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Combined exponential patterns in multiplicative $IP^{\star}$ sets

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ABSTRACT. IP sets play a fundamental role in arithmetic Ramsey theory. A subset of $\mathbb{N}$ (the set of positive integers) is called an additive IP set if it is of the form $FS(\langle x_n\rangle_{n\in\mathbb{N}}) = \{\sum_{t\in H} x_t : H \text{ is a nonempty finite subset of } \mathbb{N}\}$, whereas it is called a multiplicative IP set if it is of the form $FP(\langle x_n\rangle_{n\in\mathbb{N}}) = \{\prod_{t\in H} x_t : H \text{ is a nonempty finite subset of } \mathbb{N}\}$ for some injective sequence $\langle x_n\rangle_{n\in\mathbb{N}}$. An additive $IP^{\star}$ (resp. multiplicative $IP^{\star}$) set in $\mathbb{N}$ is a set which intersects every additive IP set (resp. multiplicative IP set). In [1], V. Bergelson and N. Hindman studied how rich additive $IP^{\star}$ sets are. They proved additive $IP^{\star}$ sets ($AIP^{\star}$ in short) contain finite sums and finite products of a single sequence. An analogous study was made by A. Sisto in [4], where he proved that multiplicative $IP^{\star}$ sets ($MIP^{\star}$ in short) contain exponential tower[1] and finite product of a single sequence. However exponential patterns can be defined in two different ways. In this article, we will prove that $MIP^{\star}$ sets contain two different exponential patterns and finite product of a single sequence. This immediately improves the result of A. Sisto. Throughout our work we will use the machinery of the algebra of the Stone-Čech Compactification of $\mathbb{N}$.

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1. Introduction

The origin of IP sets dates back to Hindman's work [2], where he proved that for any finite coloring of the set of positive integers $\mathbb{N}$, there exists a monochromatic copy of an additive IP set. Here "coloring" means disjoint partition, and a pattern being "monochromatic" means it is included in one piece of the partition. Passing to the map $n \to 2^n$ for each $n \in \mathbb{N}$, we immediately have a

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monochromatic copy of a multiplicative IP set. Let $\beta\mathbb{N}$ be the set of all ultrafilters[1] over $\mathbb{N}$, and $E(\beta\mathbb{N},+)$ (resp. $E(\beta\mathbb{N},\cdot)$) be the collection of all idempotents in $(\beta\mathbb{N},+)$ (resp. $(\beta\mathbb{N},\cdot)$). One can show that a set A is additive IP (resp. multiplicative IP set) if and only if there exists $p \in E(\beta\mathbb{N},+)$ (resp. $p \in E(\beta\mathbb{N},\cdot)$) such that $A \in p$. Hence a set A is $AIP^\star$ (resp. $MIP^\star$) if and only if $A \in p$ for all $p \in E(\beta\mathbb{N},+)$ (resp. $p \in E(\beta\mathbb{N},\cdot)$). Define $\mathcal{P}_f(\mathbb{N})$ to be the set of all nonempty finite subsets of $\mathbb{N}$. For any IP set $FS(\langle x_n\rangle_n)$, a sum subsystem of $FS(\langle x_n\rangle_n)$ is of the form $FS(\langle y_n\rangle_n)$, where for each $n \in \mathbb{N}$, y_n is defined as follows.

- There exists a sequence $\langle H_n\rangle_n$ in $\mathcal{P}_f(\mathbb{N})$ satisfying $\max H_n < \min H_{n+1}$ for all $n \in \mathbb{N}$, and
- $y_n = \sum_{t\in H_n} x_t$.

In [1], V. Bergelson and N. Hindman proved the following result, which addresses that any $AIP^\star$ set contains combined additive and multiplicative patterns.

