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Optimal connectivity results for spheres in the curve graph of low and medium complexity surfaces

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Abstract. Answering a question of Wright, we show that spheres of any radius are always connected in the curve graph of surfaces $\Sigma_{2,0}$, $\Sigma_{1,3}$, and $\Sigma_{0,6}$, and the union of two consecutive spheres is always connected for $\Sigma_{0,5}$ and $\Sigma_{1,2}$. We also classify the connected components of spheres of radius 2 in the curve graph of $\Sigma_{0,5}$ and $\Sigma_{1,2}$.

Contents

1. Introduction

1.1. Main results. Let $\Sigma = \Sigma_{g,n}$ be a connected surface with genus g and n punctures. We define the complexity of Σ to be $\xi(\Sigma) = 3g - 3 + n$. We say Σ is

- *exceptional* if $\xi(\Sigma) = 1$, i.e. $(g, n) \in \{(1, 1), (0, 4)\}$,
- *low complexity* if $\xi(\Sigma) = 2$, i.e. $(g, n) \in \{(1, 2), (0, 5)\}$,
- *medium complexity* if $\xi(\Sigma) = 3$, i.e. $(g, n) \in \{(2, 0), (1, 3), (0, 6)\}$,
- *high complexity* if $\xi(\Sigma) \geq 4$.

We now define the *curve graph* of a surface $\Sigma_{g,n}$.

Definition 1.1. Suppose α is a simple closed curve on a surface $\Sigma_{g,n}$. α is said to be an *essential* curve if it does not bound a disk (i.e. it does not bound something homeomorphic to the unit disk in $\mathbb{R}^2$).

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Definition 1.2. Suppose α is a simple closed curve on a surface $\Sigma_{g,n}$. α is said to be an *non-peripheral* curve if it does not bound a once-punctured disk.

Definition 1.3. For a surface $\Sigma_{g,n}$ with positive complexity, we define its *curve graph*, denoted $\mathcal{C}(\Sigma_{g,n})$, as follows. The vertex set of $\mathcal{C}(\Sigma_{g,n})$ is the set of isotopy classes of essential, non-peripheral simple closed curves on Σ. Suppose $\alpha, \beta \in V(\mathcal{C}(\Sigma))$. Then we define $\alpha \sim \beta$ if we can choose a representative a of α and b of β such that a, b are disjoint curves.

Let $\mathcal{C}\Sigma$ be the curve graph of some surface $\Sigma = \Sigma_{g,n}$. For any vertex $c \in \mathcal{C}\Sigma$ and radius r, let

$$S_r = S_r(c) = \{a \in \mathcal{C}\Sigma : d(a, c) = r\}$$

be the sphere of radius r about c in $\mathcal{C}\Sigma$. We will say that a sphere is connected if the induced subgraph is connected.

The main results to be proved in this paper are as follows:

Theorem 1.4. Let $\Sigma_{g,n}$ be low complexity. Fix a center $c \in \mathcal{C}\Sigma$. Then for all $r > 0$ we have that $S_r(c) \cup S_{r+1}(c)$ is connected.

Theorem 1.5. Let $\Sigma_{g,n}$ be medium complexity. Fix a center $c \in \mathcal{C}\Sigma$. Then for all $r > 0$ we have that $S_r(c)$ is connected.

In the low complexity case, we do not understand in general the connected components of S_r. However, we can understand the case of S_2.

Definition 1.6. Let $\Sigma_{g,n}$ be low complexity. Fix center $c \in \mathcal{C}\Sigma$. Let $S'_r(c)$ denote the subgraph of $S_r(c)$ generated by the set of vertices in $S_r(c)$ which are not isolated in $S_r(c)$ (i.e. $S'_r(c)$ is the subgraph of $S_r(c)$ generated by the set $\{y \in S_3(c) : \exists z \in S_3(c)$ such that $y \sim z\}$).

Theorem 1.7. Let $\Sigma_{g,n}$ be low complexity. Fix center $c \in \mathcal{C}\Sigma$. Then $S'_2(c)$ is connected.

1.2. Previous results. The main contribution of this paper is to strengthen the following theorem from [13].

Theorem 1.8 ([13], Theorem 1.1)**.** For all $r > 0$ and connected surface Σ,

(1) If Σ has high complexity, then S_r is connected.
(2) If Σ has medium complexity, then $S_r \cup S_{r+1}$ is connected.
(3) If Σ has low complexity, then $S_r \cup S_{r+1} \cup S_{r+2}$ is connected.

