Renault's Equivalence Theorem for groupoid crossed products

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ABSTRACT. We provide an exposition and proof of Renault's Equivalence Theorem for crossed products by locally Hausdorff, locally compact groupoids. Our approach stresses the bundle approach, concrete imprimitivity bimodules and is a preamble to a detailed treatment of the Brauer semigroup for a locally Hausdorff, locally compact groupoid.

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Contents

1. Introduction

Our objective in this paper is to present an exposition of the theory of groupoid actions on so-called upper-semicontinuous-C^*-bundles and to present the rudiments of their associated crossed product C^*-algebras. In particular, we shall extend the equivalence theorem from [28] and [40, Corollaire 5.4] to cover locally compact, but not necessarily Hausdorff, groupoids acting on such bundles. Our inspiration for this project derives from investigations we are pursuing into the structure of the *Brauer semigroup*, $S(G)$, of a locally compact groupoid G, which is defined to be a collection of Morita equivalence classes of actions of the groupoid on upper-semicontinuous-C^*-bundles. The semigroup $S(G)$ arises in numerous guises in the literature and one of our goals is to systematize their theory. For this purpose, we find it useful to work in the context of groupoids that are not necessarily Hausdorff. It is well-known that complications arise when one passes from Hausdorff groupoids to non-Hausdorff groupoids and *some* of them are dealt with in the literature. Likewise, conventional wisdom holds that there is no significant difference between upper-semicontinuous-C^*-bundles and ordinary C^*-bundles; one needs only to be careful. However, there are subtle points in both areas and it is fair to say that they have not been addressed or collated in a fashion that is suitable for our purposes or for other purposes where such structures arise. Consequently, we believe that it is useful and timely to write down complete details in one place that will serve the needs of both theory and applications.

The non-Hausdorff locally compact spaces that enter the theory are not arbitrary. They are what is known as locally Hausdorff. This means that each point has a Hausdorff neighborhood. Nevertheless, such a space need not have any nontrivial continuous functions. As Connes observed in [5] and [6], one has to replace continuous functions by linear combinations of functions that are continuous with compact support when restricted to certain locally Hausdorff and locally compact sets, but are not continuous globally. While at first glance, this looks like the right replacement of continuous compactly supported functions in the Hausdorff setting, it turns out that these functions are a bit touchy to work with, and there are some surprises with which one must deal. We begin our discussion, therefore, in Section 2 by reviewing the theory. In addition to recapping some of the work in the literature, we want to add a few comments of our own that will be helpful in the sequel. There are a number of "standard" results in the Hausdorff case which are considerably more subtle in the locally Hausdorff, locally compact case. In Section 3 we turn to $C_0(X)$-algebras. The key observation here is that every $C_0(X)$-algebra is actually the section algebra of an upper-semicontinuous-C^*-bundle. Since our eventual goal is the equivalence theorem (Theorem 5.5), we have to push the envelope slightly and look at upper-semicontinuous-Banach bundles over locally Hausdorff, locally compact spaces.

In Section 4, we give the definition of, and examine the basic properties of, groupoid crossed products. Here we are allowing (second countable) locally Hausdorff, locally compact groupoids acting on $C_0(G^{(0)})$-algebras. In Section 5 we state the main object of this effort: Renault's Equivalence Theorem.

Our version of the proof of the equivalence theorem requires some subtle machinations with approximate identities and Section 6 is devoted to the details. The other essential ingredients of the proof require that we talk about covariant representations of groupoid dynamical systems and prove a disintegration theorem analogous to that for ordinary groupoid representations. This we do in Section 7. With all this machinery in hand, the proof of the equivalence theorem is relatively straightforward and the remaining details are given in Section 8.

In Section 9.1 and Section 9.2 we look at two very important applications of the equivalence theorem inspired by the constructions and results in [23].

Since the really deep part of the proof of the equivalence theorem is Renault's disintegration theorem (Theorem 7.8), and since that result — particularly the details for locally Hausdorff, locally compact groupoids — is

hard to sort out of the literature, we have included a complete proof in Appendix B. Since that proof requires some gymnastics with the analogues of Radon measures on locally Hausdorff, locally compact spaces, we have also included a brief treatment of the results we need in Appendix A.

Assumptions. Because Renault's disintegration result is mired in direct integral theory, it is necessary to restrict to second countable groupoids and separable C^*-algebras for our main results. We have opted to make those assumptions throughout — at least wherever possible. In addition, we have adopted the common conventions that all homomorphisms between C^*-algebras are presumed to be $*$-preserving, and that representations of C^*-algebras are assumed to be nondegenerate.

2. Locally Hausdorff spaces, groupoids and principal G-spaces

In applications to noncommutative geometry — in particular, to the study of foliations — in applications to group representation theory, and in applications to the study of various dynamical systems, the groupoids that arise often fail to be Hausdorff. They are, however, *locally Hausdorff*, which means that each point has a neighborhood that is Hausdorff in the relative topology. Most of the non-Hausdorff, but locally Hausdorff spaces X we shall meet will, however, also be locally compact. That is, each point in X will have a Hausdorff, compact neighborhood.[1]   In such a space compact sets need not be closed, but, at least, points are closed.

Non-Hausdorff, but locally Hausdorff spaces often admit a paucity of continuous compactly supported functions. Indeed, as shown in the discussion following [21, Example 1.2], there may be no nonzero functions in $C_c(X)$. Instead, the accepted practice is to use the following replacement for $C_c(X)$ introduced by Connes in [6, 5]. If U is a Hausdorff open subset of X, then we can view functions in $C_c(U)$ as functions on X by defining them to be zero off U. Unlike the Hausdorff case, however, these extended functions may no longer be continuous, or compactly supported on X.[2]   Connes's replacement for $C_c(X)$ is the subspace, $\mathscr{C}(X)$, of the complex vector space of functions on X spanned by the elements of $C_c(U)$ for all open Hausdorff subsets U of X. Of course, if X is Hausdorff, then $\mathscr{C}(X) = C_c(X)$. The

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[1]We do not follow Bourbaki [3], where a space is compact if and only if it satisfies the every-open-cover-admits-a-finite-subcover-condition *and* is Hausdorff.

[2]Recall that the support of a function is the *closure* of the set on which the function is nonzero. Even though functions in $C_c(U)$ vanish off a compact set, the closure in X of the set where they don't vanish may not be compact.

notation $C_c(X)$ is often used in place of $\mathscr{C}(X)$. However, since elements of $\mathscr{C}(X)$ need be neither continuous nor compactly supported, the C_c notation seems ill-fitting. Nevertheless, if $f \in \mathscr{C}(X)$, then there is a compact set K_f such that $f(x) = 0$ if $x \notin K_f$. As is standard, we will say that a net $\{f_i\} \subset \mathscr{C}(X)$ converges to $f \in \mathscr{C}(X)$ in the *inductive limit topology* on $\mathscr{C}(X)$ if there is a compact set K, independent of i, such that $f_i \to f$ uniformly and each $f_i(x) = 0$ if $x \notin K$.

While it is useful for many purposes, the introduction of $\mathscr{C}(X)$ is no panacea: $\mathscr{C}(X)$ is not closed under pointwise products, in general, and neither is it closed under the process of "taking the modulus" of a function. That is, if $f \in \mathscr{C}(X)$ it need not be the case that $|f| \in \mathscr{C}(X)$ [32, p. 32]. A straightforward example illustrating the problems with functions in $\mathscr{C}(X)$ is the following.

Example 2.1. As in [21, Example 1.2], we form a groupoid G as the topological quotient of $\mathbf{Z} \times [0,1]$ where for all $t \neq 0$ we identify $(n, t) \sim (m, t)$ for all $n, m \in \mathbf{Z}$. (Thus as a set, G is the disjoint union of $\mathbf{Z}$ and $(0, 1]$.) If $f \in C[0, 1]$, then we let f^n be the function in $C(\{n\} \times [0, 1]) \subset \mathscr{C}(G)$ given by

$$f^n(m, t) := \begin{cases} f(t) & \text{if } t \neq 0, \\ f(0) & \text{if } m = n \text{ and } t = 0 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then in view of [21, Lemma 1.3], every $F \in \mathscr{C}(G)$ is of the form

$$F = \sum_{i=1}^{k} f_i^{n_i}$$

for functions $f_1, \ldots, f_k \in C[0, 1]$ and integers n_i. In particular, if $F \in \mathscr{C}(G)$ then we must have

(2.1) $$\sum_{n} F(n, 0) = \lim_{t \to 0^+} F(0, t).$$

Let $g(t) = 1$ for all $t \in [0, 1]$, and let $F \in \mathscr{C}(G)$ be defined by $F = g^1 - g^2$. Then

$$F(n, t) = \begin{cases} 1 & \text{if } t = 0 \text{ and } n = 1, \\ -1 & \text{if } t = 0 \text{ and } n = 2 \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Not only is F an example of a function in $\mathscr{C}(G)$ which is not continuous on G, but $|F| = \max(F, -F) = F^2$ fails to satisfy (2.1). Therefore $|F| \notin \mathscr{C}(G)$ even though F is. This also shows that $\mathscr{C}(G)$ is not closed under pointwise

products nor is it a lattice: if $F, F' \in \mathscr{C}(G)$, it does not follow that either $\max(F, F') \in \mathscr{C}(G)$ or $\min(F, F') \in \mathscr{C}(G)$.

We shall always assume that the locally Hausdorff, locally compact spaces X with which we deal are second countable, i.e., we shall assume there is a countable basis of open sets. Since points are closed, the Borel structure on X generated by the open sets is countably separated. Indeed, it is standard. The reason is that every second countable, compact Hausdorff space is Polish [46, Lemma 6.5]. Thus X admits a countable cover by standard Borel spaces. It follows that X can be expressed as a disjoint union of a sequence of standard Borel spaces, and so is standard.

The functions in $\mathscr{C}(X)$ are all Borel. By a *measure* on X we mean an ordinary, positive measure μ defined on the Borel subsets of X such that the restriction of μ to each Hausdorff open subset U of X is a Radon measure on U. That is, the measures we consider restrict to regular Borel measures on each Hausdorff open set and, in particular, they assign finite measure to each compact subset of a Hausdorff open set. (Recall that for second countable locally compact Hausdorff spaces, Radon measures are simply regular Borel measures.) If μ is such a measure, then every function in $\mathscr{C}(X)$ is integrable. (For more on Radon measures on locally Hausdorff, locally compact spaces, see Appendix A.2.)

Throughout, G will denote a locally Hausdorff, locally compact groupoid. Specifically we assume that G is a groupoid endowed with a topology such that:

G1: The groupoid operations are continuous.
G2: The unit space $G^{(0)}$ is Hausdorff.
G3: Each point in G has a compact Hausdorff neighborhood.
G4: The range (and hence the source) map is open.

A number of the facts about non-Hausdorff groupoids that we shall use may be found in [21]. Another helpful source is the paper by Tu [44]. Note that as remarked in [21, §1B], for each $u \in G^{(0)}$, $G^u := \{\gamma \in G : r(\gamma) = u\}$ must be Hausdorff. To see this, recall that $\{u\}$ is closed in G, and observe that

$$G *_s G = \{(\gamma, \eta) \in G \times G : s(\gamma) = s(\eta)\}$$

is closed in $G \times G$. Since $(\gamma, \eta) \mapsto \gamma\eta^{-1}$ is continuous from $G *_s G$ to G, the diagonal

$$\begin{aligned} \Delta(G^u) &:= \{(\gamma, \gamma) \in G^u \times G^u\} \\ &= \{(\gamma, \eta) \in G *_s G : \gamma\eta^{-1} = u\} \cap G^u \times G^u \end{aligned}$$

is closed in $G^u \times G^u$. Hence G^u is Hausdorff, as claimed. Of course, if G is Hausdorff, then $G^{(0)}$ is closed since $G^{(0)} = \{\gamma \in G : \gamma^2 = \gamma\}$ and convergent nets have unique limits. Conversely, if G is not Hausdorff, then to see that $G^{(0)}$ fails to be closed, let γ_i be a net in G converging to both γ and η (with $\eta \neq \gamma$). Since $G^{(0)}$ is Hausdorff by G2, we must have $s(\gamma) = s(\eta)$. Then $\gamma_i^{-1}\gamma_i \to \gamma^{-1}\eta$ (as well as to $\gamma^{-1}\gamma$). Therefore $s(\gamma_i)$ must converge to $\gamma^{-1}\eta \notin G^{(0)}$. Therefore G is Hausdorff *if and only if* $G^{(0)}$ is closed in G.

Remark 2.2. Suppose that G is a non-Hausdorff, locally Hausdorff, locally compact groupoid. Then there are distinct elements γ and η in G and a net $\{\gamma_i\}$ converging to both γ and η. Since $G^{(0)}$ is Hausdorff, $s(\gamma_i) \to u = s(\gamma) = s(\eta)$, and $r(\gamma_i) \to v = r(\gamma) = r(\eta)$. In particular, $\gamma^{-1}\eta$ is a nontrivial element of the isotropy group G_u^u. In particular, a principal locally Hausdorff, locally compact groupoid must be Hausdorff.

Since each G^u is a locally compact Hausdorff space, G^u has lots of nice Radon measures. Just as for Hausdorff locally compact groupoids, a *Haar system* on G is a family of measures on G, $\{\lambda^u\}_{u \in G^{(0)}}$, such that:

(a) For each $u \in G^{(0)}$, λ^u is supported on G^u and the restriction of λ^u to G^u is a regular Borel measure.
(b) For all $\eta \in G$ and $f \in \mathscr{C}(G)$,

$$\int_G f(\eta\gamma) \, d\lambda^{s(\eta)}(\gamma) = \int_G f(\gamma) \, d\lambda^{r(\eta)}(\gamma).$$

(c) For each $f \in \mathscr{C}(G)$,

$$u \mapsto \int_G f(\gamma) \, d\lambda^u(\gamma)$$

is continuous and compactly supported on $G^{(0)}$.

We note in passing that Renault [40, 39] and Paterson [32, Definition 2.2.2] assume that the measures in a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ have full support; i.e., they assume that $\mathrm{supp}(\lambda^u) = G^u$, whereas Khoshkam and Skandalis don't (see [21] and [22]). It is easy to see that the union of the supports of the λ^u is an invariant set for the left action of G on G (in a sense to be discussed in a moment). If this set is all of G, then we say that the Haar system is *full*. All of our groupoids will be assumed to have full Haar systems and we shall not add the adjective "full" to any Haar system we discuss. Note that if a groupoid satisfies G1, G2 and G3 and has a Haar system, then it must also satisfy G4 [32, Proposition 2.2.1].

If X is a G-space,[3] then let $G * X = \{(\gamma, x) : s(\gamma) = r(x)\}$ and define $\Theta : G * X \to X \times X$ by $\Theta(\gamma, x) := (\gamma \cdot x, x)$. We say that X is a *proper G-space* if Θ is a proper map.[4]

Lemma 2.3. *Suppose a locally Hausdorff, locally compact groupoid G acts on a locally Hausdorff, locally compact space X. Then X is a proper G-space if and only if $\Theta^{-1}(W)$ is compact in $G * X$ for all compact sets W in $X \times X$.*

Proof. If Θ is a proper map, then $\Theta^{-1}(W)$ is compact whenever W is by [3, I.10.2, Proposition 6].

Conversely, assume that $\Theta^{-1}(W)$ is compact whenever W is. In view of [3, I.10.2, Theorem 1(b)], it will suffice to see that Θ is a closed map. Let $F \subset X * G$ be a closed subset, and let $E := \Theta(F)$. Suppose that $\{(\gamma_i, x_i)\} \subset F$ and that $\Theta(\gamma_i, x_i) = (\gamma_i \cdot x_i, x_i) \to (y, x)$. Let W be a compact *Hausdorff* neighborhood of (y, x). Since F is closed, $\Theta^{-1}(W) \cap F$ is compact and eventually contains (γ_i, x_i). Hence we can pass to a subnet, relabel, and assume that $(\gamma_i, x_i) \to (\gamma, z)$ in $F \cap \Theta^{-1}(W)$. Then $(\gamma_i \cdot x_i, x_i) \to (\gamma \cdot z, z)$ in W. Since W is Hausdorff, $z = x$ and $\gamma \cdot x = y$. Therefore $(y, x) = (\gamma \cdot x, x)$ is in E. Hence E is closed. This completes the proof. $\qquad\square$

Remark 2.4. If X is Hausdorff, the proof is considerably easier. In fact, it suffices to assume only that $\Theta^{-1}(W)$ pre-compact.[5]

Definition 2.5. A G-space X is called *free* if the equation $\gamma \cdot x = x$ implies that $\gamma = r(x)$. A free and proper G-space is called a *principal G-space*.

If X is a G space, then we denote the orbit space by $G \backslash X$. The orbit map $q : X \to G \backslash X$ is continuous and open [30, Lemma 2.1]. Our next observation is that, just as in the Hausdorff case, the orbit space for a proper G-space has regularity properties comparable to those of the total space.

Lemma 2.6. *Suppose that X is a locally Hausdorff, locally compact proper G-space. Then $G \backslash X$ is a locally Hausdorff, locally compact space. In particular, if C is a compact subset of X with a compact Hausdorff neighborhood K, then $q(C)$ is Hausdorff in $G \backslash X$.*

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[3]Actions of groupoids on topological spaces are discussed in several places in the literature. For example, see [23, p. 912].

[4]Recall that a map $f : A \to B$ is proper if $f \times \mathrm{id}_C : A \times C \to B \times C$ is a a closed map for every topological space C [3, I.10.1, Definition 1]. For the case of group actions, see [3, III.4].

[5]In the Hausdorff case, "pre-compact" and "relatively compact" refer to set whose closure is compact. In potentially non-Hausdorff situations, such as here, we use "pre-compact" for a set which is *contained* in a compact set. In particular, a pre-compact set need not have compact closure. (For an example, consider [21, Example 1.2].)

Proof. It suffices to prove the last assertion. Suppose that $\{x_i\}$ is a net in C such that $G \cdot x_i$ converges to $G \cdot y$ and $G \cdot z$ for y and z in C. It will suffice to see that $G \cdot y = G \cdot z$. After passing to a subnet, and relabeling, we can assume that $x_i \to x$ in C and that there are $\gamma_i \in G$ such that $\gamma_i \cdot x_i \to y$. We may assume that $x_i, \gamma_i \cdot x \in K$. Since $\Theta^{-1}(K \times K)$ is compact and since $\{(\gamma_i, x_i)\} \subset \Theta^{-1}(K \times K)$, we can pass to a subnet, relabel, and assume that $(\gamma_i, x_i) \to (\gamma, w)$ in $\Theta^{-1}(K \times K)$. Since K is Hausdorff, we must have $w = x$. Thus $\gamma_i \cdot x_i \to \gamma \cdot x$. Since $y \in C \subset K$, we must have $\gamma \cdot x = y$. But then $G \cdot x = G \cdot y$. Similarly, $G \cdot x = G \cdot z$. Thus $G \cdot y = G \cdot z$, and we're done. $\square$

Example 2.7. If G is a locally Hausdorff, locally compact groupoid, then the left action of G on itself is free and proper. In fact, in this case, $G * G = G^{(2)}$ and Θ is homeomorphism of $G^{(2)}$ onto $G *_s G = \{(\gamma, \eta) : s(\gamma) = s(\eta)\}$ with inverse $\Phi(\beta, \alpha) = (\beta\alpha^{-1}, \alpha)$. Since Φ is continuous, $\Phi(W) = \Theta^{-1}(W)$ is compact whenever W is.

Remark 2.8. If G is a non-Hausdorff, locally Hausdorff, locally compact groupoid, then as the above example shows, G acts (freely and) properly on itself. Since this is a fundamental example — perhaps even *the* fundamental example — we will have to tolerate actions on non-Hausdorff spaces. It should be observed, however, that a *Hausdorff* groupoid G can't act properly on a non-Hausdorff space X. If G is Hausdorff, then $G^{(0)}$ is closed and $G^{(0)} * X$ is closed in $G * X$. However $\Theta(G^{(0)} * X)$ is the diagonal in $X \times X$, which if closed if and only if X is Hausdorff.

Remark 2.9. If X is a proper G-space, and if K and L are compact subsets of X, then
$$P(K, L) := \{\gamma \in G : K \cap \gamma \cdot L \neq \emptyset\}$$
is compact — consider the projection onto the first factor of the compact set $\Theta^{-1}(K \times L)$. If X is Hausdorff, the converse is true; see, for example, [1, Proposition 2.1.9]. However, the converse fails in general. In fact, if X is any non-Hausdorff, locally Hausdorff, locally compact space, then X is, of course, a G-space for the trivial group(oid) $G = \{e\}$. But in this case $\Theta(G * X) = \Delta(X) := \{(x, x) \in X \times X : x \in X\}$. But $\Delta(X)$ is closed if and only if X is Hausdorff. Therefore, if X is not Hausdorff, Θ is not a closed map, and therefore is not a proper map.[6] Of course, in this example, $P(K, L)$ is trivially compact for any K and L. In [40], it is stated that X is a proper G-space whenever $P(K, L)$ is relatively compact for all K and L compact in X. As this discussion shows, this is not true in the non-Hausdorff case. If "relatively compact" in interpreted to mean contained in a compact

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[6]Notice that $\Theta^{-1}(K \times L) = \{e\} \times K \cap L$, and $K \cap L$ need not be compact even if both K and L are.

set (as it always is here), then it can be shown that $P(K, L)$ is relatively compact for all K and L compact in X if and only if $\Theta^{-1}(W)$ is relatively compact for all compact W [43].

As Remark 2.9 illustrates, there can be subtleties involved when working with locally Hausdorff, locally compact G-spaces. We record here some technical results, most of which are routine in the Hausdorff case, which will be of use later.

Recall that a subset $U \subset G$ is called *conditionally compact* if VU and UV are pre-compact whenever V is pre-compact in G. We say that U is *diagonally compact* if UV and VU are compact whenever V is compact. If U is a diagonally compact neighborhood of $G^{(0)}$, then its interior is a conditionally compact neighborhood. We will need to see that G has a fundamental system of diagonally compact neighborhoods of $G^{(0)}$. The result is based on a minor variation, of [39, Proof of Proposition 2.1.9] and [29, Lemma 2.7] that takes into account the possibility that G is not Hausdorff.

Lemma 2.10. *Suppose that G is a locally Hausdorff, locally compact groupoid. If $G^{(0)}$ is paracompact, then G has a fundamental system of diagonally compact neighborhoods of $G^{(0)}$.*

Remark 2.11. If G is second countable, then so is $G^{(0)}$. Hence $G^{(0)}$ is always paracompact under our standing assumptions.

Proof. Let V be any neighborhood of $G^{(0)}$ in G. Since $G^{(0)}$ is paracompact, the shrinking lemma (cf. [35, Lemma 4.32]) implies that there is a locally finite cover $\{K_i\}$ of $G^{(0)}$ such that each K_i is a compact subset of $G^{(0)}$ and such that the interiors of the K_i cover $G^{(0)}$. In view of the local finiteness, any compact subset of $G^{(0)}$ meets only finitely many K_i.

Let U_i' be a compact neighborhood of K_i in G with $U_i' \subset V$. Let $U_i := U_i' \cap s^{-1}(K_i) \cap r^{-1}(K_i)$. Since $s^{-1}(K_i)$ and $r^{-1}(K_i)$ are closed, U_i is a compact set whose interior contains the interior of K_i, and

$$K_i \subset U_i \subset V \cap s^{-1}(K_i) \cap r^{-1}(K_i).$$

Therefore

$$U := \bigcup U_i$$

is a neighborhood of $G^{(0)}$. If K is any compact subset of $G^{(0)}$, then

$$U \cap s^{-1}(K) = \bigcup_{K \cap K_i \neq \emptyset} U_i \cap s^{-1}(K).$$

Since $s^{-1}(K)$ is closed and the union is finite, $U \cap s^{-1}(K)$ is compact. Similarly, $r^{-1}(K) \cap U$ is compact as well. Since $U \cdot K = (U \cap s^{-1}(K)) \cdot K$, the

former is compact as is $K \cdot U$. Thus, U is a diagonally compact neighborhood of $G^{(0)}$ contained in V. □

Remark 2.12. We have already observed that if G is not Hausdorff, then $G^{(0)}$ is not closed in G. Since points in G are closed, it nevertheless follows that $G^{(0)}$ is the intersection of all neighborhoods V of $G^{(0)}$ in G. In particular, Lemma 2.10 implies that $G^{(0)}$ is the intersection of all conditionally compact, or diagonally compact, neighborhoods of $G^{(0)}$, provided $G^{(0)}$ is paracompact.

Lemma 2.13. *Suppose that G is a locally Hausdorff, locally compact groupoid and that $K \subset G^{(0)}$ is compact. Then there is a neighborhood W of $G^{(0)}$ in G such that $WK = W \cap r^{-1}(K)$ is Hausdorff.*

Proof. Let $u \in K$ and let V_u be a Hausdorff neighborhood of u in G. Let $C_u \subset G^{(0)}$ be a closed neighborhood of u in $G^{(0)}$ such that $C_u \subset V_u$. Let $W_u := r^{-1}(G^{(0)} \setminus C_u) \cup V_u$. Then W_u is a neighborhood of $G^{(0)}$ and $W_u C_u \subset V_u$. Let $u_1, \ldots, u_n$ be such that $K \subset \bigcup_i C_{u_i}$, and let $W := \bigcap W_{u_i}$.

Suppose that γ and η are elements of $W \cdot K$ which can't be separated. Then there is a $u \in K$ such that $r(\gamma) = u = r(\eta)$ (Remark 2.2). Say $u \in C_{u_i}$. Then $\gamma, \eta \in W_{u_i}$, and consequently both are in V_{u_i}. Since the latter is Hausdorff, $\gamma = \eta$. Thus WK is Hausdorff. □

Lemma 2.14. *Suppose that G is a locally Hausdorff, locally compact groupoid and that X is a locally Hausdorff, locally compact G-space. If V is open in X and if $K \subset V$ is compact, then there is a neighborhood W of $G^{(0)}$ in G such that $W \cdot K \subset V$.*

Proof. For each $x \in K$ there is a neighborhood U_x of $r(x)$ in G such that $U_x \cdot K \subset V$. Let $x_1, \ldots, x_n$ be such that $\bigcup r(U_{x_i}) \supset r(K)$. Let $W := \bigcup U_{x_i} \cup r^{-1}(G^{(0)} \setminus r(K))$. Then W is a neighborhood of $G^{(0)}$ and $W \cdot K \subset (\bigcup U_{x_i}) \cdot K \subset V$. □

The next lemma is a good example of a result that is routine in the Hausdorff case, but takes a bit of extra care in general.

Lemma 2.15. *Suppose that G is a locally Hausdorff, locally compact groupoid and that X is a locally Hausdorff, locally compact free and proper (right) G-space. If W is a neighborhood of $G^{(0)}$ in G, then each $x \in X$ has a neighborhood V such that the inclusion $(x, x \cdot \gamma) \in V \times V$ implies that $\gamma \in W$.*

Proof. Fix $x \in X$. Let C be a compact Hausdorff neighborhood of x in X. If the lemma were false for x, then for each neighborhood V of x

such that $V \subset C$, there would be a $\gamma_V \notin W$ and a $x_V \in V$ such that $(x_V, x_V \cdot \gamma_V) \in V \times V$. This would yield a net $\{(x_V, \gamma_V)\}_{\{V \subset C\}}$. Since

$$A = \{(x, \gamma) \in X \times G : x \in C \text{ and } x \cdot \gamma \in C\}$$

is compact, we could pass to a subnet, relabel, and assume that $(x_V, \gamma_V) \to (y, \gamma)$ in A. Since C is Hausdorff and since $x_V \to x$ while $x_V \cdot \gamma_V \to x$, we would have $x = y$ and $x \cdot \gamma = x$. Therefore, we would find that $\gamma = s(x) \in W$. On the other hand, since W is open and since $\gamma_V \notin W$ for all V we would find that $\gamma \notin W$. This would be a contradiction, and completes the proof. $\square$

The next proposition is the non-Hausdorff version of Lemmas 2.9 and 2.13 from [28].

Proposition 2.16. *Suppose that G is a locally Hausdorff, locally compact groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. Let X be a locally Hausdorff, locally compact free and proper (right) G-space, let $q : X \to X/G$ be the quotient map, and let $V \subset X$ be a Hausdorff open set such that $q(V)$ is Hausdorff.*

(a) *If $\psi \in C_c(V)$, then*

$$\lambda(\psi)\big(q(x)\big) = \int_G \psi(x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma)$$

defines an element $\lambda(\psi) \in C_c\big(q(V)\big)$.

(b) *If $d \in C_c\big(q(V)\big)$, then there is a $\psi \in C_c(V)$ such that $\lambda(\psi) = d$.*

Corollary 2.17. *The map λ defined in part (a) of Proposition 2.16 extends naturally to a surjective linear map $\lambda : \mathscr{C}(X) \to \mathscr{C}(X/G)$ which is continuous in the inductive limit topology.*

Proof of Corollary 2.17. Let V be a Hausdorff open subset of X, and let $\psi \in C_c(V)$. We need to see that $\lambda(\psi) \in \mathscr{C}(X/G)$. Let W be a open neighborhood of $\mathrm{supp}_V \psi$ with a compact neighborhood contained in V.[7] Then Lemma 2.6 implies that $q(W)$ is Hausdorff, and Proposition 2.16 implies that $\lambda(\psi) \in C_c\big(q(W)\big)$. It follows that λ extends to a well-defined linear surjection. The statement about the inductive limit topology is clear. $\square$

Remark 2.18. In the language of [40], the first part of the proposition says that the Haar system on G induces a q-system on X — see [40, p. 69].

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[7]Here we use the notation supp_V to describe the support of a function on V relative to V as opposed to all of X. Recall that the support of a continuous function is the closure of the set where the function is not zero, and since X is not necessarily Hausdorff, the closure of a set relative to a subset such as V need not be the same as the closure of the subset in X.

Proof of Proposition 2.16. Let $D = \operatorname{supp}_V \psi$. Since V is locally compact Hausdorff, there is an open set W and a compact set C such that

$$D \subset W \subset C \subset V.$$

Let $\Theta : X \times G \to X \times X$ be given by $\Theta(x, \gamma) = (x, x \cdot \gamma)$. Since the G-action is proper,

$$A := \Theta^{-1}(C \times C) = \{(x, \gamma) \in X \times G : x \in C \text{ and } x \cdot \gamma \in C\}$$

is compact. Moreover, if $\{(x_i, \gamma_i)\}$ is a net in A converging to both (x, γ) and (y, η) in A, then since C is Hausdorff, we must have $x = y$. Then $\{x_i \cdot \gamma_i\}$ converges to both $x \cdot \gamma$ and $x \cdot \eta$ in the Hausdorff set C. Thus $x \cdot \gamma = x \cdot \eta$, and since the action is free, we must have $\gamma = \eta$. In sum, A is Hausdorff.

Let $F : C \times G \to \mathbf{C}$ be defined by $F(x, \gamma) = \psi(x \cdot \gamma)$. Notice that F vanishes off A. Let $K := \operatorname{pr}_2(A)$ be the projection onto the second factor; thus, K is compact in G. Unfortunately, we see no reason that K must be Hausdorff. Nevertheless, we can cover K by Hausdorff open sets $V_1, \ldots, V_n$. Let $A_j := A \cap (C \times V_j)$, let $\{f_j\}$ be a partition of unity in $C(A)$ subordinate to $\{A_j\}$ and let $F_j(x, \gamma) := f_j(x, \gamma)F(x, \gamma)$. Then $F_j \in C_c(A_i)$.

Claim 2.19. *If we extend F_j by setting to be 0 off A, we can view F_j as an element of $C_c(C \times V_j)$.*

Proof. Suppose that $\{(x_i, \gamma_i)\}$ is a net in $C \times V_i$ converging to (x, γ) in $C \times V_i$. Let

$$B := (C \times G) \cap \Theta^{-1}(X \times W) = \{(x, \gamma) : x \in C \text{ and } x \cdot \gamma \in W\}.$$

Then B is open in $C \times G$ and $B \subset A$. If $(x, \gamma) \in B$, then (x_i, γ_i) is eventually in B and $F_j(x_i, \gamma_i) \to F_j(x, \gamma)$ (since F_j is continuous on A).

On the other hand, if $(x, \gamma) \notin B$, then $F_j(x, \gamma) = 0$. If $\{F(x_i, \gamma_i)\}$ does not converge to 0, then we can pass to a subnet, relabel, and assume that there is a $\delta > 0$ such that

$$|F_j(x_i, \gamma_i)| \geq \delta \quad \text{for all } i.$$

This means that $f_j(x_i, \gamma_i) \neq 0$ for all i. Since f_j has compact support in A_j, we can pass to a subnet, relabel, and assume that $(x_i, \gamma_i) \to (y, \eta)$ in A_j. Since C is Hausdorff, $y = x$. Since V_j is Hausdorff, $\eta = \gamma$. Therefore $(x_i, \gamma_i) \to (x, \gamma)$ in A. Since F_j is continuous on A, $F_j(x, \gamma) \geq \delta$. Since $\delta > 0$, this is a contradiction. This completes the proof of the claim. $\square$

Since $C \times V_j$ is Hausdorff, we may approximate F_j in $C_c(C \times V_j)$ by sums of functions of the form $(x, \gamma) \mapsto g(x)h(\gamma)$, as in [28, Lemma 2.9] for example. Hence

$$x \mapsto \int_G F_j(x, \gamma) \, d\lambda_G^{s(x)}(\gamma)$$

is continuous.