Theorem 1.1. *Let A be an $AIP^\star$ set, and $\langle x_n\rangle_{n\in\mathbb{N}}$ be any sequence. Then there exists a sum subsystem $FS(\langle y_n\rangle_{n\in\mathbb{N}})$ of $FS(\langle x_n\rangle_{n\in\mathbb{N}})$ such that*

$$FS(\langle y_n\rangle_{n\in\mathbb{N}}) \cup FP(\langle y_n\rangle_{n\in\mathbb{N}}) \subset A.$$

An immediate question appears: what about $MIP^\star$ sets? In [4], A. Sisto was able to show that these sets contain combined multiplicative and exponential patterns. To state his theorem explicitly, we need the following definitions.

Definition 1.2. For any sequences $\langle x_t\rangle_{t=1}^\infty$, define

(1) (a) $EXP_1(\langle x_t\rangle_{t=1}^1 = \{x_1\}$,
  (b) for $n \in \mathbb{N}$, $EXP_1(\langle x_t\rangle_{t=1}^{n+1}) =$
  $\{y^{x_{n+1}} : y \in EXP_1(\langle x_t\rangle_{t=1}^n)\} \cup EXP_1(\langle x_t\rangle_{t=1}^n) \cup \{x_{n+1}\}$
  (c) $EXP_1(\langle x_t\rangle_{t=1}^\infty) = \cup_{n=1}^\infty EXP_1(\langle x_t\rangle_{t=1}^n)$.
(2) (a) $EXP_2(\langle x_t\rangle_{t=1}^1 = \{x_1\}$,
  (b) for $n \in \mathbb{N}$, $EXP_2(\langle x_t\rangle_{t=1}^{n+1}) =$
  $\{x_{n+1}^y : y \in EXP_2(\langle x_t\rangle_{t=1}^n)\} \cup EXP_2(\langle x_t\rangle_{t=1}^n) \cup \{x_{n+1}\}$
  (c) $EXP_1(\langle x_t\rangle_{t=1}^\infty) = \cup_{n=1}^\infty EXP_1(\langle x_t\rangle_{t=1}^n)$.

The following Corollary of Sisto's addresses exponential properties of $MIP^\star$ sets.

Theorem 1.3. [4, Corollary 16] *Let A be a $MIP^\star$ set. Then there exist sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ such that*

(1) $FS\left(\langle y_n\rangle_{n=1}^\infty\right) \cup EXP_1\left(\langle y_n\rangle_{n=1}^\infty\right) \subseteq A$, and
(2) $FP\left(\langle x_n\rangle_{n=1}^\infty\right) \cup EXP_2\left(\langle x_n\rangle_{n=1}^\infty\right) \subseteq A$.

A natural question appears whether it is possible to provide a joint extension of both (1) and (2) in Theorem 1.3. That means, for each $n \in \mathbb{N}$, can we choose

[1]For details on the algebra of ultrafilters we refer the book [3] of N. Hindman and D. Strauss.

$x_n = y_n$ in Theorem 1.3. In this article, we provide a partial answer to this question by proving the following theorem.

Theorem 1.4. *Let A be a $MIP^\star$ set. Then there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ such that*

$$FP\left(\langle x_n\rangle_{n=1}^{\infty}\right) \cup EXP_1\left(\langle x_n\rangle_{n=1}^{\infty}\right) \cup EXP_2\left(\langle x_n\rangle_{n=1}^{\infty}\right) \subseteq A.$$

2. Proof of Theorem 1.4

Ellis theorem [3, Theorem 2.5] tells us about the existence of idempotents in topological semigroups. It says that every compact Hausdorff right topological semigroup contains idempotents. It is a routine exercise to prove that $cl\left(E\left(\beta\mathbb{N},+\right)\right)$ is a left ideal of $\left(\beta\mathbb{N},\cdot\right)$. As left ideals contain minimal left ideals and these are closed, we can apply Ellis theorem to conclude that

$$cl\left(E\left(\beta\mathbb{N},+\right)\right)\bigcap E\left(\beta\mathbb{N},\cdot\right)\neq\emptyset.$$

To prove Theorem 1.4, we will rely on the elements of $cl\left(E\left(\beta\mathbb{N},+\right)\right)\bigcap E\left(\beta\mathbb{N},\cdot\right)$.