Our Theorem 1.4 and Theorem 1.5 strengthen the above theorem, thereby answering [13, Question 1.7]. Our Theorem 1.4 and Theorem 1.5 are sharp because S_r is never connected for $r \geq 1$ in low complexity [13, Corollary 6.12].

Our Theorem 1.7 describes the connected components of S_2 in low complexity.

1.3. Organization of the proof. In both the low and medium complexity cases for the connectivity of spheres (Theorem 1.4 and Theorem 1.5), we utilize the same proof strategy, as well as the same preliminary results from [13]. Then we modify the paths obtained in [13] in order to stay closer to S_r, with the Bounded Geodesic Image Theorem from [9] as our primary tool.

Our main contribution in the low complexity case (Theorem 1.4) is to construct improved "preliminary paths" (discussed in Section 3.4), and show this adjustment allows the argument to ultimately yield paths contained in two spheres instead of three.

In the medium complexity case (Theorem 1.5), Wright's argument included an induction on radius, for which it was crucial to use essentially non-separating curves (Definition 4.3). Since we assume Wright's result, we avoid arguing by induction, so we are able to use curves which fail to be essentially non-separating to produce paths which stay in a single sphere.

We prove Theorem 1.7 by showing that $S_2'(c)$ naturally has the structure of a $\mathbb{Z}$-bundle over $S_1(c)$. Note that $S_1(c)$ can be seen as a copy of the Farey graph because all curves in $S_1(c)$ live on a sphere with a disk removed and with three punctures, and such a sphere gives the same curve graph as $\Sigma_{0,4}$. Interestingly, the monodromy of this bundle over a Farey triangle in $S_1(c)$ is translation by 1. This $\mathbb{Z}$-bundle structure is related to some existing ideas such as a version of the Lantern relation. But as far as we know, this $\mathbb{Z}$-bundle structure has not been recorded in the literature previously, and we expect it to be of independent interest.

1.4. Motivation. This paper continues the tradition of examining the relationship between fine and coarse geometry of the curve graph. As an example, the Bounded Geodesic Image Theorem uses coarse information to deduce a precise result about the vertices on geodesics.

In particular, we can also gain a better understanding of the coarse geometry of the curve graph as a whole by understanding the fine results. This idea is exemplified in [13] where the linear connectivity of the Gromov boundary (coarse) follows from an analysis of the connectivity of S_r (fine). For previous connectivity results and other related work, see [2, 3, 4, 5, 6, 8, 7, 10, 11].

Our paper also develops techniques to perform constructions directly in the curve graph rather than spaces of lamination or Teichmüller space.

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2. Subsurface projections and the Bounded Geodesic Image Theorem

In this section, we will introduce one of our key tools, the Bounded Geodesic Image Theorem, and recall some basic facts about subsurface projections.

Let U be a subsurface of Σ and $\alpha \in \mathcal{C}\Sigma$. We say curve α *cuts* U if it is not possible to isotope α out of U. We define $\mathcal{C}(\Sigma, U)$ to be the subgraph of $\mathcal{C}\Sigma$ whose vertices are all essential non-peripheral curves that cut U, and keeping all possible edges. Note that $\mathcal{C}U$ is contained in $\mathcal{C}(\Sigma, U)$.

Given a subsurface U of Σ, there exists a subsurface projection map, denoted ρ_U, from the set of curves cutting U to finite subsets of curves on U. We will want to recall some key facts about ρ_U:

(1) The values of ρ_U are uniformly bounded in diameter.
(2) The map ρ_U is 6-Lipschitz i.e.
$$d(\rho_U(\alpha), \rho_U(\beta)) \leq 6d(\alpha, \beta).$$
(3) Define
$$d_U(\alpha, \beta) = diam(\rho_U(\alpha) \cup \rho_U(\beta)).$$
It can easily be verified that d_U satisfies the triangle inequality.

The following theorem is known as the Bounded Geodesic Image Theorem:

Theorem 2.1. [9, Theorem 3.1] Let U be a subsurface of Σ. There exists $M > 0$ such that if $d_U(\alpha, \beta) \geq M$ then every geodesic from α to β in $\mathcal{C}\Sigma$ contains a curve not cutting U.