Suppose that $\{x_i\}$ is a net in V such that $q(x_i) \to q(x)$ (with $x \in V$). If $q(x) \notin q(D)$, then since $q(D)$ is compact and hence closed in the Hausdorff set $q(V)$, we eventually have $q(x_i) \notin q(D)$. Thus we eventually have $\lambda(\psi)\big(q(x_i)\big) = 0$, and $\lambda(\psi)$ is continuous at $q(x)$. On the other hand, if $q(x) \in q(W)$, then we may as well assume that $x_i \to x$ in C. But on C,

$$x \mapsto \lambda(\psi)\big(q(x)\big) = \int_G F(x, \gamma)\, d\lambda_G^{s(x)}(\gamma) = \sum_j \int_G F_j(x, \gamma)\, d\lambda_G^{s(x)}(\gamma)$$

is continuous. This completes the proof of part (a).

For part (b), assume that $d \in C_c\big(q(V)\big)$. Then $\operatorname{supp}_{q(V)} d$ is of the form $q(K)$ for a compact set $K \subset V$. Let $g \in C_c(V)$ be strictly positive on K. Then $\lambda(g)$ is strictly positive on $\operatorname{supp}_{q(V)} d$, and $\lambda\big(g\lambda(g)d\big) = d$.   □

Lemma 2.20. *Suppose that H and G are locally Hausdorff, locally compact groupoids and that X is a (H, G)-equivalence. Let*

$$X *_s X = \{(x, y) \in X \times X : s(x) = s(y)\}.$$

Then $X *_s X$ is a principal G-space for the diagonal G-action. If $\tau(x, y)$ is the unique element in H such that $\tau(x, y) \cdot y = x$, then $\tau : X *_s X \to H$ is continuous and factors though the orbit map. Moreover, τ induces a homeomorphism of $X *_s X/G$ with H.

Proof. Clearly, $X *_s X$ is a principal G-space and τ is a well-defined map on $X *_s X$ onto H. Suppose that $\{(x_i, y_i)\}$ converges to (x, y). Passing to a subnet, and relabeling, it will suffice to show that $\{\tau(x_i, y_i)\}$ has a subnet converging to $\tau(x, y)$. Let L and K be Hausdorff compact neighborhoods of x and y, respectively. Since we eventually have $\{(\tau(x_i, y_i), y_i)\}$ in $\Theta^{-1}(K \times L)$, we can pass to a subnet, relabel, and assume that $\big(\tau(x_i, y_i), y_i\big) \to (\eta, z)$ in $\Theta^{-1}(K \times L)$. In particular, since L is Hausdorff, we must have $z = y$. Since $\eta \cdot y \in K$, $x_i \to \eta \cdot y$ and since K is Hausdorff, we must have $x = \eta \cdot y$. Thus $\eta = \tau(x, y)$. This shows that τ is continuous.

Clearly τ is G-equivariant. If $\tau(x, y) = \tau(z, w)$, then $s_X(x) = r_H\big(\tau(x, y)\big) = s_X(z)$. Since X is an equivalence, $z = x \cdot \gamma$ for some $\gamma \in G$. Similarly, $r_X(y) = s_H\big(\tau(x, y)\big) = r_X(w)$, and $y = w \cdot \gamma'$ for some $\gamma' \in G$. Therefore τ induces a bijection of $X *_s X$ onto H. To see that τ is open, and therefore a homeomorphism as claimed, suppose that $\tau(x_i, y_i) \to \tau(x, y)$. After passing to a subnet and relabeling, it will suffice to see that $\{(x_i, y_i)\}$ has a subnet converging to (x, y). Let L and K be Hausdorff compact neighborhoods of x and y, respectively. Since $\Theta^{-1}(L \times K)$ is compact, we can pass to a subnet, relabel, and assume that $\big(\tau(x_i, y_i), y_i\big) \to (\eta, z)$ in $\Theta^{-1}(L \times K)$. Since K is Hausdorff, $z = y$. On

the other hand, we must have $x_i = \tau(x_i, y_i) \cdot y_i \to \tau(x, y) \cdot y = x$. This completes the proof. $\square$

3. $C_0(X)$-algebras

A $C_0(X)$-*algebra* is a C^*-algebra A together with a nondegenerate homomorphism ι_A of $C_0(X)$ into the center of the multiplier algebra $M(A)$ of A. The map ι_A is normally suppressed and we write $f \cdot a$ in place of $\iota_A(f)a$. There is an expanding literature on $C_0(X)$-algebras which describe their basic properties; a partial list is [31, 20, 2, 10, 46]. An essential feature of $C_0(X)$-algebras is that they can be realized as sections of a bundle over X. Specifically, if $C_{0,x}(X)$ is the ideal of functions vanishing at $x \in X$, then $I_x := \overline{C_{0,x}(X) \cdot A}$ is an ideal in A, and $A(x) := A/I_x$ is called the fibre of A over x. The image of $a \in A$ in $A(x)$ is denoted by $a(x)$.

We are interested in fibred C^*-algebras as a groupoid G must act on the sections of a bundle that is fibred over the unit space (or over some G-space). In [40] and in [23], it was assumed that the algebra A was the section algebra of a C^*-bundle as defined, for example, by Fell in [12]. However recent work has made it clear that the notion of a C^*-bundle, or for that matter a Banach bundle, as defined in this way is unnecessarily restrictive, and that it is sufficient to assume only that A is a $C_0(G^{(0)})$-algebra [24, 25, 22, 21]. However, our approach here, as in [23] (and in [40]), makes substantial use of the total space of the underlying bundle. Although it predates the term "$C_0(X)$-algebra", the existence of a bundle whose section algebra is a given $C_0(X)$-algebra goes back to [16, 18, 17], and to [9]. We give some of the basic definitions and properties here for the sake of completeness.

This definition is a minor variation on [9, Definition 1.1].

Definition 3.1. An *upper-semicontinuous-Banach bundle* over a topological space X is a topological space $\mathscr{A}$ together with a continuous, open surjection $p = p_{\mathscr{A}} : \mathscr{A} \to X$ and complex Banach space structures on each fibre $\mathscr{A}_x := p^{-1}(\{x\})$ satisfying the following axioms:

B1: The map $a \mapsto \|a\|$ is upper semicontinuous from $\mathscr{A}$ to $\mathbf{R}^+$. (That is, for all $\epsilon > 0$, $\{a \in \mathscr{A} : \|a\| \geq \epsilon\}$ is closed.)

B2: If $\mathscr{A} * \mathscr{A} := \{(a, b) \in \mathscr{A} \times \mathscr{A} : p(a) = p(b)\}$, then $(a, b) \mapsto a + b$ is continuous from $\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$.

B3: For each $\lambda \in \mathbf{C}$, $a \mapsto \lambda a$ is continuous from $\mathscr{A}$ to $\mathscr{A}$.

B4: If $\{a_i\}$ is a net in $\mathscr{A}$ such that $p(a_i) \to x$ and such that $\|a_i\| \to 0$, then $a_i \to 0_x$ (where 0_x is the zero element in $\mathscr{A}_x$).

Since $\{a \in \mathscr{A} : \|a\| < \epsilon\}$ is open for all $\epsilon > 0$, it follows that whenever $a_i \to 0_x$ in $\mathscr{A}$, then $\|a_i\| \to 0$. Therefore the proof of [12, Proposition II.13.10] implies that

B3′: The map $(\lambda, a) \to \lambda a$ is continuous from $\mathbf{C} \times \mathscr{A}$ to $\mathscr{A}$.

Definition 3.2. An *upper-semicontinuous-C^*-bundle* is an upper-semicontinuous-Banach bundle $p_{\mathscr{A}} : \mathscr{A} \to X$ such that each fibre is a C^*-algebra and such that:

B5: The map $(a, b) \mapsto ab$ is continuous from $\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$.
B6: The map $a \mapsto a^*$ is continuous from $\mathscr{A}$ to $\mathscr{A}$.

If axiom B1 is replaced by

B1′: The map $a \mapsto \|a\|$ is continuous,

then $p : \mathscr{A} \to X$ is called a *Banach bundle* (or a *C^*-bundle*). Banach bundles are studied in considerable detail in §§13–14 of Chapter II of [12]. As mentioned above, the weaker notion of an upper-semicontinuous-Banach bundle is sufficient for our purposes. In fact, in view of the connection with $C_0(X)$-algebras described below, it is our opinion that upper-semicontinuous-Banach bundles, and in particular upper-semicontinuous-C^*-bundles, provide a more natural context in which to work.

If $p : \mathscr{A} \to X$ is an upper-semicontinuous-Banach bundle, then a continuous function $f : X \to \mathscr{A}$ such that $p \circ f = \mathrm{id}_X$ is called a *section*. The set of sections is denoted by $\Gamma(X; \mathscr{A})$. We say that $f \in \Gamma(X; \mathscr{A})$ *vanishes at infinity* if the the closed set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact for all $\epsilon > 0$. The set of sections which vanish at infinity is denoted by $\Gamma_0(X; \mathscr{A})$, and the latter is easily seen to be a Banach space with respect to the supremum norm: $\|f\| = \sup_{x \in X} \|f(x)\|$ (cf. [9, p. 10]); in fact, $\Gamma_0(X; \mathscr{A})$ is a Banach $C_0(X)$-module for the natural $C_0(X)$-action on sections.[8] In particular, the uniform limit of sections is a section. Moreover, if $p : \mathscr{A} \to X$ is an upper-semicontinuous-C^*-bundle, then the set of sections is clearly a $*$-algebra with respect to the usual pointwise operations, and $\Gamma_0(X; \mathscr{A})$ becomes a $C_0(X)$-algebra with the obvious $C_0(X)$-action. However, for arbitrary X, there is no reason to expect that there are any nonzero sections — let alone nonzero sections vanishing at infinity or which are compactly supported. An upper-semicontinuous-Banach bundle is said to have *enough sections* if given $x \in X$ and $a \in \mathscr{A}_x$ there is a section f such that $f(x) = a$. If X is a Hausdorff locally compact space and if $p : \mathscr{A} \to X$ is a Banach bundle, then a result of Douady and Soglio-Hérault implies there are enough

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[8]We also use $\Gamma_c(X; \mathscr{A})$ for the vector space of sections with compact support (i.e., $\{x \in X : f(x) \neq 0_x\}$ has compact closure).

sections [12, Appendix C]. Hofmann has noted that the same is true for upper-semicontinuous-Banach bundles over Hausdorff locally compact spaces [17] (although the details remain unpublished [16]). In the situation we're interested in — namely seeing that a $C_0(X)$-algebra is indeed the section algebra of an upper-semicontinuous-C^*-bundle — it will be clear that there are enough sections.

Proposition 3.3 (Hofmann, Dupré–Gillete). *If $p : \mathscr{A} \to X$ is an upper-semicontinuous-C^*-bundle over a locally compact Hausdorff space X (with enough sections), then $A := \Gamma_0(X; \mathscr{A})$ is a $C_0(X)$-algebra with fibre $A(x) = \mathscr{A}_x$. Conversely, if A is a $C_0(X)$-algebra then there is an upper-semicontinuous-C^*-bundle $p : \mathscr{A} \to X$ such that A is (isomorphic to) $\Gamma_0(X; \mathscr{A})$.*

Proof. This is proved in [46, Theorem C.26]. □

The next observation is useful and has a straightforward proof which we omit. (A similar result is proved in [46, Proposition C.24].)

Lemma 3.4. *Suppose that $p : \mathscr{A} \to X$ is an upper-semicontinuous-Banach bundle over a locally compact Hausdorff space X, and that B is a subspace of $A = \Gamma_0(X; \mathscr{A})$ which is closed under multiplication by functions in $C_0(X)$ and such that $\{f(x) : f \in B\}$ is dense in $A(x)$ for all $x \in X$. Then B is dense in A.*

As an application, suppose that $p : \mathscr{A} \to X$ is an upper-semicontinuous-C^*-bundle over a locally compact Hausdorff space X. Let $A = \Gamma_0(X; \mathscr{A})$ be the corresponding $C_0(X)$-algebra. If $\tau : Y \to X$ is continuous, then the pullback $\tau^*\mathscr{A}$ is an upper-semicontinuous-C^*-bundle over Y. If Y is Hausdorff, then as in [34], we can also form the the balanced tensor product $\tau^*(A) := C_0(Y) \otimes_{C_0(X)} A$ which is the quotient of $C_0(Y) \otimes A$ by the balancing ideal I_τ generated by

$$\{\varphi(f \circ \tau) \otimes a - \varphi \otimes f \cdot a : \varphi \in C_0(Y), f \in C_0(X) \text{ and } a \in A\}.$$

If $\varphi \in C_0(Y)$ and $a \in A$, then $\psi(\varphi \otimes a)(y) := \varphi(y)a\big(\tau(y)\big)$ defines a homomorphism of $C_0(Y) \otimes A$ into $\Gamma_0(Y; \tau^*\mathscr{A})$ which factors through $\tau^*(A)$, and has dense range in view of Lemma 3.4. As in the proof of [34, Proposition 1.3], we can also see that this map is injective and therefore an isomorphism. Since pullbacks of various sorts play a significant role in the theory, we will use this observation without comment in the sequel.

Remark 3.5. Suppose that $p : \mathscr{A} \to X$ is an upper-semicontinuous-C^*-bundle over a locally compact Hausdorff space X. If $\tau : Y \to X$ is continuous, then $f \in \Gamma_c(Y; \tau^*\mathscr{A})$ if and only if there is a continuous, compactly supported function $\check{f} : Y \to \mathscr{A}$ such that $p\big(\check{f}(y)\big) = \tau(y)$ and such that $f(y) = \big(y, \check{f}(y)\big)$. As is customary, we will not distinguish between f and $\check{f}$.

Suppose that $p : \mathscr{A} \to X$ and $q : \mathscr{B} \to X$ are upper-semicontinuous-C^*-bundles. As usual, let $A = \Gamma_0(X; \mathscr{A})$ and $B = \Gamma_0(X; \mathscr{B})$. Any continuous bundle map

(3.1)
$$\mathscr{A} \xrightarrow{\ \Phi\ } \mathscr{B}$$
$$p \searrow \quad \swarrow q$$
$$X$$

is determined by a family of maps $\Phi(x) : A(x) \to B(x)$. If each $\Phi(x)$ is a homomorphism (of C^*-algebras), then we call Φ a C^*-*bundle map*. A C^*-bundle map Φ induces a $C_0(X)$-homomorphism $\varphi : A \to B$ given by $\varphi(f)(x) = \Phi\big(f(x)\big)$.

Conversely, if $\varphi : A \to B$ is a $C_0(X)$-homomorphism, then we get homomorphisms $\varphi_x : A(x) \to B(x)$ given by $\varphi_x\big(a(x)\big) = \varphi(a)(x)$. Then $\Phi(x) := \varphi_x$ determines a bundle map $\Phi : \mathscr{A} \to \mathscr{B}$ as in (3.1). It is not hard to see that Φ must be continuous: Suppose that $a_i \to a$ in $\mathscr{A}$. Let $f \in A$ be such that $f\big(p(a)\big) = a$. Then $\varphi(f)\big(p(a)\big) = \Phi(a)$ and

$$\|\Phi(a_i) - \varphi(f)\big(p(a_i)\big)\| \le \|a_i - f\big(p(a_i)\big)\| \to 0.$$

Therefore $\Phi(a_i) \to \Phi(a)$ by the next lemma (which shows that the topology on the total space is determined by the sections).

Lemma 3.6. *Suppose that $p : \mathscr{A} \to X$ is an upper-semicontinuous-Banach-bundle. Suppose that $\{a_i\}$ is a net in $\mathscr{A}$, that $a \in \mathscr{A}$ and that $f \in \Gamma_0(X; \mathscr{A})$ is such that $f\big(p(a)\big) = a$. If $p(a_i) \to p(a)$ and if $\|a_i - f\big(p(a_i)\big)\| \to 0$, then $a_i \to a$ in $\mathscr{A}$.*

Proof. We have $a_i - f\big(p(a_i)\big) \to 0_{p(a)}$ by axiom B4. Hence

$$a_i = \big(a_i - f\big(p(a_i)\big)\big) + f\big(p(a_i)\big) \to 0_{p(a)} + a = a. \qquad \square$$

Remark 3.7. If $\Gamma_0(X; \mathscr{A})$ and $\Gamma_0(X; \mathscr{B})$ are isomorphic $C_0(X)$-algebras, then $\mathscr{A}$ and $\mathscr{B}$ are isomorphic as upper-semicontinuous-C^*-bundles. Hence in view of Proposition 3.3, every $C_0(X)$-algebra is the section algebra of a unique upper-semicontinuous-C^*-bundle (up to isomorphism).

Remark 3.8. If $\mathscr{A}$ and $\mathscr{B}$ are upper-semicontinuous-C^*-bundles over X and if $\Phi : \mathscr{A} \to \mathscr{B}$ is a C^*-bundle map such that each $\Phi(x)$ is an isomorphism, then Φ is bicontinuous and therefore a C^*-bundle isomorphism.

Proof. We only need to see that if $\Phi(a_i) \to \Phi(a)$, then $a_i \to a$. After passing to a subnet and relabeling, it suffices to see that $\{a_i\}$ has a subnet converging to a. But $p(a_i) = q\big(\Phi(a_i)\big)$ must converge to $p(a)$. Since p is open, we can pass to a subnet, relabel, and assume that there is a net

$b_i \to a$ with $p(b_i) = p(a_i)$. But we must then have $\Phi(b_i) \to \Phi(a)$. Then $\|b_i - a_i\| = \|\Phi(b_i - a_i)\| \to 0$. Therefore $b_i - a_i \to 0_{p(a)}$, and

$$a_i = (a_i - b_i) + b_i \to 0_{p(z)} + a = a. \qquad \square$$

If $p : \mathscr{E} \to X$ is an upper-semicontinuous-Banach bundle over a locally Hausdorff, locally compact space X, then as in the scalar case, there may not be any nonzero sections in $\Gamma_c(X; \mathscr{E})$. Instead, we proceed as in [40] and let $\mathscr{G}(X; \mathscr{E})$ be the complex vector space of functions from X to $\mathscr{E}$ spanned by sections in $\Gamma_c(U; \mathscr{E}|_U)$, were U is any open Hausdorff subset of X and $\mathscr{E}|_U := p^{-1}(U)$ is viewed as an upper-semicontinuous-Banach bundle over the locally compact Hausdorff space U.[9] We say $\mathscr{E}$ has enough sections if given $e \in \mathscr{E}$, there is a $f \in \mathscr{G}(X; \mathscr{E})$ such that $f\big(p(e)\big) = e$. By Hofmann's result [17], $\mathscr{E}$ always has enough sections.

Remark 3.9. Suppose that $p : \mathscr{E} \to X$ is an upper-semicontinuous-Banach bundle over a locally Hausdorff, locally compact space X. Then we say that a net $\{z_i\}_{i \in I}$ converges to z in the inductive limit topology on $\mathscr{G}(X; \mathscr{E})$ if $z_i \to z$ uniformly and if there is a compact set C in X such that all the z_i and z vanish off C. (We are not claiming that there is a topology on $\mathscr{G}(X; \mathscr{E})$ in which these are the *only* convergent nets.)

Lemma 3.10. *Suppose that Z is a locally Hausdorff, locally compact principal (right) G-bundle, that $p : \mathscr{B} \to Y$ is an upper-semicontinuous-Banach bundle and that $\sigma : Z/G \to Y$ is a continuous, open map. Let $q : Z \to Z/G$ be the orbit map. If $f \in \mathscr{G}(X; (\sigma \circ q)^* \mathscr{B})$, then the equation*

$$\lambda(f)\big(x \cdot G\big) = \int_G f(x \cdot \gamma) \, d\lambda^{s(x)}(\gamma)$$

defines an element $\lambda(f) \in \mathscr{G}(Z/G; \sigma^* \mathscr{B})$.

Proof. We can assume that $f \in \Gamma_c(V; (\sigma \circ q)^* \mathscr{B})$ with V a Hausdorff open set in Z. Using an approximation argument, it suffices to consider f of the form

$$f(x) = g(x)a\big(\sigma\big(q(x)\big)\big)$$

where $g \in C_c(V)$ and $a \in \Gamma_c(\sigma(q(V)); \mathscr{B})$. The result follows from Corollary 2.17. $\qquad \square$

Remark 3.11. The hypotheses in Lemma 3.10 may seem a bit stilted at first glance. However, they are precisely what are needed to handle induced bundle representations of groupoids. We will use this result is the situation

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[9]In the sequel, we will abuse notation a bit and simply write $\Gamma_c(U; \mathscr{E})$ rather than the more cumbersome $\Gamma_c(U; \mathscr{E}|_U)$.

where X is a principal G-space, $Z = X *_s X$, H is the associated imprimitivity groupoid $Y = H^{(0)}$, $\sigma([x,y]) = r_X(x)$ and $q = r_H$.

4. Groupoid crossed products

In this section we want to review what it means for a locally Hausdorff, locally compact groupoid G to act on a $C_0(G^{(0)})$-algebra $\mathscr{A}$ by isomorphisms. Such actions will be called *groupoid dynamical systems* and will be denoted $(\mathscr{A}, G, \alpha)$. We also discuss the associated crossed product $\mathscr{A} \rtimes_\alpha G$. Fortunately, there are several nice treatments in the literature upon which one can draw: [24, §7], [32], [25], [26, §2], [21, §1], [22, §2.4 & §3] and [40]. However, as in [24, §7], we intend to emphasize the underlying bundle structure. Otherwise, our treatment follows the excellent exposition in [21] and [22]. We remark that Renault uses the more restrictive definition of C^*-bundle in [40]. In our formulation, the groupoid analogue of a strongly continuous group of automorphisms of a C^*-algebra arises from a certain type of action of the groupoid on the total space of an upper-semicontinuous-C^*-bundle.

Definition 4.1. Suppose that G is a locally Hausdorff, locally compact groupoid and that A is a $C_0(G^{(0)})$-algebra such that $A = \Gamma_0(G^{(0)}; \mathscr{A})$ for an upper-semicontinuous-C^*-bundle $\mathscr{A}$ over $G^{(0)}$. An action α of G on A by $*$-isomorphisms is a family $\{\alpha_\gamma\}_{\gamma \in G}$ such that:

  (a) For each γ, $\alpha_\gamma : A\bigl(s(\gamma)\bigr) \to A\bigl(r(\gamma)\bigr)$ is an isomorphism.
  (b) $\alpha_{\eta\gamma} = \alpha_\eta \circ \alpha_\gamma$ for all $(\eta, \gamma) \in G^{(2)}$.
  (c) $\gamma \cdot a := \alpha_\gamma(a)$ defines a continuous action of G on $\mathscr{A}$.

The triple $(\mathscr{A}, G, \alpha)$ is called a (groupoid) dynamical system.

Our next lemma implies that our definition coincides with that in [22] (where the underlying bundle structure is not required), but first we insert a remark to help with the notation.

Remark 4.2. Suppose that $(\mathscr{A}, G, \alpha)$ is a dynamical system, and let A be the $C_0(G^{(0)})$-algebra, $\Gamma_0(G^{(0)}; \mathscr{A})$. We may pull back $\mathscr{A}$ to G with s and r to get two upper-semicontinuous-C^*-bundles $s^*\mathscr{A}$ and $r^*\mathscr{A}$ on G (See Remark 3.5). If U is a subset of G, we may restrict these bundles to U, getting bundles on U which we denote by $s|_U{}^*\mathscr{A}$ and $r|_U{}^*\mathscr{A}$. If U is open and Hausdorff in G, then we may form the $C_0(U)$-algebras, $\Gamma_0(U; s^*\mathscr{A})$ and $\Gamma_0(U; r^*\mathscr{A})$, which we denote by $s|_U{}^*(A)$ and $r|_U{}^*(A)$, respectively. Then, by the discussion in the two paragraphs preceding Lemma 3.6, we see that there bijective correspondence between bundle isomorphisms between $s|_U{}^*\mathscr{A}$ and $r|_U{}^*\mathscr{A}$ and $C_0(U)$-isomorphisms between $s|_U{}^*(A)$ and $r|_U{}^*(A)$. (See also the comments prior to Remark 3.5.)

Lemma 4.3. *Suppose that $(\mathscr{A}, G, \alpha)$ is a dynamical system and let A be the $C_0(G^{(0)})$-algebra, $\Gamma_0(G^{(0)}; \mathscr{A})$. If $U \subset G$ is open and Hausdorff, then*

$$(4.1) \qquad \alpha_U(f)(\gamma) := \alpha_\gamma\big(f(\gamma)\big)$$

defines a $C_0(U)$-isomorphism of $s|_U{}^*(A)$ onto $r|_U{}^*(A)$. If $V \subset U$ is open, then viewing $s|_V{}^*(A)$ as an ideal in $r|_U{}^*(A)$, α_V is the restriction of α_U.

Conversely, if A is a $C_0(G^{(0)})$-algebra and if for each open, Hausdorff subset $U \subset G$, there is a $C_0(U)$-isomorphism $\alpha_U : s|_U{}^*(A) \to r|_U{}^*A$ such that α_V is the restriction of α_U whenever $V \subset U$, then there are well-defined isomorphisms $\alpha_\gamma : A\big(s(\gamma)\big) \to A\big(r(\gamma)\big)$ satisfying (4.1). If in addition, $\alpha_{\eta\gamma} = \alpha_\eta \circ \alpha_\gamma$ for all $(\eta, \gamma) \in G^{(2)}$, then $(\mathscr{A}, G, \alpha)$ is a dynamical system.

Proof. If $(\mathscr{A}, G, \alpha)$ is a dynamical system, then the statements about the α_U are easily verified.

On the other hand, if the α_U are as given in the second part of the lemma, then the $\alpha_\gamma := (\alpha_U)_\gamma : A\big(s(\gamma)\big) \to A\big(r(\gamma)\big)$ are uniquely determined due to the compatibility condition on the α_U's. It only remains to check that $\gamma \cdot a := \alpha_\gamma(a)$ defines a continuous action of G on $\mathscr{A}$. Suppose that $(\gamma_i, a_i) \to (\gamma, a)$ in $\{(\gamma, a) : s(\gamma) = p(a)\}$. We need to prove that $\gamma_i \cdot a_i \to \gamma \cdot a$ in $\mathscr{A}$. We can assume that there is a Hausdorff neighborhood U of γ containing all γ_i. Let $g \in s|_U{}^*(A)$ be such that $g(\gamma) = a$. We have

$$\alpha_U(g)(\gamma_i) \to \alpha_U(g)(\gamma) := \gamma \cdot a.$$

Also

$$\|\alpha_U(g)(\gamma_i) - \gamma_i \cdot a_i\| = \|\alpha_{\gamma_i}\big(g(\gamma_i) - a_i\big)\| = \|g(\gamma_i) - a_i\| \to 0.$$

Therefore $\alpha_U(g)(\gamma_i) - \gamma_i \cdot a_i \to 0_{p(a)}$, and

$$\gamma_i \cdot a_i = \alpha_U(g)(\gamma_i) + \big(\gamma_i \cdot a_i - \alpha_U(g)(\gamma_i)\big) \to \gamma \cdot a + 0_{p(a)} = \gamma \cdot a. \qquad \square$$

In the Hausdorff case, the crossed product is a completion of the compactly supported sections $\Gamma_c(G; r^*\mathscr{A})$ of the pullback of $\mathscr{A}$ via the range map $r : G \to G^{(0)}$. In the non-Hausdorff case, we must find a substitute for $\Gamma_c(G; r^*\mathscr{A})$. As in Section 3, we let $\mathscr{G}(G; r^*\mathscr{A})$ be the subspace of functions from G to $\mathscr{A}$ spanned by elements in $\Gamma_c(U; r^*\mathscr{A})$ for all open, Hausdorff sets $U \subset G$. (Elements in $\Gamma_c(U; r^*\mathscr{A})$ are viewed as functions on G as in the definition of $\mathscr{C}(G)$.)

Proposition 4.4. *If G is a locally Hausdorff, locally compact groupoid with Haar system $\{\lambda^u\}$, then $\mathscr{G}(G; r^*\mathscr{A})$ becomes a $*$-algebra with respect to the operations*

$$f * g(\gamma) = \int_G f(\eta)\alpha_\eta\big(g(\eta^{-1}\gamma)\big) \, d\lambda^{r(\gamma)}(\eta) \quad and \quad f^*(\gamma) = \alpha_\gamma\big(f(\gamma^{-1})^*\big).$$

The proof of the proposition is fairly routine, the only real issue being to see that the formula for $f * g$ defines an element of $\mathscr{G}(G; r^*\mathscr{A})$. For this, we need a preliminary observation.

Lemma 4.5. *Suppose that U and W are Hausdorff open subsets of G. Let $U *_r W = \{(\eta, \gamma) : r(\eta) = r(\gamma)\}$, and let*

$$r^*\mathscr{A} = \{(\eta, \gamma, a) : r(\eta) = r(\gamma) = p(a)\}$$

be the pullback. If $F \in \Gamma_c(U *_r W; r^*\mathscr{A})$, then

$$f(\gamma) = \int_G F(\eta, \gamma) \, d\lambda^{r(\gamma)}(\eta)$$

defines a section in $\Gamma_c(W; r^*\mathscr{A})$.

Proof. If $F_i \to F$ in the inductive limit topology on $\Gamma_c(U *_r W; r^*\mathscr{A})$, then it is straightforward to check that $f_i \to f$ in the inductive limit topology on $\Gamma_c(W; r^*\mathscr{A})$. Thus it will suffice to consider F of the form $(\eta, \gamma) \mapsto h(\eta, \gamma) a(r(\gamma))$ for $h \in C_c(U *_r W)$ and $a \in \Gamma_c(G^{(0)}; \mathscr{A})$. Since $U *_r W$ is closed in $U \times W$, we can assume that h is the restriction of $H \in C_c(U \times W)$. Since sums of the form $(\eta, \gamma) \mapsto h_1(\eta) h_2(\gamma)$ are dense in $C_c(U *_r W)$, we may as well assume that $H(\eta, \gamma) = h_1(\eta) h_2(\gamma)$ for $h_1 \in C_c(U)$ and $h_2 \in C_c(W)$. But then

$$f(\gamma) = h_2(\gamma) a(r(\gamma)) \int_G h_1(\eta) \, d\lambda^{r(\gamma)}(\eta),$$

which is clearly in $\Gamma_c(W; r^*\mathscr{A})$. $\square$

Proof of Proposition 4.4. To see that convolution is well-defined, it suffices to see that $f * g \in \mathscr{G}(G; r^*\mathscr{A})$ when $f \in \Gamma_c(U; r^*\mathscr{A})$ and $g \in \Gamma_c(V; r^*\mathscr{A})$ for Hausdorff open sets U and V. We follow the argument of [21, p. 52–3]. Since U is Hausdorff, and therefore regular, there is an open set U_0 and a compact set K_f such that

$$\operatorname{supp} f \subset U_0 \subset K_f \subset U.$$

Given $\gamma \in \operatorname{supp} g$, we have $K_f \gamma \gamma^{-1} \subset U$. Even if $K_f \gamma \gamma^{-1}$ is empty, using the local compactness of G and the continuity of multiplication, we can find an open set W_γ and a compact set K_γ such that $\gamma \in W_\gamma \subset K_\gamma$ with

$$K_f K_\gamma K_\gamma^{-1} \subset U.$$

Claim 4.6. $U_0 W_\gamma$ *is Hausdorff.*

Proof. Suppose that $\eta_i \gamma_i$ converges to both α and β in $U_0 W_\gamma$. Since $W_\gamma \subset K_\gamma$, we can pass to a subnet, relabel, and assume that $\gamma_i \to \gamma \in K_\gamma$. Then η_i converges to both $\alpha \gamma^{-1}$ and $\beta \gamma^{-1}$. This the latter are both in

$U_0 W_\gamma K_\gamma^{-1} \subset K_f K_\gamma K_\gamma^{-1} \subset U$, and since U is Hausdorff, we must have $\alpha \gamma^{-1} = \beta \gamma^{-1}$. But then $\alpha = \beta$. This proves the claim. $\qquad\square$

Since $\operatorname{supp} g$ is compact, we can find open sets $U_1, \ldots, U_n$ and $W_1, \ldots, W_n$ such that $\operatorname{supp} g \subset \bigcup W_i$, $\operatorname{supp} f \subset U_i$ and $U_i W_i$ is Hausdorff. If we let $U' := \bigcap U_i$ and use a partition of unity to write $g = \sum g_i$ with each $\operatorname{supp} g_i \subset V_i$, then we can view $f \in C_c(U')$ and replace g by g_i. Thus we may assume that $f \in \Gamma_c(U; r^*\mathscr{A})$ and $g \in \Gamma_c(V; r^*\mathscr{A})$ for Hausdorff open sets U and V with UV Hausdorff as well. Next observe that $(\eta, \gamma) \mapsto (\eta, \eta\gamma)$ is a homeomorphism of $B := \{(\eta, \gamma) \in U \times V : s(\eta) = r(\gamma)\}$ onto an *open* subset B' of $U *_r UV$. On B', we can define a continuous function with compact support by

$$(\eta_1, \gamma_1) \mapsto f(\eta_1) \alpha_{\eta_1}\big(g(\eta_1^{-1}\gamma_1)\big).$$

Extending this function to be zero off the open subset B', we get a section (as in Remark 3.5) $\varphi \in \Gamma_c(U *_r UV; r^*\mathscr{A})$ such that

$$\varphi(\eta_1, \gamma_1) = f(\eta_1)g(\gamma_1) \quad \text{for all } (\eta_1, \gamma_1) \in B.$$

It follows from Lemma 4.5, that

$$f * g \in \Gamma_c(UV; r^*\mathscr{A}) \subset \mathscr{G}(G; r^*\mathscr{A}).$$

The remaining assertions required to prove the proposition are routine to verify. $\qquad\square$

If $f \in \mathscr{G}(G; r^*\mathscr{A})$, then $\gamma \mapsto \|f(\gamma)\|$ is upper-semicontinuous on open Hausdorff subsets, and is therefore integrable on G with respect to any Radon measure. Thus we can define the *I-norm* by

$$(4.2) \quad \|f\|_I = \max\left\{ \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| \, d\lambda^u(\gamma), \ \sup_{u \in G^{(0)}} \int_G \|f(\gamma)\| \, d\lambda_u(\gamma) \right\}.$$

The *crossed product* $\mathscr{A} \rtimes_\alpha G$ is defined to be the enveloping C^*-algebra of $\mathscr{G}(G; r^*\mathscr{A})$. Specifically, we define the (universal) C^*-norm by

$$\|f\| :=$$

$$\sup\{\, \|L(f)\| : L \text{ is a } \|\cdot\|_I\text{-decreasing } *\text{-representation of } \mathscr{G}(G; r^*\mathscr{A}) \,\}.$$

Then $\mathscr{A} \rtimes_\alpha G$ is the completion of $\mathscr{G}(G; r^*\mathscr{A})$ with respect to $\|\cdot\|$. (The notation $A \rtimes_\alpha G$ would also be appropriate; it is used in [22] for example. Since our approach is a bundle one, using a notation that includes the bundle seems appropriate.)

