

Bounds on torsion of CM abelian varieties over a p -adic field with values in a field of p -power roots

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ABSTRACT. Let p be a prime number and M the extension field of a p -adic field K obtained by adjoining all p -power roots of all elements of K . In this paper, we show that there exists a constant C , depending only on K and an integer $g > 0$, which satisfies the following property: If $A_{/K}$ is a g -dimensional CM abelian variety, then the order of the torsion subgroup of $A(M)$ is bounded by C .

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1. Introduction

Let p be a prime number. Let K be a number field (= a finite extension of \mathbb{Q}) or a p -adic field (= a finite extension of \mathbb{Q}_p). Let A be an abelian variety defined over K of dimension g . It follows from the Mordell-Weil theorem and the main theorem of [Ma] that the torsion subgroup $A(K)_{\text{tors}}$ of $A(K)$ is finite. The following question is quite natural and has been studied extensively:

Question. What can be said about the size of the order of $A(K)_{\text{tors}}$?

If K is a number field of degree d and A is an elliptic curve (i.e., $g = 1$), it is really surprising that there exists a constant $B(d)$, depending only on the degree d , such that $\#A(K)_{\text{tors}} < B(d)$. An explicit formula for such a constant $B(d)$ is first given by Merel [Me]. After that, Oesterlé and Parent [Pa] give a refinement of Merel's bound, independently. (Oesterlé's proof was unpublished until Derickx transcribed it in his Ph.D Thesis; see [DKSS, Section 6] for the published article). The amazing point here is that the constant $B(d)$ is uniform in the sense that it depends not on the number field K but on the degree $d = [K : \mathbb{Q}]$.

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Such uniform boundedness results are not known for abelian varieties of dimension greater than one. Next we consider the case where K is a p -adic field. As remarked by Cassels, the “uniform boundedness theorem” for p -adic base fields would be false (cf. Lemma 17.1 and p.264 of [Ca]). For abelian varieties A over K with anisotropic reduction, Clark and Xarles [CX] give an upper bound of the order of $A(K)_{\text{tors}}$ in terms of g , p and some numerical invariants of K . This includes the case in which A has potentially good reduction.

We are interested in the order of $A(L)_{\text{tors}}$ for certain algebraic extensions L of K of infinite degree. Now we suppose that K is a p -adic field. There are not so many known L so that $A(L)_{\text{tors}}$ is finite. Imai [Im] showed that $A(L)_{\text{tors}}$ is finite if A has potential good reduction and $L = K(\mu_{p^\infty})$, where μ_{p^∞} is the set of p -power root of unity. The author [Oz] showed that Imai’s finiteness result holds even if we replace $K(\mu_{p^\infty})$ with a certain type of a Lubin-Tate extension field of a p -adic field. The result [KT] of Kubo and Taguchi is also interesting. They showed that the torsion subgroup of $A(K(\sqrt[p^\infty]{K}))$ is finite, where A is an abelian variety over K with potential good reduction and $K(\sqrt[p^\infty]{K})$ is the extension field of K obtained by adjoining all p -power roots of all elements of K . Our main theorem is motivated by the result of Kubo and Taguchi. The goal of this paper is to show that, under the assumption that A has complex multiplication, the order of $A(K(\sqrt[p^\infty]{K}))_{\text{tors}}$ is “uniformly” bounded. (Here we say that A has complex multiplication if there exists a ring homomorphism $F \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \text{End}_{\overline{K}} A$ for some algebraic number field F of degree $2g$.)

Theorem 1.1. *There exists a constant $C(K, g)$, depending only on a p -adic field K and an integer $g > 0$, which satisfies the following property: If A is a g -dimensional abelian variety over K with complex multiplication, then we have*

$$\#A\left(K(\sqrt[p^\infty]{K})\right)_{\text{tors}} < C(K, g).$$

The theorem above gives a global result: For any integer $d > 0$, we denote by $\mathbb{Q}_{\leq d}$ the composite of all number fields of degree $\leq d$. If we fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, then $\mathbb{Q}_{\leq d}$ is embedded into the composite field of all p -adic fields of degree $\leq d$, which is a finite extension of \mathbb{Q}_p . If we denote by $\mathbb{Q}_{\leq d, p}$ the extension field of $\mathbb{Q}_{\leq d}$ obtained by adjoining all p -power roots of all elements of $\mathbb{Q}_{\leq d}$, then the following is an immediate consequence of our main theorem.

Corollary 1.2. *There exists a constant $C(d, g, p)$, depending only on positive integers d, g and a prime number p , which satisfies the following property: If A is a g -dimensional abelian variety over $\mathbb{Q}_{\leq d}$ with complex multiplication, then we have*

$$\#A(\mathbb{Q}_{\leq d, p})_{\text{tors}} < C(d, g, p).$$

The organization of the paper is as follows. Section 2.1 is a preliminary of the proof of Theorem 1.1. Some results related with characters and p -adic Hodge theory are given there. In Section 2.2, we give a proof of Theorem 1.1. Here is a sketch of our proof: If we denote by ρ the Galois representation given by the

p -adic Tate module of A , we will find that it is enough to give an upper bound of the minimal value of $v_p(\det(\rho - I))$ (here, v_p is the normalized p -adic valuation). For this, we first reduce an argument to the case where A has both good reduction and complex multiplication over the base field. If this is the case, the (semi-simplification of the) representation ρ is given by some crystalline characters ψ_1, \dots, ψ_{2g} , and then we see that it suffices to give a bound of the minimal value of $\sum_{i=1}^{2g} v_p(\psi_i - 1)$. We obtain this bound by applying the results given in Section 2.1 with careful treatments for the Hodge-Tate type of ψ_i .

