\}$, we choose the branch of $\sqrt{z+i}$ whose value at 0 is $e^{\pi i/4}$. Describe your method carefully, and include verification of all relevant limit statements.

8. Find an <u>explicit</u> series representation for a meromorphic function on $\mathbf{C}$, which is holomorphic on $\mathbf{C} - \{1, 2, 3, \ldots\}$, and which has at each point $z = n \in \mathbf{N}$ a simple pole with residue n. Include proofs of all required convergence statements.

9. Prove that all holomorphic automorphisms of $\mathbf{C}$ (i.e. holomorphic maps $f : \mathbf{C} \to \mathbf{C}$ which are one-to-one and onto) are precisely the linear functions $f(z) = a + bz$ for arbitrary $a, b \in \mathbf{C}$.