Example 4.7. As in the case of ordinary C^*-dynamical systems (see [46, Example 2.33]), the a groupoid C^*-algebra $C^*(G, \lambda)$ is a degenerate case of the groupoid crossed product. Let G be a locally Hausdorff, locally compact

groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$. Let $\mathscr{A} = \mathscr{T}_{G^{(0)}}$ be the trivial bundle $G^{(0)} \times \mathbf{C}$. Then G acts by isomorphisms on $\mathscr{T}_{G^{(0)}}$ by left translation:

$$\mathrm{lt}_\gamma^G\big(s(\gamma), z\big) = \big(r(\gamma), z\big).$$

Then it is routine to check that $\big(\mathscr{T}_{G^{(0)}}, G, \mathrm{lt}^G\big)$ is a dynamical system with $\mathscr{T}_{G^{(0)}} \rtimes_{\mathrm{lt}^G} G$ isomorphic to the groupoid C^*-algebra $C^*(G, \lambda)$.

Since a group is a groupoid whose unit space is a single point, we can view ordinary dynamical systems and their crossed products as trivial examples of groupoid dynamical systems and crossed products. A more interesting class examples arise as follows.

Example 4.8. Suppose that A is a $C_0(X)$-algebra and that X is a locally compact (Hausdorff) $\mathfrak{G}$-space for a locally compact group $\mathfrak{G}$. Suppose that

$$\check{\alpha} : \mathfrak{G} \to \mathrm{Aut}\, A$$

is a strongly continuous automorphism group such that

$$(4.3) \qquad\qquad \check{\alpha}_s(\varphi \cdot a) = \mathrm{lt}_s(\varphi) \cdot \check{\alpha}_s(a),$$

where $\mathrm{lt}_s(\varphi)(x) = \varphi(s^{-1} \cdot x)$. For example, if $\mathrm{Prim}\, A$ is Hausdorff, then we can let $X := \mathrm{Prim}\, A$. Then A is a $C_0(X)$-algebra (via the Dauns–Hofmann Theorem), and X is naturally a $\mathfrak{G}$-space such that (4.3) holds (see [46, Lemma 7.1]).

Let $G = \mathfrak{G} \times X$ be the transformation groupoid, and note that $A = \Gamma_0(X; \mathscr{A})$ for an upper-semicontinuous-bundle $\mathscr{A}$ [46, Theorem C.26]. Then we get a groupoid dynamical system $(\mathscr{A}, G, \alpha)$ where

$$(4.4) \qquad\qquad \alpha_{(s,x)}\big(a(s^{-1} \cdot x)\big) = \check{\alpha}_s(a)(x).$$

Then it is a matter of checking that $\mathscr{A} \rtimes_\alpha G$ is isomorphic to the ordinary crossed product $A \rtimes_{\check{\alpha}} \mathfrak{G}$ via the map that sends $f \in C_c(G, A)$ to $\widetilde{f} \in \Gamma_c(G; r^* \mathscr{A})$ given by

$$(4.5) \qquad\qquad \widetilde{f}(s, x) = \Delta(s)^{\frac{1}{2}} f(s)(x),$$

where Δ is the modular function on $\mathfrak{G}$.[10]   For example,

$$\Delta(s)^{\frac{1}{2}} f * g(s)(x) = \int_{\mathfrak{G}} \Delta(r)^{\frac{1}{2}} f(r) \check{\alpha}_r \big(\Delta(r^{-1}s)^{\frac{1}{2}} g(r^{-1}s) \big) \, ds \, (x)$$

$$= \int_{\mathfrak{G}} \Delta(r)^{\frac{1}{2}} f(r)(x) \check{\alpha}_r \big(\Delta(r^{-1}s)^{\frac{1}{2}} g(s^{-1}r) \big)(x) \, ds$$

$$= \int_{G} \widetilde{f}(r,x) \alpha_{(s,r)} \big(\widetilde{g}(s^{-1}r, s^{-1} \cdot x) \big) \, ds$$

$$= \widetilde{f} * \widetilde{g}(s,x).$$

5. Renault's Equivalence Theorem

In this section, we want to extend Renault's definition [40, Definition 5.3] of an equivalence between two dynamical systems $(H, \mathscr{B}, \beta)$ and $(\mathscr{A}, G, \alpha)$ to the setting of upper-semicontinuous-C^*-bundles, and to give a precise statement of the his Equivalence Theorem in this context. In doing so, we also give an explicit description of the pre-imprimitivity bimodule between $\mathscr{G}(H; r^*\mathscr{B})$ and $\mathscr{G}(G; r^*\mathscr{A})$.

Definition 5.1. An *equivalence* between dynamical systems $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$ is an upper-semicontinuous-Banach bundle $p_{\mathscr{E}} : \mathscr{E} \to X$ over a (H, G)-equivalence X together with $B\big(r(x)\big) - A\big(s(x)\big)$-imprimitivity bimodule structures on each fibre $\mathscr{E}_x$ and commuting (continuous) H- and G-actions on the left and right, respectively, of $\mathscr{E}$ such that the following additional properties are satisfied.

(a) (Continuity) The maps induced by the imprimitivity bimodule inner products from $\mathscr{E} * \mathscr{E} \to \mathscr{B}$ and $\mathscr{E} * \mathscr{E} \to \mathscr{A}$ are continuous as are the maps $\mathscr{B} * \mathscr{E} \to \mathscr{E}$ and $\mathscr{E} * \mathscr{A} \to \mathscr{E}$ induced by the imprimitivity bimodule actions.

(b) (Equivariance) The groupoid actions are equivariant with respect to the bundle map $p_{\mathscr{E}} : \mathscr{E} \to X$; that is, $p_{\mathscr{E}}(\eta \cdot e) = \eta \cdot p_{\mathscr{E}}(e)$ and $p_{\mathscr{E}}(e \cdot \gamma) = p_{\mathscr{E}}(e) \cdot \gamma$.

---

[10]The modular function is introduced simply because it is traditional to use the modular function as part of the definition of the involution on $C_c(\mathfrak{G}, A) \subset A \rtimes_{\check{\alpha}} \mathfrak{G}$ and we need $f \mapsto \widetilde{f}$ to be $*$-preserving. The indicated map on dense subalgebras is isometric because representations which are continuous in the inductive limit topology are in fact bounded with respect to the universal norms. For the same reason, it is possible, although not traditional, to define the involution on ordinary crossed products without the modular function. In the latter case, we could dispense with the modular function in (4.5).

(c) (Compatibility) The groupoid actions are compatible with the imprimitivity bimodule structure:

$$\begin{aligned}
{}_{\mathscr{B}}\langle \eta \cdot e \,,\, \eta \cdot f \rangle &= \beta_\eta\big({}_{\mathscr{B}}\langle e \,,\, f \rangle\big) & \eta \cdot (b \cdot e) &= \beta_\eta(b) \cdot (\eta \cdot e) \\
\langle e \cdot \gamma \,,\, f \cdot \gamma \rangle_{\mathscr{A}} &= \alpha_\gamma^{-1}\big(\langle e \,,\, f \rangle_{\mathscr{A}}\big) & (e \cdot a) \cdot \gamma &= (e \cdot \gamma) \cdot \alpha_\gamma^{-1}(a).
\end{aligned}$$

(d) (Invariance) The H-action commutes with the $\mathscr{A}$-action on $\mathscr{E}$ and the G-action commutes with the $\mathscr{B}$-action. That is, $\eta \cdot (e \cdot a) = (\eta \cdot e) \cdot a$ and $(b \cdot e) \cdot \gamma = b \cdot (e \cdot \gamma)$.

Lemma 5.2. *As a consequence of invariance we have*

$$
{}_{\mathscr{B}}\langle e \cdot \gamma \,,\, f \cdot \gamma \rangle = {}_{\mathscr{B}}\langle e \,,\, f \rangle \quad \text{and} \quad \langle \eta \cdot e \,,\, \eta \cdot f \rangle_{\mathscr{A}} = \langle e \,,\, f \rangle_{\mathscr{A}}
$$

for all $e, f \in \mathscr{E}$, $\eta \in H$ and $\gamma \in G$.

Proof. If $g \in \mathscr{E}$, then using invariance, we have

$$\begin{aligned}
{}_{\mathscr{B}}\langle e \,,\, f \rangle \cdot (g \cdot \gamma) &= \big({}_{\mathscr{B}}\langle e \,,\, f \rangle \cdot g\big) \cdot \gamma \\
&= \big(e \cdot \langle f \,,\, g \rangle_{\mathscr{A}}\big) \cdot \gamma \\
&= (e \cdot \gamma) \cdot \langle f \cdot \gamma \,,\, g \cdot \gamma \rangle_{\mathscr{A}} \\
&= {}_{\mathscr{B}}\langle e \cdot \gamma \,,\, f \cdot \gamma \rangle \cdot (g \cdot \gamma).
\end{aligned}$$

The first equation follows and the second follows by symmetry. $\qquad\square$

Remark 5.3. Since our inner products are full, the converse of Lemma 5.2 holds as well. That is, if the inner products are invariant under the "other" groupoid action, then invariance holds.

Example 5.4. An important and instructive example of Definition 5.1 is to see that $(\mathscr{A}, G, \alpha)$ is equivalent to itself via $p : r^*\mathscr{A} \to G$. Recall that

$$
r^*\mathscr{A} := \{(\gamma, a) \in G \times \mathscr{A} : r(\gamma) = p_{\mathscr{A}}(a)\}.
$$

We equip the fibre over γ with a $A\big(r(\gamma)\big) - A\big(s(\gamma)\big)$-imprimitivity bimodule structure as follows:

$$\begin{aligned}
{}_{A(r(\gamma))}\big\langle (\gamma \,,\, a), (\gamma, b)) \big\rangle &:= ab^* & a \cdot (\gamma, b) &:= (\gamma, ab) \\
\big\langle (\gamma \,,\, a), (\gamma, b)) \big\rangle_{A(s(\gamma))} &:= \alpha_\gamma^{-1}(a^*b) & (\gamma, b) \cdot a &= (\gamma, b\alpha_\gamma(a)).
\end{aligned}$$

We let G act on the right and left of $r^*\mathscr{A}$ as follows:

$$
\beta \cdot (\gamma, a) := (\beta\gamma, ab) \qquad (\gamma, a) \cdot \beta = (\gamma\beta, a).
$$

At this point, it is a simple matter to verify that axioms (a)–(d) of Definition 5.1 are satisfied.

Theorem 5.5 ([40, Corollaire 5.4]). *Suppose that G and H are second countable locally Hausdorff, locally compact groupoids with Haar systems $\{\lambda_G^u\}_{u \in G^{(0)}}$ and $\{\lambda_H^v\}_{v \in H^{(0)}}$, respectively. If $p_{\mathscr{E}} : \mathscr{E} \to X$ is an equivalence between $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$, then $\mathsf{X}_0 = \mathscr{G}(X; \mathscr{E})$ becomes a $\mathscr{B} \rtimes_\beta H - \mathscr{A} \rtimes_\alpha G$-pre-imprimitivity bimodule with respect to the following operations:*

$$(5.1) \quad {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z , w \rangle\!\rangle(\eta) := \int_{G^{\mathscr{B}}} \big\langle z(\eta \cdot x \cdot \gamma) , \eta \cdot w(x \cdot \gamma) \big\rangle \, d\lambda_G^{s(x)}(\gamma),$$

$$(5.2) \quad f \cdot z(x) := \int_H f(\eta) \cdot \big(\eta \cdot z(\eta^{-1} \cdot x) \big) \, d\lambda_H^{r(x)}(\eta),$$

$$(5.3) \quad z \cdot g(x) := \int_G \big(z(x \cdot \gamma) \cdot \gamma^{-1} \big) \cdot \alpha_\gamma\big(g(\gamma^{-1}) \big) \, d\lambda_G^{s(x)}(\gamma) \;\; and$$

$$(5.4) \quad \langle\!\langle w , v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}(\gamma) := \int_H \big\langle w(\eta^{-1} \cdot y \cdot \gamma^{-1}) , v(\eta^{-1} \cdot y) \cdot \gamma^{-1} \big\rangle_{\mathscr{A}} \, d\lambda_H^{r(y)}(\eta).$$

Remark 5.6. Since X is a (H, G)-equivalence, the equation $r(x) = r(y)$ implies that $y = x \cdot \gamma'$ for some $\gamma' \in G$. Thus in (5.1) we are free to choose any $x \in r_X^{-1}\big(s_H(\eta)\big)$. On the other hand, we can replace x in (5.1) by $y := \eta \cdot x$ and obtain

$$(5.1') \quad {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z , w \rangle\!\rangle(\eta) := \int_{G^{\mathscr{B}}} \big\langle z(y \cdot \gamma) , \eta \cdot w(\eta^{-1} \cdot y \cdot \gamma) \big\rangle \, d\lambda_G^{s(y)}(\gamma),$$

where any $y \in r_X^{-1}\big(r_G(\eta)\big)$ will do. Similarly, in (5.4), we are free to choose any $y \in s_X^{-1}\big(s_G(\gamma)\big)$.

Remark 5.7. Checking that (5.1)–(5.4) take values in the appropriate spaces of functions is a bit fussy in the non-Hausdorff case. We can suppose that $z \in \Gamma_c(U; \mathscr{E})$ and $w \in \Gamma_c(V; \mathscr{E})$, where U and V are Hausdorff open subsets of X. Then $U *_s V$ is a Hausdorff open subset of $X *_s X$. Let $q : X *_s X \to H$ be the "orbit" map (cf. Lemma 2.20). We get an element $f \in \Gamma_c(U *_s V; (r_H \circ q)^* \mathscr{B})$ defined by

$$f(x, y) := {}_{\mathscr{B}}\big\langle z(x) , \tau(x, y) \cdot w(y) \big\rangle,$$

where $\tau(x, y)$ is defined as in Lemma 2.20. Then the obscure hypotheses of Lemma 3.10 have been cooked up so that we can conclude that

$$ {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z , w \rangle\!\rangle(\sigma) = \lambda(f)\big(q(\sigma \cdot x, x)\big) $$

defines an element in $\Gamma_c(q(U *_s V); r_H^* \mathscr{B})$ as required.

To see that (5.2) defines an element of $\mathscr{G}(X; \mathscr{E})$, we proceed exactly as in the proof for the convolution in Proposition 4.4. We assume $f \in \Gamma_c(V; r^* \mathscr{B})$ and $z \in \Gamma_c(U; \mathscr{E})$. Using partitions of unity, we can assume that the open

set $V \cdot U$ is Hausdorff. The map $(\eta, x) \mapsto (\eta, \eta \cdot x)$ is a homeomorphism of $B = \{(\eta, x) \in V \times U : s(\eta) = r(x)\}$ onto an open subset B' of $V *_r V \cdot U = \{(\sigma, y) \in V \times V \cdot U : r(\sigma) = r(y)\}$. Hence the integrand in (5.2) is a section $h \in \Gamma_c(V *_r V \cdot U; r^* \mathscr{E})$. An argument analogous to that in Lemma 4.5 shows that $f \cdot z \in \mathscr{G}(X; \mathscr{E})$.

Remark 5.8. It is worth noting that, in Example 5.4, the inner-products and actions set out in (5.1)–(5.4) are the natural ones:

$$_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z, w \rangle\!\rangle = z * w^* \qquad\qquad f \cdot z = f * z$$

$$\langle\!\langle w, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} = w^* * v \qquad\qquad z \cdot g = z * g.$$

For example, we start with (5.1):

$$_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z, w \rangle\!\rangle(\eta) = \int_{G^{A(r(\gamma))}} \langle z(\eta \cdot x \cdot \gamma), \eta \cdot w(x \cdot \gamma)\rangle \, d\lambda_G^{s(x)}(\gamma)$$

$$= \int_G z(\eta\gamma)\eta \cdot w(\gamma)^* \, d\lambda_G^{s(\eta)}(\gamma)$$

$$= \int_G z(\gamma)\alpha_\eta\big(w(\eta^{-1}\gamma)^*\big) \, d\lambda_G^{r(\eta)}(\gamma)$$

$$= \int_G z(\gamma)\alpha_\gamma\big(w^*(\gamma^{-1}\eta)\big) \, d\lambda_G^{r(\eta)}(\gamma)$$

$$= z * w^*(\eta).$$

Similarly, we can start with (5.2):

$$f \cdot z(x) = \int_G f(\eta) \cdot \big(\eta \cdot z(\eta^{-1} \cdot x)\big) \, d\lambda_G^{r(x)}(\eta)$$

$$= \int_G f(\eta)\alpha_\eta\big(z(\eta^{-1}x)\big) \, d\lambda_G^{r(x)}(\eta)$$

$$= f * z(x).$$

We can do the same with (5.3) and (5.4), or appeal to symmetry (as described below).

It will be helpful to see that equivalence of dynamical systems is completely symmetric. Let $\mathscr{E}$ be an equivalence between $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$. Let $\mathscr{E}^*$ be the underlying topological space of $\mathscr{E}$, let $\flat : \mathscr{E} \to \mathscr{E}^*$ be the identity map and define $p^* : \mathscr{E}^* \to X^{\mathrm{op}}$ by $p^*\big(\flat(e)\big) = \big(p(e)\big)^{\mathrm{op}}$. Then as a Banach space, the fibre $\mathscr{E}^*_{x^{\mathrm{op}}} = \mathscr{E}_x$ and we can give $\mathscr{E}^*_{x^{\mathrm{op}}}$ the dual $A\big(r(x^{\mathrm{op}})\big) - B\big(s(x^{\mathrm{op}})\big)$-imprimitivity bimodule structure of the dual module $(\mathscr{E}_x)^*$. Then $p^* : \mathscr{E}^* \to X^{\mathrm{op}}$ is a $(\mathscr{A}, G, \alpha) - (\mathscr{B}, H, \beta)$ equivalence. Furthermore, if we define $\Phi : \mathscr{G}(X; \mathscr{E}) \to \mathscr{G}(X^{\mathrm{op}}; \mathscr{E}^*)$ by $\Phi(f)(x^{\mathrm{op}}) := \flat\big(f(x)\big)$,

then we can easily compute that

$$_{\mathscr{A}\rtimes_\alpha G}\langle\!\langle \Phi(z) \, , \, \Phi(w)\rangle\!\rangle(\gamma)$$

$$= \int_H {}_\mathscr{A}\langle \Phi(z)(\gamma\cdot x^{\mathrm{op}}\cdot\eta) \, , \, \gamma\cdot\Phi(w)(x^{\mathrm{op}}\cdot\eta)\rangle \, d\lambda_H^{s(x^{\mathrm{op}})}(\eta)$$

$$= \int_H {}_\mathscr{A}\langle {}^\flat\big(z(\eta^{-1}\cdot x\cdot\gamma^{-1})\big) \, , \, \gamma\cdot{}^\flat\big(w(\eta^{-1}\cdot x)\big)\rangle \, d\lambda_H^{r(x)}$$

$$= \int_G {}_\mathscr{A}\langle {}^\flat\big(z(\eta^{-1}\cdot x\cdot\gamma^{-1})\big)\big) \, , \, {}^\flat\big(w(\eta^{-1}\cdot x)\cdot\gamma^{-1}\big)\rangle \, d\lambda_H^{r(x)}$$

$$= \langle z(\eta^{-1}\cdot x\cdot\gamma^{-1})\big) \, , \, w(\eta^{-1}\cdot x)\cdot\gamma^{-1}\rangle_\mathscr{A} \, d\lambda_H^{r(x)}$$

$$= \langle\!\langle z \, , \, w\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G}.$$

Equally exciting computations give us the following lemma.

Lemma 5.9. *With* $\Phi:\mathscr{G}(X;\mathscr{E})\to\mathscr{G}(X^{\mathrm{op}};\mathscr{E}^*)$ *defined as above, we have*

$$_{\mathscr{A}\rtimes_\alpha G}\langle\!\langle \Phi(z) \, , \, \Phi(w)\rangle\!\rangle = \langle\!\langle z \, , \, w\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G} \qquad \langle\!\langle \Phi(w) \, , \, \Phi(v)\rangle\!\rangle_{\mathscr{B}\rtimes_\beta H} = {}_{\mathscr{B}\rtimes_\beta H}\langle\!\langle w \, , \, v\rangle\!\rangle$$

$$g\cdot\Phi(z) = \Phi(z\cdot g^*) \qquad\qquad\qquad \Phi(z)\cdot f = \Phi(f^*\cdot z).$$

This lemma can be very useful. For example, once we show that ${}_{\mathscr{B}\rtimes_\beta H}\langle\!\langle z \, , \, z\rangle\!\rangle$ is positive for all z, it follows by symmetry that

$$_{\mathscr{A}\rtimes_\alpha G}\langle\!\langle \Phi(z) \, , \, \Phi(z)\rangle\!\rangle > 0$$

for all z. But by Lemma 5.9, we must have $\langle\!\langle z \, , \, z\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G}$ positive.

Now for example, we show that the left-inner product respects left-module action:

$$_{\mathscr{B}\rtimes_\beta H}\langle\!\langle f\cdot z \, , \, w\rangle\!\rangle(\eta) = \int_G {}_\mathscr{B}\langle f\cdot z(\eta\cdot x\cdot\gamma) \, , \, \eta\cdot w(x\cdot\gamma)\rangle \, d\lambda_G^{s(x)}(\gamma)$$

$$= \int_H\int_G {}_\mathscr{B}\langle f(\sigma)\cdot\big(\sigma\cdot z\cdot(\sigma^{-1}\eta\cdot x\cdot\gamma)\big) \, , \, \eta\cdot w(x\cdot\gamma)\rangle\lambda_G^{s(x)}(\gamma) \, d\lambda_H^{r(\eta)}(\sigma)$$

$$= \int_H f(\sigma)\beta_\sigma\Big(\int_G {}_\mathscr{B}\langle z(\sigma^{-1}\eta\cdot x\cdot\gamma) \, , \, \sigma^{-1}\eta\cdot w(x\cdot\gamma)\rangle \, d\lambda_G^{s(x)}(\gamma)\Big)d\lambda_H^{r(\eta)}(\sigma)$$

$$= \int_H f(\sigma)\beta_\sigma\Big({}_{\mathscr{B}\rtimes_\beta H}\langle\!\langle z \, , \, w\rangle\!\rangle(\sigma^{-1}\gamma)\Big)d\lambda_H^{r(\eta)}(\sigma)$$

$$= f*{}_{\mathscr{B}\rtimes_\beta H}\langle\!\langle z \, , \, w\rangle\!\rangle(\eta).$$

By symmetry and by applying Lemma 5.9, it also follows that

$$\langle\!\langle w \, , \, v\cdot g\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G} = \langle\!\langle w \, , \, v\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G}*g.$$

Similar computations show that (5.2) defines a left-action, and it automatically follows that (5.3) is a right action by symmetry.

Next we check that

$$
\text{(5.5)} \qquad {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z \, , \, w \rangle\!\rangle \cdot v = z \cdot \langle\!\langle w \, , \, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}.
$$

(For the sake of honesty, not to mention motivation, we should admit that we started with (5.1) and (5.2), and then used (5.5) to compute what (5.3) and (5.4) should be.) Anyway, to check (5.5) we compute

$$
{}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z \, , \, w \rangle\!\rangle \cdot v(x) = \int_H {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z \, , \, w \rangle\!\rangle \cdot \big(\eta \cdot v(\eta^{-1} \cdot x)\big) \, d\lambda_H^{r(x)}(\eta)
$$

$$
= \int_H \int_G {}^{\mathscr{B}}\langle z(\eta \cdot y \cdot \gamma) \, , \, \eta \cdot w(y \cdot \gamma) \rangle \cdot \big(\eta \cdot v(\eta^{-1} \cdot x)\big) d\lambda_G^{s(y)}(\gamma) \, d\lambda_H^{s(x)}(\eta)
$$

which, after replacing y by $\eta^{-1} \cdot x$ and taking advantage of invariance (Definition 5.1(d)), is

$$
= \int_H \int_G {}^{\mathscr{B}}\langle z(x \cdot \gamma) \, , \, \eta \cdot w(\eta^{-1} \cdot x \cdot \gamma) \rangle \cdot \big(\eta \cdot v(\eta^{-1} \cdot x) \cdot \gamma\big) \cdot \gamma^{-1}
$$
$$
d\lambda_G^{s(y)}(\gamma) \, d\lambda_H^{s(x)}(\eta)
$$

which, since $\mathscr{E}_{x \cdot \gamma}$ is an imprimitivity bimodule, is

$$
= \int_H \int_G \big(z(x \cdot \gamma) \cdot \langle \eta \cdot w(\eta^{-1} \cdot x \cdot \gamma) \, , \, \eta \cdot v(\eta^{-1} \cdot x) \cdot \gamma \rangle_{\mathscr{A}}\big) \cdot \gamma^{-1}
$$
$$
d\lambda_G^{s(y)}(\gamma) \, d\lambda_H^{s(x)}(\eta)
$$

which, in view of Lemma 5.2, is

$$
= \int_H \int_G \big(z(x \cdot \gamma) \cdot \gamma^{-1}\big) \cdot \alpha_\gamma\big(\langle w(\eta^{-1} \cdot x \cdot \gamma) \, , \, v(\eta^{-1} \cdot x) \cdot \gamma \rangle_{\mathscr{A}}\big)
$$
$$
d\lambda_G^{s(y)}(\gamma) \, d\lambda_H^{s(x)}(\eta)
$$

$$
= \int_G \big(z(x \cdot \gamma) \cdot \gamma^{-1}\big) \cdot \alpha_\gamma\big(\langle\!\langle w \, , \, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}(\gamma^{-1})\big) \, d\lambda_G(\gamma)
$$
$$
= z \cdot \langle\!\langle w \, , \, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}(x).
$$

Example 5.10 (The scalar case). Suppose that G and H are second countable locally Hausdorff, locally compact groupoids with Haar systems $\{\lambda^u\}_{u \in G^{(0)}}$ and $\{\beta^v\}_{v \in H^{(0)}}$, respectively. Then if X is a (H, G)-equivalence, we can make $\mathscr{T}_X = X \times \mathbf{C}$ into a $(\mathscr{T}_{H^{(0)}}, H, \mathrm{lt}^H) - (\mathscr{T}_{G^{(0)}}, G, \mathrm{lt}^G)$-equivalence in the obvious way. Then Theorem 5.5 implies that $C^*(H, \beta) \cong \mathscr{T}_{H^{(0)}} \rtimes_{\mathrm{lt}^H} H$

and $C^*(G, \lambda) \cong \mathscr{T}_{G^{(0)}} \rtimes_{\mathrm{lt} G} G$ are Morita equivalent. Therefore we recover the main theorem from [28].

Example 5.11 (Morita equivalence over T)**.** Let $p_{\mathscr{A}} : \mathscr{A} \to T$ and $p_{\mathscr{B}} : \mathscr{B} \to T$ be upper-semicontinuous-C^*-bundles over a locally compact Hausdorff space T. As usual, let $A = \Gamma_0(T; \mathscr{A})$ and $B = \Gamma_0(T; \mathscr{B})$ be the associated $C_0(T)$-algebras. We can view the topological space T as a groupoid — the so-called *co-trivial groupoid* — and then we get dynamical systems $(\mathscr{A}, T, \mathrm{id})$ and $(\mathscr{B}, T, \mathrm{id})$. If $q : \mathscr{X} \to T$ is a $(\mathscr{A}, T, \mathrm{id}) - (\mathscr{B}, T, \mathrm{id})$-equivalence, then in the case $p_{\mathscr{A}}$ and $p_{\mathscr{B}}$ are C^*-bundles, q is what we called an $\mathscr{A} - \mathscr{B}$-imprimitivity bimodule bimodule in [23, Definition 2.17].[11] As in [23, Proposition 2.18], $\mathsf{X} := \Gamma_0(T; \mathscr{X})$ is a $A -_T B$-imprimitivity bimodule. Just as in the Banach bundle case, the converse holds: if X is a $A -_T B$-imprimitivity bimodule, then there is an upper-semicontinuous-Banach bundle $q : \mathscr{X} \to T$ such that $X \cong \Gamma_0(T; \mathscr{X})$. In the Banach bundle case, this follows from [12, Theorem II.13.18 and Corollary II.14.7]. The proof in the upper-semicontinuous-Banach bundle case is similar (and invokes [9, Proposition 1.3]).

Example 5.12 (Raeburn's Symmetric Imprimitivity Theorem)**.** Perhaps *the* fundamental Morita equivalence result for ordinary crossed products is the Symmetric Imprimitivity Theorem due to Raeburn [33]. We want to see here that, at least in the separable case, the result follows from Theorem 5.5. We follow the notation and treatment from [46, Theorem 4.1]. The set-up is as follows. We have commuting free and proper actions of locally compact groups $\mathfrak{K}$ and $\mathfrak{H}$ on the left and right, respectively, of a locally compact space P together with commuting actions $\check{\alpha}$ and $\check{\beta}$ on a C^*-algebra D. In order to apply the equivalence theorem, we assume that $\mathfrak{K}$, $\mathfrak{H}$ and P are second countable and that D is separable.

Then, as in [46, §3.6], we can form the induced algebras $B := \mathrm{Ind}_{\mathfrak{H}}^P(D, \check{\beta})$ and $A := \mathrm{Ind}_{\mathfrak{K}}^P(D, \check{\alpha})$, and the diagonal actions

$$\check{\sigma} : \mathfrak{K} \to \mathrm{Aut}\,\mathrm{Ind}_{\mathfrak{H}}^P(D, \check{\beta}) \quad \text{and} \quad \check{\tau} : \mathfrak{H} \to \mathrm{Aut}\,\mathrm{Ind}_{\mathfrak{K}}^P(D, \check{\alpha})$$

defined in [46, Lemma 3.54]. The Symmetric Imprimitivity Theorem implies that

(5.6) $\mathrm{Ind}_{\mathfrak{H}}^P(D, \check{\beta}) \rtimes_{\check{\sigma}} \mathfrak{K}$ is Morita equivalent to $\mathrm{Ind}_{\mathfrak{K}}^P(D, \check{\alpha}) \rtimes_{\check{\tau}} \mathfrak{H}$.

Since $B = \mathrm{Ind}_{\mathfrak{H}}^P(D, \check{\beta})$ is clearly a $C_0(P/\mathfrak{H})$-algebra, $B = \Gamma_0(P/\mathfrak{H}; \mathscr{B})$ for an upper-semicontinuous-C^*-bundle $\mathscr{B}$. The fibre $B(p \cdot \mathfrak{H})$ over $p \cdot \mathfrak{H} \in P/gH$ can be identified with $\mathrm{Ind}_{\mathfrak{H}}^{p \cdot \mathfrak{H}}(D, \beta)$. Of course, for any $q \in p \cdot \mathfrak{H}$, the map $f \mapsto$

---

[11]In [23, Definition 2.17], the hypothesis that the inner products should be continuous on $\mathscr{X} * \mathscr{X}$ was inadvertently omitted.

$f(q)$ identifies $B(p \cdot \mathfrak{H})$ with A. However, this identification is not natural, and we prefer to view $B(p \cdot \mathfrak{H})$ as functions on $p \cdot \mathfrak{H}$. It will be convenient to denote elements of $\mathscr{B}$ as pairs $(p \cdot \mathfrak{H}, f)$ where f is an appropriate function on $p \cdot \mathfrak{H}$. As in Example 4.8, we can realize $\mathrm{Ind}_{\mathfrak{H}}^P(D, \check{\beta}) \rtimes_{\check{\sigma}} \mathfrak{K}$ as a groupoid crossed product $\mathscr{B} \rtimes_\sigma H$, where H is the transformation groupoid $H := \mathfrak{K} \times P/\mathfrak{H}$ and $\sigma_{(t, p \cdot \mathfrak{H})}$ is defined as follows. Given $f \in \mathrm{Ind}_{\mathfrak{H}}^P(\beta)$, we can view $f|_{t^{-1} \cdot p \cdot \mathfrak{H}}$ as an element of $B(t^{-1} \cdot p \cdot \mathfrak{H})$, and we get an element of $B(p \cdot \mathfrak{H})$ by

$$\sigma_{(t, p \cdot \mathfrak{H})}(f)(q) = \check{\alpha}_t\big(f(t^{-1} \cdot q)\big).$$

In a similar way, we can realize $\mathrm{Ind}_{\mathfrak{K}}^P(D, \check{\alpha}) \rtimes_{\check{\tau}} \mathfrak{H}$ as a groupoid crossed product $\mathscr{A} \rtimes_\tau G$ where G is the transformation groupoid $G := \mathfrak{K} \backslash P \times \mathfrak{H}$ and $\tau_{(h, K \cdot p)}$ is given by

$$\tau_{(h, K \cdot p)}(f)(q) = \check{\beta}_h\big(f(q \cdot h)\big).$$

We want to derive (5.6) from the equivalence theorem by showing that the trivial bundle $\mathscr{E} = P \times A$ is a $(\mathscr{B}, H, \tau) - (\mathscr{A}, G, \sigma)$ equivalence. We have to equip $\mathscr{E}_p = \{(p, a) : a \in D\}$ with a $B(p \cdot \mathfrak{H}) - A(\mathfrak{K} \cdot p)$-imprimitivity bimodule structure and specify the H and G actions on $\mathscr{E}$. Standard computations show that we get an imprimitivity bimodule structure using the following inner-products and actions:

$$_{B(\mathfrak{K} \cdot p)}\big\langle (p\,,a), (p,b) \big\rangle(p \cdot h) = \check{\beta}_h^{-1}(ab^*) \qquad (p \cdot \mathfrak{H}, f) \cdot (p, a) = (p, f(p)a)$$

$$\big\langle (p\,,a), (p,b) \big\rangle_{A(\mathfrak{K} \cdot p)}(t \cdot p) = \check{\alpha}_t(a^*b) \qquad (p, a) \cdot (\mathfrak{K} \cdot p, f) = (p, af(p)).$$

The H and G actions are given by

$$(t, p \cdot \mathfrak{H}) \cdot (t^{-1} \cdot p, a) = (p, \check{\alpha}_t(a)) \qquad (p, a) \cdot (h, \mathfrak{K} \cdot p) = (p \cdot h, \check{\beta}_h^{-1}(a)).$$

Since P is a (H, G)-equivalence, it is now simply a matter of checking axioms (a), (b), (c) and (d) of Definition 5.1.