Notation : Throughout this paper, a p -adic field means a finite extension of \mathbb{Q}_p in a fixed algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p . If F is an algebraic extension of \mathbb{Q}_p , we denote by \mathcal{O}_F and \mathbb{F}_F the ring of integers of F and the residue field of F , respectively. We denote by G_F the absolute Galois group of F and also denote by Γ_F the set of \mathbb{Q}_p -algebra embeddings of F into $\overline{\mathbb{Q}_p}$. We put $d_F = [F : \mathbb{Q}_p]$. For an algebraic extension F'/F , we denote by $e_{F'/F}$ and $f_{F'/F}$ the ramification index of F'/F and the extension degree of the residue field extension of F'/F , respectively. We set $e_F := e_{F/\mathbb{Q}_p}$ and $f_F := f_{F/\mathbb{Q}_p}$, and also set $q_F := p^{f_F}$. If F is a p -adic field, we denote by F^{ab} and F^{ur} the maximal abelian extension of F and the maximal unramified extension of F , respectively.

2. Proof

2.1. Some technical tools. We denote by v_p the p -adic valuation on a fixed algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p normalized by $v_p(p) = 1$. Let K be a p -adic field. For any continuous character χ of G_K , we often regard χ as a character of $\text{Gal}(K^{\text{ab}}/K)$. We denote by Art_K the local Artin map $K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$ with arithmetic normalization. We set $\chi_K := \chi \circ \text{Art}_K$. We denote by \widehat{K}^\times the profinite completion of K^\times . Note that the local Artin map induces a topological isomorphism $\text{Art}_K : \widehat{K}^\times \xrightarrow{\sim} \text{Gal}(K^{\text{ab}}/K)$. For a finite extension K'/K , we denote by $N_{K'/K}$ the norm map from K' to K .

Proposition 2.1. *Let K and k be p -adic fields. We denote by k_π the Lubin-Tate extension¹ of k associated with a uniformizer π of k . (If $k = \mathbb{Q}_p$ and $\pi = p$, then we have $k_\pi = \mathbb{Q}_p(\mu_{p^\infty})$.) Let $\chi_1, \dots, \chi_n : G_K \rightarrow \overline{\mathbb{Q}_p}^\times$ be continuous characters. Then we have*

$$\begin{aligned} & \text{Min} \left\{ \sum_{i=1}^n v_p(\chi_i(\sigma) - 1) \mid \sigma \in G_{Kk_\pi} \right\} \\ & \leq \text{Min} \left\{ \sum_{i=1}^n v_p(\chi_{i,K} \circ N_{Kk/K}(\omega) - 1) \mid \omega \in N_{Kk/k}^{-1}(\pi^{f_{Kk/k}\mathbb{Z}}) \right\}. \end{aligned}$$

¹See [Yo] for Lubin-Tate extensions.

Proof. We have a topological isomorphism $\text{Art}_k^{-1} : \text{Gal}(k^{\text{ab}}/k) \xrightarrow{\sim} \widehat{k}^\times$ and $\text{Art}_k^{-1}(\text{Gal}(k^{\text{ab}}/k^{\text{ur}})) = \mathcal{O}_k^\times$. We denote by M the maximal unramified extension of k contained in Kk . Since the group $\text{Art}_k^{-1}(\text{Gal}(k^{\text{ab}}/M))$ contains \mathcal{O}_k^\times and is a subgroup of $\widehat{k}^\times = \pi^{\widehat{\mathbb{Z}}} \times \mathcal{O}_k^\times$ of index $[M : k]$, we see $\text{Art}_k^{-1}(\text{Gal}(k^{\text{ab}}/M)) = \pi^{[M:k]\widehat{\mathbb{Z}}} \times \mathcal{O}_k^\times$. (Here, $\pi^{\widehat{\mathbb{Z}}}$ is the closure of the subgroup $\pi^{\mathbb{Z}}$ of \widehat{k}^\times generated by π , which is topologically isomorphic to $\widehat{\mathbb{Z}}$. We write $\pi^{n\widehat{\mathbb{Z}}}$ for the n -th power of $\pi^{\widehat{\mathbb{Z}}}$ for any integer n .) On the other hand, we have $\text{Art}_k^{-1}(\text{Gal}(k^{\text{ab}}/k_\pi)) = \pi^{\widehat{\mathbb{Z}}}$. Thus we obtain $\text{Art}_k^{-1}(\text{Gal}(k^{\text{ab}}/Mk_\pi)) = \pi^{[M:k]\widehat{\mathbb{Z}}}$. Denote by $\text{Res}_{Kk/k}$ the natural restriction map $\text{Gal}((Kk)^{\text{ab}}/Kk) \rightarrow \text{Gal}(k^{\text{ab}}/k)$. Then one can check that it follows $\text{Res}_{Kk/k}^{-1}(\text{Gal}(k^{\text{ab}}/Mk_\pi)) = \text{Gal}((Kk)^{\text{ab}}/Kk_\pi)$. Thus it follows that the group $\text{Art}_{Kk}^{-1}(\text{Gal}((Kk)^{\text{ab}}/Kk_\pi))$ coincides with $N_{Kk/k}^{-1}(\pi^{[M:k]\widehat{\mathbb{Z}}})$. Therefore, if we take any $\omega \in N_{Kk/k}^{-1}(\pi^{[M:k]\widehat{\mathbb{Z}}})$, we have

$$\begin{aligned} & \text{Min} \left\{ \sum_{i=1}^n v_p(\chi_i(\sigma) - 1) \mid \sigma \in G_{Kk_\pi} \right\} \\ &= \text{Min} \left\{ \sum_{i=1}^n v_p(\chi_i(\sigma) - 1) \mid \sigma \in \text{Gal}((Kk)^{\text{ab}}/Kk_\pi) \right\} \\ &= \text{Min} \left\{ \sum_{i=1}^n v_p(\chi_{i,K} \circ N_{Kk/K} \circ \text{Art}_{Kk}^{-1}(\sigma) - 1) \mid \sigma \in \text{Gal}((Kk)^{\text{ab}}/Kk_\pi) \right\} \\ &\leq \sum_{i=1}^n v_p(\chi_{i,K} \circ N_{Kk/K}(\omega) - 1). \end{aligned}$$

