PhD PRELIM EXAM IN COMPLEX ANALYSIS
August 2009

Let $\mathbb{C}$ denote the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk. A *holomorphic* function is just another name for a (complex) *analytic* function.

1. Find ALL values of i^i and $\sqrt{i}$ in the form $x + iy$ or (x, y), $x, y \in \mathbb{R}$.

2. Find the Laurent series of the function
$$f(z) = \frac{z}{(z-1)(z-2)}$$
in the following regions: (1) $0 < |z-1| < 1$, (2) $|z-2| > 1$.

3. Let $f(z)$ be a holomorphic function in $\mathbb{D}$ which extends continuously to $\overline{\mathbb{D}}$ and dA be area measure. Show that
$$f(z) = \frac{1}{\pi} \int_{\overline{D}} \frac{f(w)\, dA(w)}{(1 - z\,\bar{w})^2}$$
for all $z \in \mathbb{D}$.

4. Suppose $f(z)$ is an entire function and the real part of $f(z)$ is never zero. Show that f must be a constant.

5. a) State the Schwarz Lemma for $\mathbb{D}$ and prove it, assuming power series expansion and maximum modulus principle.

   b) Show that every $h \in Aut(\mathbb{D})$ (i.e. biholomorphic map of $\mathbb{D}$ onto itself) is of the form
$$h(z) = e^{i\theta} \frac{z - a}{1 - \overline{a}\, z}$$
for some $\theta \in [0, 2\pi]$ and $a \in \mathbb{D}$. You may use the Schwarz Lemma, but must prove all other assertions you make.

6. Let $G \subset \mathbb{C}$ be open and simply connected, and $A \subset G$ a discrete subset of G. Prove that a holomorphic function f on $G \setminus A$ has an antiderivative on $G \setminus A$ (i.e., there is F holomorphic on $G \setminus A$ with $F' = f$ on $G \setminus A$) if and only if $res_a(f) = 0$ for all $a \in A$.

7. Let $f(z)$ be a holomorphic function in $\mathbb{D}$ which extends continuously to $\overline{\mathbb{D}}$ which satisfies $|f(z)| < 1$ for all $z \in \partial\mathbb{D}$. Show that there is exactly one point $w \in \mathbb{D}$ such that $f(w) = w$.

8. Let $\mathcal{F} = \{f : f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ with } |a_n| \leq n \text{ for all } n = 0, 1, 2, ...\}$.

   a) Prove that every $f \in \mathcal{F}$ defines a holomorphic function on $\mathbb{D}$.

   b) Prove that $\mathcal{F}$ is a compact subset of the set of all holomorphic functions on $\mathbb{D}$ in the topology of uniform convergence on compact subsets of $\mathbb{D}$.

9. Let $\Gamma = \{\omega \in \mathbb{C} : \omega = m + in \text{ for all } m, n \in \mathbb{Z}\}$. Carefully prove that the series

$$P(z) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Gamma \\ \omega \neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

   defines a meromorphic function on $\mathbb{C}$. Identify the region where $P(z)$ is holomorphic.