Checking part (a) (Continuity) at first seems awkward because the bundles $\mathscr{A}$ and $\mathscr{B}$ are only specified indirectly. However we can do what we need using sections. For example, we have the following observation.

Lemma 5.13. *The map $\mathscr{E} * \mathscr{E} \to \mathscr{B}$ is continuous if and only if $p \mapsto {}_{\mathscr{B}}\big\langle f(p)\,, g(p)\big\rangle$ is in $\Gamma_c(P; r_P^* \mathscr{B})$ for all $f, g \in \Gamma_c(P; \mathscr{E})$.*

Proof. The $(\Longrightarrow)$ direction is immediate. For the other direction, assume that $a_i \to a$ and $b_i \to b$ in $\mathscr{E}$ with $p(a_i) = x_i = p(b_i)$ converging to $p(a) = x = p(b)$. Then we need to see that ${}_{\mathscr{B}}\langle a_i\,, b_i \rangle \to {}_{\mathscr{B}}\langle a\,, b \rangle$ in $\mathscr{B}$. For this, it suffices to find a section $F \in \Gamma_c(P; r_P^* \mathscr{B})$ such that $F(x) = {}_{\mathscr{B}}\langle a\,, b \rangle$ and such that $\|F(x_i) - {}_{\mathscr{B}}\langle a_i\,, b_i \rangle\| \to 0$ (see Lemma 3.6). Thus, we can take $F(p) := {}_{\mathscr{B}}\big\langle f(p)\,, g(p)\big\rangle$, where $f(x) = a$ and $g(x) = b$. $\qquad \square$

However, even with Lemma 5.13 in hand, there is still a bit of work to do. Let $P *_r P := \{(p, q) \in P \times P : p \cdot \mathfrak{H} = q \cdot \mathfrak{H}\}$. Then $P *_r P$ is locally compact and the properness of the action implies that there is a continuous map $\theta : P *_r P \to \mathfrak{H}$ such that $q \cdot \theta(q, p) = p$. We know from [34], for example, that $r_P^*(B) = \Gamma_0(P; r_P^* \mathscr{B})$ is (isomorphic to) the balanced tensor product

$$C_0(P) \otimes_{C_0(P/\mathfrak{H})} B.$$

Therefore sections in $\Gamma_0(P; r_P^* \mathscr{B})$ are given by continuous bounded D-valued functions on $P *_r P$ such that $p \mapsto \|F(p, \cdot)\|$ vanishes at infinity on P and such that

$$F(p, q \cdot h) = \beta_h^{-1}\big(F(p, q)\big).$$

Therefore if $f, g \in C_c(P, D)$, then we get a section F in $\Gamma_0(P; r_P^* \mathscr{B})$ by defining

$$F(p, q) = {}_{\mathscr{B}}\big\langle f(p), g(p) \big\rangle(q) = {}_{\mathscr{B}}\big\langle f(p), g(p) \big\rangle\big(p \cdot \theta(p, q)\big) = \beta_{\theta(q,p)}\big(f(p)g(p)^*\big).$$

Then the continuity of the map from $\mathscr{E} * \mathscr{E} \to \mathscr{B}$ can be derived easily from Lemma 5.13. The rest of part (a) follows similarly.

Part (b) (Equivariance) is built in, and both part (c) (Compatibility) and part (d) (Invariance) follow from straightforward computations. Thus $\mathscr{E}$ is the desired equivalence.

We will return to the proof of the equivalence theorem in §8. In the meantime, we need to build up a bit of technology. In particular, we need some special approximate identities, and we need to know that representations of crossed products are the integrated form of covariant representations in a manner that parallels that for ordinary dynamical systems.

6. Approximate identities

In this section, we assume throughout that $\mathscr{E}$ implements an equivalence between the groupoid dynamical systems $(H, \mathscr{B}, \beta)$ and $(\mathscr{A}, G, \alpha)$ as laid out in Definition 5.1. Notice that since H and G are possibly non-Hausdorff locally Hausdorff, locally compact groupoids, we have to allow that our (H, G)-equivalence X may not be Hausdorff as well.

Lemma 6.1. *Let* $B = \Gamma_0(H^{(0)}; \mathscr{B})$ *act on* $\mathscr{G}(X; \mathscr{E})$ *in the natural way:* $b \cdot z(x) := b\big(r(x)\big) \cdot z(x)$. *If* $\{b_l\}$ *is an approximate identity for* B, *then for all* $z \in \mathscr{G}(X; \mathscr{E})$, $b_l \cdot z$ *converges to* z *in the inductive limit topology.*

Proof. Fix $\epsilon > 0$ and a Hausdorff open set $U \subset X$. Let $z \in \Gamma_c(U; \mathscr{E})$. It will suffice to see that there is an l_0 such that $l \geq l_0$ implies that

$$\|b_l\big(r(x)\big) \cdot z(p) - z(p)\| < \epsilon \quad \text{for all } p.$$

Let C be a compact subset of U such that z vanishes off U. Since $\mathscr{E}_x$ is a left Hilbert $B\big(r(x)\big)$-module, $b_l\big(r(x)\big) \cdot z(x)$ converges to $z(x)$ for each x.[12] Since $e \mapsto \|e\|$ is upper semicontinuous, there is a cover of C by open sets $V_1, \ldots V_n$ such that $V_i \subset U$ and such that there is a $a_i \in \Gamma_0(H^{(0)}; \mathscr{B})$ such that

$$\|a_i\big(r(x)\big) \cdot z(x) - z(x)\| < \delta \quad \text{for all } x \in V_i,$$

where $\delta = \min(\epsilon/3, \epsilon/(3\|z\|_\infty + 1))$. Let $\varphi_i \in C_c^+(U)$ be such that $\operatorname{supp}\varphi_i \subset V_i$ and such that $\sum \varphi_i(x) = 1$ if $x \in C$. Define $a \in \Gamma_c(U; r^*\mathscr{B})$ by

$$a(x) = \sum_i a_i\big(r(x)\big)\varphi_i(x).$$

Then for all $x \in X$,

(6.1) $$\|a(x) \cdot z(x) - z(x)\| < \delta.$$

We can find a l_0 such that $l \geq l_0$ implies

$$\|b_l\big(r(x)\big)a_i\big(r(x)\big) - a_i\big(r(x)\big)\| < \frac{\epsilon}{3} \quad \text{for all } i \text{ and all } x.$$

Then

(6.2) $$\|b_l\big(r(x)\big) \cdot a(x) - a(x)\| \leq \sum_i \|b_l\big(r(x)\big)a_i\big(r(x)\big) - a_i\big(r(x)\big)\|\varphi_i(x) < \frac{\epsilon}{3}.$$

If $l \geq l_0$, we have $\|b_l\big(r(x)\big) \cdot z(x) - z(x)\|$ bounded by

$$\|b_l\big(r(x)\big)\big(z(x) - a(x) \cdot z(x)\big)\| + \|\big(b_l\big(r(x)\big)a(x) - a(x)\big) \cdot z(x)\|$$
$$+ \|a(x) \cdot z(x) - z(x)\|$$

Since $\|b_l\big(r(x)\big)\| \leq 1$ for and x, and in view of (6.1) and (6.2), the above is bounded by ϵ. This completes the proof. $\qquad\square$

Lemma 6.2. *Suppose that U is a Hausdorff open subset of X. Then $\Gamma_c(U; \mathscr{E})$ becomes a left pre-Hilbert $\Gamma_0(U; r^*\mathscr{B})$-module where the left action and pre-inner product are given by*

$$b \cdot z(x) := b(x)z(x) \quad \text{and} \quad {}_{\Gamma_0(U;r^*\mathscr{B})}\langle z, w\rangle(x) := {}_{B(r(x))}\langle z(x), w(x)\rangle.$$

Proof. The only issues are the positivity of the inner product and the density of the span of the range of the inner product. But since every irreducible representation of the $C_0(U)$-algebra $\Gamma_0(U; r^*\mathscr{B})$ factors through a fibre, to show positivity it will suffice to see that for each x, ${}_{B(r(x))}\langle z(x), w(x)\rangle \geq 0$

---

[12]The relative topology on $\mathscr{E}_x$ is the Banach space topology [9, p. 10].

in $B(r(x))$. However, this follows since $\mathscr{E}$ is an equivalence. Furthermore, the ideal

$$I_x = \text{span}\left\{ {}_{\Gamma_0(U;r^*\mathscr{B})}\langle z , w\rangle(x) : z, w \in \Gamma_c(U;\mathscr{E}|_U) \right\}$$

$$= \text{span}\left\{ {}_{B(r(x))}\langle z(x) , w(x)\rangle : z, w \in \Gamma_c(U;\mathscr{E}|_U) \right\}$$

is dense in the fibre $\Gamma_0(U;r^*\mathscr{B})(x)$ over x. Since the ideal I spanned by the inner product is a $C_0(X)$-module, it follows that I is dense in $\Gamma_0(U;r^*\mathscr{B})$.

$\square$

We also need the following observation which was used in [45, p. 75] with an inadequate reference.

Lemma 6.3. *Suppose that* X *is a full right Hilbert A-module. Then sums of the form*

$$\sum_{i=1}^{n} \langle x_i , x_i\rangle_A$$

are dense in A^+.

Remark 6.4. We'll actually need the left-sided version of the result. But this follows immediately by taking the dual module. (Note that a sum is really required; think of the usual $\mathcal{K}(\mathcal{H})$-valued inner product on a Hilbert space $\mathcal{H}$.)

Proof. Fix $a \in A^+$. Then $a = b^*b$ and since X is full, we can approximate b by a sum

$$\sum_{i=1}^{r} \langle x_i , y_i\rangle_A .$$

Therefore we can approximate a by

(6.3) $$\sum_{ij} \langle x_j , y_j\rangle_A^* \langle x_i , y_i\rangle_A = \sum_{ij} \langle x_i \cdot \langle x_j , y_j\rangle_A , y_i\rangle_A$$

$$= \sum_{ij} \langle \Theta_{x_i,x_j}(y_j) , y_i\rangle_A .$$

But $M := (\Theta_{x_i,x_j})$ is a positive matrix in $M_r(\mathcal{K}(\mathsf{X}))$ ([35, Lemma 2.65]). Thus there is a matrix $(T_{ij}) \in M_r(\mathcal{K}(\mathsf{X}))$ such that

$$\Theta_{x_i,x_j} = \sum_{i=1}^{r} T_{ik}^* T_{jk} .$$

Then (6.3) equals

$$\sum_{ijk} \langle T_{jk}(y_j)\, ,\, T_{ik}(y_i) \rangle_A = \sum_k \langle z_k\, ,\, z_k \rangle_A,$$

where

$$z_k = \sum_i T_{ik}(y_i).$$

This completes the proof.   □

Corollary 6.5. *Suppose that b is a positive element in $B = \Gamma_0(H^{(0)}; \mathscr{B})$, that C is compact subset of a Hausdorff open subset U of X. If $\epsilon > 0$, then there are $z_1, \dots, z_n \in \Gamma_c(U; \mathscr{E})$ such that*

$$\left\| b(r(x)) - \sum_{i=1}^n {}_{B(r(x))}\langle z_i(x)\, ,\, z_i(x) \rangle \right\| < \epsilon \quad \text{for all } x \in C.$$

Proof. There is a $d \in \Gamma_0(U; r^*\mathscr{B})$ such that $d(x) = b(r(x))$ for all $x \in C$. In view of Lemma 6.2, Lemma 6.3 implies that there are z_i such that

$$\left\| d(x) - \sum_i {}_{B(r(x))}\langle z_i(x)\, ,\, z_i(x) \rangle \right\| < \epsilon$$

for all x. This suffices.   □

Since we plan to build an approximate identity, we need to recognize one when we see one.

Proposition 6.6. *Let $\{b_l\}_{l \in L}$ be an approximate identity for $B = \Gamma_0(H^{(0)}; \mathscr{B})$. Suppose that for each 4-tuple (K, U, l, ϵ) consisting of a compact subset $K \subset H^{(0)}$, a conditionally compact neighborhood U of $H^{(0)}$ in H, $l \in L$ and $\epsilon > 0$ there is a*

$$e = e_{(K,U,l,\epsilon)} \in \mathscr{G}(H; r^*\mathscr{B})$$

such that:

(a) $\operatorname{supp} e \subset U$.

(b) $\displaystyle \int_H \|e(\eta)\| \, d\lambda_H^u(\eta) \le 4$ *for all $u \in K$.*

(c) $\displaystyle \left\| \int_H e(\eta) \, \lambda_H^u(\eta) - b_l(u) \right\| < \epsilon$ *for all $u \in K$.*

Then $\{e_{(K,U,l,\epsilon)}\}$, directed by increasing K and l, and decreasing U and ϵ, is an approximate identity in the inductive limit topology for both the left action of $\mathscr{G}(H; r^*\mathscr{B})$ on itself, and of $\mathscr{G}(H; r^*\mathscr{B})$ on $\mathscr{G}(X; \mathscr{E})$.

Proof. In view of Example 5.4, it suffices to treat just the case of the action of $\mathscr{G}(H; r^*\mathscr{B})$ on $\mathscr{G}(X; \mathscr{E})$. Let V be a Hausdorff open subset of X and let $z \in \Gamma_c(V; \mathscr{E})$. It will suffice to see that $e_m \cdot z \to z$ in the inductive limit topology.

Let $K_1 := \operatorname{supp}_V z$. Lemma 2.14 implies that there is a diagonally compact neighborhood W_1 of $H^{(0)}$ in H such that $K_2 := W_1 \cdot K_1 \subset V$. Using Lemma 2.13, and shrinking W_1 a bit if necessary, we can also assume that $W_1 r(K_2)$ is Hausdorff.

Claim 6.7. *There is a conditionally compact neighborhood U_1 of $H^{(0)}$ in H such that $U_1 \subset W_1$ and such that $\eta \in U_1$ implies that*

$$(6.4) \qquad \|\eta \cdot z(\eta^{-1} \cdot x) - z(x)\| < \epsilon \quad \text{for all } x \in X.$$

Proof. Notice that if the left-hand side of (6.4) is nonzero, then we must have x in the compact set K_2. Therefore if the claim were false, then for each $U \subset W_1$ there would be a $\eta_U \in U$ and a $x_U \in K_2$ such that

$$(6.5) \qquad \|\eta_U \cdot z(\eta_U^{-1} \cdot x_U) - z(x_U)\| \geq \epsilon.$$

Since we must also have each η_U in the compact set $W_1 \cdot r(K_2)$, and since each x_U is in the compact set K_2, there are subnets $\{\eta_{U_a}\}$ and $\{x_{U_a}\}$ converging to $\eta \in W_1 r(K_2)$ and $x \in K_2$, respectively. For any $U \subset W_1$, we eventually have η_{U_a} in $U r(K_2) \subset W_1 r(K_2)$. Since $W_1 r(K_2)$ is Hausdorff, we must have $\eta \in U r(K_2)$ for all U. Therefore $\eta \in r(K_2)$ in view of Remark 2.12. Therefore $\{\eta_{U_a}^{-1} \cdot x_{U_a}\}$ converges to x in V. Thus $\eta_{U_a} \cdot z(\eta_{U_a}^{-1} \cdot x_{U_a}) \to z(x)$ in $\mathscr{E}$. Since $e \mapsto \|e\|$ is upper semicontinuous, this eventually contradicts (6.5). This completes the proof of the claim. $\square$

Lemma 6.1 implies that we can choose l_1 such that $l \geq l_1$ implies

$$\|b_l(r(x))z(x) - z(x)\| < \epsilon \quad \text{for all } x \in X.$$

If $e = e_{(K,U,l,\epsilon)}$ with $K \supset r(K_2)$, $U \subset U_1$ and $l \geq l_1$, then $\|e \cdot z(x) - z(x)\| = 0$ if $r(x) \notin K$ and if $r(x) \in K$ we compute that

$$\|e \cdot z(x) - z(x)\| \leq \left\| \int_H e(\eta)\big(\eta \cdot z(\eta^{-1} \cdot x) - z(x)\big)\, d\lambda_H^{r(x)}(\eta) \right\|$$

$$+ \left\| \left(\int_H e(\eta)\, d\lambda_H^{r(x)}(\eta) - b_l(r(x)) \right) z(x) \right\|$$

$$+ \left\| b_l(r(x))z(x) - z(x) \right\|$$

$$\leq 4\epsilon + \epsilon\|z\|_\infty + \epsilon.$$

Since $\operatorname{supp}(e \cdot z) \subset K_2$, this suffices. $\square$

Now we can state and prove the key result we require on approximate identities. It is a natural extension of [28, Proposition 2.10] to our setting. In fact, we will make considerable use of the constructions from [28].

Proposition 6.8. *There is a net $\{e_\lambda\}$ in $\mathscr{G}(H; r^*\mathscr{B})$ consisting of elements of the form*

$$e_\lambda = \sum_{i=1}^{n_\lambda} {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z_i^\lambda , z_i^\lambda \rangle\!\rangle$$

with each $z_i^\lambda \in \mathscr{G}(X; \mathscr{E})$, which is an approximate identity for the left action of $\mathscr{G}(H; r^*\mathscr{B})$ on itself and on $\mathscr{G}(X; \mathscr{E})$.

Proof. We will apply Proposition 6.6. Let $\{b_l\}$ be as in that proposition, and let (K, U, l, ϵ) be given.

Let $O_1, \ldots, O_n$ be pre-compact Hausdorff open sets in X such that $\{r(O_i)\}$ cover K. Let $\{h_i\} \subset C_c^+(H^{(0)})$ be such that $\operatorname{supp} h_i \subset r(O_i)$ and such that

$$\sum_{i=1}^n h_i(u) = 1 \quad \text{if } u \in K, \text{ and} \quad \sum_{i=1}^n h_i(u) \leq 1 \quad \text{for all } u.$$

Let C_i be a compact set in O_i such that

$$(6.6) \qquad\qquad r(C_i) = K \cap \operatorname{supp} h_i.$$

Notice that $\bigcup r(C_i) = K$, and that there are compact neighborhoods D_i of C_i such that $D_i \subset O_i$.

For each i, we will produce e_i, which is a sum of inner-products required in the proposition, with the additional properties that:

(P1) $\operatorname{supp} e_i \subset U$.
(P2) If $u \in K$, then

$$\int_H \|e_i(\eta)\| \, d\lambda_H^u(\eta) \leq 2\Big(h_i(u) + \frac{1}{n}\Big).$$

(P3) If $u \in K$, then

$$\left\|\int_H e_i(\eta) \, d\lambda_H^u(\eta) - h_i(u)b_l(u)\right\| < \frac{\epsilon}{n}.$$

Then if $e := \sum e_i$, we certainly have $\operatorname{supp} e \subset U$. Furthermore, if $u \in K$, then

$$\int_H \|e(\eta)\| \, d\lambda_H^u(\eta) \leq \sum_{i=1}^n \int_H \|e_i(\eta)\| \, d\lambda_H^u(\eta)$$

$$\leq \sum_{i=1}^n 2\left(h_i(u) + \frac{1}{n}\right)$$

$$\leq 4.$$

Moreover, if $u \in K$, then

$$\left\| \int_H e(\eta) \, d\lambda_H^u - b_l(u) \right\| \leq \sum_{i=1}^n \left\| \int_H e_i(\eta) \, d\lambda_H^u(\eta) - h_i(u)b_l(u) \right\| < \epsilon.$$

Therefore it will suffice to produce e_i's as described above.

Fix i, and let $\delta = \min\left(\frac{1}{2}, \frac{2}{n}, \frac{\epsilon}{5}\right)$. Use Corollary 6.5 to find $z_j \in \Gamma_c(O_i; \mathscr{E})$ such that

(6.7) $$\left\| h_i\big(r(x)\big)b_l\big(r(x)\big) - \sum_{j=1}^m {}_{B(r(x))}\big\langle z_j(x) \, , \, z_j(x) \big\rangle \right\| < \delta \quad \text{for all } x \in D_i.$$

To make some of the formulas in the sequel a little easier to digest, we introduce the notation

$$\Upsilon(\eta, y) := \sum_{j=1}^m {}_{B(r(x))}\big\langle z_j(y) \, , \, \eta \cdot z_j(\eta^{-1} \cdot y) \big\rangle.$$

Notice that the summation in (6.7) is $\Upsilon\big(r(x), x\big)$.

Claim 6.9. *There is a conditionally compact neighborhood W of $H^{(0)}$ in H such that $W \subset U$ and such that $\eta \in W$ implies that*

(6.8) $$\|\Upsilon(\eta, y) - \Upsilon\big(r(y), y\big)\| < \delta \quad \text{for all } y \in X.$$

Proof. The proof follows the lines of the proof of Claim 6.7. We just sketch the details here.

Let K_0 be a compact subset of O_i such that for all $1 \leq j \leq m$ we have $z_j(x) = 0$ if $x \notin K_0$. Let W_1 be a diagonally compact neighborhood of $H^{(0)}$ in H such that $W_1 \cdot K_0 \subset O_i$ and such that $W_1 r(K_0)$ is Hausdorff. If the claim where false, then for each $W \subset W_1$ we could find an $x_W \in W_1 \cdot K_0$ and an $\eta_W \in W \cap \big(W_1 r(K_0)\big)$ such that

(6.9) $$\|\Upsilon(\eta_W, x_W) - \Upsilon\big(r(x_W), x_W\big)\| \geq \delta > 0.$$

We could then pass to subnet, relabel, and assume that $x_W \to x \in W_1 \cdot K_0$ and that $\eta_W \to r(x)$. Since the net would eventually fall in O_i, $\Upsilon(\eta_W, x_W) \to \Upsilon\big(r(x), x\big)$, which would eventually contradict (6.9).   □

We repeat some of the constructions from [28, Proposition 2.10] — taking care to remain in the Hausdorff realm. Let $V_1, \dots, V_k$ be pre-compact open sets contained in D_i which cover C_i, and such that $(x, x \cdot \eta) \in V_j \times V_j$ implies that $\eta \in W$.

Since $\{r(V_j)\}$ covers $r(C_i)$, there are $d_j \in C_c^+(H^{(0)})$ such that $\operatorname{supp} d_i \subset r(V_i)$, $\sum_j d_j(u) = 1$ if $u \in r(C_i)$, and $\sum_j d_j(u) \le 1$ for all u. Since the G-action on X is free and proper, Proposition 2.16 implies that there are $\psi_j \in C_c^+(V_j)$ such that

$$ d_j\big(r(x)\big) = \int_G \psi_j(x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma). $$

Since the V_j are all contained in D_i, there is a constant M such that

$$ M := \sup_{x \in X} \sum_{j=1}^{k} \int_G \mathbb{1}_{V_i}(x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma). $$

(To see this, let $\xi_j \in C_c^+(O_i)$ be such that $\xi_j(x) = 1$ for all $x \in V_j$. Proposition 2.16 implies that $\lambda(\xi_j) \in C_c(X/G)$. Then $M \le \sum_{j=1}^{k} \|\lambda(\xi_j)\|_\infty$.)

Using [28, Lemma 2.14], we can find $\varphi_j \in C_c^+(O_i)$ with $\operatorname{supp} \varphi_j \subset V_j$ such that

$$ (6.10) \qquad \left| \psi_j(x) - \varphi_j(x) \int_H \varphi_j(\eta^{-1} \cdot x) \, d\lambda_H^{r(x)}(\eta) \right| < \frac{\delta}{M}. $$

The point is that

$$ \left| \int_H \int_G \sum_{j=1}^{k} \varphi_j(x \cdot \gamma) \varphi_j(\eta^{-1} \cdot x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^{r(x)}(\eta) - \sum_{j=1}^{k} d_j\big(r(x)\big) \right| $$

$$ = \left| \int_H \int_G \sum_{j=1}^{k} \varphi_j(x \cdot \gamma) \varphi_j(\eta^{-1} \cdot x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^{r(x)}(\eta) \right. $$

$$ \left. - \sum_{j=1}^{k} \int_G \psi_j(x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \right| $$

$$ = \left| \int_G \left(\sum_{i=1}^{k} \varphi_j(x \cdot \gamma) \int_H \varphi_j(\eta^{-1} \cdot x \cdot \gamma) \, d\lambda_H^{r(x)}(\eta) - \psi_j(x \cdot \gamma) \right) d\lambda_G^{s(x)}(\gamma) \right| $$

$$ \le \frac{\delta}{M} \int_G \mathbb{1}_{V_j}(x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) < \delta $$

To make the formulas easier to read, let

$$F(\eta, y) := \sum_{j=1}^{k} \varphi_j(y)\varphi_j(\eta^{-1} \cdot y).$$

Notice that our choice of V_j's implies that

(6.11) $$F(\eta, y) = 0 \quad \text{if } \eta \notin W \text{ or } y \notin D_i.$$

Then the above calculation implies that if $r(x) \in r(C_i)$, then

(6.12) $$\left| \int_H \int_G F(\eta, x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^{r(x)}(\eta) - 1 \right| < \delta,$$

while $\delta < \frac{1}{2}$ implies that we always have

(6.13) $$0 \le \int_H \int_G F(\eta, x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^{r(x)}(\eta) < 1 + \delta \le 2.$$

Define

$$w_{jp}(x) := \varphi_j(x)z_p(x) \quad \text{and} \quad e_i(\eta) := \sum_{jp} {}_{\mathscr{B} \rtimes_\beta H}\langle\!\langle w_{jp} , w_{jp} \rangle\!\rangle(\eta).$$

Using $(5.1')$, we have

$$e_i(\eta) = \int_G F(\eta, x \cdot \gamma)\Upsilon(\eta, x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma).$$

If $\eta \in W$, then we chose W such that

(6.14) $$\|\Upsilon(\eta, y) - \Upsilon\big(r(y), y\big)\| < \delta \quad \text{for all } y.$$

On the other hand, if $y \in D_i$, then we also have

(6.15) $$\|\Upsilon\big(r(y), y\big) - h_i\big(r(y)\big)b\big(r(y)\big)\| < \delta.$$

Since we always have $\|b_l(u)\| \le 1$, it follows that

(6.16) $$\|\Upsilon(\eta, y)\| \le h_i\big(r(y)\big) + 2\delta \le h_i\big(r(y)\big) + \frac{1}{n}$$

provided $\eta \in W$ and $y \in D_i$.

Next we want to see that e_i satisfies Properties (P1)–(P3). Since (6.11) implies that $\operatorname{supp} e_i \subset W$ and since we chose $W \subset U$, condition (P1) is clearly satisfied.

On the other hand, if $u \in K$ and $r(x) = u$, then

$$\int_H \|e_i(\eta)\| \, d\lambda_H^u(\eta) \le \int_H \int_G F(\eta, x \cdot \gamma)\|\Upsilon(\eta, x \cdot \gamma)\| \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^u(\eta)$$

which, since $F(\eta, x \cdot \gamma) = 0$ unless $\eta \in W$ and $x \cdot \gamma \in D_i$ allows us to use (6.16), is

$$\leq \left(h_i(u) + \frac{1}{n}\right) \int_G \int_H F(\eta, x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^u(\eta)$$

which, by (6.13), is

$$\leq 2\left(h_i(u) + \frac{1}{n}\right).$$

Thus, (P2) is verified.

Similarly,

$$\left\| \int_H e_i(\eta) \, d\lambda_H^u(\eta) - h_i(u)b_l(u) \right\|$$

$$= \left\| \int_H \int_G F(\eta, x \cdot \gamma)\Upsilon(\eta, x \cdot \gamma) \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^u(\eta) - h_i\big(r(x)\big)b_l\big(r(x)\big) \right\|$$

$$\leq \int_H \int_G F(\eta, x \cdot \gamma)\|\Upsilon(\eta, x \cdot \gamma) - h_i\big(r(x)\big)b_l\big(r(x)\big)\| \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^u(\eta)$$

$$+ \int_H \int_G |F(\eta, x \cdot \gamma) - 1| \, d\lambda_G^{s(x)}(\gamma) \, d\lambda_H^u(\eta)\|b_l\big(r(x)\big)\|.$$

Keeping in mind that $F(\eta, x \cdot \gamma)$ vanishes off $W \times D_i$, the first of these integrals is bounded by 4δ in view of (6.13), (6.14) and (6.15). Using (6.12) and the fact that $\|b_l\| \leq 1$, the second integral is bounded by δ. Our choice of δ implies that $5\delta < \epsilon$. Therefore (P3) is satisfied, and the proposition follows from Proposition 6.6. $\qquad\square$

7. Covariant representations

A critical ingredient in understanding groupoid crossed products (or groupoid C^*-algebras for that matter) is Renault's Proposition 4.2 in his 1987 *Journal of Operator Theory* paper [40] (cf. Theorem 7.8). To appreciate it fully, and to make the necessary adjustments to generalize it to crossed products (Theorem 7.12), we review unitary representations of groupoids.

Let μ be a Radon measure on $G^{(0)}$. We get Radon measures ν and ν^{-1} on G via the equations

$$(7.1) \qquad \nu(f) := \int_{G^{(0)}} \int_G f(\gamma) \, d\lambda^u(\gamma) \, d\mu(u) \quad \text{for } f \in \mathscr{C}(G),$$

$$(7.2) \qquad \nu^{-1}(f) := \int_{G^{(0)}} \int_G f(\gamma) \, d\lambda_u(\gamma) \, d\mu(u) \quad \text{for } f \in \mathscr{C}(G).$$

In the event ν and ν^{-1} are equivalent measures, we say that μ is *quasi-invariant*. The modular function $d\nu/d\nu^{-1}$ is denoted by Δ. Thus

$$(7.3) \qquad \int_{G^{(0)}} \int_G f(\gamma)\Delta(\gamma)\, d\lambda_u(\gamma)\, d\mu(u) = \int_{G^{(0)}} \int_G f(\gamma)\, d\lambda^u(\gamma)\, d\mu(u).$$

Remark 7.1. Of course Δ is only determined ν-almost everywhere. However, Δ can always be chosen to be a homomorphism from G to the positive reals, $\mathbf{R}_\times^+$. The details are spelled out in the proof of [27, Theorem 3.15]. The idea is this: Owing to [14, Corollary 3.14] and [39, Proposition I.3.3], *any* choice of the Radon–Nikodym derivative Δ is what is called an almost everywhere homomorphism of G into $\mathbf{R}_\times^+$. This means that the set of points $(\gamma_1, \gamma_2) \in G^{(2)}$ such that $\Delta(\gamma_1\gamma_2) \neq \Delta(\gamma_1)\Delta(\gamma_2)$ is a null set with respect to the measure

$$\nu^{(2)} := \int_{G^{(0)}} \lambda_u \times \lambda^u \, d\mu(u).$$

Since G is σ-compact, [36, Theorem 5.2] and [38, Theorem 3.2] together imply that any almost everywhere homomorphism from G to any analytic *groupoid* is equal to a homomorphism almost everywhere.

As noted in [27, Remark 3.18], quasi-invariant measures are easy to come by. Let μ_0 be any probability measure on $G^{(0)}$ and let $\nu_0 := \mu_0 \circ \lambda$ be as in (7.1). Then ν_0 is σ-finite and is equivalent to a probability measure ν on G. As show in [39, pp. 24–25], $\mu = s_*\nu$ (that is, $\mu(E) = \nu\big(s^{-1}(E)\big)$ is quasi-invariant, and it is also equivalent to μ_0 if μ_0 was quasi-invariant to begin with.

Given a quasi-invariant measure, the next step on the way to building unitary representations of groupoids is a *Borel Hilbert Bundle over a space* X. As explained in [27], these are nothing more or less than the total space of a direct integral of Hilbert spaces *a la* Dixmier. (See also [37, p. 264] and [46, Appendix F]) We start with a collection

$$\mathscr{H} := \{\mathcal{H}(x)\}_{x \in X}$$

of complex Hilbert spaces. Then the total space is the disjoint union

$$X * \mathscr{H} := \{(x, h) : h \in \mathcal{H}(x)\},$$

and we let $\pi : X * \mathscr{H} \to X$ be the obvious map.