□

We recall an observation of Conrad for crystalline characters. We often use p -adic Hodge theory. For the basic notion of p -adic Hodge theory, it is helpful for the reader to refer [Fo1] and [Fo2]. Let B_{cris} be the Fontain's p -adic period ring and set $D_{\text{cris}}^K(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$ for any \mathbb{Q}_p -representation V of G_K . Let us denote by K_0 the maximal unramified subextension of K/\mathbb{Q}_p and denote by φ_{K_0} the Frobenius map of K_0 , that is, the (unique) lift of the p -th power map on the residue field of K_0 . Since $B_{\text{cris}}^{G_K} = K_0$, $D_{\text{cris}}^K(V)$ is a K_0 -vector space. Moreover, $D_{\text{cris}}^K(V)$ is a filtered φ -module over K ; it is of finite dimension over K_0 , it is equipped with a bijective φ_{K_0} -semi-linear Frobenius operator φ and it is equipped with with a decreasing exhaustive and separated filtration on $D_{\text{cris}}^K(V) \otimes_{K_0} K$. On the other hand, we denote by \underline{K}^\times the Weil restriction $\text{Res}_{K/\mathbb{Q}_p}(\mathbb{G}_m)$. This is an algebraic torus such that, for a \mathbb{Q}_p -algebra R , the R -valued points $\underline{K}^\times(R)$ of \underline{K}^\times is $\mathbb{G}_m(R \otimes_{\mathbb{Q}_p} K)$.

Proposition 2.2 ([Co, Proposition B.4]). *Let K and F be p -adic fields, and let $\chi : G_K \rightarrow F^\times$ be a continuous character. We denote by $F(\chi)$ the \mathbb{Q}_p -representation*

of G_K underlying a 1-dimensional F -vector space endowed with an F -linear action by G_K via χ .

- (1) χ is crystalline² if and only if there exists a (necessarily unique) \mathbb{Q}_p -homomorphism $\chi_{\text{alg}} : \underline{K}^\times \rightarrow \underline{F}^\times$ such that χ_K and χ_{alg} (on \mathbb{Q}_p -points) coincides on \mathcal{O}_K^\times ($\subset K^\times = \underline{K}^\times(\mathbb{Q}_p)$).
- (2) Let K_0 be the maximal unramified subextension of K/\mathbb{Q}_p . Assume that χ is crystalline and let χ_{alg} be as in (1). (Note that χ^{-1} is also crystalline.) Then, the filtered φ -module $D_{\text{cris}}^K(F(\chi^{-1})) = (B_{\text{cris}} \otimes_{\mathbb{Q}_p} F(\chi^{-1}))^{G_K}$ over K is free of rank 1 over $K_0 \otimes_{\mathbb{Q}_p} F$ and its K_0 -linear endomorphism φ^{f_K} is given by the action of the product $\chi_K(\pi_K) \cdot \chi_{\text{alg}}^{-1}(\pi_K) \in F^\times$. Here, π_K is any uniformizer of K .

We define some notations for later use. It is helpful for the readers to refer [Se, Section III, A4 and A5] and [Co, Appendix B]. Assume that K is a Galois extension of \mathbb{Q}_p . Let $\chi : G_K \rightarrow K^\times$ be a crystalline character. Let $\chi_{\text{LT}} : I_K \rightarrow K^\times$ be the restriction to the inertia I_K of the Lubin-Tate character³ associated with any choice of uniformizer of K (it depends on the choice of a uniformizer of K , but its restriction to the inertia subgroup does not). By definition, the character χ_{LT} is characterized by $\chi_{\text{LT}} \circ \text{Art}_K(x) = x^{-1}$ for any $x \in \mathcal{O}_K^\times$. (We remark that χ_{LT} is the restriction to I_K of the p -adic cyclotomic character if $K = \mathbb{Q}_p$.) Then, since χ is crystalline, we have

$$\chi = \prod_{\sigma \in \Gamma_K} \sigma^{-1} \circ \chi_{\text{LT}}^{h_\sigma}$$

on the inertia I_K for some (unique) integer h_σ . Equivalently, the character χ_{alg} (appeared in Proposition 2.2) on \mathbb{Q}_p -points is given by

$$\chi_{\text{alg}}(x) = \prod_{\sigma \in \Gamma_K} (\sigma^{-1}x)^{-h_\sigma}$$

for $x \in K^\times$. We say that $\mathbf{h} = (h_\sigma)_{\sigma \in \Gamma_K}$ is the *Hodge-Tate type* of χ . Note that $\{h_\sigma \mid \sigma \in \Gamma_K\}$ as a set is the set of Hodge-Tate weights of $K(\chi)$, that is, $C \otimes_{\mathbb{Q}_p} K(\chi) \simeq \bigoplus_{\sigma \in \Gamma_K} C(h_\sigma)$ where C is the completion of $\overline{\mathbb{Q}_p}$ (cf. [Se, Chapter III, A5, Lemma 1 and Theorem 2]).

For any set of integers $\mathbf{h} = (h_\sigma)_{\sigma \in \Gamma_K}$ indexed by Γ_K , we define a continuous character $\psi_{\mathbf{h}} : \mathcal{O}_K^\times \rightarrow \mathcal{O}_K^\times$ by

$$\psi_{\mathbf{h}}(x) = \prod_{\sigma \in \Gamma_K} (\sigma^{-1}x)^{-h_\sigma}. \quad (1)$$

Lemma 2.3. For $1 \leq i \leq r$, let $\mathbf{h}_i = (h_{i,\sigma})_{\sigma \in \Gamma_K}$ be a set of integers. For each i , assume that

²This means that the \mathbb{Q}_p -representation $F(\chi)$ of G_K is crystalline.

³Our Lubin-Tate character here is the character “ χ_E ” in [Se, Section III, A4]. See also Proposition 4 of *loc. cit.*

- (a) $\sum_{\sigma \in \Gamma_K} h_{i,\sigma}$ is not zero, and
- (b) $h_{i,\sigma} \neq h_{i,\tau}$ for some $\sigma, \tau \in \Gamma_K$.

Then, there exists an element ω of $\ker N_{K/\mathbb{Q}_p}$ such that $\psi_{\mathbf{h}_1}(\omega), \dots, \psi_{\mathbf{h}_r}(\omega)$ are of infinite orders.