From here on, M will refer to the constant required for Theorem 2.1, which can be taken independent of Σ and U [12].

3. Low complexity

Throughout this section, we deal with $\Sigma = \Sigma_{0,5}$. Assume that a center vertex $c \in \mathcal{C}\Sigma_{0,5}$ is fixed and let $S_r = S_r(c)$.

3.1. Organization. The outcome of this section is to prove Theorem 1.4. We do so by first taking arbitrary $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$ and constructing a preliminary path, described in Proposition 3.7, connecting b to b'. We then offer Lemma 3.21 to serve a similar function as [13, Lemma 6.16] to push this path up to $S_r \cup S_{r+1}$ using Dehn twists, by observing that vertices on this preliminary path only enter $S_3(a)$ when they are close to $S_{r-1} \cup S_r$. This adjustment is sufficient in proving the path stays within two consecutive spheres rather than three.

3.2. Definitions.

Definition 3.1. A vertex $x \in S_r$ has *unique backtracking* if it has a unique neighbor in S_{r-1}, i.e. there is a unique $y \in S_{r-1}$ such that $x \sim y$.

Definition 3.2. A vertex $x \in S_r$ has *no sidestepping* if it does not have any neighbor in S_r, i.e. there is no $y \in S_r$ such that $x \sim y$.

Definition 3.3. A vertex $x \in S_r$ is *forward facing* if it has unique backtracking and no sidestepping.

3.3. Pentagons in $\mathcal{C}\Sigma_{0,5}$. It is important to note that $\mathcal{C}\Sigma$ contains no cycles of length 3 or 4 [13, Lemma 6.1]. Thus, we often study paths on $\mathcal{C}\Sigma$ by using pentagons.

Definition 3.4. Label the 5 punctures of Σ with the elements of $\mathbb{Z}/5\mathbb{Z}$. The 5 tuple of curves $(a_1, a_2, a_3, a_4, a_5)$ is a *pentagon* if for $i \in \mathbb{Z}/5\mathbb{Z}$:

(1) a_i goes around punctures i and $i+1$,
(2) the intersection number between a_i, a_{i+1} and a_i, a_{i-1} is 2, and
(3) the intersection number between a_i, a_{i+2} and a_i, a_{i-2} is 0.

To obtain a 5-cycle from a pentagon with vertices $(a_1, a_2, a_3, a_4, a_5)$, we can traverse the curves in the following order: $(a_1, a_3, a_5, a_2, a_4)$. We use the following lemmas to find pentagons in $\mathcal{C}\Sigma_{0,5}$.

Lemma 3.5. [13, Lemma 6.5] Suppose $a_1, a_3 \in S_{r-1}$ are adjacent. Then there are curves $a_2, a_3, a_5 \in S_r \cup S_{r+1}$ such that $(a_1, a_2, a_3, a_4, a_5)$ is a pentagon.

Lemma 3.6. [13, Lemma 6.6] Suppose $a_1 \in S_{r-1}$ and $a_3, a_4 \in S_r \cap S_1(a_1)$ have $i(a_3, a_4) = 2$. Then there exist $a_2, a_5 \in S_r \cup S_{r+1}$ such that $(a_1, a_2, a_3, a_4, a_5)$ is a pentagon.

3.4. Preliminary path construction.

Proposition 3.7. Suppose $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$. Then there exists a path γ from b to b' contained in $S_1(a) \cup S_2(a) \cup S_3(a)$ such that the following hold for all vertices v on the path γ:

(1) If $v \in S_3(a)$, then $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$.
(2) If $v \in S_1(a)$, then $v \in S_{r+1}$.

First we recall the following lemmas:

Lemma 3.8. [13, Lemma 6.10] For any $a \in \mathcal{C}\Sigma_{0,5}$ and $x \in S_1(a)$, x is forward facing with respect to a.

Lemma 3.9. [13, Lemma 6.13] For any $a \in \mathcal{C}\Sigma_{0,5}$, $S_1(a) \cup S_2(a)$ is connected.

Definition 3.10. Suppose $\Sigma = \Sigma_{g,n}$ is a surface and $v \in \mathcal{C}\Sigma$ is a vertex in its curve graph. For all natural numbers n, let $B_n(v)$ denote the set of vertices of $\mathcal{C}\Sigma$ that is of distance at most n from v (the distance is computed using the graph metric of $\mathcal{C}\Sigma$).

