Complex analysis preliminary examination
August 31, 2000

1. Let f be a harmonic function in the unit disk. Given that e^f is harmonic prove that eithr f is holomorphic, or $\bar{f}$ is holomorphic.

2. Let f_n, $n = 0, 1, ...$ be a sequence of functions analytic in a closed bounded domain $\bar{\Omega}$ with smooth boundary. Suppose that $f_n \to f_0$ as $n \to \infty$ uniformly on compact subsets of Ω. Does this imply that

$$\int_{\partial\Omega} |f_n(z)| ds(z) \to \int_{\partial\Omega} |f_0(z)| ds(z),$$

where ds is the linear Lebesgue measure on $\partial\Omega$. Prove, or give a counterexample.

3. If f is analytic in the unit disk Δ, continuous in $\bar{\Delta}$ and maps Δ into itself, prove that for every point $a \in \Delta$

$$f^{(3)}(a) \leq \frac{6(1+|a|^2)}{(1-|a|^2)^3}.$$

Hint: Use Cauchy formula.

4. Prove that

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}.$$

5. Let B be a finite Blaschke product. Prove that

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{B(z)} = \bar{B'}(0).$$

6. Let f be analytic and absolutely integrable with respect to the Lebesgue area measure $dA = dxdy$ in the unit disk Δ. Prove that for $a \in \Delta$

$$\int_{\Delta} f(z) \frac{\bar{z}}{(1 - a\bar{z})^3} dA(z) = \frac{1}{2} f'(a).$$

7. Let f be analytic in the closed unt disk and $|f(z)|$ is constant for $|z| = 1$. Prove that $arg(f(e^{i\theta}))$ is a monotone function of θ.

8. Let f be a smooth bounded function in the unit disk Δ and $F(z)$ be given by

$$F(z) = \frac{1}{\pi} \int_{\Delta} \frac{f(w) dA(w)}{w - z}.$$

Prove that F(z) is analytic outside of $\bar{\Delta}$ and for every $z \in \Delta$

$$\frac{\partial F}{\partial \bar{z}} = f(z).$$

Hint: Use Cauchy-Green formula.