Proof. For any character χ on \mathcal{O}_K^\times , we denote by χ' the restriction of χ to $1 + p^2\mathcal{O}_K$. To show the lemma, it suffices to show

$$\ker N'_{K/\mathbb{Q}_p} \not\subset \bigcup_{i=1}^r \ker \psi'_{\mathbf{h}_i}. \tag{2}$$

(In fact, any non-trivial element of $\text{Im } \psi'_{\mathbf{h}_i}$ is of infinite order since $\text{Im } \psi'_{\mathbf{h}_i}$ is a subgroup of a torsion free group $1 + p^2\mathcal{O}_K$.) Since $N'_{K/\mathbb{Q}_p} (1 + p^2\mathcal{O}_K)$ is an open subgroup of \mathbb{Z}_p^\times , we see that the dimension⁴ of $\ker N'_{K/\mathbb{Q}_p}$ is $d_K - 1$. We claim that $\dim \ker \psi_{\mathbf{h}_i} < d_K - 1$. By the assumption (a), we see that $\text{Im } \psi'_{\mathbf{h}_i}$ contains an open subgroup H of \mathbb{Z}_p^\times . Thus we have $\dim \ker \psi'_{\mathbf{h}_i} = d_K - \dim \text{Im } \psi'_{\mathbf{h}_i} \leq d_K - 1$. If we assume $\dim \ker \psi'_{\mathbf{h}_i} = d_K - 1$, then $\dim \text{Im } \psi'_{\mathbf{h}_i} = 1$ and thus H is a finite index subgroup of $\text{Im } \psi'_{\mathbf{h}_i}$. It follows that there exists an open subgroup U of \mathcal{O}_K^\times such that $\psi_{\mathbf{h}_i}$ restricted to U has values in \mathbb{Z}_p^\times . By [Oz, Lemma 2.4], we obtain that $h_{i,\sigma} = h_{i,\tau}$ for any $\sigma, \tau \in \Gamma_K$ but this contradicts the assumption (b) in the statement of the lemma. Thus we conclude that $\dim \ker \psi'_{\mathbf{h}_i} < d_K - 1$.

Now we fix an isomorphism $\iota : 1 + p^2\mathcal{O}_K \simeq \mathbb{Z}_p^{\oplus d_K}$ of topological groups. We define vector subspaces N and P_i of $\mathbb{Q}_p^{\oplus d_K}$ by $N := \iota(\ker N'_{K/\mathbb{Q}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $P_i := \iota(\ker \psi'_{\mathbf{h}_i}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We know that $\dim_{\mathbb{Q}_p} N = d_K - 1$ and $\dim_{\mathbb{Q}_p} P_i < d_K - 1$. Assume that (2) does not hold, that is, $\ker N'_{K/\mathbb{Q}_p} \subset \bigcup_{i=1}^r \ker \psi'_{\mathbf{h}_i}$. Then we have $N \subset \bigcup_{i=1}^r P_i$. This implies $N = \bigcup_{i=1}^r (N \cap P_i)$. By the lemma below, we find that $N = N \cap P_i \subset P_i$ for some i but this contradicts the fact that $\dim_{\mathbb{Q}_p} N > \dim_{\mathbb{Q}_p} P_i$. \square

Lemma 2.4. *Let V be a vector space over a field F of characteristic zero. Let W_1, \dots, W_r be vector subspaces of V . If $V = \bigcup_{i=1}^r W_i$, then $V = W_i$ for some i .*

Proof. We show by induction on r . The cases $r = 1, 2$ are clear. Assume that the lemma holds for r and suppose $V = \bigcup_{i=1}^{r+1} W_i$. We assume both $W_1 \not\subset \bigcup_{i=2}^{r+1} W_i$ and $W_{r+1} \not\subset \bigcup_{i=1}^r W_i$ holds. Then there exist elements $\mathbf{x}_1 \in W_1 \setminus \bigcup_{i=2}^{r+1} W_i$ and $\mathbf{x}_{r+1} \in W_{r+1} \setminus \bigcup_{i=1}^r W_i$. It is not difficult to check that we have $\lambda \mathbf{x}_1 + \mathbf{x}_{r+1} \notin W_1 \cup W_{r+1}$ for any $\lambda \in F^\times$. Hence there exists an integer $2 \leq j_n \leq$

⁴If a profinite group G has an open subgroup U which is isomorphic to $\mathbb{Z}_p^{\oplus d}$, then d does not depend on the choice of U and we say that d is the dimension of G . For example, $\dim \mathbb{Z}_p^{\oplus d} = d$. Note that the dimension of G is zero if and only if G is finite. See [DDMS] for general theories of dimensions of p -adic analytic groups.

r for each integer $n > 0$ such that $n\mathbf{x}_1 + \mathbf{x}_{r+1} \in W_{j_n}$. Take any integers $0 < \ell < k$ so that $j_\ell = j_k (= j)$. Then $(k - \ell)\mathbf{x}_1 = (k\mathbf{x}_1 + \mathbf{x}_{r+1}) - (\ell\mathbf{x}_1 + \mathbf{x}_{r+1}) \in W_j$. Since F is of characteristic zero, we have $\mathbf{x}_1 \in W_j$ but this contradicts the fact that $\mathbf{x}_1 \notin \bigcup_{i=2}^{r+1} W_i$. Therefore, either $W_1 \subset \bigcup_{i=2}^{r+1} W_i$ or $W_{r+1} \subset \bigcup_{i=1}^r W_i$ holds. This shows that $V = \bigcup_{i=2}^{r+1} W_i$ or $V = \bigcup_{i=1}^r W_i$ and the induction hypothesis implies $V = W_i$ for some i . \square

Finally we describe the following consequence of p -adic Hodge theory, which is well-known for experts.

Proposition 2.5. *Let X be a proper smooth variety with good reduction over a p -adic field K . Then we have*

$$\det(T - \varphi^{f_K} | D_{\text{cris}}^K(H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)) = \det(T - \text{Frob}_K^{-1} | H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_\ell))$$

for any prime $\ell \neq p$. Here, Frob_K stands for the arithmetic Frobenius of K .