Definition 7.2. Let $\mathscr{H} = \{\mathcal{H}(x)\}_{x \in X}$ be a family of Hilbert spaces. Then $(X * \mathscr{H}, \pi)$ is an analytic (standard) Borel Hilbert Bundle if $X * \mathscr{H}$ has an analytic (standard) Borel structure such that:

(a) E is a Borel subset of X if and only if $\pi^{-1}(E)$ is Borel in $X * \mathscr{H}$.
(b) There is sequence $\{f_n\}$ of sections such that:

(i) The maps $\widetilde{f}_n : X * \mathscr{H} \to \mathbf{C}$ are each Borel where
$$\widetilde{f}_n(x,h) := \big(f_n(x) \mid h\big).$$

(ii) For each n and m,
$$x \mapsto \big(f_n(x)|f_m(x)\big)$$
is Borel.

(iii) The functions $\{\widetilde{f}_n\}$, together with π, separate points of $X * \mathscr{H}$.

Remark 7.3. A section $f : X \to X * \mathscr{H}$ is Borel if and only if
$$x \mapsto \big(f(x)|f_n(x)\big)$$
is Borel for all n. In particular, if $B(X * \mathscr{H})$ is the set of Borel sections and if $f \in B(X * \mathscr{H})$, then $x \mapsto \|f(x)\|$ is Borel. If μ is a measure on X, then the quotient $L^2(X * \mathscr{H}, \mu)$ of
$$\mathcal{L}^2(X * \mathscr{H}, \mu) = \big\{ f \in B(X * \mathscr{H}) : x \mapsto \|f(x)\|^2 \text{ is integrable} \big\},$$

where functions agreeing μ-almost everywhere are identified, is a Hilbert space with the obvious inner product. Thus $L^2(X * \mathscr{H}, \mu)$ is nothing more than the associated *direct integral*
$$\int_X^{\oplus} \mathcal{H}(x) \, d\mu(x).$$

Definition 7.4. If $X * \mathscr{H}$ is a Borel Hilbert Bundle, then its *isomorphism groupoid* is the groupoid
$$\mathrm{Iso}(X * \mathscr{H}) := \big\{ (x, V, y) : V \in U(\mathcal{H}(y), \mathcal{H}(x)) \big\}$$
with the weakest Borel structure such that
$$(x, V, y) \mapsto \big(V f(y)|g(x)\big)$$
is Borel for all $f, g \in B(X * \mathscr{H})$.

As a Borel space, $\mathrm{Iso}(X * \mathscr{H})$ is analytic or standard whenever X has the same property.

With the preliminaries in hand, we have the machinery to make the basic definition for the analogue of a unitary representation of a group. Note that we must fix a Haar system in order to make sense of quasi-invariant measures.

Definition 7.5. A unitary representation of a groupoid G with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ is a triple $(\mu, G^{(0)} * \mathscr{H}, L)$ consisting of a quasi-invariant measure μ on $G^{(0)}$, a Borel Hilbert bundle $G^{(0)} * \mathscr{H}$ over $G^{(0)}$ and a Borel homomorphism $\hat{L} : G \to \mathrm{Iso}(G^{(0)} * \mathscr{H})$ such that
$$\hat{L}(\gamma) = \big(r(\gamma), L_\gamma, s(\gamma)\big).$$

Recall that the $\|\cdot\|_I$-norm was defined in (4.2).

Proposition 7.6. *If $(\mu, G^{(0)} * \mathscr{H}, L)$ is a unitary representation of a locally Hausdorff, locally compact groupoid G, then we obtain a $\|\cdot\|_I$-norm bounded representation of $\mathscr{C}(G)$ on*

$$\mathcal{H} := \int_{G^{(0)}}^{\oplus} \mathcal{H}(u)\, d\mu(x) = L^2(G^{(0)} * \mathscr{H}, \mu),$$

called the integrated form of $(\mu, G^{(0)} * \mathscr{H}, L)$, determined by

$$(7.4) \qquad \big(L(f)h \mid k\big) = \int_G f(\gamma)\big(L_\gamma\big(h(s(\gamma))\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma).$$

Remark 7.7. Equation (7.4) is convenient as it avoids dealing with vector-valued integration. However, it is sometimes more convenient in computations to realize that (7.4) is equivalent to

$$(7.5) \qquad L(f)h(u) = \int_G f(\gamma)L_\gamma\big(h(s(\gamma))\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(\gamma).$$

These sorts of vector-valued integrals are discussed in [46, §1.5]. In any event, showing that L is a homomorphism of $\mathscr{C}(G)$ into $B(\mathcal{H})$ is fairly straightforward and requires only that we recall that Δ is a homomorphism (at least almost everywhere). The quasi-invariance, in the form of Δ, is used to show that L is $*$-preserving. These assertions will follow from the more general results for covariant representations proved in Proposition 7.11.

We turn our attention now to the principal result in the theory: [40, Proposition 4.2]. A proof in the Hausdorff case is given in [27]. This result provides very general conditions under which a representation of a groupoid C^*-algebra is the integrated form of a unitary representation of the groupoid. In fact, it covers representations of $\mathscr{C}(G)$ acting on pre-Hilbert spaces. A complete proof will be given in Appendix B, but for the remainder of this section, we will show how it may be extended to representations of groupoid crossed products $\mathscr{G}(G; r^*\mathscr{A})$ in the setting of not-necessarily-Hausdorff locally compact groupoids acting on upper-semicontinuous-C^*-bundles (see Theorem 7.12).

Theorem 7.8 (Renault's Proposition 4.2). *Suppose that $\mathcal{H}_0$ is a dense subspace of a complex Hilbert space $\mathcal{H}$. Let L be a homomorphism from $\mathscr{C}(G)$ into the algebra of linear maps on $\mathcal{H}_0$ such that:*

(a) *$\{L(f)h : f \in \mathscr{C}(G) \text{ and } h \in \mathcal{H}_0\}$ is dense in $\mathcal{H}$.*

(b) *For each $h, k \in \mathcal{H}_0$,*

$$f \mapsto \big(L(f)h \mid k\big)$$

is continuous in the inductive limit topology on $\mathscr{C}(G)$.

(c) *For $f \in \mathscr{C}(G)$ and $h, k \in \mathcal{H}_0$ we have*

$$\big(L(f)h \mid k\big) = \big(h|L(f^*)k\big).$$

Then each $L(f)$ is bounded and extends to an operator $\overline{L}(f)$ on $\mathcal{H}$ of norm at most $\|f\|_I$. Furthermore, $\overline{L}$ is a representation of $\mathscr{C}(G)$ on $\mathcal{H}$ and there is a unitary representation $(\mu, G^{(0)} * \mathscr{H}, U)$ of G such that $\mathcal{H} \cong L^2(G^{(0)} * \mathscr{H}, \mu)$ and $\overline{L}$ is (equivalent to) the integrated form of $(\mu, G^{(0)} * \mathscr{H}, U)$.

Returning to the situation where we have a covariant system $(\mathscr{A}, G, \alpha)$, let $(\mu, G^{(0)} * \mathscr{H}, U)$ be a unitary representation and let

$$\mathcal{H} = \int_{G^{(0)}}^{\oplus} \mathcal{H}(u) \, d\mu(u) = L^2(G^{(0)} * \mathscr{H}, \mu)$$

be the associated Hilbert space. Recall that $D \in B(\mathcal{H})$ is called *diagonal* if there is a bounded Borel function $\varphi \in L^\infty(\mu)$ such that $D = L_\varphi$, where by definition

$$L_\varphi h(u) = \varphi(u) h(u).$$

The set of diagonal operators $\mathcal{D}$ is an abelian von Neumann subalgebra of $B(\mathcal{H})$. The general theory of direct integrals is based on the following basic observations (see for example [46, Appendix F]). An operator T belongs to $\mathcal{D}'$ if and only if there are operators $T(u) \in B(\mathcal{H}(u))$ such that

$$Th(u) = T(u)\big(h(u)\big)$$

for μ-almost every $u \in G^{(0)}$ [46, Theorem F.21]. Moreover, if $A := \Gamma_0(G^{(0)}; \mathscr{A})$ and if $M : A \to B(\mathcal{H})$ is a representation such that $M(A) \subset \mathcal{D}'$, then there are representations $M_u : A \to B(\mathcal{H}(u))$ such that

(7.6) $M(a)h(u) = M_u\big(a\big)\big(h(u)\big)$ for μ-almost all u.

Of course, the M_u are only determined up to a μ-null set, and it is customary to write

$$M = \int_{G^{(0)}}^{\oplus} M_u \, d\mu(u).$$

An important example for the current discussion occurs when we are given a $C_0(G^{(0)})$-linear representation $M : A \to B(L^2(G^{(0)} * \mathscr{H}, \mu))$: that is,

(7.7) $M(\varphi \cdot a) = L_\varphi M(a).$

Then it is easy to see that $M(A) \subset \mathcal{D}'$. *In addition*, it is not hard to see that (7.7) implies that for each u, $\ker M_u \supset I_u$, where I_u is the ideal of sections in A vanishing at u. In particular, we can view M_u as a representation of the fibre $A(u)$. Thus (7.6) becomes

(7.8) $M(a)h(u) = M_u\big(a(u)\big)\big(h(u)\big).$

The remainder of this section is devoted to modifying the discussion contained in [40] to cover the setting of upper-semicontinuous-Banach bundles. Although this is straightforward, we sketch the details for convenience.

Definition 7.9. A covariant representation $(M, \mu, G^{(0)} * \mathscr{H}, U)$ of $(\mathscr{A}, G, \alpha)$ consists of a unitary representation $(\mu, G^{(0)} * \mathscr{H}, U)$ and a $C_0(G^{(0)})$-linear representation $M : A \to B\big(L^2(G^{(0)} * \mathscr{H}, \mu)\big)$ decomposing as in (7.8) such that there is a ν-null set N such that for all $\gamma \notin N$,

$$(7.9) \qquad U_\gamma M_{s(\gamma)}(b) = M_{r(\gamma)}\big(\alpha_\gamma(b)\big)U_\gamma \quad \text{for all } b \in A\big(s(\gamma)\big).$$

Remark 7.10. Suppose that $(M, \mu, G^{(0)} * \mathscr{H}, U)$ is a covariant representation of $(\mathscr{A}, G, \alpha)$ as above. Then by definition, the set Σ of $\gamma \in G$ such that (7.9) holds in ν-conull. Since U and α are bona fide homomorphisms, it is not hard to see that Σ is closed under multiplication. By a result of Ramsay's ([36, Lemma 5.2] or [27, Lemma 4.9]), there is a μ-conull set $V \subset G^{(0)}$ such that $G|_V \subset \Sigma$.

Proposition 7.11. *If $(M, \mu, G^{(0)} * \mathscr{H}, U)$ is a covariant representation of $(\mathscr{A}, G, \alpha)$, then there is a $\| \cdot \|_I$-norm decreasing $*$-representation R of $\mathscr{G}(G; r^*\mathscr{A})$ given by*

$$(7.10) \quad \big(R(f)h \mid k\big) = \int_G \big(M_{r(\gamma)}\big(f(\gamma)\big)U_\gamma h\big(s(\gamma)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}} \, d\nu(\gamma)$$

or

$$(7.11) \qquad R(f)h(u) = \int_G M_u\big(f(\gamma)\big)U_\gamma h\big(s(\gamma)\big)\Delta(\gamma)^{-\frac{1}{2}} \, d\lambda^u(\gamma).$$

Proof. Clearly, (7.10) and (7.11) define the same operator. Using (7.10), the quasi-invariance of μ and the usual Cauchy–Schwartz inequality in $L^2(\nu)$ we have

$$\big|\big(R(f)h \mid k\big)\big| \le \int_G \|f(\gamma)\|\|h\big(s(\gamma)\big)\|\|k\big(r(\gamma)\big)\|\Delta(\gamma)^{-\frac{1}{2}} \, d\nu(\gamma)$$

$$\le \left(\int_G \|f(\gamma)\|\|h\big(s(\gamma)\big)\|^2\Delta(\gamma)^{-1} \, d\nu(\gamma)\right)^{\frac{1}{2}}$$

$$\left(\int_G \|f(\gamma)\|\|k\big(r(\gamma)\big)\|^2 \, d\nu(\gamma)\right)^{\frac{1}{2}}$$

$$\le \left(\|f\|_I\|h\|^2\right)^{\frac{1}{2}}\left(\|f\|_I\|k\|^2\right)^{\frac{1}{2}}$$

$$= \|f\|_I\|h\|\|k\|.$$

Therefore R is bounded as claimed.

To see that R is multiplicative, we invoke Remark 7.10 to find μ-conull set $V \subset G^{(0)}$ such that (7.9) holds for all $\gamma \in G|_V$. Then if $u \in V$, we have

$$R(f * g)(h)(u) = \int_G M_u\big(f * g(\gamma)\big)U_\gamma h\big(s(\gamma)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(\gamma)$$

$$= \int_G \int_G M_u\big(f(\eta)\alpha_\eta\big(g(\eta^{-1}\gamma)\big)\big)U_\gamma h\big(s(\gamma)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(\eta)\, d\lambda^u(\gamma)$$

$$= \int_G M_u\big(f(\eta)\big)\int_G M_u\big(\alpha_\eta\big(g(\eta^{-1}\gamma)\big)\big)U_\gamma h\big(s(\gamma)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(\gamma)\, d\lambda^u(\eta)$$

$$= \int_G M_u\big(f(\eta)\big)\int_G M_u\big(\alpha_\eta\big(g(\gamma)\big)\big)U_{\eta\gamma} h\big(s(\gamma)\big)\Delta(\eta\gamma)^{-\frac{1}{2}}\, d\lambda^{s(\eta)}(\gamma)\, d\lambda^u(\eta).$$

Now since $U_{\eta\gamma} = U_\eta U_\gamma$, $\Delta(\eta\gamma) = \Delta(\eta)\Delta(\gamma)$ and since

$$U_\eta M_{s(\eta)}(a) = M_u\big(\alpha_\eta(a)\big)U_\eta$$

because $u \in V$, we get

$$\int_G M_u\big(f(\eta)\big)U_\eta\left(\int_G M_u\big(g(\gamma)\big)U_\gamma h\big(s(\gamma)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^{s(\eta)}(\gamma)\right)\Delta(\eta)^{-\frac{1}{2}}\, d\lambda^u(\eta)$$

$$= \int_G M_u\big(f(\eta)\big)U_\eta R(g)h\big(s(\eta)\big)\Delta(\eta)^{-\frac{1}{2}}\, d\lambda^u(\eta)$$

$$= R(f)R(g)h(u).$$

We also have to see that R is $*$-preserving. This will require the quasi-invariance of μ.

$$\big(R(f^*)h \mid k\big) = \int_G \big(M_{r(\gamma)}\big(f^*(\gamma)\big)U_\gamma h\big(s(\gamma)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

$$= \int_G \big(M_{r(\gamma)}\alpha_\gamma\big(f(\gamma)\big)^* U_\gamma h\big(s(\gamma)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

which, since $\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$ is invariant under inversion, is

$$= \int_G \big(M_{s(\gamma)}\big(\alpha_\gamma^{-1}\big(f(\gamma)\big)\big)^* U_\gamma^* h\big(r(\gamma)\big) \mid k\big(s(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

Now

$$U_\gamma M_{s(\gamma)}\big(\alpha_\gamma^{-1}\big(a\big(r(\gamma)\big)\big)\big) = M_{r(\gamma)}\big(a\big(r(\gamma)\big)\big)U_\gamma$$

for ν-almost all γ. Thus,

$$\big(R(f^*)h \mid k\big) = \int_G \big(h\big(r(\gamma)\big) \mid M_{s(\gamma)}\big(f(\gamma)\big)U_\gamma k\big(s(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

$$= \big(h \mid R(f)k\big). \qquad \square$$

The previous result admits a strong converse in the spirit of Renault's Theorem 7.8. The extra generality will be used in the proof of the equivalence theorem (Theorem 5.5).

Theorem 7.12 ([40, Lemme 4.6]). *Suppose that $\mathcal{H}_0$ is a dense subspace of a complex Hilbert space $\mathcal{H}$ and that π is a homomorphism from $\mathscr{G}(G; r^*\mathscr{A})$ to the algebra linear operators on $\mathcal{H}_0$ such that:*

(a) $\operatorname{span}\{\pi(f)h : f \in \mathscr{G}(G; r^*\mathscr{A}) \text{ and } h \in \mathcal{H}_0\}$ *is dense in $\mathcal{H}$.*
(b) *For each $h, k \in \mathcal{H}_0$,*

$$f \mapsto \big(\pi(f)h \mid k\big)$$

is continuous in the inductive limit topology.
(c) *For each $f \in \mathscr{G}(G; r^*\mathscr{A})$ and all $h, k \in \mathcal{H}_0$*

$$\big(\pi(f)h \mid k\big) = \big(h \mid \pi(f^*)k\big).$$

Then each $\pi(f)$ is bounded and extends to a bounded operator $\Pi(f)$ on $\mathcal{H}$ such that Π is a representation of $\mathscr{G}(G; r^*\mathscr{A})$ satisfying $\|\Pi(f)\| \leq \|f\|_I$. Furthermore, there is a covariant representation $(M, \mu, G^{(0)} * \mathscr{H}, L)$ such that Π is equivalent to the corresponding integrated form.

Proof. Let $\mathcal{H}_{00} = \operatorname{span}\{\pi(f)h : f \in \mathscr{G}(G; r^*\mathscr{A}) \text{ and } h \in \mathcal{H}_0\}$. The first-order of business is to define actions of $\mathscr{C}(G)$ and $A := \Gamma_0(G^{(0)}; \mathscr{A})$ on $\mathcal{H}_{00}$. If $\varphi \in \mathscr{C}(G)$, $a \in A$ and $f \in \mathscr{G}(G; r^*\mathscr{A})$, then we define elements of $\mathscr{G}(G; r^*\mathscr{A})$ as follows:

$$(7.12) \qquad \varphi \cdot f(\gamma) := \int_G \varphi(\eta) \alpha_\eta\big(f(\eta^{-1}\gamma)\big)\, d\lambda^{r(\gamma)}(\eta),$$

$$(7.13) \qquad a \cdot f(\gamma) := a\big(r(\gamma)\big)f(\gamma) \quad \text{and}$$

$$(7.14) \qquad f \cdot a(\gamma) := f(\gamma)\alpha_\gamma\big(a\big(s(\gamma)\big)\big).$$

Note that if $\varphi_i \to \varphi$ and $f_i \to f$ in the inductive limit topology then $\varphi_i \cdot f_i \to \varphi \cdot f$ in the inductive limit topology.
Suppose that

$$\sum_i \pi(f_i)h_i = 0$$

in $\mathcal{H}_{00}$. As a special case of Proposition 6.8, we know that there is an approximate identity $\{e_j\}$ in $\mathscr{G}(G; r^*\mathscr{A})$ for the inductive limit topology.

Thus we have

$$\sum_i \pi(\varphi \cdot f_i)h_i = \lim_j \sum_i \pi\big(\varphi \cdot (e_j * f_i)\big)h_i$$

$$= \lim_j \pi(\varphi \cdot e_i)\Big(\sum_i \pi(f_i)h_i\Big)$$

$$= 0.$$

Therefore we can define a linear operator $L(\varphi)$ on $\mathcal{H}_{00}$ by

$$L(\varphi)\pi(f)h := \pi(\varphi \cdot f)h.$$

It is fairly straightforward to check that L satisfies (a), (b) & (c) of Theorem 7.8. Thus Renault's Proposition 4.2 (Theorem 7.8) applies and there is a unitary representation $(\mu, G^{(0)} * \mathcal{H}, L)$ of G such that $\mathcal{H} = L^2(G^{(0)} * \mathcal{H}, \mu)$ and such that the original map L is the integrated form of $(\mu, G^{(0)} * \mathcal{H}, L)$.

The action of $A = \Gamma_0(G^{(0)}; \mathscr{A})$ on $\mathscr{G}(G; r^*\mathscr{A})$ given by (7.13) easily extends to $\widetilde{A}$. Since $\widetilde{A}$ is a unital C^*-algebra,

(7.15) $$k := \big(\|a\|^2 1_A - a^*a\big)^{\frac{1}{2}}$$

is an element of $\widetilde{A}$ for all $a \in A$. Since it is easy to check that

$$\big(\pi(a \cdot f)h \mid \pi(g)h\big) = \big(h \mid \pi\big((a \cdot f)^* * g\big)k\big)$$

$$= \big(\pi(f)h \mid \pi(a^* \cdot g)k\big),$$

we can use (7.15) to show that

$$\Big\|\sum \pi(a \cdot f_i)h_i\Big\|^2 = \|a\|^2\Big\|\sum \pi(f_i)h_i\Big\|^2 - \Big\|\sum \pi(k \cdot f_i)(h_i)\Big\|^2.$$

It follows that

$$M(a)\pi(f)h := \pi(a \cdot f)h$$

defines a bounded operator on $\mathcal{H}_{00}$ which extends to a bounded operator $M(a)$ on $\mathcal{H}$ with $\|M(a)\| \le \|a\|$. In particular, $M : A \to B(\mathcal{H})$ is a $C_0(G^{(0)})$-linear representation of A on $\mathcal{H}$. Therefore M decomposes as in (7.8).

If $\varphi \in \mathscr{C}(G)$ and $a \in A$, then we define two *different* elements of $\mathscr{G}(G; r^*\mathscr{A})$ by

$$a \otimes \varphi(\gamma) = a\big(r(\gamma)\big)\varphi(\gamma) \quad \text{and} \quad \varphi \otimes a(\gamma) = \varphi(\gamma)\alpha_\gamma\big(a\big(s(\gamma)\big)\big).$$

If $g \in \mathscr{G}(G; r^*\mathscr{A})$, then

$$(a \otimes \varphi) * g(\gamma) = \int_G a\big(r(\eta)\big)\varphi(\eta)\alpha_\eta\big(g(\eta^{-1}\gamma)\big)\, d\lambda^{r(\gamma)}(\eta)$$

$$= a\big(r(\gamma)\big) \int_G \varphi(\eta)\alpha_\eta\big(g(\eta^{-1}\gamma)\big)\, d\lambda^{r(\gamma)}(\eta)$$

$$= a \cdot \varphi \cdot g(\gamma).$$

Thus

(7.16) $$\pi\big((a \otimes \varphi) * g\big) = M(a)L(\varphi)\pi(g).$$

And a similar computation shows that

(7.17) $$\pi(\varphi \otimes a) = L(\varphi)M(a).$$

We conclude that for $h, k \in \mathcal{H}_{00}$,

$$(\pi(a \otimes \varphi)h|k) = (M(a)L(\varphi)h|k)$$

$$= \int_G \varphi(\gamma)\big(M_{r(\gamma)}\big(a\big(r(\gamma)\big)\big)u_\gamma\big(h\big(s(\gamma)\big)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

(7.18) $$= \int_G \big(M_{r(\gamma)}\big(a \otimes \varphi(\gamma)\big)U_\gamma\big(h\big(s(\gamma)\big)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma).$$

Similarly,

$$(\pi(\varphi \otimes a)h|k) = \big(L(\varphi)M(a)H \mid k\big)$$

(7.19) $$= \int_G \varphi(\gamma)\big(U_\gamma M_{s(\gamma)}\big(a\big(s(\gamma)\big)\big)\big(h\big(s(\gamma)\big)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

Since $\mathrm{span}\{a \otimes \varphi\}$ is dense in $\mathscr{G}(G; r^*\mathscr{A})$, (7.18) must hold for all $f \in \mathscr{G}(G; r^*\mathscr{A})$. In particular, it must hold for $f = \varphi \otimes a$, and (7.19) must coincide with

$$\int_G \varphi(\gamma)\big(M_{r(\gamma)}\big(\alpha_\gamma\big(a\big(s(\gamma)\big)\big)\big)U_\gamma\big(h\big(s(\gamma)\big)\big) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma)$$

for all $a \in A$ and $\varphi \in \mathscr{C}(G)$.

For each $a \in A$, let

(7.20) $$V(\gamma) := U_\gamma M_{s(\gamma)}\big(a\big(s(\gamma)\big)\big) - M_{r(\gamma)}\big(\alpha_\gamma\big(a\big(s(\gamma)\big)\big)\big)U_\gamma.$$

Then

$$\int_G \varphi(\gamma)\big(V(\gamma)h\big(s(\gamma) \mid k\big(r(\gamma)\big)\big)\Delta(\gamma)^{-\frac{1}{2}}\, d\nu(\gamma) = 0$$

for all $h, k \in L^2(G^{(0)} * \mathscr{H}, \mu)$ and $\varphi \in \mathscr{C}(G)$. In particular, for each $h, k \in L^2(G^{(0)} * \mathscr{H}, \mu)$, there is a ν-null set $N(h, k)$ such that $\gamma \notin N(h, k)$ implies

that

$$(7.21) \qquad \big(V(\gamma)h\big(s(\gamma)\big) \mid k\big(r(\gamma)\big)\big) = 0.$$

Since $L^2(G^{(0)} * \mathscr{H}, \mu)$ is separable, there is a ν-null set N such that $\gamma \notin N$ implies (7.21) holds for all h and k. In other works, $V(\gamma) = 0$ for ν-almost all γ.

Therefore there is a ν-null set $N(a)$ such that $\gamma \notin N(a)$ implies that

$$(7.22) \qquad U_\gamma M_{s(\gamma)}\big(a\big(s(\gamma)\big)\big) = M_{r(\gamma)}\big(\alpha_\gamma\big(a\big(s(\gamma)\big)\big)\big)U_\gamma.$$

Since A is separable, and $a \mapsto a(u)$ is a surjective homomorphism of A onto $A(u)$, there is a ν-null set N such that (7.22) holds for all $a \in A$ and $\gamma \notin N$.

It follows that $(M, \nu, G^{(0)} * \mathscr{H}, L)$ is covariant and that π is the restriction of its integrated form to $\mathcal{H}_{00}$. The rest is easy. $\qquad\square$

8. Proof of the equivalence theorem

The discussion to this point provides us with the main tools we need to complete the proof of Theorem 5.5. Another key observation is that the inner products and actions are continuous with respect to the inductive limit topology. Since this is slightly more complicated in the not necessarily Hausdorff setting, we include a statement and proof for convenience.

Lemma 8.1. *The actions and inner products on the pre-$\mathscr{B} \rtimes_\beta H - \mathscr{A} \rtimes_\alpha G$-imprimitivity bimodule $\mathsf{X}_0 := \mathscr{G}(X; \mathscr{E})$ of Theorem 5.5 are continuous in the inductive limit topology. In particular, if $v_i \to v$ and $w_i \to w$ in the inductive limit topology on $\mathscr{G}(X; \mathscr{E})$ and if $f_i \to f$ in the inductive limit topology on $\mathscr{G}(H; r^*\mathscr{B})$, then:*

(a) *$f_i \cdot w_i \to f \cdot w$ in the inductive limit topology on $\mathscr{G}(X; \mathscr{E})$.*
(b) *$\langle\!\langle w_i , v_i \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} \to \langle\!\langle w , v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}$ in the inductive limit topology on $\mathscr{G}(G; r^*\mathscr{A})$.*

Proof. By symmetry, it suffices to just check (a) and (b). Let K_v, K_w and K_f be compact sets such that $v(x) = 0$ if $x \notin K_v$, $w(x) = 0$ if $x \notin K_w$ and $f(\eta) = 0$ if $\eta \notin K_f$. Then $f \cdot w(x) = 0$ if $x \notin K_f \cdot K_v$. Using Lemma 5.2, we see that $\|\eta \cdot w(x)\| = \|w(x)\|$, and thus

$$\|f \cdot w\|_\infty \leq \|f\|_\infty \|w\|_\infty \sup_{u \in H^{(0)}} \lambda_H^u(K_f).$$

Now establishing (a) is straightforward.

To prove (b), notice that as in Lemma 2.20, there is a continuous map $\sigma : X *_r X \to G$ which induces a homeomorphism of $H \backslash X *_r X$ onto G such that $x \cdot \sigma(x, y) = y$. (In particular, $\sigma(y \cdot \gamma^{-1}, y) = \gamma$.) Thus $K_r \sigma(K_w *_r K_v)$ is compact and $\langle\!\langle w , v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}(\gamma) = 0$ if $\gamma \notin K_r$. Also, there is a compact set

K_1 such that $s_X(K_1) = s_G(K_r)$. Thus if the integral in (5.4) is nonzero, we can assume that $y \in K_1$. Since the H-action is proper,

$$K_0 := \{\eta \in H : \eta \cdot K_v \cap K_1 \neq 0\}$$

is compact. Since the G-action on $\mathscr{E}$ is isometric,

$$\|\langle\!\langle w , v\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}\|_\infty \leq \|w\|_\infty \|v\|_\infty \sup_{u \in H^{(0)}} \lambda_H^u(K_0),$$

and the rest is straightforward. $\qquad\square$

We have already observed in Remark 5.7 that (5.1)–(5.4) are well-defined and take values in the appropriate functions spaces. To complete the proof, we are going to apply [35, Definition 3.9]. We have also already checked the required algebraic identities in parts (a) and (d) of that Definition. All that remains in order to verify (a) is to show that inner products are positive. This and the density of the range of the inner products (a.k.a. part (b)) follow from Proposition 6.8, Lemma 8.1 and symmetry by standard means (cf. e.g., [46, p. 115], or [41] or the discussion following Lemma 2 in [13]).

To establish the boundedness of the inner products, we need to verify that

(8.1) $$\langle\!\langle f \cdot z , f \cdot z\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} \leq \|f\|^2_{\mathscr{B} \rtimes_\beta H} \langle\!\langle z , z\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} \quad \text{and}$$

(8.2) $$_{\mathscr{B} \rtimes_\beta H}\langle\!\langle z \cdot g , z \cdot g\rangle\!\rangle \leq \|g\|^2_{\mathscr{A} \rtimes_\alpha G} {_{\mathscr{B} \rtimes_\beta H}}\langle\!\langle z , z\rangle\!\rangle.$$

By symmetry, it is enough to prove (8.1).

But if ρ is a state on $\mathscr{A} \rtimes_\alpha G$, then

$$(\cdot \mid \cdot)_\rho := \rho\big(\langle\!\langle \cdot , \cdot\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}\big)$$

makes $\mathscr{G}(X;\mathscr{E})$ a pre-Hilbert space. Let $\mathcal{H}_0$ be the dense image of $\mathscr{G}(X;\mathscr{E})$ in the Hilbert space completion $\mathcal{H}_\rho$. The left action of $\mathscr{G}(H;r_H^*\mathscr{B})$ on $\mathscr{G}(X;\mathscr{E})$ gives a homomorphism π of $\mathscr{G}(H;r_H^*\mathscr{B}) \subset \mathscr{B} \rtimes_\beta H$ into the linear operators on $\mathcal{H}_0$. We want to check that the requirements (a)–(c) of Theorem 7.12 are satisfied.

Notice that if $g_i \to g$ in the inductive limit topology on $\mathscr{G}(G;r^*\mathscr{A})$, then $\|g_i - g\|_I \to 0$ and $g_i \to g$ in the C^*-norm. Thus, $\rho(g_i) \to \rho(g)$. If $f_i \to f$ in the inductive limit topology on $\mathscr{G}(H;r^*\mathscr{B})$, then Lemma 8.1 implies that $\langle\!\langle f_i \cdot w , v\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} \to \langle\!\langle f \cdot w , v\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}$ in the inductive limit topology. Therefore $\big(\pi(f_i)v \mid w\big)_\rho \to \big(\pi(f)v \mid w\big)_\rho$. This establishes requirement (b) of Theorem 7.12. Requirement (a) follows in a similar way using the approximate identity for $\mathscr{G}(H;r^*\mathscr{B})$ as constructed in Proposition 6.8. To see that (c) holds, we just need to observe that

(8.3) $$\langle\!\langle f \cdot w , v\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} = \langle\!\langle w , f^* \cdot v\rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}.$$

We could verify (8.3) directly via a complicated computation. However, notice that (8.3) holds for all f in the span of the left inner product as in the proof of [35, Lemma 3.7]. However, Proposition 6.8 implies that given any $f \in \mathscr{G}(H; r^*\mathscr{B})$, there is a net $\{f_i\}$ in the span of the inner product such that $f_i \to f$ (and therefore $f_i^* \to f^*$) in the inductive limit topology. Then by Lemma 8.1,

$$\langle\!\langle f \cdot w \,,\, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} = \lim_i \langle\!\langle f_i \cdot w \,,\, v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G} = \lim_i \langle\!\langle w \,,\, f_i^* \cdot v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}$$
$$= \langle\!\langle w \,,\, f^* \cdot v \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}.$$

Since the requirements of Theorem 7.12 are satisfied, it follows that π is bounded with respect to the C^*-norm on $\mathscr{G}(H; r_H^* \mathscr{B})$. In particular,

$$\rho\big(\langle\!\langle f \cdot z \,,\, f \cdot z \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}\big) \le \|f\|_{\mathscr{B} \rtimes_\beta H}^2 \rho\big(\langle\!\langle z \,,\, z \rangle\!\rangle_{\mathscr{A} \rtimes_\alpha G}\big).$$

As this holds for all ρ, (8.1) follows, and this completes the proof.

9. Applications

The equivalence theorem is a powerful tool, and we plan to make considerable use of it in a subsequent paper on the equivariant Brauer semigroup of a groupoid, extending the results in [19] to the groupoid setting. Here we want to remark that a number of the constructions and results in [23] can be succinctly described in terms of equivalences and the equivalence theorem.

9.1. Morita equivalent actions. Our first application, which asserts that Morita equivalent dynamical systems induce Morita equivalent crossed products, is the natural generalization to the setting of groupoids of the main results in [8] and [4]. The key definition is lifted directly from [23, Definition 3.1]; the only difference is that we allow the weaker notion of Banach bundle and dynamical system.