Lemma 3.11. [13, Lemma 6.14] Suppose $x \in S_r$ is forward facing and $y, y' \in S_1(x) \cap S_{r+1}$. Then there exists a path from y to y' in $(S_{r+1} \cup S_{r+2}) \cap B_2(x)$.

Lemma 3.11 gives us the following corollary.

Corollary 3.12. Suppose x_{j-1}, x_j, x_{j+1} is a path in $S_1(a) \cup S_2(a)$ with $x_j \in S_1(a)$. Then there exists a path from x_{j-1} to x_{j+1} contained in $(S_2(a) \cup S_3(a)) \cap B_2(x_j)$.

Proof. This statement is exactly the conclusion of Lemma 3.11 with a as the center, $x = x_j$, $y = x_{j-1}$, and $y' = x_{j+1}$, so we only need to check the conditions are satisfied.

First, we see x_j is forward facing with respect to a because $x_j \in S_1(a)$ by assumption and by Lemma 3.8, every vertex in $S_1(a)$ is forward facing with respect to a.

Second, we have $x_{j-1}, x_{j+1} \in S_1(x_j)$ because x_{j-1}, x_j, x_{j+1} is a path by assumption.

Third, we observe $x_{j-1}, x_{j+1} \in S_1(a) \cup S_2(a)$ and $S_1(a)$ is totally disconnected because $\mathcal{C}\Sigma_{0,5}$ has no triangles. Now x_{j-1}, x_{j+1} are adjacent to $x_j \in S_1(a)$. Thus, $x_{j-1}, x_{j+1} \notin S_1(a)$ so $x_{j-1}, x_{j+1} \in S_2(a)$. This verifies the conditions of Lemma 3.11. □

Now we have the tools to construct the preliminary path as stated in Proposition 3.7.

Proof of Proposition 3.7. By Lemma 3.9, $S_1(a) \cup S_2(a)$ is connected. Since $b, b' \in S_1(a)$ this implies there exists a path $b = x_0, ..., x_l = b'$ contained in $S_1(a) \cup S_2(a)$. Now for each $x_j \in (S_{r-1} \cup S_r) \cap S_1(a)$ replace the path segment x_{j-1}, x_j, x_{j+1} with the path $x_{j-1} = x_j^0, x_j^1, ..., x_j^k = x_{j+1}$ for some $k \geq 0$ given by Corollary 3.12. Call this path γ. First we observe by construction that γ has no vertex in $(S_{r-1} \cup S_r) \cap S_1(a)$.

Now we check that γ satisfies the conclusions of Proposition 3.7 by utilizing the following three sublemmas. The first sublemma will show that γ is contained in $S_1(a) \cup S_2(a) \cup S_3(a)$.

Sublemma 3.13. The path γ is contained in $S_1(a) \cup S_2(a) \cup S_3(a)$.

Proof. By construction the vertices in γ are either in $S_1(a) \cup S_2(a)$ or in $(S_2(a) \cup S_3(a)) \cap B_2(x_j)$ for some $j \leq l$. □

The second sublemma will establish part (1) of Proposition 3.7, namely, if $v \in S_3(a)$ then $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$.

Sublemma 3.14. If v is a vertex in γ and $v \in S_3(a)$, then $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$.

Proof. The original path $b = x_0, ..., x_l = b'$ is contained in $S_1(a) \cup S_2(a)$ so if $v \in S_3(a)$, then v must have been obtained from replacing the segment x_{j-1}, x_j, x_{j+1} with the path $x_{j-1} = x_j^0, x_j^1, ..., x_j^k = x_{j+1}$ for some $k \geq 0$. In particular, $v = x_j^i$ for some $i \leq k$. By Corollary 3.12, $v = x_j^i \in (S_2(a) \cup S_3(a)) \cap B_2(x_j)$ so $d(v, x_j) \leq 2$. Additionally, $x_j \in (S_{r-1} \cup S_r) \cap S_1(a)$. Thus $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$. □

The final sublemma will establish part (2) of Proposition 3.7, namely, if $v \in S_1(a)$ then $v \in S_{r+1}$.