Proof. Let Y be the special fiber of a proper smooth model of X over the integer ring of K . By the crystalline conjecture shown by Faltings [Fa] (cf. [Ni], [Tsu]), we have an isomorphism $D_{\text{cris}}^K(H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)) \simeq K_0 \otimes_{W(\mathbb{F}_{q_K})} H_{\text{cris}}^i(Y/W(\mathbb{F}_{q_K}))$ of φ -modules over K_0 . It follows from Corollary 1.3 of [CLS] (cf. [KM, Theorem 1] and [Na, Remark 2.2.4 (4)]) that the characteristic polynomial of $K_0 \otimes_{W(\mathbb{F}_{q_K})} H_{\text{cris}}^i(Y/W(\mathbb{F}_{q_K}))$ for the (f_K -iterate) Frobenius action coincides with $\det(T - \text{Frob}_K^{-1} | H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_\ell))$ for any prime $\ell \neq p$. Thus the result follows. \square

2.2. Proof of the main theorem. Let A be a g -dimensional abelian variety over K with complex multiplication. We denote by L the field obtained by adjoining to K all points of $A[12]$. It follows from [Si, Theorem 4.1] that endomorphisms of A are defined over L . By Raynaud's criterion of semi-stable reduction [Gr, Proposition 4.7], A has semi-stable reduction over L . Moreover, A has good reduction over L since A has complex multiplication [ST, Section 2, Corollary 1]. Since the extension degree of L over K is at most the order of $GL_{2g}(\mathbb{Z}/12\mathbb{Z})$ and there exist only finitely many p -adic fields of a given degree, we immediately reduce a proof of Theorem 1.1 to showing the following:

Proposition 2.6. *There exists a constant $\hat{C}(K, g)$, depending only on a p -adic field K and an integer $g > 0$, which satisfies the following property: Let A be a g -dimensional abelian variety over K with the properties that A has good reduction over K and $\text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field of degree $2g$. Then we have*

$$\#A \left(K(\sqrt[p^\infty]{K}) \right)_{\text{tors}} < \hat{C}(K, d).$$

Proof. Since there exist only finitely many p -adic field of a given degree, replacing K by a finite extension, we may assume the following hypothesis:

- (H) K is a Galois extension of \mathbb{Q}_p and K contains all p -adic fields of degree $\leq 2g$.

In the rest of the proof, we set $M := K(\sqrt[p^\infty]{K})$. Let A be a g -dimensional abelian variety over K with the properties that A has good reduction over K and $F := \text{End}_K(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field of degree $2g$. Let $T = T_p(A) := \varprojlim_n A[p^n]$ be the p -adic Tate module of A and $V = V_p(A) := T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Then V is a free $F_p := F \otimes \mathbb{Q}_p$ -module of rank one and the representation $\rho : G_K \rightarrow GL_{\mathbb{Z}_p}(T) \subset GL_{\mathbb{Q}_p}(V)$ defined by the G_K -action on T has values in $GL_{F_p}(V) = F_p^\times$. In particular, ρ is an abelian representation. The representation V is a Hodge-Tate representation with Hodge-Tate weights 0 (multiplicity g) and 1 (multiplicity g). Moreover, V is crystalline since A has good reduction over K . Fix an isomorphism $\iota : T \xrightarrow{\sim} \mathbb{Z}_p^{\oplus 2g}$ of \mathbb{Z}_p -modules. We have an isomorphism $\hat{\iota} : GL_{\mathbb{Z}_p}(T) \simeq GL_{2g}(\mathbb{Z}_p)$ relative to ι . We abuse notation by writing ρ for the composite map $G_K \rightarrow GL_{\mathbb{Z}_p}(T) \simeq GL_{2g}(\mathbb{Z}_p)$ of ρ and $\hat{\iota}$. Now let $P \in T$ and denote by \bar{P} the image of P in $T/p^n T$. By definition, we have $\iota(\sigma P) = \rho(\sigma)\iota(P)$ for $\sigma \in G_K$. Suppose that $\bar{P} \in (T/p^n T)^{G_M}$. This implies $\sigma P - P \in p^n T$ for any $\sigma \in G_M$. This is equivalent to saying that $(\rho(\sigma) - I)\iota(P) \in p^n \mathbb{Z}_p^{\oplus 2g}$ where I is the identity matrix, and this in particular implies $\det(\rho(\sigma) - I)\iota(P) \in p^n \mathbb{Z}_p^{\oplus 2g}$ for any $\sigma \in G_M$. If we denote by M_{ab} the maximal abelian extension of K contained in M , it holds that $\rho(G_M) = \rho(G_{M_{\text{ab}}})$ since $\rho(G_K)$ is abelian. Thus we have

$$\det(\rho(\sigma) - I)\iota(P) \in p^n \mathbb{Z}_p^{\oplus 2g} \quad \text{for any } \sigma \in G_{M_{\text{ab}}}. \tag{3}$$

On the other hand, we set $G := \text{Gal}(M/K)$ and $H := \text{Gal}(M/K(\mu_{p^\infty}))$. Let $\chi_p : G_K \rightarrow \mathbb{Z}_p^\times$ be the p -adic cyclotomic character. Since we have $\sigma\tau\sigma^{-1} = \tau^{\chi_p(\sigma)}$ for any $\sigma \in G$ and $\tau \in H$, we see $(G, G) \supset (G, H) \supset H^{\chi_p(\sigma)-1}$ (here, (\cdot, \cdot) stands for the commutator). Hence we have a natural surjection

$$H/H^{\chi_p(\sigma)-1} \twoheadrightarrow \overline{H/(G, G)} = \text{Gal}(M_{\text{ab}}/K(\mu_{p^\infty})) \quad \text{for any } \sigma \in G. \tag{4}$$

Let ν be the smallest p -power integer with the properties that $\nu > 1$ and $\chi_p(G_K) \supset 1 + \nu\mathbb{Z}_p$. Then (4) gives the fact that $\text{Gal}(M_{\text{ab}}/K(\mu_{p^\infty}))$ is of exponent ν , that is, $\sigma \in G_{K(\mu_{p^\infty})}$ implies $\sigma^\nu \in G_{M_{\text{ab}}}$. Hence it follows from (3) that, for any point $P \in T$ such that its image \bar{P} in $T/p^n T$ is fixed by G_M , we have

$$\det(\rho(\sigma)^\nu - I)\iota(P) \in p^n \mathbb{Z}_p^{\oplus 2g} \quad \text{for any } \sigma \in G_{K(\mu_{p^\infty})}. \tag{5}$$

Claim. There exists a constant $C_0(K, g)$, depending only on K and g such that

$$v_p(\det(\rho(\sigma_0)^\nu - I)) \leq C_0(K, g)$$

for some $\sigma_0 \in G_{K(\mu_{p^\infty})}$.