Proof of Theorem 1.4: Let $p \in cl\left(E\left(\beta\mathbb{N},+\right)\right)\bigcap E\left(\beta\mathbb{N},\cdot\right)$, and A be a $MIP^\star$ set. As $A \in p$, and $p = p\cdot p$, denote by $A^\star = \left\{x\in A : x^{-1}A\in p\right\}\in p$. Choose $x_1 \in A^\star$. Then by [3, Lemma 4.14] $x_1^{-1}A^\star \in p$. As A is a $MIP^\star$ set, we have $B_1 = \{n : n^{x_1}\in A\}$ is a $MIP^\star$ set. Also by [4, Lemma 13], the set $C_1 = \left\{m : x_1^m \in A\right\}$ is $AIP^\star$ set. Set

$$D_1 = B_1 \cap A^\star \cap x_1^{-1}A^\star \in p.$$

As $p \in cl\left(E\left(\beta\mathbb{N},+\right)\right)$, we have $C_1 \cap D_1 \neq \emptyset$. Let $x_2 \in C_1\cap D_1$. Then $x_2 \in B_1$ and this implies $x_2^{x_1} \in A$. As $x_2 \in C_1$ and so $x_1^{x_2} \in A$. Again $x_2 \in A^\star \cap x_1^{-1}A^\star$, this implies $\{x_1, x_2, x_1x_2\} \subset A$. Hence $\{x_1^{x_2}, x_2^{x_1}\} \subset A$, and $\{x_1, x_2, x_1x_2\} \subset A^\star$.

Inductively assume that for some $N\in\mathbb{N}$, we have $x_1, x_2, \ldots, x_N$ such that

(1) $EXP_1\left(\langle x_n\rangle_{n=1}^{N}\right)\bigcup EXP_2\left(\langle x_n\rangle_{n=1}^{N}\right)\subset A$ and
(2) $FP\left(\langle x_n\rangle_{n=1}^{N}\right)\subset A^\star$.

For each $z\in EXP_1\left(\langle x_n\rangle_{n=1}^{N}\right)$, let $B_z = \{n : z^n\in A\}$ is a $MIP^\star$set. For each $z \in EXP_2\left(\langle x_n\rangle_{n=1}^{N}\right)$, let $C_z = \{m : m^z \in A\}$ is an $AIP^\star$ set. Hence $\bigcap_{z\in EXP_2\left(\langle x_n\rangle_{n=1}^{N}\right)} C_z$ is an $AIP^\star$ set. So

$$D_{N+1} = \bigcap_{z\in EXP_1\left(\langle x_n\rangle_{n=1}^{N}\right)} B_z \cap A^\star \cap \bigcap_{y\in FP\left(\langle x_n\rangle_{n=1}^{N}\right)} y^{-1}A^\star \in p.$$

Again $p\in cl\left(E\left(\beta\mathbb{N},+\right)\right)$, hence

$$E_{N+1} = \bigcap_{z\in EXP_2\left(\langle x_n\rangle_{n=1}^{N}\right)} C_z \cap D_{N+1} \neq \emptyset,$$

and let $x_{n+1}\in E_{N+1}$. Then, $z^{x_{N+1}}\in A$ for all $z\in EXP_1\left(\langle x_i\rangle_{i=1}^{N}\right)$, and $x_{N+1}^{y}\in A$ for all $y \in EXP_2\left(\langle x_i\rangle_{i=1}^{N}\right)$. Again $x_{N+1} \in A^\star\cap\bigcap_{y\in FP\left(\langle x_n\rangle_{n=1}^{N}\right)} y^{-1}A^\star$ implies $FP\left(\langle x_i\rangle_{i=1}^{N+1}\right)\subset A^\star$.

Hence we have

(1) $EXP_1\left(\langle x_n\rangle_{n=1}^{N+1}\right)\bigcup EXP_2\left(\langle x_n\rangle_{n=1}^{N+1}\right)\subset A$, and

(2) $FP\left(\langle x_n\rangle_{n=1}^{N+1}\right)\subset A^\star$.

This completes the induction. □

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