Definition 9.1. Let G be a locally Hausdorff, locally compact groupoid and suppose that G acts on two upper-semicontinuous-C^*-bundles over $G^{(0)}$, $\mathscr{A}$ and $\mathscr{B}$. Then the two dynamical systems $(\mathscr{A}, G, \alpha)$ and $(\mathscr{B}, G, \alpha)$ are called *Morita equivalent* if there is an $\mathscr{A} - \mathscr{B}$-imprimitivity bimodule bimodule $\mathscr{X}$ over $G^{(0)}$ (see Example 5.11), and a G-action on $\mathscr{X}$ such that $x \mapsto V_\gamma(x) := \gamma \cdot x$ is an isomorphism and such that

$$_{\mathscr{A}}\langle V_\gamma(x) \,,\, V_\gamma(y) \rangle = \alpha_\gamma\big(_{\mathscr{A}}\langle x \,,\, y \rangle\big) \quad \text{and} \quad \langle V_\gamma(x) \,,\, V_\gamma(y) \rangle_{\mathscr{B}} = \beta_\gamma\big(_{\mathscr{A}}\langle x \,,\, y \rangle\big).$$

We considered the equivalence relation of Morita equivalence of dynamical systems in [23]. However, we did not consider the corresponding crossed products. But in the situation of Definition 9.1, there is an equivalence between $(\mathscr{A}, G, \alpha)$ and $(\mathscr{B}, G, \beta)$. Then Theorem 5.5 implies that the crossed

products are Morita equivalent and provides a concrete imprimitivity bimodule. This generalizes [8] and [4]. It is instructive to work out the details. We let

$$\mathscr{E} := r_G^* \mathscr{X} = G * \mathscr{X} = \{(\gamma, x) : s(\gamma) = p_{\mathscr{X}}(x)\}$$

with $p_{\mathscr{E}}$ given by $(\gamma, x) \mapsto \gamma$, and we view G as a (G, G)-equivalence. Note that $\mathscr{E}_x$ is naturally identified with $\mathsf{X}(r(\gamma))$, which is given to be a $A(r(\gamma)) - B(r(\gamma))$-imprimitivity bimodule. However, β_γ^{-1} is an isomorphism of $B(r(\gamma))$ onto $B(s(\gamma))$, and so we obtain a $A(r(\gamma)) - B(s(\gamma))$-imprimitivity bimodule via composition. Thus we have

$$_{\mathscr{A}}\langle (\gamma, x) , (\gamma, y) \rangle := {}_{\mathscr{A}}\langle x , y \rangle \qquad \langle (\gamma, x) , (\gamma, y) \rangle_{\mathscr{B}} := \beta_\gamma^{-1}\big(\langle x , y \rangle_{\mathscr{B}}\big)$$
$$a \cdot (\gamma, x) := (\gamma, a \cdot x) \qquad\qquad (\gamma, x) \cdot b := \big(\gamma, x \cdot \beta_\gamma(b)\big).$$

We define commuting G-actions on the right and the left by

$$\sigma \cdot (\gamma, x) := \big(\sigma\gamma, V_\sigma(x)\big) \quad \text{and} \quad (\gamma, x) \cdot \sigma := (\gamma\sigma, x).$$

Recall Definition 5.1. Clearly continuity and equivariance are satisfied. For compatibility, we check:

$$_{\mathscr{A}}\langle \sigma \cdot (\gamma, x) , \sigma \cdot (\gamma, y) \rangle = {}_{\mathscr{A}}\langle V_\sigma(x) , V_\sigma(y) \rangle$$
$$= \alpha_\gamma\big({}_{\mathscr{A}}\langle (\gamma, x) , (\gamma, y) \rangle\big),$$

while

$$\langle (\gamma, x) \cdot \sigma , (\gamma, y) \cdot \sigma \rangle_{\mathscr{B}} = {}_{\mathscr{B}}\langle V_{\sigma^{-1}\gamma^{-1}}(x) , V_{\sigma^{-1}\gamma^{-1}}(y) \rangle$$
$$= \beta_\sigma^{-1}\big(\langle V_{\gamma^{-1}}(x) , V_{\gamma^{-1}}(y) \rangle_{\mathscr{B}}\big)$$
$$= \beta_\sigma^{-1}\big(\langle (\gamma, x) , (\gamma, y) \rangle_{\mathscr{B}}\big).$$

There are equally exciting computations involving the actions:

$$\sigma \cdot \big(a \cdot (\gamma, x)\big) = \sigma \cdot (\gamma, a \cdot x)$$
$$= \big(\sigma\gamma, V_\sigma(a \cdot x)\big)$$
$$= \big(\sigma\gamma, \alpha_\sigma(a) \cdot V_\sigma(x)\big)$$
$$= \alpha_\sigma(a) \cdot \big(\sigma \cdot (\gamma, x)\big),$$

while

$$\big((\gamma, x) \cdot b\big) \cdot \sigma = \big(\gamma, x \cdot \beta_\gamma(b)\big) \cdot \sigma$$
$$= \big(\gamma\sigma, x \cdot \beta_\gamma(b)\big)$$
$$= (\gamma\sigma, x) \cdot \beta_\sigma^{-1}(b)$$
$$= \big((\gamma, x) \cdot \sigma\big) \cdot \beta_\sigma^{-1}(b).$$

We also have to check invariance:

$$
\begin{aligned}
\sigma \cdot \big((\gamma, x) \cdot b\big) = \sigma \cdot \big(\gamma, x \cdot \beta_\gamma(b)\big) \\
= \big(\sigma\gamma, V_\sigma\big(x \cdot \beta_\gamma(b)\big)\big) \\
= \big(\sigma\gamma, V_\gamma(x) \cdot \beta_{\sigma\gamma}(b)\big) \\
= \big(\sigma\gamma, V_\sigma(x)\big) \cdot b \\
= \big(\sigma \cdot (\gamma, x)\big) \cdot b,
\end{aligned}
$$

while

$$
\begin{aligned}
a \cdot \big((\gamma, x) \cdot \sigma\big) = a \cdot (\gamma\sigma, x) \\
= (\gamma\sigma, a \cdot x) \\
= (\gamma, a \cdot x) \cdot \sigma \\
= \big(a \cdot (\gamma, x)\big) \cdot \sigma.
\end{aligned}
$$

Thus $\mathscr{E}$ is a $(\mathscr{A}, G, \alpha) - (\mathscr{B}, G, \beta)$-equivalence and $\mathscr{A} \rtimes_\alpha G$ is Morita equivalent to $\mathscr{B} \rtimes_\beta G$ via the completion of the pre-imprimitivity bimodule $\mathsf{X}_0 = \mathscr{G}(G; \mathscr{E})$. Of course, each section of $\mathscr{E}$ is of the form $z(\gamma) = (\gamma, \check{z}(\gamma))$ where $\check{z} : G \to \mathscr{X}$ is a continuous function satisfying the appropriate properties. Naturally, we want to identify X_0 with these functions. Then the appropriate inner products and actions are given by

$$
(9.1) \qquad {}_{\mathscr{A} \rtimes_\alpha G}\langle\!\langle z \, , \, w \rangle\!\rangle(\eta) = \int_G {}_{\mathscr{A}}\big\langle z(\eta\gamma) \, , \, V_\eta(w(\gamma)) \big\rangle \, d\lambda^{s(\eta)}(\gamma)
$$

$$
(9.2) \qquad f \cdot z(\gamma) = \int_G f(\eta) \cdot V_\eta\big(z(\eta^{-1}\gamma)\big) \, d\lambda^{r(\gamma)}(\eta)
$$

$$
(9.3) \qquad \langle\!\langle z \, , \, w \rangle\!\rangle_{\mathscr{B} \rtimes_\beta G}(\gamma) = \int_G \beta_\eta\big(\langle z(\eta^{-1}) \, , \, w(\eta^{-1}\gamma)\rangle_{\mathscr{B}}\big) \, d\lambda^{r(\gamma)}(\eta) \text{ and}
$$

$$
(9.4) \qquad z \cdot g(\eta) = z(\gamma) \cdot \beta_\gamma\big(g(\gamma^{-1}\eta)\big) \, d\lambda^{r(\eta)}(\gamma).
$$

These equations are verified as follows: for (9.1), we have

$$
\begin{aligned}
{}_{\mathscr{A} \rtimes_\alpha G}\langle\!\langle z \, , \, w \rangle\!\rangle(\eta) = \int_G {}_{\mathscr{A}}\big\langle z(\eta\gamma) \, , \, \eta \cdot w(\gamma) \big\rangle \, d\lambda^{s(\eta)}(\gamma) \\
= \int_G {}_{\mathscr{A}}\big\langle (\eta\gamma, \bar{z}(\eta\gamma)) \, , \, (\eta\gamma, V_\eta(\overline{w}(\gamma))) \big\rangle \, d\lambda^{s(\eta)}(\gamma) \\
= \int_G {}_{\mathscr{A}}\big\langle \bar{z}(\eta\gamma) \, , \, V_\eta\big(\overline{w}(\gamma)\big) \big\rangle \, d\lambda^{s(\eta)}(\gamma).
\end{aligned}
$$

And

$$f \cdot z(\gamma) = \int_G f(\eta) \cdot \left(\eta \cdot z(\eta^{-1}\gamma)\right) d\lambda^{r(\gamma)}(\eta)$$

$$= \int_G f(\eta) \cdot \left(\gamma, V_\eta\left(\bar{z}(\eta^{-1}\gamma)\right)\right) d\lambda^{r(\gamma)}(\eta)$$

$$= \int_G \left(\gamma, f(\eta) \cdot V_\eta\left(\bar{z}(\eta^{-1}\gamma)\right)\right) d\lambda^{r(\gamma)}(\eta)$$

gives (9.2). Equations (9.3) and (9.4) follow from similar computations.

9.2. Equivalence and the basic construction. In [23], we introduced the Brauer Group $\mathrm{Br}(G)$ of a groupoid second countable locally compact Hausdorff groupoid. One of the basic results is that if X is a (H, G)-equivalence, then there is a natural isomorphism φ^X of $\mathrm{Br}(G)$ onto $\mathrm{Br}(H)$. The map φ^X is defined via the "basic construction" which associates a dynamical system $(\mathscr{A}^X, H, \alpha^X)$ to any given dynamical system $(\mathscr{A}, G, \alpha)$ [23, Proposition 2.15]. (In [23], we worked with C^*-bundles rather than upper-semicontinuous-C^*-bundles, but the construction is easily modified to handle the more general bundles we are working with in this paper.) We briefly recall the details. The pullback

$$s_X^* \mathscr{A} = \{(x, a) \in X \times \mathscr{A} : s_X(x) = p(a)\}$$

is a right G-space:

$$(x, a) \cdot \gamma = \left(x \cdot \gamma, \alpha_\gamma^{-1}(a)\right).$$

Using the proof of [23, Proposition 2.5], we can show that the quotient $\mathscr{A}^X := s_X^* \mathscr{A}/G$ is an upper-semicontinuous-C^*-bundle. If we denote the image of (x, a) in $\mathscr{A}^X$ by $[x, a]$, then the action of H is given by

$$\alpha^X(\eta)[x, a] := [\eta \cdot x, a].$$

Our goal in this section is the use the equivalence theorem to see that $\mathscr{A} \rtimes_\alpha G$ is Morita equivalent to $\mathscr{A}^X \rtimes_{\alpha^X} H$ (*and* to exhibit the equivalence bimodule). As a special case, we see that the isomorphism $\varphi^X : \mathrm{Br}(G) \to \mathrm{Br}(H)$ induces a Morita equivalence of the corresponding dynamical systems.

Here we let

(9.5) $$\mathscr{E} = s_X^* \mathscr{A} = \{(x, a) : s(x) = p_{\mathscr{A}}(a)\}.$$

Then $\mathscr{E}_x$ is easily identified with $A(s(x))$, with an $A^X(r(x)) - A(s(x))$-imprimitivity bimodule structure given as follows. First, since x is given, it is not hard to identify

(9.6) $$A^X(r(x)) = \{[x \cdot \gamma, a] : s(\gamma) = p_{\mathscr{A}}(a)\}$$

with $A(s(x))$ via $[x \cdot \gamma, a] \mapsto \alpha_\gamma(a)$. Then the imprimitivity bimodule structure is just the usual $_{A(s(x))}A(s(x))_{A(s(x))}$ one. Thus

$$_{\mathscr{A}^X}\langle (x,a) , (x,b) \rangle := [x, ab^*] \qquad\qquad \langle (x,a) , (x,b) \rangle_{\mathscr{A}} := a^*b$$
$$[x \cdot \gamma, b] \cdot (x, a) := (x, \alpha_\gamma(b)a) \qquad\qquad (x,a) \cdot b := (x, ab).$$

The H- and G-actions on $\mathscr{E}$ are given by

$$\eta \cdot (x, a) := (\eta \cdot x, a) \quad \text{and} \quad (x, a) \cdot \gamma := (x \cdot \gamma, \alpha_\gamma^{-1}(a)).$$

It remains to check conditions (a)–(d) of Definition 5.1. We start with continuity. Clearly the maps $\mathscr{E} * \mathscr{E} \to \mathscr{A}^X$, $\mathscr{E} * \mathscr{E} \to \mathscr{A}$ and $\mathscr{E} * \mathscr{A} \to \mathscr{E}$ are continuous. Showing that $\mathscr{A}^X * \mathscr{E} \to \mathscr{E}$ is continuous requires a little fussing. Suppose that $(x_i, a_i) \to (x_0, a_0)$ in $\mathscr{E}$ while $[y_i, b_i] \to [y_0, b_0]$ in A^X with $x_i \cdot G = y_i \cdot G$ for all i. We need to see that $[y_i, b_i] \cdot (x_i, a_i) \to [y_0, b_0] \cdot (x_0, a_0)$. It will suffice to see that a subnet has this property.[13]   Also, we may as well let $y_i = x_i$ for all i. Then we can pass to a subnet, and relabel, so that there are γ_i such that $(x_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}(b_i)) \to (x_0, b_0)$. Since $x_i \to x_0$ and since the G-action on X is proper, we can pass to another subnet, relabel, and assume that $\gamma_i \to s(x_0)$. Thus $b_i = \alpha_{\gamma_i} \circ \alpha_{\gamma_i}^{-1}(b_i) \to b_0$ and

$$[x_i, b_i] \cdot (x_i, a_i) = (x_i, b_i a_i) \to (x_0, b_0 a_0)$$

as required.

Equivariance is clear and invariance follows from some unexciting computations. For example,

$$[x, b] \cdot ((x, a) \cdot \gamma) = [x, b] \cdot (x \cdot \gamma, \alpha_\gamma^{-1}(a))$$
$$= [x \cdot \gamma, \alpha_{\gamma^{-1}}(b)] \cdot (x \cdot \gamma, \alpha_\gamma^{-1}(a))$$
$$= (x \cdot \gamma, \alpha_\gamma^{-1}(ba))$$
$$= ([x, b] \cdot (s, a)) \cdot \gamma.$$

To check compatibility, notice that

$$_{\mathscr{A}^X}\langle \eta \cdot (x, a) , \eta \cdot (x, b) \rangle = [\eta \cdot x, ab^*]$$
$$= \alpha_\eta^X(_{\mathscr{A}^X}\langle (x, a) , (x, b) \rangle).$$

Similarly,

$$\langle (x, a) \cdot \gamma , (x, b) \cdot \gamma \rangle_{\mathscr{A}} = \alpha_\gamma^{-1}(\langle (x, a) , (x, b) \rangle_{\mathscr{A}}).$$

---

[13] To show that a given net $\{x_i\}$ converges to x, it suffices to see that every subnet has a subnet converging to x. In the case here, we can simply begin by replacing the given net with a subnet and then relabeling. Then it does suffice to find a convergent subnet.

The fact that the actions are compatible are easy, but we remark that it also follows from invariance and Lemma 5.2:

$$\eta \cdot \big({}_{\mathscr{A}^X}\langle z \, , \, w\rangle \cdot v \big) = \eta \cdot \big(z \cdot \langle w \, , \, v\rangle_{\mathscr{A}} \big)$$
$$= (\eta \cdot z) \cdot \big(\langle \eta \cdot w \, , \, \eta \cdot v\rangle_{\mathscr{A}} \big)$$
$$= {}_{\mathscr{A}^X}\langle \eta \cdot z \, , \, \eta \cdot w\rangle \cdot (\eta \cdot v)$$
$$= \alpha_\eta^X \big({}_{\mathscr{A}^X}\langle z \, , \, w\rangle \big) \cdot (\eta \cdot v).$$

The fullness of the inner products gives

$$\eta \cdot (a \cdot v) = \alpha_\eta^X(a) \cdot (\eta \cdot v).$$

Before writing down the corresponding pre-imprimitivity bimodule structure on $\mathscr{G}(H; r_H^* \mathscr{A}^X)$, a few comments about the nature of sections of $r_H^* \mathscr{A}^X$ will be helpful. First recall that $\mathscr{A}^X = X * \mathscr{A}/G$ and that we can identify H with $X * X/G$ via $\eta \mapsto [\eta \cdot x, x]$ (with any $x \in r^{-1}(s(x))$). Thus

$$r_H^* \mathscr{A}^X = \big\{ \big([x,y],[z,a]\big) : x \cdot G = z \cdot G \text{ and } s(z) = p_{\mathscr{A}}(a) \big\}.$$

If $X * X * \mathscr{A} = \{(x,y,a) : s(x) = s(y) = p_{\mathscr{A}}(a)\}$, then $X * X * \mathscr{A}/G = \mathscr{A}^{X*X}$ is a C^*-bundle over H which is isomorphic to $r_H^* \mathscr{A}^X$. Consequently, $f \in \mathscr{G}(H; r_H^* \mathscr{A}^X)$ must be of the form

$$f\big([x,y]\big) = \big([x,y],[x,y,\widetilde{f}(x,y)]\big)$$

for a function $\widetilde{f} : X * X \to \mathscr{A}$ such that $p_{\mathscr{A}}\big(\widetilde{f}(x,y)\big) = s(x)$, such that $\widetilde{f}(x \cdot \gamma, y \cdot \gamma) = \alpha_\gamma^{-1}\big(\widetilde{f}(x,y)\big)$ and such that $\operatorname{supp} f/G$ is compact. In fact, $\widetilde{f}$ must also be continuous. Let $(x_i, y_i) \to (x,y)$ in $X * X$. Again, it will be enough to see that $\widetilde{f}(x_i, y_i)$ has a subnet converging to $\widetilde{f}(x,y)$. Since $[x_i, y_i, \widetilde{f}(x_i, y_i)] \to [x, y, \widetilde{f}(x,y)]$, we can pass to a subnet, relabel, and find γ_i such that $\big(x_i \cdot \gamma_i, y_i \cdot \gamma_i, \alpha_{\gamma_i}^{-1}\big(\widetilde{f}(x_i, y_i)\big)\big) \to \big(x, y, \widetilde{f}(x,y)\big)$. Since the G-action is proper, we can pass to another subnet, relabel, and assume that $\gamma_i \to s(x)$. It follows that $\widetilde{f}(x_i, y_i) \to f(x,y)$ as required. Thus we will often identify f and $\widetilde{f}$. Moreover, we will view $\mathscr{A}^X \rtimes_{\alpha^X} H$ as the completion of the set $C_\alpha(X * X; \mathscr{A})$ of functions with the above properties.

If $z \in \mathsf{X}_0 := \mathscr{G}(X; s_X^* \mathscr{A})$, then $z(x) = (x, \check{z}(x))$ for some continuous function $\check{z} : G \to \mathscr{A}$ such that $p_{\mathscr{A}}\big(\check{z}(x)\big) = s(x)$. Consequently,

$$_{\mathscr{A}^X\rtimes_{\alpha X}H}\langle\!\langle z\,,\,w\rangle\!\rangle(\eta) = \int_G {}_{\mathscr{A}^X}\langle z(\eta\cdot x\cdot\gamma)\,,\,\eta\cdot w(x\cdot\gamma)\rangle\,d\lambda_G^{s(x)}(\gamma)$$

$$= \int_G [\eta\cdot x\cdot\gamma, \check{z}(\eta\cdot x\cdot\gamma)\check{w}(x\cdot\gamma)^*]\,d\lambda_G^{s(x)}(\gamma)$$

$$= \int_G [\eta\cdot x, \alpha_\gamma^{-1}\big(\check{z}(\eta\cdot x\cdot\gamma)\check{w}(x\cdot\gamma)^*\big)\,d\lambda_G^{s(x)}(\gamma).$$

Thus identifying z and $\check{z}$, we have

$$(9.7)\qquad {}_{\mathscr{A}^X\rtimes_{\alpha X}H}\langle\!\langle z\,,\,w\rangle\!\rangle(x,y) = \int_G \alpha_\gamma^{-1}\big(z(x\cdot\gamma)w(y\cdot\gamma)^*\big)\,d\lambda_G^{(s(x))}(\gamma)$$

as a function on $X * X$. To work out the left-action of $f \in C_\alpha(X * X; \mathscr{A})$, notice that

$$f\cdot z(x) = \int_H f(\eta)\cdot\big(\eta\cdot z(\eta^{-1}\cdot x)\big)\,d\lambda_H^{r(x)}(\eta)$$

$$= f(\eta)\cdot[x, \check{z}(\eta^{-1}\cdot x)]\,d\lambda_H^{r(x)}(\eta)$$

$$= \int_H [x, \eta^{-1}\cdot x, \widetilde{f}(x,\eta^{-1}\cdot x)]\cdot[x, \check{z}(\eta^{-1}\cdot x)]\,d\lambda_H^{r(x)}(\eta)$$

$$= \int_H [x, \widetilde{f}(x,\eta^{-1}\cdot x)\check{z}(\eta^{-1}\cdot x)]\,d\lambda_H^{r(x)}(\eta).$$

Thus, after identifications, the correct formula is

$$(9.8)\qquad f\cdot z(x) = \int_H f(x,\eta^{-1}\cdot x)z(\eta^{-1}\cdot x)\,d\lambda_H^{r(x)}(\eta).$$

Similar, but less involved, considerations show that

$$(9.9)\quad \langle\!\langle z\,,\,w\rangle\!\rangle_{\mathscr{A}\rtimes_\alpha G}(x,x\cdot\gamma) = \int_H z(\eta^{-1}\cdot x)^*\alpha_\gamma\big(w(\eta^{-1}\cdot x\cdot\gamma)\big)\,d\lambda_H^{r(x)}(\eta),$$

$$(9.10)\qquad\qquad z\cdot g(x) = \int_G \alpha_\gamma\big(z(x\cdot\gamma)\cdot g(\gamma^{-1})\big)\,d\lambda_G^{s(x)}(\gamma).$$

Appendix A. Radon measures

In the proof of the disintegration theorem, we will need some facts about *complex Radon measures* and "Radon" measures on locally Hausdorff, locally compact spaces that are a bit beyond the standard measure theory courses we all teach — although much of what we need in the Hausdorff case can be found in authorities like [11, Chap. 4]. (In particular, complex Radon measures are defined in [11, Definition 4.3.1], and in the Hausdorff

case, the Radon–Nikodym Theorem we need can be sorted out from [11, §§4.15.7–9].)

A.1. Radon measures: the Hausdorff case.

To start with, let X be a locally compact *Hausdorff* space. For simplicity, we will assume that X is second countable. A (positive) *Radon measure* on X is a regular Borel measure associated to a positive linear functional $\mu : C_c(X) \to \mathbf{C}$ via the Riesz Representation Theorem. It is standard practice amongst the cognoscenti to identify the measure and the linear functional, and we will do so here — cognoscente or not. Additionally, we don't add the adjective "positive" unless we're trying to be pedantic. Notice that if μ is a Radon measure on X, then $\mu : C_c(X) \to \mathbf{C}$ is continuous in the inductive limit topology. Thus we define a *complex Radon measure* on X to be a linear functional $\nu : C_c(X) \to \mathbf{C}$ which is continuous in the inductive limit topology.[14] If ν is actually bounded with respect to the supremum norm on $C_c(X)$, so that ν extends to a bounded linear functional on $C_0(X)$, then ν is naturally associated to a bona fide complex measure on X (whose total variation norm coincides with the norm of ν as a linear functional) [42, Theorem 6.19]. However, in general, a complex Radon measure need not be bounded. Nevertheless, we want to associate a measure of sorts (that is, a set function) to ν. The problem is that for complex measures, it doesn't make sense to talk about sets of infinite measure so we can't expect to get a set function defined on unbounded sets in the general case.

Let $\varphi = \mathrm{Re}\,\nu$, the real linear functional on $C_c(X)$ (viewed as a real vector space). Fix $f \in \mathbf{C}_c^+(X)$ and consider

$$(A.1) \qquad \{\varphi(g) \in \mathbf{R} : |g| \le f\}.$$

If (A.1) were not bounded, then we could find g_n such that $|g_n| \le \frac{1}{n}f$ and such that $|\varphi(g_n)| \ge n$. This gives us a contradiction since $g_n \to 0$ in the inductive limit topology. Consequently, φ is *relatively bounded* as defined in [15, Definition B.31], and [15, Theorem B.36] implies that $\varphi = \mu_1 - \mu_2$ where each μ_i is a positive linear functional on $C_c(X)$; that is, each μ_i is a Radon measure. Applying the same analysis to the complex part of ν, we find that there are Radon measures μ_i such that $\nu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$,

---

[14] As we shall see in the next paragraph, a complex Radon measure must be relatively bounded. Hence, if X is compact, then ν is always bounded as a linear functional on $C(X)$, and we're back in the standard textbooks.

and for each $f \in C_c(X)$, we have

$$(A.2) \quad \nu(f) = \int_X f(x)\, d\mu_1(x) - \int_X f(x)\, d\mu_2(x)$$
$$+ i \int_X f(x)\, d\mu_3(x) - i \int_X f(x)\, d\mu_4(x).$$

Although in general the μ_i will not be finite measures — so that it makes no sense to talk about $\mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ as a complex measure on X — we nevertheless want a "measure theory" associated to ν. (Since we are assuming that X is second countable, Radon measures are necessarily σ-finite.) In particular, we can define $\mu_0 := \mu_1 + \mu_2 + \mu_3 + \mu_4$. Then $\mu_i \ll \mu$ for all i and there are Borel functions $h_i : X \to [0, \infty)$ such that $\mu_i = h_i \mu$. Since the h_i are finite-valued, we can define a **C**-valued Borel function by $h = h_1 - h_2 + ih_3 - ih_4$. For each $f \in C_c(X)$, we have

$$\nu(f) = \int_X f(x)h(x)\, d\mu_0(x).$$

We can write $h(x) = \rho(x)p(x)$ for a nonnegative Borel function p and a unimodular Borel function ρ. Replacing $p\mu_0$ by μ, we then have

$$(A.3) \qquad \nu(f) = \int_X f(x)\rho(x)\, d\mu(x) \quad \text{for all } f \in C_c(X).$$

If, for example, X is compact, then it is well-known that the measure μ appearing in (A.3) is unique, and that ρ is determined μ-almost everywhere. If X is second countable, and therefore σ-compact, then we see that μ and ρ satisfy the same uniqueness conditions. As in the compact case, we will write $|\nu|$ for μ and call $|\nu|$ the *total variation* of ν.

Since Radon measures are finite on compact subsets, we can certainly make perfectly good sense out of $\nu(f)$ for any $f \in \mathcal{B}_c^b(X)$ — that is, for any bounded Borel function f which vanishes outside a compact set — simply by using (A.3). (In fact, we can make sense out of $\nu(f)$ whenever $f \in \mathcal{L}^1(|\nu|)$.) In particular, if B is a pre-compact[15] Borel set in X, then we will happily write $\nu(B)$ for $\nu(\mathbb{1}_B)$. We say that a Borel set (possibly not pre-compact) is *locally ν-null* if $\nu(B \cap K) = 0$ for all compact sets $K \subset X$.

We will also need a version of the Radon–Nikodym Theorem for our complex Radon measures. Specifically, we suppose that μ is a Radon measure and that ν is a complex Radon measure such that $\nu \ll \mu$ — that is, $\mu(B) = 0$ implies B is locally ν-null. If $\nu \ll \mu$ and if $\mu(E) = 0$, then for each Borel

---

[15]We say that a set is *pre-compact* if it is contained in a compact subset. Alternatively, if X is Hausdorff, B is pre-compact if its closure is compact.

set $F \subset E$, we have

$$\int_F \rho(x)\, d|\nu|(x) = 0.$$

It follows that $\rho(x) = 0$ for $|\nu|$-almost all $x \in E$. Since $|\rho(x)| = 1 \neq 0$ for all x, we must have $|\nu|(E) = 0$. That is $\nu \ll \mu$ if and only if $|\nu| \ll \mu$. Therefore there is a Borel function $\varphi : X \to [0, \infty)$ such that

$$\nu(f) = \int_X f(x)\rho(x)\, d|\nu|(x) = \int_X f(x)\rho(x)\varphi(x)\, d\mu(x),$$

and we call $\frac{d\nu}{d\mu} := \varphi\rho$ the Radon–Nikodym derivative of ν with respect to μ. Of course, $\frac{d\nu}{d\mu}$ is determined μ-almost everywhere.

A.2. Radon measures on locally Hausdorff, locally compact spaces. Now we want to consider functionals on $\mathscr{C}(X)$ where X is a locally Hausdorff, locally compact space. The situation is more complicated because we will not be able to invoke [15, Theorem B.36] since the vector space $\mathscr{C}(X)$ *need not* have the property that $f \in \mathscr{C}(X)$ implies $|f| \in \mathscr{C}(X)$, and hence $\mathscr{C}(X)$ need not be a lattice. This troubling possibility was illustrated in Example 2.1.

Consider a second countable locally Hausdorff, locally compact space X. As in the Hausdorff case, a Radon measure on X starts life as a linear functional $\mu : \mathscr{C}(X) \to \mathbf{C}$ which is positive in the usual sense: $f \geq 0$ should imply that $\mu(f) \geq 0$. To produced a bona fide Borel measure on X corresponding to μ, we will need the following straightforward observation.

Lemma A.1. *Suppose that $(X, \mathcal{M})$ is a Borel space, that $\{U_i\}$ is a cover of X by Borel sets and that μ_i are Borel measures on U_i such that if B is a Borel set in $U_i \cap U_j$, then $\mu_i(B) = \mu_j(B)$. Then there is a Borel measure μ on X such that $\mu|_{U_i} = \mu_i$ for all i.*

Furthermore, if $\{U_j'\}$ and μ_j' is another such family of measures resulting in a Borel measure μ', and if the μ_i and μ_j' agree on overlaps as above, then $\mu = \mu'$.

Sketch of the Proof. As usual, we can find pairwise disjoint Borel sets $B_i \subset U_i$ such that for each n, $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n U_i$. Then we define μ by

$$\mu(B) := \sum_{i=1}^n \mu_i(B \cap B_i).$$

Suppose that B is the countable *disjoint* union $\bigcup_{k=1}^{\infty} E_k$. Then, since the μ_i are each countably additive,

$$\mu(B) = \sum_{i=1}^{n} \mu_i(B \cap B_i) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mu_i(E_k \cap B_i) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \mu_i(E_k \cap B_i)$$

$$= \sum_{k=1}^{\infty} \mu(E_k).$$

Therefore μ is a measure.

If $B \subset U_k$, then

$$\mu(B) = \sum \mu_i(B \cap B_i)$$

which, since $B_i \cap U_k = B_i \cap \bigcup_{j=1}^{k} B_j$ and since the B_j are pairwise disjoint, is

$$= \sum_{i=1}^{k} \mu_i(B \cap B_i)$$

which, since $B \cap B_j \subset U_j \cap U_k$ is

$$= \sum_{i=1}^{k} \mu_k(B \cap B_j) = \mu_k\left(B \cap \bigcup_{j=1}^{k} B_j\right)$$

$$= \mu_k(B).$$

Thus $\mu|_{U_k} = \mu_k$ as claimed.

The proof of uniqueness is straightforward. □

If μ is a Radon measure on $\mathscr{C}(X)$, we can let $\{U_i\}$ be a countable open cover of X by Hausdorff open sets. We can let $\mu_i := \mu|_{C_c(U_i)}$. Then the μ_i are measures as in Lemma A.1, and there is a measure $\bar{\mu}$ on X such that $\bar{\mu}|_{U_i} = \mu_i$. If $f \in \mathscr{C}(X)$, then by [21, Lemma 1.3], we can write $f = \sum f_i$, where each $f_i \in C_c(U_i)$ and only finitely many f_i are nonzero. Then

$$\mu(f) = \sum \mu_i(f_i)$$

$$= \sum \int_X f_i(x)\, d\mu_i(X)$$

$$= \sum \int_X f_i(x)\, d\bar{\mu}(x)$$

$$= \int_X f(x)\, d\bar{\mu}(x).$$

Moreover, $\bar{\mu}$ does not depend on the cover $\{U_i\}$. In the sequel, we will drop the "bar" and write simply "μ" for both the linear functional and the measure as in the Hausdorff case.

Suppose that ν and μ are Radon measures on $\mathscr{C}(X)$ and that we use the same letters for the associated measures on X. As expected, we write $\nu \ll \mu$ if $\mu(E) = 0$ implies $\nu(E) = 0$. Let $\{U_i\}$ be a countable cover of X by Hausdorff open sets, and let ν_i and μ_i be the associated (honest) Radon measures on U_i. Clearly we have $\nu_i \ll \mu_i$ and we can let $\rho_i = \frac{d\nu_i}{d\mu_i}$ be the Radon–Nikodym derivative. The usual uniqueness theorems imply that $\rho_i = \rho_j$ μ-almost everywhere on $U_i \cap U_j$. A standard argument, as in the proof of Lemma A.1, implies that there is a Borel function $\rho : X \to [0, \infty)$ such that $\rho = \rho_i$ μ-almost everywhere on U_i. Then if $f = \sum f_i \in \mathscr{C}(X)$, we have

$$\nu(f) = \sum \nu_i(f_i) = \sum \int_X f_i(x)\rho_i(x)\, d\mu_i(x)$$
$$= \sum \int_X f_i(x)\rho(x)\, d\mu(x) = \int_X f(x)\rho(x)\, d\mu(x)$$
$$= \mu(f\rho).$$

Naturally, we call ρ the Radon–Nikodym derivative of ν with respect to μ.