Admitting this claim, we can finish the proof of Proposition 2.6 immediately: It follows from Claim 2.2 and (5) that $(T/p^n T)^{G_M} \subset p^{n-C_0(K, g)} T/p^n T$ for $n > C_0(K, g)$. Setting $C(K, g)_p := p^{C_0(K, g)2g}$, we obtain

$$\#A(M)[p^n] = \#(T/p^n T)^{G_M} \leq \#(T/p^{C_0(K, g)} T) = C(K, g)_p,$$

which shows $\#A(M)[p^\infty] \leq C(K, g)_p$. On the other hand, we remark that Kubo and Taguchi showed in [KT, Lemma 2.3] that the residue field \mathbb{F}_M of M is finite. The reduction map induces an injection from the prime-to- p part of $A(M)$ into $\overline{A}(\mathbb{F}_M)$ where \overline{A} is the reduction of A . If we denote by q the order of \mathbb{F}_M , it follows from the Weil bound that $\#\overline{A}(\mathbb{F}_M) \leq (1 + \sqrt{q})^{2g}$. Therefore, setting $C(K, g) := C(K, g)_p \cdot (1 + \sqrt{q})^{2g}$, we conclude that $\#A(M)_{\text{tors}} \leq C(K, g)$. This finishes the proof of the proposition.

It suffices to show Claim 2.2. Since the action of G_K on V factors through an abelian quotient of G_K , it follows from Schur's lemma that each Jordan Hölder factor of $V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ is of dimension one. Let $\psi_1, \dots, \psi_{2g} : G_K \rightarrow \overline{\mathbb{Q}_p}^\times$ be the characters associated with the Jordan Hölder factors of $V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$. Since K contains all p -adic fields of degree $\leq 2g$, we know that each ψ_i has values in K^\times (in fact, for any $\sigma \in G_K$, we know that $\psi_1(\sigma), \dots, \psi_{2g}(\sigma)$ are the roots of the polynomial $\det(T - \sigma | V) \in \mathbb{Q}_p[T]$ of degree $2g$). In the rest of the proof, we regard ψ_i as a character $G_K \rightarrow K^\times$ of G_K with values in K^\times . We remark that each ψ_i is a crystalline character since V is crystalline. Furthermore, we have

$$v_p(\det(\rho(\sigma)^\nu - I)) = v_p\left(\prod_{i=1}^{2g} (\psi_i^\nu(\sigma) - 1)\right) = \sum_{i=1}^{2g} v_p(\psi_i^\nu(\sigma) - 1)$$

for any $\sigma \in G_{K(\mu_{p^\infty})}$. Hence it follows from Proposition 2.1 that we have

$$\begin{aligned} & \text{Min} \left\{ v_p(\det(\rho(\sigma)^\nu - I) \mid \sigma \in G_{K(\mu_{p^\infty})} \right\} \\ & \leq \text{Min} \left\{ \sum_{i=1}^{2g} v_p(\psi_{i,K}^\nu(p\omega)^{-1} - 1) \mid \omega \in \ker N_{K/\mathbb{Q}_p} \right\}. \end{aligned} \quad (6)$$

Note that we have

$$\begin{aligned} \psi_{i,K}(p\omega)^{-1} &= \psi_{i,K}(\pi_K^{-e_K} \cdot \pi_K^{e_K} p^{-1}) \cdot \psi_{i,K}(\omega)^{-1} \\ &= \psi_{i,K}(\pi_K)^{-e_K} \psi_{i,\text{alg}}(\pi_K^{e_K} p^{-1}) \cdot \psi_{i,K}(\omega)^{-1} \\ &= \alpha_i^{-e_K} \cdot \psi_{i,\text{alg}}(p)^{-1} \cdot \psi_{i,K}(\omega)^{-1} \end{aligned} \quad (7)$$

for $\omega \in \ker N_{K/\mathbb{Q}_p}$ where $\alpha_i := \psi_{i,K}(\pi_K) \psi_{i,\text{alg}}(\pi_K)^{-1}$.

Lemma 2.7. *Let the notation be as above. Let A^\vee be the dual abelian variety of A , and let \overline{A} and \overline{A}^\vee be the reductions of A and A^\vee , respectively.*

- (1) α_i is a root of the characteristic polynomial of the geometric Frobenius endomorphism of $\overline{A}/\mathbb{F}_K$.
- (2) $\alpha_i^{-1} q_K$ is a root of the characteristic polynomial of the geometric Frobenius endomorphism of $\overline{A}^\vee/\mathbb{F}_K$.

Proof. In this proof, we denote by W^\vee the dual representation

$$W^\vee = \text{Hom}_{\mathbb{Q}_p}(W, \mathbb{Q}_p)$$

for any p -adic representation W . As $K(\psi_i^{-1})$ is a subquotient of $V_p(A)^\vee \otimes_{\mathbb{Q}_p} K$, it follows from Proposition 2.2 that α_i is a root of the characteristic polynomial $f(T) := \det(T - \varphi^{f_K} | D_{\text{cris}}^K(V_p(A)^\vee))$ of the K_0 -linear endomorphism φ^{f_K} , the f_K -th iterate of the Frobenius φ , on the K_0 -vector space $D_{\text{cris}}^K(V_p(A)^\vee)$. We find that

$$\begin{aligned} f(T) &= \det(T - \varphi^{f_K} | D_{\text{cris}}^K(H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_p)) \\ &= \det(T - \text{Frob}_K^{-1} | H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)) = \det(T - \text{Frob}_K | V_\ell(\overline{A})) \end{aligned}$$

for any prime $\ell \neq p$ where Frob_K stands for the arithmetic Frobenius. The second equality follows from Proposition 2.5. The last term above coincides with the characteristic polynomial of the geometric Frobenius endomorphism of $\overline{A}/\mathbb{F}_K$. This shows (1). On the other hand, it follows from Proposition 2.2 again that α_i^{-1} is a root of $\det(T - \varphi^{f_K} | D_{\text{cris}}^K(V_p(A)))$. Since $V_p(A)(-1) \simeq V_p(A^\vee)^\vee$, we see that $\alpha_i^{-1}q_K$ is a root of $f^\vee(T) := \det(T - \varphi^{f_K} | D_{\text{cris}}^K(V_p(A^\vee)^\vee))$. Now the same argument of the proof of (1) with replacing A by A^\vee gives a proof of (2). \square

