Preliminary Examination in Complex Analysis

June 6, 1994

1. Show that every function that is meromorphic on the extended complex plane is rational.

2. Show that if u is a real-valued harmonic function in a domain $\Omega \subset \mathbf{C}$ such that u^2 is harmonic in Ω, then u is constant.

3. By using complex integration verify the formula $\displaystyle\int_0^{2\pi} \frac{1}{a+\sin\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2-1}}$ where $a > 1$.

4. Let $f(z) = e^z + z$ and for $0 \le \theta < 2\pi$ let $R(\theta) = \{z : z = re^{i\theta},\ r \ge 0\}$. Show that $\displaystyle\lim_{\substack{z\to\infty \\ z\in R(\theta)}} |f(z)| = \infty$ for all θ. Does this imply that $\displaystyle\lim_{z\to\infty} \frac{1}{f(z)} = 0$? Explain.

5. Suppose that f is an analytic function in $\mathcal{H} = \{z : \text{Im } z > 0\}$ and $\text{Im } f(z) > 0$ for $z \in \mathcal{H}$. Show that $|f'(z)| \le \dfrac{\text{Im } f(z)}{\text{Im } z}$ for $z \in \mathcal{H}$, and determine when equality holds in this inequality.

6. Let Δ denote the open unit disk in $\mathbf{C}$ and let $A = \{z : \frac{3}{4} < |z| < 1\}$. Show that the function $f(z) = \dfrac{1}{z-\frac{1}{2}}$ cannot be approximated uniformly on compact subsets of A by functions analytic in Δ.

7. For $|z| < 1$ let $f(z) = \dfrac{1}{1-z}\exp\left[-\dfrac{1}{1-z}\right]$, and for $0 \le \theta < 2\pi$ let $\ell_\theta = \{z : z = re^{i\theta}, 0 \le r < 1\}$. Show that f is bounded on each set ℓ_θ. Is f bounded in Δ? Explain.

8. Find all conformal automorphisms of the annulus $\{z : 1 < |z| < 2\}$.