Sublemma 3.15. The only vertices in γ which are in $S_1(a)$ are also in S_{r+1}.

Proof. Let v be a vertex on γ such that $v \in S_1(a)$. Now $a \in S_r$ so $v \in S_{r-1} \cup S_r \cup S_{r+1}$. But by construction γ has no vertices in $(S_{r-1} \cup S_r) \cap S_1(a)$ because any such vertices in the original path were replaced by a path in $S_2(a) \cup S_3(a)$. Thus, $v \in S_{r+1}$. □

Since we have verified the conclusions of Proposition 3.7 for arbitrary $a \in S_r$ and $b, b' \in S_1(a) \cap S_{r+1}$, this finishes the proof. □

3.5. Some Results on Dehn Twists. In this section, we prove some results about Dehn twists that we will later use. We first fix some important notation.

Remark 3.16. Suppose $a, b \in \mathcal{C}\Sigma_{0,5}$. Let $T_a(b)$ denote the left Dehn twist of b around a. Henceforth, we will refer to a left Dehn twist as a Dehn twist.

In addition, we use d_a to denote the distance between the projections to the curve graph of the annular subsurface associated to an element a of $\mathcal{C}\Sigma_{0,5}$.

We make use of the following fact, which was proven in [9, Equation 2.6].

Proposition 3.17. Suppose a, b are vertices in $\mathcal{C}\Sigma_{0,5}$ such that $d(a, b) \geq 2$. Then

$$\lim_{N\to\infty} d_a(b, T_a^N(b)) = \infty. \tag{1}$$

Lemma 3.18. Suppose $a \in S_r$ and $d(b, a) \geq 2$. Then there exists a positive integer $N(a, b)$, such that for all $N' \geq N(a, b)$, we have $d_a(T_a^{N'}(b), c) \gg M$.

Proof. For all integers m, we have

$$d_a(T_a^m(b), c) \geq d_a(T_a^m(b), b) - d_a(b, c), \tag{2}$$

where $d_a(b, c)$ is a constant. Thus, the lemma follows from Proposition 3.17. □

Remark 3.19. For the rest of Section 3.5, we will continue to use $N(a, b)$ to denote the constant in Lemma 3.18. Note that $N(a, b)$ depends on a, b.

Corollary 3.20. Suppose $a \in S_r$ and $d(b, a) \geq 2$. If $N' \geq N(a, b)$, then

$$d(c, T_a^{N'}(b)) \geq r. \tag{3}$$

Proof. By Lemma 3.18 and Theorem 2.1, any geodesic from $T_a^{N'}(b)$ to c must contain a vertex that lies in $B_1(a)$. This implies that $d(T_a^{N'}(b), c) \geq r$. □

3.6. Main lemma. In this section, we make a critical improvement to [13, Lemma 6.16] by proving Lemma 3.21. These two results are almost the same, except that we construct a path such that every vertex x_i is at most $r+2$ distance away from the center c (property 2 in Lemma 3.21), whereas [13, Lemma 6.16] does not prove this upper bound.

Lemma 3.21. Suppose $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$. Then there exists a path $b, x_1, \ldots, x_l, b'$ with four properties:

(1) $1 \leq d(x_i, a) \leq 3$.
(2) $r \leq d(x_i, c) \leq r + 2$.
(3) If $d(x_i, c) = r$, then $d(x_i, a) = 2$, there exists a unique vertex z adjacent to both x_i and a, $z \in S_{r-1}$, and z is the unique backtrack of x_i.
(4) If $d(x_i, c) = r$ and if a has unique backtracking, then x_i has no sidestepping.

Proof. We first construct a path and then prove that it satisfies the four listed properties.

We begin by considering the path α from Proposition 3.7. Let $b, y_1, \dots, y_l, b'$ be the vertices of the path. By Lemma 3.18, for all i such that $d(y_i, a) \geq 2$, there exists a positive integer $N(a, y_i)$ such that if $N' \geq N(a, y_i)$, then $d_a(T_a^{N'}(y_i), c) \gg M$. Take $N = \max_i(N(y_i, a))$. Let γ be the path obtained by applying T_a^N to α. The vertices of γ are then

$$b, T_a^N(y_1), \dots, T_a^N(y_l), b'. \tag{4}$$

Let $x_i = T_a^N(y_i)$ for all $1 \leq i \leq l$.