By a *complex Radon measure* on $\mathscr{C}(X)$, we mean a linear functional $\nu : \mathscr{C}(X) \to \mathbf{C}$ which is continuous in the inductive limit topology. Since $\mathscr{C}(X)$ is not a lattice, the usual proofs that ν decomposes into a linear combination of (positive) Radon measures fail (for example, the proof of [11, Theorem 4.3.2] requires that $\min(f, g) \in \mathscr{C}(X)$ when $f, g \in \mathscr{C}(X)$, and Example 2.1 shows this need not be the case), and we have been unable to supply a "non-Hausdorff" proof. Nevertheless, we can employ the techniques of Lemma A.1 to build what we need from an open cover $\{U_i\}$ of X by Hausdorff subsets. By restriction, we get complex Radon measures ν_i on $C_c(U_i)$. As above there are essentially unique unimodular functions ρ_i such that

$$\nu_i(f) = \int_X f(x)\rho_i(x)\, d|\nu_i|(x) \quad \text{for all } f \in C_c(U_i).$$

Standard uniqueness arguments imply that $|\nu_i|(B) = |\nu_j|(B)$ for Borel sets $B \subset U_i \cap U_j$. We can let $|\nu|$ be the corresponding measure on X. Then $\rho_i(x) = \rho_j(x)$ for $|\nu|$-almost every $x \in B$, and we can define a Borel function $\rho : X \to \mathbf{T}$ such that $\rho(x) = \rho_i(x)$ for $|\nu|$-almost $x \in U_i$. The measure $|\nu|$ and the $|\nu|$-equivalence class of ρ are independent of $\{U_i\}$, and

$$\nu(f) = \int_X f(x)\rho(x)\, d|\nu|(x) \quad \text{for all } f \in \mathscr{C}(X).$$

Suppose that μ is a (positive) Radon measure on $\mathscr{C}(X)$ and that ν is a complex Radon measure on $\mathscr{C}(X)$. As expected, we write $\nu \ll \mu$ if every μ-null set is locally ν-null. Let $\{U_i\}$ be as above. Clearly $\nu_i \ll \mu_i$ and therefore $|\nu_i| \ll \mu_i$. It follows that $|\nu| \ll \mu$. Arguing as above, there is a $\mathbf{C}$-valued Borel function ρ that acts as a Radon–Nikodym derivative for ν with respect to μ; that is,

$$(A.4) \qquad \nu(f) = \int_X f(x)\rho(x)\, d\mu(x) \quad \text{for all } f \in \mathscr{C}(X).$$

Using (A.4) and the continuity of ν, it is not hard to see that $|\nu|$ is continuous in the inductive limit topology and therefore a Radon measure.

Appendix B. Proof of the disintegration theorem

In this section, we want to give a proof of Renault's disintegration theorem (Theorem 7.8). Let L, $\mathcal{H}$, $\mathcal{H}_0$ and $\mathcal{H}_{00}$ be as in the statement of Theorem 7.8. In particular, if $\text{Lin}(\mathcal{H}_0)$ is the collection of linear operators on the vector space $\mathcal{H}_0$, then $L : \mathscr{C}(G) \to \text{Lin}(\mathcal{H}_0)$ is a homomorphism satisfying conditions (a), (b) and (c) of Theorem 7.8. If $\mathcal{H}_0'$ is a dense subspace of a Hilbert space $\mathcal{H}'$, then we say that $L' : \mathscr{C}(G) \to \text{Lin}(\mathcal{H}_0')$ is *equivalent* to L is there is a unitary $U : \mathcal{H} \to \mathcal{H}'$ intertwining L and L' as well as the dense subspaces $\mathcal{H}_0$ and $\mathcal{H}_0'$.

The first step in the proof will be to produce the measure μ that appears in the direct integral in the disintegration. This is straightforward and is done in the next proposition. The real work will be to show that the measure is quasi-invariant.

Proposition B.1. *Suppose that $L : \mathscr{C}(G) \to \text{Lin}(\mathcal{H}_0)$ is as above. Then there is a representation $M : C_0(G^{(0)}) \to B(\mathcal{H})$ such that for all $h \in C_0(G^{(0)})$, $f \in \mathscr{C}(G)$ and $\xi \in \mathcal{H}_0$ we have*

$$(B.1) \qquad M(h)L(f)\xi = L\big((h \circ r) \cdot f\big)\xi.$$

In particular, after replacing L by an equivalent representation, we may assume that $\mathcal{H} = L^2(G^{(0)} * \mathscr{V}, \mu)$ for a Borel Hilbert bundle $G^{(0)} * \mathscr{V}$ and a finite Radon measure μ on $G^{(0)}$ such that

$$M(h)\xi(u) = h(u)\xi(u) \quad \text{for all } h \in C_0(G^{(0)}) \text{ and } \xi \in L^2(G^{(0)} * \mathscr{V}, \mu).$$

Proof. We can easily make sense of $(h \circ r) \cdot f$ for $h \in C_0(G^{(0)})^{\sim}$.[16] Furthermore, we can compute that

$$\big(L((h \circ r) \cdot f)\xi \mid L(g)\eta\big) = \big(L(f)\xi \mid L((\bar{h} \circ r) \cdot g)\eta\big).$$

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[16] As usual, if A is C^*-algebra, then $\widetilde{A}$ is equal to A if $1 \in A$ and A with a unit adjoined otherwise.

Then, if $k \in C_0(G^{(0)})^\sim$ is such that

$$\|h\|_\infty^2 1 - |h|^2 = |k|^2,$$

we can compute that

$$\|h\|_\infty^2 \left\|\sum_{i=1}^n L(f_i)\xi_i\right\|^2 - \left\|\sum_{i=1}^n L\big((h \circ r) \cdot f_i\big)\xi_i\right\|^2$$

$$= \sum_{ij} \big(L\big(((\|h\|_\infty^2 1 - |h|^2) \circ r) \cdot f_i\big)\xi_i | L(f_j)\xi_j\big)$$

$$= \left\|\sum_{i=1}^n L\big((k \circ r) \cdot f_i\big)\xi_i\right\|^2$$

$$\geq 0.$$

Since $\mathcal{H}_{00}$ is dense in $\mathcal{H}$, it follows that there is a well-defined bounded operator $M(h)$ on all of $\mathcal{H}$ satisfying (B.1). It is not hard to see that M is a $*$-homomorphism. To see that M is a representation, by convention, we must also see that M is nondegenerate. But if $f \in C_c(V) \subset \mathscr{C}(G)$, then $r\big(\mathrm{supp}_V f\big)$ is compact in $G^{(0)}$. Hence there is a $h \in C_0(G^{(0)})$ such that $M(h)f = f$. From this, it is straightforward to see that M is nondegenerate and therefore a representation.

Since M is a representation of $C_0(G^{(0)})$, it is equivalent to a multiplication representation on $L^2(G^{(0)} * \mathscr{V}, \mu)$ for an appropriate Borel Hilbert bundle $G^{(0)} * \mathscr{V}$ and finite Radon measure μ — for example, see [46, Example F.25]. The second assertion follows, and this completes the proof. $\qquad\square$

Lemma B.2. *If $\mathcal{H}_{00}'$ is a dense subspace of $\mathcal{H}_{00}$, then*

$$\mathrm{span}\{L(f)\xi : f \in \mathscr{C}(G) \text{ and } \xi \in \mathcal{H}_{00}'\}$$

is dense in $\mathcal{H}$.

Proof. In view of Proposition 6.8, there is a self-adjoint approximate identity $\{e_i\}$ for $\mathscr{C}(G)$ in the inductive limit topology. Then if $L(f)\xi \in \mathcal{H}_{00}$, we see that

$$\|L(e_i)L(f)\xi - L(f)\xi\|^2 =$$
$$\big(L(f^* * e_i * e_i * f)\xi \mid \xi\big) - 2\,\mathrm{Re}\big(L(f^* * e_i * f)\xi \mid \xi\big) + \big(L(f^* * f)\xi \mid \xi\big),$$

which tends to zero since L is continuous in the inductive limit topology (by part (b) of Theorem 7.8). It follows that $\mathcal{H}_{00}' \subset \overline{\mathrm{span}}\{L(f)\xi : \xi \in \mathcal{H}_{00}' \text{ and } f \in \mathscr{C}(G)\}$. Since $\mathcal{H}_{00}'$ is dense, the result follows. $\qquad\square$

The key to Renault's proof, which we are following here, is to realize $\mathcal{H}$ as the completion of (a quotient of) the algebraic tensor product $\mathscr{C}(G) \odot \mathcal{H}_0$ which has a natural fibring over $G^{(0)}$.

Lemma B.3. *Then there is a positive sesquilinear form $\langle \cdot\, , \cdot \rangle$ on $\mathscr{C}(G) \odot \mathcal{H}_0$ such that*

(B.2) $$\langle f \otimes \xi\, , g \otimes \eta \rangle = \big(L(g^* * f)\xi \mid \eta\big).$$

Furthermore, the Hilbert space completion $\mathcal{K}$ of $\mathscr{C}(G) \odot \mathcal{H}_0$ is isomorphic to $\mathcal{H}$. In fact, if $[f \otimes \xi]$ is the class of $f \otimes \xi$ in $\mathcal{K}$, then $[f \otimes \xi] \mapsto L(f)\xi$ is well-defined and induces an isomorphism of $\mathcal{K}$ with $\mathcal{H}$ which maps the quotient $\mathscr{C}(G) \odot \mathcal{H}_0/\mathcal{N}$, where $\mathcal{N}$ is the subspace $\mathcal{N} = \{\sum_i f_i \otimes \xi_i : \sum_i L(f_i)\xi_i = 0\}$ of vectors in $\mathscr{C}(G) \odot \mathcal{H}_0$ of length zero, onto $\mathcal{H}_{00}$.

Proof. Using the universal properties of the *algebraic* tensor product, as in the proof of [35, Proposition 2.64] for example, it is not hard to see that there is a unique sesquilinear form on $\mathscr{C}(G) \odot \mathcal{H}_0$ satisfying (B.2).[17] Thus to see that $\langle \cdot\, , \cdot \rangle$ is a pre-inner product, we just have to see that it is positive. But

(B.3) $$\left\langle \sum_i f_i \otimes \xi_i, \sum_i f_i \otimes \xi_i \right\rangle = \sum_{ij}\big(L(f_j^* * f_i)\xi_i \mid \xi_j\big)$$
$$= \sum_{ij}\big(L(f_i)\xi_i \mid L(f_j)\xi_i\big)$$
$$= \left\| \sum_i L(f_i)\xi_i \right\|^2.$$

As in [35, Lemma 2.16], $\langle \cdot\, , \cdot \rangle$ defines an inner-product on $\mathscr{C}(G) \odot \mathcal{H}_0/\mathcal{N}$, and $[f_i \otimes \xi] \mapsto L(f_i)\xi$ is well-defined in view of (B.3). Since this map has range $\mathcal{H}_{00}$ and since $\mathcal{H}_{00}$ is dense in $\mathcal{H}$ by part (a) of Theorem 7.8, the map extends to an isomorphism of $\mathcal{K}$ onto $\mathcal{H}$ as claimed. $\square$

From here on, using Lemma B.3, we will normally identify $\mathcal{H}$ with $\mathcal{K}$, and $\mathcal{H}_{00}$ with $\mathscr{C}(G) \odot \mathcal{H}_0/\mathcal{N}$. Thus we will interpret $[f \otimes \xi]$ as a vector in

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[17]For fixed g and η, the left-hand side of (B.2) is bilinear in f and ξ. Therefore, by the universal properties of the algebraic tensor product, (B.2) defines linear map $m(g, \eta) :$ $\mathscr{C}(G) \odot \mathcal{H}_0 \to \mathbf{C}$. Then $(g, \eta) \mapsto \overline{m(g, \eta)}$ is a bilinear map into the space $\mathrm{CL}(\mathscr{C}(G) \odot \mathcal{H}_0)$ of conjugate linear functionals on $\mathscr{C}(G) \odot \mathcal{H}_0$. Then we get a linear map $N : \mathscr{C}(G) \odot \mathcal{H}_0 \to$ $\mathrm{CL}(\mathscr{C}(G) \odot \mathcal{H}_0)$. We can then define $\langle \alpha\, , \beta \rangle := \overline{N(\beta)(\alpha)}$. Clearly $\alpha \mapsto \langle \alpha\, , \beta \rangle$ is linear and it is not hard to check that $\overline{\langle \alpha\, , \beta \rangle} = \langle \beta\, , \alpha \rangle$.

$\mathcal{H}_{00} \subset \mathcal{H}_0 \subset \mathcal{H}$. Then we have

(B.4) $$L(g)[f \otimes \xi] = [g * f \otimes \xi] \quad \text{and}$$

(B.5) $$M(h)[f \otimes \xi] = [(h \circ r) \cdot f \otimes \xi],$$

where M is the representation of $C_0(G^{(0)})$ defined in Proposition B.1, $g \in \mathscr{C}(G)$ and $h \in C_0(G^{(0)})$.

Remark B.4. In view of Proposition B.1, M extends to a $*$-homomorphism of $\mathcal{B}_c^b(G)$ into $B(\mathcal{H})$ such that $M(h) = 0$ if $h(u) = 0$ for μ-almost all u (where μ is the measure defined in that proposition). However, at this point, we can not assert that (B.5) holds for any $h \notin C_0(G^{(0)})$.

Showing that μ is quasi-invariant requires that we extend equations (B.4) and (B.5) to a larger class of functions. This can't be done without also enlarging the domain of definition of L. This is problematic as we don't as yet know that each $L(f)$ is bounded in any sense, and $\mathcal{H}_0$ is not complete. We'll introduce only those functions we absolutely need.

Definition B.5. Suppose that V is an open Hausdorff set in G. Let $\mathcal{B}_c^1(V)$ be the collection of bounded Borel functions on V which are the pointwise limit of a uniformly bounded sequence $\{f_n\} \subset C_c(V)$ such that there is a compact set $K \subset V$ such that $\operatorname{supp} f_n \subset K$ for large n. We let $\mathscr{B}^1(G)$ be the vector space spanned by the $\mathcal{B}_c^1(V)$ for all $V \subset G$ open and Hausdorff.

It is important to note that $\mathscr{B}^1(G)$ is not a very robust class of functions on G. In particular, it is not closed under the type of convergence used in its definition. Nevertheless, its elements are all integrable with respect to any Radon measure on G, and the following lemma is an easy consequence of the dominated convergence theorem applied to the total variation measure.

Lemma B.6. *Suppose that σ is a complex Radon measure on $\mathscr{C}(G)$ such that*

(B.6) $$\sigma(f) = \int_G f(\gamma)\rho(\gamma) \, d|\sigma|(\gamma)$$

for a unimodular function ρ and total variation $|\sigma|$ (see Appendix A.2). Then σ extends to a linear functional on $\mathscr{B}^1(G)$ such that (B.6) holds and such that if $\{f_n\}$ is a uniformly bounded sequence in $\mathscr{B}^1(G)$ converging pointwise to $f \in \mathscr{B}^1(G)$ with supports eventually contained in a fixed compact set, then $\sigma(f_n) \to \sigma(f)$.

Sketch of the Proof. Since $|\sigma|$ is a Radon measure, (B.6) makes good sense for any $f \in \mathscr{B}^1(G)$. Thus σ extends as claimed. The rest is an easy consequence of the dominated convergence theorem applied to $|\sigma|$. $\square$

Corollary B.7. *If* $f, g \in \mathscr{B}^1(G)$, *then*

$$f * g(\gamma) := \int_G f(\eta)g(\eta^{-1}\gamma) \, d\lambda^{r(\gamma)}(\eta)$$

defines an element $f * g$ *of* $\mathscr{B}^1(G)$.

Proof. As in the proof of Proposition 4.4, we can assume that there are Hausdorff open sets U and V such that UV is Hausdorff and such that $f \in \mathcal{B}_c^1(U)$ while $g \in \mathcal{B}_c^1(V)$. Let $\{f_n\}$ and $\{g_n\}$ be uniformly bounded sequences in $C_c(U)$ and $C_c(V)$, respectively, such that $f_n \to f$ and $g_n \to g$ pointwise and with supports contained in a fixed compact set. Since

$$|f_n * g_n(\gamma)| \leq \|f_n\|_\infty \|g_n\|_\infty \sup_{u \in G^{(0)}} \lambda^u\big((\operatorname{supp} f_n)(\operatorname{supp} g_n)\big),$$

it follows that $\{f_n * g_n\}$ is a uniformly bounded sequence in $C_c(UV)$, all of whose supports are in a fixed compact set, converging pointwise to $f * g$. Thus $f * g \in \mathcal{B}_c^1(UV) \subset \mathscr{B}^1(G)$. $\qquad\square$

In view of the continuity assumption on L, we can define a complex Radon measure $L_{\xi,\eta}$ on $\mathscr{C}(G)$ via

$$L_{\xi,\eta}(f) := \big(L(f)\xi \mid \eta\big)$$

for each ξ and η in $\mathcal{H}_0$. Keep in mind that we can extend $L_{\xi,\eta}$ to a linear functional on all of $\mathscr{B}^1(G)$.

Lemma B.8. *There is a positive sesquilinear form on* $\mathscr{B}^1(G) \odot \mathcal{H}_0$, *extending that on* $\mathscr{C}(G) \odot \mathcal{H}_0$, *such that*

$$\langle f \otimes \xi \,,\, g \otimes \eta \rangle = L_{\xi,\eta}(g^* * f) \quad \text{for all } f, g \in \mathscr{C}(G) \text{ and } \xi, \eta \in \mathcal{H}_0.$$

In particular, if

$$\mathcal{N}_b := \left\{ \sum_i f_i \otimes \xi \in \mathscr{C}(G) \odot \mathcal{H}_0 : \left\langle \sum_i f_i \otimes \xi, \sum_i f_i \otimes \xi_i \right\rangle = 0 \right\}$$

is the subspace of vectors of zero length, then the quotient $\mathscr{B}^1(G) \odot \mathcal{H}_0 / \mathcal{N}_b$ *can be identified with a subspace of* $\mathcal{H}$ *containing* $\mathcal{H}_{00} := \mathscr{C}(G) \odot \mathcal{H}_0 / \mathcal{N}$.

Remark B.9. As before, we will write $[f \otimes \xi]$ for the class of $f \otimes \xi$ in the quotient $\mathscr{B}^1(G) \odot \mathcal{H}_0 / \mathcal{N}_b \subset \mathcal{H}$.

Proof. Just as in Lemma B.3, there is a well-defined sesquilinear form on $\mathscr{B}^1(G) \odot \mathcal{H}_0$ extending that on $\mathscr{C}(G) \odot \mathcal{H}_0$. (Note that the right-hand side of (B.2) can be rewritten as $L_{\xi,\eta}(g^* * f)$.) In particular, we have

$$\left\langle \sum_i f_i \otimes \xi_i, \sum_j g_j \otimes \eta_j \right\rangle = \sum_{ij} L_{\xi_i, \eta_j}(g_j^* * f_i).$$

We need to see that the form is positive. Let $\alpha := \sum_i f_i \otimes \xi_i$, and let $\{f_{i,n}\}$ be a uniformly bounded sequence in $\mathscr{C}(G)$ converging pointwise to f_i with all the supports contained in a fixed compact set. Then for each i and j, $f_{j,n}^* * f_{i,n} \to f_j^* * f_i$ in the appropriate sense. In particular, Lemma B.6 implies that

$$\langle \alpha \, , \, \alpha \rangle = \sum_{ij} L_{\xi_i, \xi_j}(f_j^* * f_i)$$

$$= \lim_n \sum_{ij} L_{\xi_i, \xi_j}(f_{j,n}^* * f_{i,n})$$

$$= \lim_n \langle \alpha_n \, , \, \alpha_n \rangle,$$

where $\alpha_n := \sum_i f_{i,n} \otimes \xi_i$. Since $\langle \cdot \, , \, \cdot \rangle$ is positive on $\mathscr{C}(G) \odot \mathcal{H}_0$ by Lemma B.3, we have $\langle \alpha_n \, , \, \alpha_n \rangle \geq 0$ and we've shown that $\langle \cdot \, , \, \cdot \rangle$ is still positive on $\mathscr{B}^1(G) \odot \mathcal{H}_0$.

Clearly the map sending the class $f \otimes \xi + \mathcal{N}$ to $f \otimes \xi + \mathcal{N}_b$ is isometric and therefore extends to an isometric embedding of $\mathcal{H}$ into the Hilbert space completion $\mathcal{H}_b$ of $\mathscr{B}^1(G) \odot \mathcal{H}_0$ with respect to $\langle \cdot \, , \, \cdot \rangle$. However if $g \otimes \xi \in \mathscr{B}^1(G) \odot \mathcal{H}_0$ and if $\{g_n\}$ is a sequence in $\mathscr{C}(G)$ such that $g_n \to g$ in the usual way, then

$$\|(g_n \otimes \xi + \mathcal{N}_b) - (g \otimes \xi + \mathcal{N}_b)\|^2 = L_{\xi, \xi}(g_n^* * g_n - g_n^* * g - g * g_n^* + g^* * g),$$

and this tends to zero by Lemma B.6. Thus the image of $\mathcal{H}$ in $\mathcal{H}_b$ is all of $\mathcal{H}_b$. Consequently, we can identify the completion of $\mathscr{B}^1(G) \odot \mathcal{H}_0$ with $\mathcal{H}$ and $\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b$ with a subspace of $\mathcal{H}$ containing $\mathcal{H}_{00}$. $\square$

The "extra" vectors provided by $\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b$ are just enough to allow us to use a bit of general nonsense about unbounded operators to extend the domain of each $L(f)$. More precisely, for $f \in \mathscr{C}(G)$, we can view $L(f)$ as an operator in $\mathcal{H}$ with domain $\mathcal{D}(L(f)) = \mathcal{H}_{00}$. Then using part (c) of Theorem 7.8, we see that

$$L(f^*) \subset L(f)^*.$$

This implies that $L(f)^*$ is a densely defined operator. Hence $L(f)$ is closable [7, Proposition X.1.6]. Consequently, the closure of the graph of $L(f)$ in $\mathcal{H} \times \mathcal{H}$ is the graph of the closure $\overline{L(f)}$ of $L(f)$ [7, Proposition X.1.4].

Suppose that $g \in \mathscr{B}^1(G)$. Let $\{g_n\}$ be a uniformly bounded sequence in $\mathscr{C}(G)$ all with supports in a fixed compact set such that $g_n \to g$ pointwise. Then

(B.7) $\|[g_n \otimes \xi] - [g \otimes \xi]\|^2 = L_{\xi, \xi}(g_n^* * g_n - g^* * g_n - g_n^* * g + g * g).$

However $\{g_n^* * g_n - g^* * g_n - g_n^* * g + g * g\}$ is uniformly bounded and converges pointwise to zero. Since the supports are all contained in a fixed compact set, the left-hand side of (B.7) tends to zero by Lemma B.6. Similarly,

$$\|[f * g_n \otimes \xi] - [f * g \otimes \xi]\|^2 \to 0.$$

If follows that

$$\big([g_n \otimes \xi, L(f)[g_n \otimes \xi]\big) \to \big([g \otimes \xi], [f * g \otimes \xi]\big)$$

in $(\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b) \times (\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b) \subset \mathcal{H} \times \mathcal{H}$. Therefore $[g \otimes \xi] \in \mathcal{D}\big(\overline{L(f)}\big)$ and $\overline{L(f)}[g \otimes \xi] = [f * g \otimes \xi]$. We have proved the following.

Lemma B.10. *For each $f \in \mathscr{C}(G)$, $L(f)$ is a closable operator in $\mathcal{H}$ with domain $\mathcal{D}(L(f)) = \mathcal{H}_{00} = \mathscr{C}(G) \odot \mathcal{H}_0/\mathcal{N}$. Furthermore $\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b$ belongs to $\mathcal{D}\big(\overline{L(f)}\big)$, and*

$$\overline{L(f)}[g \otimes \xi] = [f * g \otimes \xi] \quad \text{for all } f \in \mathscr{C}(G), \ g \in \mathscr{B}^1(G) \text{ and } \xi \in \mathcal{H}_0.$$

Now can extend L a bit.

Lemma B.11. *For each $f \in \mathscr{B}^1(G)$, there is a well-defined operator $L_b(f) \in \operatorname{Lin}(\mathscr{B}^1(G) \odot \mathcal{H}_0)/\mathcal{N}_b)$ such that*

(B.8) $$L_b(f)[g \otimes \xi] = [f * g \otimes \xi].$$

If $f \in \mathscr{C}(G)$, then $L_b(f) \subset \overline{L(f)}$.

Proof. To see that (B.8) determines a well-defined operator, we need to see that

(B.9) $$\sum_i [g_i \otimes \xi_i] = 0 \quad \text{implies} \quad \sum_i [f * g_i \otimes \xi_i] = 0.$$

However,

(B.10) $$\left\| \sum_i [f * g_i \otimes \xi_i] \right\|^2 = \sum_{ij} L_{\xi_i, \xi_j}(g_j^* * f^* * f * g_i).$$

Since $f \in \mathscr{B}^1(G)$, we can approximate the right-hand side of (B.10) by sums of the form

(B.11) $$\sum_{ij} L_{\xi_i, \xi_j}(g_j^* * h^* * h * g_i),$$

where $h \in \mathscr{C}(G)$. But (B.11) equals

$$\left\| \overline{L(h)} \sum_i [g_i \otimes \xi_i] \right\|^2$$

which is zero if the left-hand side of (B.9) is zero. Hence the right-hand side of (B.9) is also zero and $L_b(f)$ is well-defined.

If $f \in \mathscr{C}(G)$, then $L_b(f) \subset \overline{L(f)}$ by Lemma B.10. $\qquad\square$

The previous gymnastics have allowed us to produce some additional vectors in $\mathcal{H}$ and to extend slightly the domain of L. The next lemma provides the technical assurances that, despite the subtle definitions above, our new operators act via the formulas we expect.

Lemma B.12. *Suppose that $f \in \mathscr{B}^1(G)$ and that k is a bounded Borel function on $G^{(0)}$ which is the pointwise limit of a uniformly bounded sequence from $C_0(G^{(0)})$. Then for all $g, h \in \mathscr{C}(G)$ and $\xi, \eta \in \mathcal{H}_0$, we have the following.*

(a)
$$\big(L_b(f)[g \otimes \xi] \mid [h \otimes \eta]\big) = \big([f * g \otimes \xi] \mid [h \otimes \eta]\big)$$
$$= L_{\xi,\eta}(h^* * f * g)$$
$$= L_{[g\otimes\xi],[h\otimes\eta]}(f)$$

(b)
$$\big(M(k)[g \otimes \xi] \mid [h \otimes \eta]\big) = L_{\xi,\eta}(h^* * ((k \circ r) \cdot g))$$
$$= \big([(k \circ r) \cdot g \otimes \xi] \mid [h \otimes \eta]\big)$$
$$= \big(M(k)L(g)\xi \mid L(h)\eta\big)$$

(c) $\qquad \big(M(k)L_b(f)[g \otimes \xi] \mid [h \otimes \eta]\big) = \big(L_b((k \circ r) \cdot f)[g \otimes \xi] \mid [h \otimes \eta]\big).$

Proof. We start with (a). The first equality is just the definition of $L_b(f)$. The second follows from the definition of the inner product on $\mathscr{B}^1(G) \odot \mathcal{H}_0/\mathcal{N}_b$. If f is in $\mathscr{C}(G)$, then the third equation holds just by untangling the definition of the complex Radon measure $L_{\xi,\eta}$ and using the continuity in the inductive limit topology. The third equality holds for $f \in \mathscr{B}^1(G)$ by applying the continuity assertion in Lemma B.6.

Part (b) is proved similarly. The first equation holds if $k \in C_0(G^{(0)})$ by definition of $M(k)$ and $L_{\xi,\eta}$. If $\{k_n\} \subset C_0(G^{(0)})$ is a bounded sequence converging pointwise to k, then $M(k_k) \to M(k)$ in the weak operator topology by the dominated convergence theorem. On the other hand $h^* * (k_n \circ r) \cdot g \to h^* * (k \circ r)g$ in the required way. Thus $L_{\xi,\eta}(h^* * (k_n \circ r) \cdot g) \to L_{\xi,\eta}(h^* * (k \circ r)g)$ by Lemma B.6. Thus the first equality is valid. The second equality is clear if $k \in C_0(G^{(0)})$ and passes to the limit as above. The third equality is simply our identification of $[g \otimes \xi]$ with $L(g)\xi$ as in Lemma B.3.

For part (c), first note that if $f_n \to f$ and $k_n \to k$ are uniformly bounded sequences converging pointwise with supports in fixed compact sets independent of n, then $(k \circ r) \cdot f = \lim_n (k_n \circ r) \cdot f_n$. It follows that $(k \circ r) \cdot f \in \mathscr{B}^1(G)$.

Also, $[f \otimes \xi] = \lim[f_n \otimes \xi]$, and since $M(k)$ is bounded, part (b) implies that

$$
\begin{aligned}
M(k)[f \otimes \xi] &= \lim_n M(x)[f_n \otimes \xi] \\
&= \lim_n [(k \circ r) \cdot f_n \otimes \xi] \\
&= [(k \circ r) \cdot f \otimes \xi].
\end{aligned}
$$

Since it is not hard to verify that $M(k)^*[f \otimes \xi] = (\bar{k} \circ r) \cdot f \otimes \xi]$, we can compute that

$$
\begin{aligned}
\big(M(k)L_b(f)[g \otimes \xi] \mid [h \otimes \eta]\big) &= \big([f * g \otimes \xi] \mid (\bar{k} \circ r) \cdot h \otimes \eta]\big) \\
&= \big([k \circ r) \cdot (f * g) \otimes \xi] \mid [h \otimes \eta]\big) \\
&= \big([((k \circ r) \cdot f) * g \otimes \xi] \mid [h \otimes \eta]\big) \\
&= \big(L_b((k \circ r) \cdot f)[g \otimes \xi] \mid [h \otimes \eta]\big). \qquad \square
\end{aligned}
$$

Proposition B.13. *Let μ be the Radon measure on $G^{(0)}$ associated to L by Proposition B.1. Then μ is quasi-invariant.*

Proof. We need to show that measures ν and ν^{-1} (defined in (7.1) and (7.2), respectively) are equivalent. Therefore, we have to show that if A is pre-compact in G, then $\nu(A) = 0$ if and only if $\nu(A^{-1}) = 0$. Since $(A^{-1})^{-1} = A$, it's enough to show that A ν-null implies that A^{-1} is too. Further, we can assume that $A \subset V$, where V is open and Hausdorff. Since $\nu|_V$ is regular, we may as well assume that A is a G_δ-set so that $f := \mathbb{1}_A$ is in $\mathcal{B}^1_c(V) \subset \mathscr{B}^1(G)$. Let $\widetilde{f}(x) = f(x^{-1})$. We need to show that $\widetilde{f}(x) = 0$ for ν-almost all x. Since A is a G_δ, we can find a sequence $\{f_n\} \subset C^+_c(V)$ such that $f_n \searrow f$.

If k is any function in $\mathscr{C}(G)$, then $kf\bar{k} = |k|^2 f \in \mathscr{B}^1(G)$ and vanishes ν-almost everywhere. By the monotone convergence theorem,

$$
\lambda(kf\bar{k})(u) := \int_G |k(\gamma)|^2 f(\gamma) \, d\lambda^u(\gamma)
$$

defines a function in $\mathcal{B}^1_c(G^{(0)})$ which is equal to 0 for μ-almost all u. In particular, $M(\lambda(kf\bar{k})) = 0$.

It then follows from part (b) of Lemma B.12 that

$$
(B.12) \qquad 0 = \big(M\big(\lambda(kf\bar{k})\big)L(g)\xi \mid L(g)\xi\big) = L_{\xi,\xi}(g^* * \big(\lambda(kf\bar{k}) \circ r\big) \cdot g)
$$

for all $g \in \mathscr{C}(G)$ and $\xi \in \mathcal{H}_0$. On the other hand, if (B.12) holds for all $g, k \in \mathscr{C}(G)$ and $\xi \in \mathcal{H}_0$, then we must have $M\big(\lambda(kf\bar{k})\big) = 0$ for all $k \in \mathscr{C}(G)$. Since $f(\gamma) \geq 0$ everywhere, this forces $|k(\gamma)|^2 f(\gamma) = 0$ for ν-almost all γ. Since k is arbitrary, we conclude that $f(\gamma) = 0$ for ν-almost

all γ. Therefore it will suffice to show that

(B.13) $\qquad L_{\xi,\xi}\big(g^* * \big(\lambda(k\widetilde{f}\bar{k}) \circ r\big) \cdot g\big) = 0$ for all $g, k \in \mathscr{C}(G)$ and $\xi \in \mathcal{H}_0$,

where we have replaced f with $\widetilde{f}$ in the right-hand side of (B.12). First, we compute that with f in (B.12) we have

$$g^* * \big(\lambda(kf\bar{k}) \circ r\big) \cdot g(\sigma) = \int_G \overline{g(\gamma^{-1})}\big(\lambda(kf\bar{k}) \circ r\big) \cdot g(\gamma^{-1}\sigma)\, d\lambda^{r(\sigma)}(\gamma)$$

$$= \int_G \overline{g(\gamma^{-1})}\lambda(kf\bar{k})\big(s(\gamma)\big)g(\gamma^{-1}\sigma)\, d\lambda^{r(\sigma)}(\gamma)$$

$$= \int_G \int_G \overline{g(\gamma^{-1})k(\eta)}f(\eta)k(\eta)g(\gamma^{-1}\sigma)\, d\lambda^{s(\gamma)}(\eta)\, d\lambda^{r(\sigma)}(\gamma)$$

which, after sending $\eta \mapsto \gamma^{-1}\eta$ and using left-invariance of the Haar system, is

$$= \int_G \int_G \overline{g(\gamma^{-1})k(\gamma^{-1}\eta)}f(\gamma^{-1}\eta)k(\gamma^{-1}\eta)g(\gamma^{-1}\sigma)\, d\lambda^{r(\sigma)}(\eta)\, d\lambda^{r(\sigma)}(\gamma)$$

After defining $F(\gamma, \eta) := k(\gamma^{-1}\eta)g(\gamma^{-1})$ and $f \cdot F(\gamma, \eta) := f(\gamma^{-1}\eta)F(\gamma, \eta)$ for $(\gamma, \eta) \in G *_r G$, this gives

(B.14) $\qquad \displaystyle\int_G \int_G \overline{F(\gamma, \eta)}f \cdot F(\sigma^{-1}\gamma, \sigma^{-1}\eta)\, d\lambda^{r(\sigma)}(\eta)\, d\lambda^{r(\sigma)}(\gamma).$

We will have to look at integrals of the form (B.14) in some detail. First note that if U and V are Hausdorff open sets in G, then $U *_r V$ is a Hausdorff open set in $G *_r G$. Thus if $g, k \in \mathscr{C}(G)$, then $F(\gamma, \eta) := k(\gamma^{-1}\eta)g(\gamma^{-1})$ defines an element $F \in \mathscr{C}(G *_r G)$.[18]

Lemma B.14. *Suppose that $F_1, F_2 \in \mathscr{C}(G *_r G)$. Then*

$$\sigma \mapsto \int_G \int_G \overline{F_1(\gamma, \eta)}F_2(\sigma^{-1}\gamma, \sigma^{-1}\eta)\, d\lambda^{r(\sigma)}(\eta)\, d\lambda^{r(\sigma)}(\gamma)$$

defines an element of $\mathscr{C}(G)$ which we denote by $\overline{F}_1 *_{\lambda*\lambda} F_2$.

