Complex Analysis Ph.D. Prelim

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1. Show that $u = e^x(x\ \cos y - y\ \sin y)$ is harmonic in the complex plane in 2 ways:

   A. From the definition of harmonic.

   B. By exhibiting an entire function f such that $u = Ref$.

2. A. State Schwarz's Lemma.

   B. State the Riemann Mapping Theorem.

   C. Prove uniqueness in the Riemann Mapping Theorem.

3. Let a and b be real numbers with $a > b > 1$.

   A. Show that b^z can be defined as an entire function such that $b^0 = 1$.

   C. Let n be a positive integer. Show that the equation $b^z = a\ z^n$ has n solutions in $|z| < 1$.

4. Let $C_{\mathcal{E}} = \{\mathcal{E}e^{i\theta} : 0 \leq \theta \leq \pi\}$ denote the semicircle traversed <u>clockwise</u>.

   A. Calculate $\displaystyle\int_{C_{\mathcal{E}}} \frac{1}{z}\, dz.$

   B. Determine $\displaystyle\lim_{\mathcal{E}\to 0} \int_{C_{\mathcal{E}}} \frac{1}{z(z^2+1)} dz.$

   C. Show that $\displaystyle\lim_{\mathcal{E}\to 0} \int_{C_{\mathcal{E}}} \frac{e^{iz}}{z(z^2+1)} dz = -\pi i$ [Consider $e^{iz} - 1$.]

5. Map the region bounded by the circles $|z| = 1$ and $|z+1| = 2$ conformally onto the open unit disk.

6. A. Determine the region of convergence of the series

$$1 + \frac{2z}{1+z} + \frac{3z^2}{(1+z)^2} + \ldots + \frac{(n+1)z^n}{(1+z)^n} + \ldots$$

   B. By summing the series show that the series actually represents a polynomial in its region of a convergence.

7. Let Ω be a <u>bounded</u> domain. Let $\mathcal{F}$ be the family of functions which are analytic in Ω and map Ω into itself.

   A. Show that $\mathcal{F}$ has locally bounded derivatives.

   B. Is $\mathcal{F}$ closed in the topology of uniform convergence on compact subsets of Ω? Justify your answer.

8. Let $I = \displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx.$

   A. Explain why I is absolutely convergent.

   B. Incorporate the semicircle and the result from Problem 4 in a contour integration argument to evaluate I.