Proposition 3.7, as well as the fact that Dehn twists preserve distance (Remark 3.16), verifies property (1) above.

Now we verify property (2). We first claim that for all i, $d(x_i, c) \geq r$. Let us fix some i. If $d(y_i, a) \geq 2$, then Corollary 3.20 implies that $d(x_i, c) = d(T_a^N(y_i), c) \geq r$. On the other hand, $d(y_i, a) \geq 1$ by construction of α. So the only remaining case to consider is if $d(y_i, a) = 1$, then the assumptions on the path α imply that $y_i \in S_{r+1}$. So $d(x_i, c) = d(T_a^N(y_i), c) = d(y_i, c) \geq r$.

Next, we claim that for all i, $d(x_i, c) \leq r+2$. This follows from the observation that if $y_i \in S_3(a)$, then by assumptions on the path α, there exists $z_i \in S_1(a) \cap (S_r \cup S_{r-1})$ such that $d(y_i, z_i) \leq 2$. But since Dehn twists preserve distances and fix vertices adjacent to the center of the twist,

$$d(x_i, z_i) = d(T_a^N(y_i), T_a^N(z_i)) = d(y_i, z_i). \tag{5}$$

And so, $d(x_i, z_i) \leq 2$. So,

$$d(x_i, c) \leq d(x_i, z_i) + d(c, z_i) \leq r + 2. \tag{6}$$

This finishes the verification of property (2).

To verify property (3), we suppose $d(x_i, c) = r$. Recall that by definition, $x_i = T_a^N(y_i)$. If $d(y_i, a) = 1$, then by construction of α, we have $y_i \in S_{r+1}$. Since T_a^N fixes y_i, we conclude that $x_i = T_a^N(y_i)$ belongs to S_{r+1}. This contradicts the assumption that $d(x_i, c) = r$. So we must have $d(y_i, a) \geq 2$.

And so by Lemma 3.18 and Theorem 2.1, every geodesic from $x_i = T_a^N(y_i)$ to c must pass through $B_1(a)$. Let ζ be one such geodesic and z be one vertex in $\zeta \cap B_1(a)$. Since $d(x_i, a) \geq 2$, z must belong to S_{r-1}, implying that $d(x_i, a) = 2$. By construction, z is a vertex adjacent to both x_i and z. It is the unique such vertex because $\mathcal{C}\Sigma_{0,5}$ has no quadrilaterals.

To finish verifying property (3), it remains to show that z is the unique backtrack of x_i. Let z' be a backtrack of x_i. There is a geodesic $\tilde{\zeta}$ connecting z to c that passes through z'. By the Bounded Geodesic Image Theorem, $\tilde{\zeta}$ must intersect $B_1(a)$. Since $z' \in S_{r-1}$, z' must in fact belong to $B_1(a)$. Because $\Sigma_{0,5}$ has no quadrilaterals, z and z' must coincide. This verifies property (3).

To verify property (4), assume a has unique backtracking and $x_i \in S_r$. Suppose for the sake of contradiction that s is a sidestep of x_i. We note that s is not adjacent to a because otherwise x_i, s, a, z would form a quadrilateral, a contradiction. s is also not equal to a, since otherwise x_i, a, z form a triangle, a contradiction.

Let z be the unique neighbor of x_i and a constructed during the verification of property (3). During the verification of property (3), we proved that $d(y_i, a) \geq 2$. So by Lemma 3.18, $d_a(x_i, c) \gg M$. Additionally, since $d(x_i, s) = 1$, by the coarse-Lipschitz property of d_a, we have $d_a(x_i, s)$ is bounded. So by the triangle inequality, $d_a(s, c) \gg M$. By Theorem 2.1, we know that every geodesic from s to c passes through $B_1(a)$.

Let η be one such geodesic. Since $s \in S_r$ and s is not adjacent or equal to a, we have $\eta \cap B_1(a) \subset S_{r-1}$. But since a has unique backtracking, the only vertex in $B_1(a) \cap S_{r-1}$ is z. This shows that η must pass through z. But then s, z, x_i form a triangle, a contradiction. This proves property (4). □

3.7. Proving Theorem 1.4. Before we begin the proof of Theorem 1.4, we will need to make use of the following lemmas. Essentially, these lemmas modify the paths constructed in Lemma 3.21 so that they lie in $S_{r+1}(c)$ and S_{r+2}. Lemma 3.23 will play a crucial role in the proof of Theorem 1.4.