We continue the proof of Proposition 2.6. Let $\mathbf{h}_i = (h_{i,\sigma})_{\sigma \in \Gamma_K}$ be the Hodge-Tate type of ψ_i . Then we have $h_{i,\sigma} \in \{0, 1\}$ for any i and σ . Reordering, we may suppose there is an r for which we have the following:

- (I) $\mathbf{h}_i \neq (0)_{\sigma \in \Gamma_K}, (1)_{\sigma \in \Gamma_K}$ for $1 \leq i \leq r$, and
- (II) $\mathbf{h}_i = (0)_{\sigma \in \Gamma_K}$ or $\mathbf{h}_i = (1)_{\sigma \in \Gamma_K}$ for $r + 1 \leq i \leq 2g$.

Consider the case $\mathbf{h}_i = (0)_{\sigma \in \Gamma_K}$. If this is the case, ψ_i is unramified. This implies that $\psi_{i,\text{alg}}$ (on \mathbb{Q}_p -points) is trivial. Take any $\omega \in \ker N_{K/\mathbb{Q}_p}$ and consider the p -adic value $v_p(\psi_{i,K}^\vee(p\omega)^{-1} - 1)$. By (7), we have

$$\psi_{i,K}^\vee(p\omega)^{-1} = \alpha_i^{-ve_K}. \tag{8}$$

We remark that the right hand side is independent of the choice of $\omega \in \ker N_{K/\mathbb{Q}_p}$ and α_i must be a p -adic unit (since so is the left hand side).

Next, consider the case $\mathbf{h}_i = (1)_{\sigma \in \Gamma_K}$. If this is the case, we have $\psi_i = \chi_p$ on I_K , that is, $\psi_{i,\text{alg}}$ (on \mathbb{Q}_p -points) is N_{K/\mathbb{Q}_p}^{-1} . Take any $\omega \in \ker N_{K/\mathbb{Q}_p}$ and consider the p -adic value $v_p(\psi_{i,K}^\vee(p\omega)^{-1} - 1)$. By (7), we have

$$\psi_{i,K}^\vee(p\omega)^{-1} = (\alpha_i^{-e_K} \cdot N_{K/\mathbb{Q}_p}(p))^\vee = (\alpha_i^{-1}q_K)^{ve_K}. \tag{9}$$

We remark that the last term is independent of the choice of $\omega \in \ker N_{K/\mathbb{Q}_p}$.

Suppose $r + 1 \leq i \leq 2g$. Let L be the unramified extension of K of degree ve_K . Denote by $f_i(T)$ the characteristic polynomial of the Frobenius endomorphism of $\overline{A}/\mathbb{F}_L$ or that of $\overline{A^\vee}/\mathbb{F}_L$ if $\mathbf{h}_i = (0)_{\sigma \in \Gamma_K}$ or $\mathbf{h}_i = (1)_{\sigma \in \Gamma_K}$, respectively. We also set $\alpha := \psi_{i,K}^\vee(p\omega)$ or $\alpha := \psi_{i,K}^\vee(p\omega)^{-1}$ if $\mathbf{h}_i = (0)_{\sigma \in \Gamma_K}$ or $\mathbf{h}_i = (1)_{\sigma \in \Gamma_K}$, respectively. It follows from (8), (9) and Lemma 2.7 that α is a unit root of $f_i(T)$. Writing $f_i(T) = (T - 1)g_i(T) + f_i(1)$ for some $g_i(T) \in \mathbb{Z}[T]$, we obtain $0 = f_i(\alpha) =$

$(\alpha - 1)g_i(\alpha) + f_i(1)$. This gives

$$v_p(\alpha^{-1} - 1) = v_p(\alpha - 1) \leq v_p((\alpha - 1)g_i(\alpha)) = v_p(f_i(1)).$$

On the other hand, $f_i(1)$ coincides with the order of $\overline{A}(\mathbb{F}_{q_L})$ or $\overline{A}^\vee(\mathbb{F}_{q_L})$. Hence it follows from the Weil bound that $f_i(1) \leq (1 + \sqrt{q_L})^{2g} \leq (1 + \sqrt{p}^{vd_K})^{2g}$, which gives an inequality $v_p(f_i(1)) \leq \log_p(1 + \sqrt{p}^{vd_K})^{2g}$. Therefore, setting $C_2(K, g) := \log_p(1 + \sqrt{p}^{vd_K})^{2g}$, we obtain

$$v_p(\psi_{i,K}^\nu(p\omega)^{-1} - 1) \leq C_2(K, g)$$

for $r + 1 \leq i \leq 2g$.

Suppose $1 \leq i \leq r$. We define a subset $\mathcal{R} = \mathcal{R}(K, g)$ of $\overline{\mathbb{Q}}_p$ by taking the set consisting of all $\alpha \in \overline{\mathbb{Q}}_p$ that are a root of a polynomial in $\mathbb{Z}[T]$ of degree at most $2g$ and also a q_K -Weil integer⁵ of weight 1. We also define $\mathcal{R}' = \mathcal{R}'(K, g) := \{(\alpha^{-e_K} p^h)^\nu \mid \alpha \in \mathcal{R}, 0 < h < d_K\}$. Then, both \mathcal{R} and \mathcal{R}' are finite sets and depend only on K and g . Furthermore, Lemma 2.7 and the Weil Conjecture imply that each α_i is an element of \mathcal{R} . Thus, setting

$$\gamma_i := \alpha_i^{-e_K} \cdot \psi_{i,\text{alg}}(p)^{-1} = \alpha_i^{-e_K} \cdot p^{\sum_{\sigma \in \Gamma_K} h_{i,\sigma}},$$

