Complex Analysis Prelim

January, 2005

Throughout let $D = \{z : |z| < 1\}$ and $\mathbf{C}$ = complex plane.

1. Let $u(x,y) = \dfrac{y}{x^2+y^2}$. Show that u is harmonic in the punctured plane $\mathbf{C}\backslash\{0\}$ in 2 ways:

   A. From the definition of harmonic.

   B. By finding a function f, analytic in $\mathbf{C}\backslash\{0\}$ with $u = \text{Re} f$.

2. Construct a conformal map of the region $D\backslash\{z : |z+\frac{1}{2}| \leq \frac{1}{2}\}$ onto the region $\mathbf{C}\backslash\{z : Re\ z \leq 0\}$.

3. Let α be a real number and consider the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 - 2x + \alpha}$. Determine for what α the integral converges and, in those cases, determine its value. Include the details of your contour argument.

4. Let $f(z) = \cos(i\, z^3)$. Determine $Z(f) = \{z : f(z) = 0\}$. Indicate with a picture where the solutions lie in $\mathbf{C}$.

5. Let $p(z) = 3z^{15} + 4z^8 + 6z^5 + 19z^4 + 3z + 1$. Show that $p(z)$ has 4 zeros for $|z| < 1$ and 11 zeros for $1 < |z| < 2$.

6. Let $f : D \to D$ be analytic and satisfy $f(\frac{1}{2}) = \frac{1}{2}$ and $f'(\frac{1}{2}) = -1$. Find an explicit formula for f.

7. Let f and g be analytic in a nonempty connected open set U and satisfy $|f| = |g|$ there. What else can you deduce about the relationship between f and g. Justify your answer.

8. Let f be analytic in D and satisfy $|f(z)| \leq \dfrac{1}{1-|z|}$ there. Show that $|f'(0)| \leq 4$.

9. Let $\{b_n\}$ be a sequence of complex numbers such that $\limsup |b_n|^{\frac{1}{n}} = 1$. Let $\mathcal{F}$ be the family of function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which are analytic in D and satisfy $|a_n| \leq |b_n|$, $n = 0, 1, 2, \ldots$ Prove that $\mathcal{F}$ is a compact family in the topology of uniform convergence on compact sets in D.

10. A. State carefully the Riemann Mapping Theorem.

    B. Let f be a conformal map from D onto D satisfying $f(0) = 0$ and $|f'(0)| = 1$. Using only the Riemann Mapping Theorem show that $f(e^{i\theta} z) = e^{i\theta} f(z)$ for every real number θ.

    C. Deduce that there is a real number θ_0 such that $f(z) = e^{i\theta_0} z$.