Lemma 3.22. Suppose $a \in S_r$ has unique backtracking and $b, b' \in S_{r+1}$ are both adjacent to a. Then there exists a path from b to b' entirely in $(S_{r+1} \cup S_{r+2}) \cap B_4(a)$.

Proof. Consider the path from b to b' given by Lemma 3.21. Each vertex on this path that lies in S_r is forward facing and also in $B_2(a)$. Forward facing vertices have no side stepping, so this path has no adjacent vertices in S_r. Thus, we can apply Lemma 3.11 to each vertex in S_r to obtain the appropriate path in $(S_{r+1} \cup S_{r+2}) \cap B_4(a)$. □

Lemma 3.23. Suppose $a \in S_r$ and $b, b' \in S_{r+1}$ are both adjacent to a. Then there exists a path from b to b' entirely in $(S_{r+1} \cup S_{r+2}) \cap B_6(a)$.

Proof. Lemma 3.21 gives a path from b to b' in $(S_r \cup S_{r+1} \cup S_{r+2}) \cap B_3(a)$ such that each vertex on this path that lies in S_r has unique backtracking and is in $B_2(a)$. By Lemma 3.5, we can modify the path at each pair of adjacent vertices that lie in S_r to obtain a new path in $(S_r \cup S_{r+1} \cup S_{r+2}) \cap B_4(a)$ with the additional assumption that no two adjacent vertices are in S_r. Now we can apply Lemma 3.22 to each vertex in S_r to obtain the appropriate path in $(S_{r+1} \cup S_{r+2}) \cap B_6(a)$. □

Next, we recall [13, Lemma 2.1], which states the sufficient conditions for the connectivity of spheres.

Lemma 3.24. [13, Lemma 2.1] Let Γ be an arbitrary graph and fix $c \in \Gamma$. Fix $w > 0$, and let $r > 0$ be arbitrary. Suppose the following conditions hold:

(1) For every $z \in S_r(c)$ and $x, y \in S_{r+1}(c) \cap B_1(z)$ there exists a path

$$x = x_0, x_1, \dots, x_l = y$$

with

$$x_i \in S_{r+1}(c) \cup \cdots \cup S_{r+w}(c)$$

for $0 \leq i \leq l$.

(2) For every adjacent pair $x, y \in S_r(c)$ there exists a path

$$x = x_0, x_1, \ldots, x_l = y$$

with

$$x_i \in S_{r+1}(c) \cup \cdots \cup S_{r+w}(c)$$

for $0 < i < l$.

Then $S_r(c) \cup S_{r+1}(c) \cup \cdots \cup S_{r+w-1}(c)$ is connected.

Proof of Theorem 1.4. Since the curve graphs $\mathcal{C}\Sigma_{0,5}$ and $\mathcal{C}\Sigma_{1,2}$ for the low complexity surfaces are isomorphic, it suffices to prove Theorem 1.4 for $\mathcal{C}\Sigma_{0,5}$. The result follows immediately from combining Lemma 3.24 with Lemma 3.23 and Lemma 3.5. □

4. Medium complexity

Throughout this section, we assume Σ is medium complexity. Again we fix a center vertex c and let $S_r = S_r(c)$. In this section, we upgrade the results from [13, Theorem 1.1] to prove Theorem 1.5: S_r is connected for medium complexity surfaces.

4.1. Organization. We use [13, Theorem 1.1 (2)] that $S_r \cup S_{r+1}$ is connected and begin with a path in $S_r \cup S_{r+1}$. Then we use the definition $\mathcal{O}(z)$, introduced by Wright, as a tool to push the path into S_{r+1} by allowing the path to contain vertices which need not be essentially non-separating.

4.2. Essentially non-separating curves.

Definition 4.1. Let Σ be a surface and U be a subsurface. We call U a *pair of pants* if U is of genus 0 with 3 boundary components.

Definition 4.2. A curve on Σ is called a *pants curve* if it bounds a genus 0 subsurface with 2 punctures.