Proof. We can take $F_i \in C_c(U_i *_r V_i)$ with each U_i and V_i open and Hausdorff. As in the proof of Proposition 4.4, we can assume that $U_1 U_2^{-1}$ and $V_1 V_2^{-1}$ are Hausdorff. Note that

$$\|\overline{F}_1 *_{\lambda*\lambda} F_2\|_\infty \le \|F_1\|_\infty \|F_2\|_\infty \sup_{u \in G^{(0)}} \lambda^u(K_1)\lambda^u(K_2)$$

---

[18]For example, we can assume that $k \in C_c(U)$ and $g \in C_c(V)$ with U and V both open and Hausdorff. A partition of unity argument as in the proof of Proposition 4.4, allows us to assume that VU is Hausdorff. Then observe that $\operatorname{supp} F \subset VU *_r V^{-1}$.

whenever supp $F_1 \subset K_1 *_r K_2$. Thus to see that

$$\overline{F}_1 *_{\lambda * \lambda} F_2 \in C_c(U_1 U_2^{-1} \cap V_1 V_2^{-1}),$$

it will suffice to consider only those F_i is dense subspaces of $C_c(U_1 *_r V_1)$ and $C_c(U_2 *_r V_2)$. In particular, we can assume that each F_i is of the form $F_i(\gamma, \eta) = k_i(\eta)g(\gamma^{-1})$. But then

$$\overline{F}_1 *_{\lambda * \lambda} F_2(\sigma) = \overline{k}_1 * \widetilde{k}_2(\sigma)g_1^* * g_2(\sigma),$$

and the result follows. □

Lemma B.15. *Functions of the form*

(B.15) $(\gamma, \eta) \mapsto k(\gamma^{-1}\eta)g(\gamma^{-1})$ *with* $k, g \in \mathscr{C}(G)$

span a dense subspace of $\mathscr{C}(G *_r G)$ *in the inductive limit topology.*

Proof. We have already noted that functions of the form given in (B.15) determine elements $\theta_{k,g}$ in $\mathscr{C}(G *_r G)$. Furthermore, arguing as in the proof of Proposition 4.4, it will suffice to show that we can approximate functions $\theta \in C_c(U *_r V)$ with U and V open, Hausdorff and such that UV is Hausdorff. Then the span of functions $\theta_{k,g}$ with $k \in C_c(UV)$ and $g \in C_c(V^{-1})$ is dense in $C_c(U *_r V)$ in the inductive limit topology by the Stone–Weierstrass Theorem. □

Let $\mathscr{A}_0 \subset \Gamma_c(G *_r G; \iota^* \mathscr{B})$ be the dense subspace of functions of the form considered in Lemma B.15. We continue to write f for the characteristic function of our fixed pre-compact, ν-null set. Then we know from (B.12) that

(B.16) $L_{\xi, \xi}(\overline{F} *_{\lambda * \lambda} (f \cdot F)) = 0$ for all $F \in \mathscr{A}_0$.

It is not hard to check that, if $f' \in \mathcal{B}_c^1(G)$, then $F *_{\lambda * \lambda} (f' \cdot F) \in \mathscr{B}^1(G)$ and that if $F_n \to F$ in the inductive limit topology in $\mathscr{C}(G *_r G)$, then $\{\overline{F}_n *_{\lambda * \lambda} (f' \cdot F)\}$ is uniformly bounded and converges pointwise to $\overline{F} *_{\lambda * \lambda} (f' \cdot F)$. In particular the continuity of the $L_{\xi, \xi}$ (see Lemma B.6) implies that (B.16) holds for all $F \in \mathscr{C}(G *_r G)$. But if we define $\widetilde{F}(x, y) := F(y, x)$, then we see from the definition that

$$\overline{\widetilde{F}} *_{\lambda * \lambda} (f \cdot \widetilde{F}) = \overline{F} *_{\lambda * \lambda} (\widetilde{f} \cdot F)),$$

where we recall that $\widetilde{f}(x) := f(x^{-1})$. Thus

$$L_{\xi, \xi}(\overline{F} *_{\lambda * \lambda} (\widetilde{f} \cdot F))) = 0$$ for all $F \in \mathscr{C}(G *_r G)$.

Since the above holds in particular for $F \in \mathscr{A}_0$, this implies (B.13), and completes the proof. □

To define the Borel Hilbert bundle we need, we need to see that the complex Radon measures $L_{\xi,\eta}$ defined above are absolutely continuous with respect to the measure ν. In order to prove this, we need to restrict ξ and η to lie in $\mathcal{H}_{00}$, and to employ Lemma B.12.

Lemma B.16. *Let a and b be vectors in $\mathcal{H}_{00}$ (identified with $\mathscr{C}(G)\odot\mathcal{H}_0/\mathcal{N}$). Let $L_{a,b}$ be the complex Radon measure given by*

$$L_{a,b}(f) := \big(L(f)a \mid b\big).$$

Then $L_{a,b}$ is absolutely continuous with respect to the measure ν defined in (7.1).[19]

Proof. It is enough to show that if M is a pre-compact ν-null set and if $f := \mathbb{1}_M$, then $L_{a,b}(f) = 0$. We can also assume that $M \subset V$ where V is a Hausdorff open set. Since $\nu|_V$ is a Radon measure, and therefore regular, we may as well assume that M is a G_δ-set. Then $f \in \mathcal{B}_c^1(V) \subset \mathscr{B}^1(G)$.

On the other hand,

$$0 = \int_{G^{(0)}} \int_G f(\gamma)\,d\lambda^u(\gamma)\,d\mu(u),$$

so there is a μ-null set $N \subset G^{(0)}$ such that $\lambda^u(M \cap G^u) = 0$ if $u \notin N$. As above, we can assume that N is a G_δ set. Then for any $g \in \mathscr{C}(G)$, we have

$$f * g(\gamma) = \int_G f(\eta)g(\eta^{-1}\gamma)\,d\lambda^{r(\gamma)}(\eta) = 0$$

whenever $r(\gamma) \notin N$. Since $\operatorname{supp}\lambda^{r(\gamma)} = G^{r(\gamma)}$, it follows that for all $\gamma \in G$ (without exception),

(B.17) $\qquad f * g(\gamma) = \mathbb{1}_N\big(r(\gamma)\big)f * g(\gamma) = \big((\mathbb{1}_N \circ r) \cdot f\big) * g(\gamma).$

Since $a, b \in \mathcal{H}_{00}$, it suffices to consider $a = [g \otimes \xi]$ and $b = [h \otimes \eta]$ (with $g, h \in \mathscr{C}(G)$ and $\xi, \eta \in \mathcal{H}_0$). Note that f and $\mathbb{1}_N$ satisfy the hypotheses of Lemma B.12. Therefore, by part (a) of that lemma,

$$L_{[g\otimes\xi,h\otimes\eta]}(f) = \big([f * g \otimes \xi] \mid [h \otimes \eta]\big)$$

which, by (B.17), is

$$= \big([((\mathbb{1}_N \circ r) \cdot f) * g \otimes \xi] \mid [h \otimes \eta]\big)$$

which, by part (a) of Lemma B.12, is

$$= \big(L_b((\mathbb{1}_N \circ r) \cdot f)[g \otimes \xi] \mid [h \otimes \eta]\big)$$

---

[19]Absolute continuity of complex Radon measures on locally Hausdorff, locally compact spaces is discussed in Appendix A.2.

which, by part (c) of Lemma B.12, is

$$= \big(M(\mathbb{1}_N) L_b(f)[g \otimes \xi] \mid [h \otimes \eta] \big).$$

Since $M(\mathbb{1}_N) = 0$, the last inner product is zero as desired. This completes the proof. $\qquad\square$

Since the measures ν and ν_0 are equivalent, for each $\xi, \eta \in \mathcal{H}_{00}$, we can, in view of Lemma B.16, let $\rho_{\xi,\eta}$ be the Radon–Nikodym derivative of $L_{\xi,\eta}$ with respect to ν_0. Then for each $\xi, \eta \in \mathcal{H}_{00}$, we have

$$\big(L(f)\xi \mid \eta \big) = L_{\xi,\eta}(f) = \int_G f(x)\, dL_{\xi,\eta}(x)$$

$$= \int_G f(x)\rho_{\xi,\eta}(x)\Delta(x)^{-\frac{1}{2}}\, d\nu(x)$$

$$= \int_{G^{(0)}} \int_G f(x)\rho_{\xi,\eta}(x)\Delta(x)^{-\frac{1}{2}}\, d\lambda^u(x)\, d\mu(u).$$

Our next computation serves to motivate the construction in Lemma B.17. If $\xi, \eta \in \mathcal{H}_{00}$, then we can apply Lemma B.16 and compute that

$$\big(L(f)\xi \mid L(g)\eta \big) = \big(L(g^* * f)\xi \mid \eta \big) = L_{\xi,\eta}(g^* * f)$$

$$= \int_{G^{(0)}} \int_G g^* * f(\gamma)\rho_{\xi,\eta}(\gamma)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(\gamma)\, d\mu(u)$$

$$= \int_{G^{(0)}} \int_G \int_G \overline{g(\eta^{-1})} f(\eta^{-1}\gamma)\rho_{\xi,\eta}(\gamma)\Delta(\gamma)^{-\frac{1}{2}}\, d\lambda^u(u)\, d\lambda^u(\gamma)\, d\mu(u)$$

which, by Fubini and sending $\gamma \mapsto \eta\gamma$, is

$$= \int_{G^{(0)}} \int_G \int_G \overline{g(\eta^{-1})} f(\gamma)\rho_{\xi,\eta}(\eta\gamma)\Delta(\eta\gamma)^{-\frac{1}{2}}\, d\lambda^{s(\eta)}(\gamma)\, d\lambda^u(\eta)\, d\mu(u)$$

which, after sending $\eta \mapsto \eta^{-1}$, and using the symmetry of ν_0, is

$$= \int_{G^{(0)}} \int_G \int_G \overline{g(\eta)} f(\gamma)\rho_{\xi,\eta}(\eta^{-1}\gamma)\Delta(\eta)^{-\frac{1}{2}}\Delta(\gamma)^{-\frac{1}{2}}$$

$$d\lambda^u(\gamma)\, d\lambda^u(\eta)\, d\mu(u).$$

Since it is not clear to what extent $\rho_{\xi,\eta}$ is a sesquilinear function of (ξ, η), we fix once and for all a countable orthonormal basis $\{\zeta_i\}$ for $\mathcal{H}_{00}$. (Actually, any countable linearly independent set whose span is dense in $\mathcal{H}_{00}$ will do.) We let

$$\mathcal{H}'_{00} := \operatorname{span}\{\zeta_i\}.$$

To make the subsequent formulas a bit easier to read, we will write ρ_{ij} in place of the Radon–Nikodym derivative ρ_{ζ_i,ζ_j}. The linear independence of the ζ_i guarantees that each $\alpha \in \mathscr{C}(G) \odot \mathcal{H}'_{00}$ can be written *uniquely* as

$$\alpha = \sum_i f_i \otimes \zeta_i$$

where all but finitely many f_i are zero.

Lemma B.17. *For each $u \in G^{(0)}$, there is a sesquilinear form $\langle \cdot \, , \cdot \rangle_u$ on $\mathscr{C}(G) \odot \mathcal{H}'_{00}$ such that*

$$(\text{B.18}) \quad \langle f \otimes \zeta_i \, , g \otimes \zeta_j \rangle_u = \int_G \int_G \overline{g(\eta)} f(\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\eta\gamma)^{-\frac{1}{2}} \, d\lambda^u(\gamma) \, d\lambda^u(\eta).$$

Furthermore, there is a μ-conull set $F \subset G^{(0)}$ such that $\langle \cdot \, , \cdot \rangle_u$ is a pre-inner product for all $u \in F$.

Remark B.18. As mentioned earlier, we fixed the ζ_i because it isn't clear that the right-hand side of (B.18) is linear in ζ_i or conjugate linear in ζ_j.

Proof. Given $\alpha = \sum_i f_i \otimes \zeta_i$ and $\beta = \sum_j g_j \otimes \zeta_j$, we get a well-defined form via the definition

$$\langle \alpha \, , \beta \rangle_u = \sum_{ij} \int_G \int_G \overline{g_j(\eta)} f_i(\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\eta\gamma)^{-\frac{1}{2}} \, d\lambda^u(\gamma) \, d\lambda^u(\eta).$$

This clearly satisfies (B.18), and is linear in α and conjugate linear in β. It only remains to provide a conull Borel set F such that $\langle \cdot \, , \cdot \rangle_u$ is positive for all $u \in F$.

However, (B.18) was inspired by the calculation preceding the lemma. Hence if $\alpha := \sum_i f_i \otimes \zeta_i$, then

$$(\text{B.19}) \quad \left\| \sum L(f_i)\zeta_i \right\|^2 = \sum_{ij} \big(L(f_i)\zeta_i \mid L(f_j)\zeta_j \big)$$

$$= \sum_{ij} \big(L(f_j^* * f_i)\zeta_i \mid \zeta_j \big)$$

$$= \sum_{ij} \int_{G^{(0)}} \int_G \int_G \overline{f_j(\eta)} f_i(\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\gamma\eta)^{-\frac{1}{2}}$$

$$d\lambda^u(\gamma) \, d\lambda^u(\eta) \, d\mu(u)$$

$$= \sum_{ij} \int_{G^{(0)}} \langle f_i \otimes \zeta_i \, , f_j \otimes \zeta_j \rangle_u \, d\mu(u)$$

$$= \int_{G^{(0)}} \langle \alpha \, , \alpha \rangle_u \, d\mu(u).$$

Thus, for μ-almost all u, we have $\langle \alpha , \alpha \rangle_u \geq 0$. The difficulty is that the exceptional null set depends on α. However, there is a sequence $\{f_i\} \subset \mathscr{C}(G)$ which is dense in $\mathscr{C}(G)$ in the inductive limit topology. Let $\mathscr{A}_0$ be the rational vector space spanned by the countable set $\{f_i \otimes \zeta_j\}_{i,j}$. Since $\mathscr{A}_0$ is countable, there is a μ-conull set F such that $\langle \cdot , \cdot \rangle_u$ is a positive $\mathbf{Q}$-sesquilinear form on $\mathscr{A}_0$. However, if $g_i \to g$ and $h_i \to h$ in the inductive limit topology in $\mathscr{C}(G)$, then, since $\lambda^u \times \lambda^u$ is a Radon measure on $G^u \times G^u$, we have $\langle g_i \otimes \zeta_j , h_i \otimes \zeta_k \rangle_u \to \langle g \otimes \zeta_j , h \otimes \zeta_k \rangle_u$. It follows that for all $u \in F$, $\langle \cdot , \cdot \rangle_u$ is a positive sesquilinear form (over $\mathbf{C}$) on the complex vector space generated by

$$\{f \otimes \zeta_i : f \in \mathscr{C}(G)\}.$$

However, as that is all $\mathscr{C}(G) \odot \mathcal{H}'_{00}$, the proof is complete. $\square$

We need the following technical result which is a rather specialized version of the Tietze Extension Theorem for locally Hausdorff, locally compact spaces.

Lemma B.19. *Suppose that $g \in C_c(G^u)$ for some $u \in G^{(0)}$. Then there is a $f \in \mathscr{C}(G)$ such that $f|_{G^u} = g$.*

Proof. There are Hausdorff open sets $V_1, \ldots, V_n$ such that $\operatorname{supp} g \subset \bigcup V_i$. Then, using a partition of unity, we can find $g_i \in C_c(G^u)$ such that $\operatorname{supp} g_i \subset V_i$ and such that $\sum g_i = g$. By the Tietze Extension Theorem, there are $f_i \in C_c(V_i)$ such that $f_i|_{G^u} = g_i$. Then $f := \sum f_i$ does the job. $\square$

Note that for *any* $u \in G^{(0)}$, the value of $\langle f \otimes \zeta_i , g \otimes \zeta_j \rangle_u$ depends only on $f|_{G^u}$ and $g|_{G^u}$. Furthermore, using our specialized Tietze Extension result above, we can view $\langle \cdot , \cdot \rangle_u$ as a sesquilinear form on $C_c(G^u)$. (Clearly, since G^u is Hausdorff, each $f \in \mathscr{C}(G)$ determines an element of $C_c(G^u)$. We need Lemma B.19 to know that every function in $C_c(G^u)$ arises in this fashion.) In particular, if $f \in \mathscr{C}(G)$ and $\sigma \in G$, then we let $\check{u}(\sigma)f$ be any element of $\mathscr{C}(G)$ such that

$$(\check{u}(\sigma)f)(\gamma) = \Delta(\sigma)^{\frac{1}{2}} f(\sigma^{-1}\gamma) \quad \text{for all } \gamma \in G^{r(\sigma)}.$$

Of course, $\check{u}(\sigma)f$ is only well-defined on $G^{r(\sigma)}$.

The next computation is critical to what follows. We have

$$(\text{B.20}) \quad \langle \check{u}(\sigma)f \otimes \zeta_i , g \otimes \zeta_j \rangle_{r(\sigma)}$$
$$= \int_G \int_G \overline{g(\eta)} f(\sigma^{-1}\gamma) \rho_{ij}(\gamma^{-1}\eta) \Delta(\sigma^{-1}\gamma\eta)^{-\frac{1}{2}} \, d\lambda^{r(\sigma)}(\eta) \, d\lambda^{r(\sigma)}(\gamma)$$

which, after sending $\gamma \mapsto \sigma\gamma$, is

$$= \int_G \int_G \overline{g(\eta)} f(\gamma) \rho_{ij}(\gamma^{-1}\sigma^{-1}\eta) \Delta(\gamma\eta)^{-\frac{1}{2}} \, d\lambda^{r(\sigma)}(\eta) \, d\lambda^{s(\sigma)}(\gamma)$$

which, after sending $\eta \mapsto \sigma\eta$, is

$$= \int_G \int_G \overline{g(\sigma\eta)} f(\gamma) \rho_{ij}(\gamma^{-1}\eta) \Delta(\sigma)^{-\frac{1}{2}} \Delta(\gamma\eta)^{-\frac{1}{2}} \, d\lambda^{s(\sigma)}(\eta) \, d\lambda^{s(\sigma)}(\gamma)$$

$$= \int_G \int_G \big(\check{u}(\sigma^{-1})g\big)(\eta) f(\gamma) \Delta(\gamma\eta)^{-\frac{1}{2}} \, d\lambda^{s(\sigma)}(\eta) \, d\lambda^{s(\sigma)}(\gamma)$$

$$= \big\langle f \otimes \zeta_i \,,\, \check{u}(\sigma^{-1})g \otimes \eta \big\rangle_{s(\sigma)}.$$

Recall that G acts continuously on the left of $G^{(0)}$: $\gamma \cdot s(\gamma) = r(\gamma)$. In particular, if C is compact in G and if K is compact in $G^{(0)}$, then

$$C \cdot K = \{\gamma \cdot u : (\gamma, u) \in G^{(2)} \cap (C \times K)\}$$

is compact. If $U \subset G^{(0)}$, then we say that U is *saturated* if U is G-invariant. More simply, U is saturated if $s(x) \in U$ implies $r(x)$ is in U. If $V \subset G^{(0)}$, then its *saturation* is the set $[V] = G \cdot V$ which is the smallest saturated set containing V.

The next result is a key technical step in our proof and takes the place of the Ramsay selection theorems ([38, Theorem 3.2] and [36, Theorem 5.1]) used in Muhly's and Renault's proof.

Lemma B.20. *We can choose the μ-conull Borel set $F \subset G^{(0)}$ in Lemma B.17 to be saturated for the G-action on $G^{(0)}$.*

Proof. Let F be the Borel set from Lemma B.17. We want to see that $\langle \cdot \,,\, \cdot \rangle_v$ is positive for all v in the saturation of F. To this end, suppose that $u \in F$ and that $\sigma \in G$ is such that $s(\sigma) = u$ and $r(\sigma) = v$. Then

$$\gamma \mapsto \Delta(\sigma)^{\frac{1}{2}} f(\sigma^{-1}\gamma)$$

is in $C_c(G^v)$, and such functions span a dense subspace of $C_c(G^v)$ in the inductive limit topology. Moreover, as we observed at the end of the proof of Lemma B.17,

$$\langle f_i \otimes \zeta_j \,,\, g_i \otimes \zeta_k \rangle_v \to \langle f \otimes \zeta_j \,,\, g \otimes \zeta_k \rangle_v$$

provided $f_i \to f$ and $g_i \to g$ in the inductive limit topology in $C_c(G^v)$. Therefore, to show that $\langle \cdot \,,\, \cdot \rangle_v$ is positive, it will suffice to check on vectors of the form $\alpha := \sum_i \check{u}(\sigma)(f_i) \otimes \zeta_i$. Then using the key calculation that begins

at (B.20), we have

(B.21)
$$\langle \alpha \, , \, \alpha \rangle_v = \sum_{ij} \langle \check{u}(\sigma^{-1}\sigma)f_i \otimes \zeta_i \, , \, f_j \otimes \zeta_j \rangle_u.$$

$$= \sum_{ij} \langle f_i \otimes \zeta_i \, , \, f_j \otimes \zeta_j \rangle_u.$$

$$= \left\langle \sum_i f_i \otimes \zeta_i, \sum_i f_i \otimes \zeta_i \right\rangle_u$$

which is positive since $u \in F$.

It only remains to verify that the saturation of F is Borel. Since μ is a Radon measure — and therefore regular — we can shrink F a bit, if necessary, and assume it is σ-compact. Say $F = \bigcup K_n$. On the other hand, G is second countable and therefore σ-compact. If $G = \bigcup C_m$, then $[F] = \bigcup C_m \cdot K_n$. Since each $C_m \cdot K_n$ is compact, $[F]$ is σ-compact and therefore Borel. This completes the proof. □

From here on, we will assume that F is saturated. In view of Lemma B.17, for each $u \in F$ we can define $\mathcal{H}(u)$ to be the Hilbert space completion of $\mathscr{C}(G) \odot \mathcal{H}'_{00}$ with respect to $\langle \cdot \, , \, \cdot \rangle_u$. We will denote the image of $f \otimes \zeta_i$ in $\mathcal{H}(u)$ by $f \otimes_u \zeta_i$. Since the complement of F is μ-null and also saturated, what we do off F has little consequence. In particular, G is the disjoint union of $G|_F$ and the ν-null set $G|_{G^{(0)} \setminus F}$.[20] Nevertheless, for the sake of nicety, we let $\mathcal{V}$ be a Hilbert space with an orthonormal basis $\{e_{ij}\}$ doubly indexed by the same index sets as for $\{f_i\}$ and $\{\zeta_j\}$, and set $\mathcal{H}(u) = \mathcal{V}$ if $u \in G^{(0)} \setminus F$. We then let

$$G^{(0)} * \mathscr{H} = \{(u, h) : u \in F \text{ and } h \in \mathcal{H}(u)\},$$

and define $\Phi_{ij} : F \to F * \mathscr{H}$ by

$$\Phi_{ij}(u) := \begin{cases} f_i \otimes_u \zeta_j & \text{if } u \in F \text{ and} \\ e_{ij} & \text{if } u \notin F. \end{cases}$$

(Technically, $\Phi_{ij}(u) = (u, f_i \otimes_u \zeta_j)$ — at least for $u \in F$ — but we have agreed to obscure this subtlety.) Then [46, Proposition F.8] implies that we can make $G^{(0)} * \mathscr{H}$ into a Borel Hilbert bundle over $G^{(0)}$ in such a way that

---

[20]The saturation of F is critical to what follows. If F is not saturated, then in general G is not the union of $G|_F$ and $G|_{G^{(0)} \setminus F}$. But as F *is* saturated, note that a homomorphism $\varphi : G|_F \to H$ can be trivially extended to a homomorphism on all of G by letting φ be suitably trivial on $G|_{G^{(0)} \setminus F}$. This is certainly not the case if F is not saturated.

the $\{\Phi_{ij}\}$ form a fundamental sequence (see [46, Definition F.1]). Note that if $f \otimes \zeta_i \in \mathscr{C}(G) \odot \mathcal{H}'_{00}$ and if $\Phi(u) := f \otimes_u \zeta_i$, then

$$u \mapsto \big\langle \Phi(u) \, , \, \Phi_{ij}(u) \big\rangle_u$$

is Borel on F.[21] It follows that Φ is a Borel section of $G^{(0)} * \mathcal{H}$ and defines a class in $L^2(G^{(0)} * \mathcal{H}, \mu)$.

Furthermore, (B.21) shows that for each $\sigma \in G|_F$, there is a unitary $U_\sigma : \mathcal{H}\big(s(\sigma)\big) \to \mathcal{H}\big(r(\sigma)\big)$ characterized by

$$U_\sigma(f \otimes_{s(\sigma)} \zeta_i) = \check{u}(\sigma) f \otimes_{r(\sigma)} \zeta_i.$$

If $\sigma \notin G|_F$, then $\mathcal{H}\big(s(\sigma)\big) = \mathcal{H}\big(r(\sigma)\big) = \mathcal{V}$, and we can let U_σ be the identity operator.

Lemma B.21. *The map $\hat{U}$ from G to $\mathrm{Iso}(G^{(0)} * \mathcal{H})$ defined by $\hat{U}(\sigma) := \big(r(\sigma), U_\sigma, s(\sigma)\big)$ is a Borel homomorphism. Hence $(\mu, G^{(0)} * \mathcal{H}, \hat{U})$ is a unitary representation of G on $L^2(G^{(0)} * \mathcal{H}, \mu)$.*

Proof. If $\sigma \in G|_F$, then

$$\big(U_\sigma \Phi_{ij}\big(s(\sigma)\big) \mid \Phi_{kl}\big(r(\sigma)\big)\big) =$$
$$\int_G \int_G \overline{f_k(\eta)} f_i(\sigma^{-1}\gamma) \rho_{jl}(\eta^{-1}\gamma) \Delta(\sigma^{-1}\gamma\eta)^{-\frac{1}{2}} \, d\lambda^{r(\sigma)} \, d\lambda^{r(\sigma)}.$$

Thus $\sigma \mapsto \big(U_\sigma \Phi_{ij}\big(s(\sigma)\big) \mid \Phi_{kl}\big(r(\sigma)\big)\big)$ is Borel on F by Fubini's Theorem. Since it is clearly Borel on the complement of F, $\hat{U}$ is Borel. The algebraic properties are straightforward. For example, assuming that $\gamma \in G^{r(\sigma)}$, we have on the one hand,

$$\big(\check{u}(\sigma\eta) f\big)(\gamma) = \Delta(\sigma\eta)^{\frac{1}{2}} f\big((\sigma\eta)^{-1}\gamma\big),$$

while

$$\big(\check{u}(\sigma)\check{u}(\eta) f\big)(\gamma) = \Delta(\sigma)^{\frac{1}{2}} \big(\check{u}(\eta) f\big)(\sigma^{-1}\gamma)$$
$$= \Delta(\sigma\eta)^{\frac{1}{2}} f\big(\eta^{-1}\sigma^{-1}\gamma\big).$$

It follows that $\hat{U}$ is multiplicative on $G|_F$. Of course, it is clearly multiplicative on the complement (which is $G|_{G^{(0)} \setminus F}$ since F is saturated). $\square$

Lemma B.22. *Each $f \otimes \zeta_i \in \mathscr{C}(G) \odot \mathcal{H}'_{00}$ determines a Borel section $\Phi(u) := f \otimes_u \zeta_i$ in $\mathcal{L}^2(G^{(0)} * \mathcal{H}, \mu)$ whose class in $L^2(G^{(0)} * \mathcal{H}, \mu)$ depends only on the class of $[f \otimes \zeta_i] \in \mathscr{C}(G) \odot \mathcal{H}'_{00}/\mathcal{N} \subset \mathscr{C}(G) \odot \mathcal{H}_0/\mathcal{N} = \mathcal{H}_{00}$. Furthermore, there*

---

[21]We can define $\Phi(u)$ to be zero off F. We are going to continue to pay as little attention as possible to the null complement of F in the sequel.

is a unitary isomorphism V of $\mathcal{H}$ onto $L^2(G^{(0)} * \mathcal{H}, \mu)$ such that $V(L(f)\zeta_i) = [\Phi]$.

Proof. We have already seen that Φ is in $\mathcal{L}^2(F * \mathcal{H}, \mu)$. More generally, the computation (B.19) in the proof of Lemma B.17 shows that if $\alpha = \sum_i f_i \otimes \zeta_i$ and $\Psi(u) := \sum_i f_i \otimes_u \zeta_i$, then

$$\|\Psi\|_2^2 = \left\| \sum_i L(f_i)\zeta_i \right\|^2$$

Thus there is a well-defined isometric map V as in the statement of lemma mapping $\mathrm{span}\{L(f)\zeta_i : f \in \mathscr{C}(G)\}$ onto a dense subspace of $L^2(F * \mathscr{H}, \mu)$. Since $\mathcal{H}'_{00}$ is dense in $\mathcal{H}_{00}$, and therefore in $\mathcal{H}$, the result follows by Lemma B.2. $\qquad\square$

The proof of Theorem 7.8 now follows almost immediately from the next proposition.

Proposition B.23. *The unitary V defined in Lemma B.22 intertwines L with a representation L' which in the integrated form of the unitary representation $(\mu, G^{(0)} * \mathscr{H}, U)$ from Lemma B.21.*

Proof. We have $L'(f_1) = VL(f_1)V^*$. On the one hand,

$$
\begin{aligned}
\big(L(f_1)[f \otimes \zeta_i] \mid [g \otimes \zeta_j]\big)_{\mathcal{H}} &= \big(VL(f_1)[f \otimes \zeta_i] \mid V[g \otimes \zeta_j]\big) \\
&= \big(L'(f_1)V[f \otimes \zeta_1] \mid V[g \otimes \zeta_j]\big).
\end{aligned}
$$

But the left-hand side is

$$
\begin{aligned}
\big(L(f_1 * f)\zeta_i \mid L(g)\zeta_j\big) &= L_{\zeta_i, \zeta_j}(g^* * f_1 * f) \\
&= \int_{G^{(0)}} \int_G \int_G \overline{g(\eta)} f_1 * f(\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\eta\gamma)^{-\frac{1}{2}} \, d\lambda^u(\gamma) \, d\lambda^u(\eta) \, d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G \int_G \overline{g(\eta)} f_1(\sigma) f(\sigma^{-1}\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\eta\gamma)^{-\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad d\lambda^u(\sigma) \, d\lambda^u(\gamma) \, d\lambda^u(\eta) \, d\mu(u) \\
&= \int_{G^{(0)}} \int_G \int_G \int_G f_1(\sigma) \overline{g(\eta)} \big(\check{u}(\sigma)f\big)(\gamma) \rho_{ij}(\eta^{-1}\gamma) \Delta(\eta\gamma)^{-\frac{1}{2}} \Delta(\sigma)^{-\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad d\lambda^u(\sigma) \, d\lambda^u(\gamma) \, d\lambda^u(\eta) \, d\mu(u) \\
&= \int_F \int_G f_1(\sigma) \big\langle \check{u}(\sigma)f \otimes \zeta_i \,,\, g \otimes \zeta_j \big\rangle_u \Delta(\sigma)^{-\frac{1}{2}} \, d\lambda^u(\sigma) \, d\mu(u) \\
&= \int_F \int_G f_1(\gamma) \big\langle U_\gamma(f \otimes_{s(\gamma)} \zeta_i) \,,\, g \otimes_u \zeta_j \big\rangle_u \Delta(\sigma)^{-\frac{1}{2}} \, d\lambda^u(\sigma) \, d\mu(u)
\end{aligned}
$$

$$= \int_G f_1(\sigma) \big\langle U_\sigma V[f \otimes \zeta_i]\big(s(\sigma)\big) \, , \, V[g \otimes \zeta_j]\big(r(\sigma)\big) \big\rangle_{r(\sigma)} \Delta(\sigma)^{-\frac{1}{2}} \, d\nu(\sigma).$$

Thus L' is the integrated form as claimed. □

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