we have $\gamma_i^\nu \in \mathcal{R}'$. We consider the continuous character $\psi_{\mathbf{h}_i} : \mathcal{O}_K^\times \rightarrow \mathcal{O}_K^\times$ defined in (1). The character $\psi_{i,\text{alg}}$ (on \mathbb{Q}_p -points) restricted to \mathcal{O}_K^\times coincides with $\psi_{\mathbf{h}_i}$. By Lemma 2.3, there exists an element $\omega = \omega(K; \mathbf{h}_1, \dots, \mathbf{h}_r)$ of $\ker N_{K/\mathbb{Q}_p}$ such that $\psi_{\mathbf{h}_1}^\nu(\omega), \dots, \psi_{\mathbf{h}_r}^\nu(\omega)$ are of infinite order. Since \mathcal{R}' is finite, there exists an integer r such that $\psi_{\mathbf{h}_1}^\nu(\omega^r), \dots, \psi_{\mathbf{h}_r}^\nu(\omega^r)$ are not contained in \mathcal{R}' . Putting $\omega_0 = \omega^r$, it holds that

- ω_0 is an element of $\ker N_{K/\mathbb{Q}_p}$. Furthermore, ω_0 depends only on K, g and $\mathbf{h}_1, \dots, \mathbf{h}_r$, and
- $\psi_{\mathbf{h}_1}^\nu(\omega_0), \dots, \psi_{\mathbf{h}_r}^\nu(\omega_0)$ are not contained in \mathcal{R}' .

Now we define a constant $C(K, g, \mathbf{h}_1, \dots, \mathbf{h}_r)$ by

$$C(K, g, \mathbf{h}_1, \dots, \mathbf{h}_r) = \text{Max} \left\{ \sum_{i=1}^r v_p(\gamma_i' \psi_{\mathbf{h}_i}^\nu(\omega_0)^{-1} - 1) \mid \gamma_i' \in \mathcal{R}' \right\}.$$

⁵We say that α is a q_K -Weil integer of weight w if α is an algebraic integer such that $|\iota(\alpha)| = q_K^{w/2}$ for any embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$.

By construction of ω_0 , we see that the constant above is finite and depends only on $K, g, \mathbf{h}_1, \dots, \mathbf{h}_r$. We find that

$$\begin{aligned} & \text{Min} \left\{ \sum_{i=1}^{2g} v_p(\psi_{i,K}^\vee(p\omega)^{-1} - 1) \mid \omega \in \ker N_{K/\mathbb{Q}_p} \right\} \\ & \leq \sum_{i=1}^{2g} v_p(\psi_{i,K}^\vee(p\omega_0)^{-1} - 1) \\ & = \sum_{i=1}^r v_p(\gamma_i^\vee \psi_{\mathbf{h}_i}^\vee(\omega_0)^{-1} - 1) + \sum_{i=r+1}^{2g} v_p(\psi_{i,K}^\vee(p\omega_0)^{-1} - 1) \\ & \leq C(K, g, \mathbf{h}_1, \dots, \mathbf{h}_r) + (2g - r)C_2(K, g) \leq C_0(K, g). \end{aligned} \tag{10}$$

Here,

$$\begin{aligned} C_0(K, g) & := \text{Max}\{C(K, g, \mathbf{h}_1, \dots, \mathbf{h}_r) + (2g - r)C_2(K, g) \\ & \quad \mid 0 \leq r \leq 2g, \mathbf{h}_1, \dots, \mathbf{h}_r : \text{Case (I)}\} \end{aligned}$$

(if $r = 0$, we consider the constant $C(K, g, \mathbf{h}_1, \dots, \mathbf{h}_r)$ as zero). By construction, the constant $C_0(K, g)$ is finite and depends only on K and g . By (6) and (10), we conclude that $C_0(K, g)$ defined here satisfies the desired property of Claim 2.2. This is the end of the proof of Proposition 2.6. \square

We end this paper with the following remarks.

- Remark 2.8.** (1) We do not know the explicit description of the bound $C(K, g)$ in Theorem 1.1.
 (2) We do not know whether we can remove the sentence “with complex multiplication” from the statement of Theorem 1.1 or not.
 (3) Let K be a p -adic field. Let $\pi = \pi_0$ be a uniformizer of K and π_n a p^n -th root of π such that $\pi_{n+1}^p = \pi_n$ for any $n \geq 0$. We set $K_\infty := K(\pi_n \mid n \geq 0)$. The field K_∞ is clearly a subfield of $K(\sqrt[p^\infty]{K})$. It is well-known that K_∞ is one of key ingredients in (integral) p -adic Hodge theory since K_∞ is familiar to the theory of norm fields. We can check the equality

$$A(K_\infty)_{\text{tors}} = A(K)_{\text{tors}}$$

holds for any abelian variety A over K with good reduction. It should be remarked that we do not need CM assumption here and the main theorem of [CX] gives an explicit bound on the order of $A(K)_{\text{tors}}$. The proof for the above equality is as follows: It follows from the criterion of Néron-Ogg-Shafarevich [ST, Theorem 1] that the inertia subgroup I_K of G_K acts trivially on the prime-to- p part of $A(\overline{K})_{\text{tors}}$. Since K_∞ is totally ramified over K , we obtain the fact that the prime-to- p parts of $A(K)_{\text{tors}}$ and $A(K_\infty)_{\text{tors}}$ coincide with each other. On the

other hand, we consider the following natural maps.

$$A(K)[p^n] \simeq \mathrm{Hom}_{G_K}(\mathbb{Z}/p^n\mathbb{Z}, A(\overline{K})[p^n]) \\ \xrightarrow{\iota} \mathrm{Hom}_{G_{K_\infty}}(\mathbb{Z}/p^n\mathbb{Z}, A(\overline{K})[p^n]) \simeq A(K_\infty)[p^n].$$

Since A has good reduction, the injection ι above is bijective (cf. [Br, Theorem 3.4.3] for $p > 2$; [Ki], [La], [Li] for $p = 2$). This implies $A(K_\infty)[p^\infty] = A(K)[p^\infty]$.

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