Definition 4.3. A curve on $\mathcal{C}\Sigma$ is *essentially non-separating* if it is non-separating or a pants curve. A two-component multi-curve $\alpha \cup \beta$ is *essentially non-separating* if α and β themselves are essentially non-separating, and either

(1) $\alpha \cup \beta$ is non-separating,
(2) at least one of α or β is a pants curve, or
(3) $\alpha \cup \beta$ bounds a genus 0 subsurface with 1 puncture.

For $c \in \mathcal{C}\Sigma$, we can define $\mathcal{C}_c\Sigma$ as the subgraph of $\mathcal{C}\Sigma$ whose vertex set is the union between the singleton set $\{c\}$ and the set of all essentially non-separating curves on $\mathcal{C}_c\Sigma$. Disjoint curves α and β are joined by an edge if either $\alpha \cup \beta$ is essentially non-separating or they have different distances to c.

To fix notation, let $S_r^c = S_r \cap \mathcal{C}_c\Sigma$.

Remark 4.4. [13, Lemma 5.2] Wright showed that S_r^c coincides with the sphere of radius r in $\mathcal{C}_c\Sigma$.

We now recall the following results:

Lemma 4.5. $S_r^c \cup S_{r+1}^c$ is connected.

Proof. [13, Proposition 5.4] verifies that the sufficient conditions for the connectivity of spheres in Lemma 3.24 hold in $\mathcal{C}_c\Sigma$ with $w = 2$. □

Lemma 4.6. [13, Lemma 5.3] Suppose Σ has medium complexity. For all $x \in S_r$, then either $x \in S_r^c$ or there exists $x' \in S_r^c \cap S_1(x)$.

4.3. Definition and properties of $\mathcal{O}(z)$. In order to prove Theorem 1.5, we make use of the following definition and prove several of its properties.

Definition 4.7. For any $z \in S_r^c$, define

$$\mathcal{O}(z) = \{a \in S_1(z) \cap \mathcal{C}_c\Sigma \,:\, d_U(a, c) > M\}$$

where U is the unique component of $\Sigma - z$ that is not a pair of pants. Observe that $\mathcal{O}(z) \subseteq S_{r+1}^c$.

Recalling [13, Lemma 7.2], we know we can connect any essentially non-separating curve to $\mathcal{O}(z)$:

Lemma 4.8. [13, Lemma 7.2] Let $z \in S_r^c$ and U be the unique connected component of $\Sigma - z$ that is not a pair of pants. Then for all $N > 0$, any $x \in S_1(z) \cap S_{r+1}^c$ can be connected to some $e \in \mathcal{O}(z)$ by a path in $S_1(z) \cap S_{r+1}^c$. Moreover, e can be taken such that $d_U(e, c) > N$.

Additionally, we will make use of the following lemma:

Lemma 4.9. Let $z \in S_r^c$ and $a, b \in \mathcal{O}(z)$. Then a, b can be connected by a path contained entirely in S_{r+1}.

Proof. Let U be the unique connected component of $\Sigma - z$ that is not a pair of pants. Observe that the subsurface projection $\rho_U(c)$ is a finite set with diameter bounded by some constant k (see Section 2). Thus, there exists $c' \in \rho_U(c)$ such that $d_{\mathcal{C}\Sigma}(c, c') \leq k$. Since $a, b \in \mathcal{O}(z)$, both $d_U(a, c), d_U(b, c) \geq M + 1$, so by the triangle inequality,

$$a, b \in \bigcup_{r'=M+1+k}^{\infty} S_{r'}(c'), \tag{7}$$

where each $S_{r'}(c')$ is a sphere in $\mathcal{C}U$. This union is a subgraph of $\mathcal{C}U$. It is connected because U is low complexity, and so Theorem 1.4 gives that $S_{M+1+k}(c') \cup S_{M+2+k}(c')$ is a connected subset of $\mathcal{C}U$. Thus, we can find a path in $\mathcal{C}U$

$$a = p_0, \ldots, p_l = b$$

such that each $d_U(p_i, c') \geq M+1+k$. Then by the triangle inequality, $d_U(p_i, c) > M$.

Applying Theorem 2.1 for all $0 < i < l$, every geodesic from p_i to c must go through z, as z is the only vertex not cutting U since it is essentially non-separating. By construction, for all i, p_i lies entirely within U and so $d(z, p_i) = 1$. Since $d(z, c) = r$ and , $d(p_i, c) = r + 1$ for all i, as desired. □

4.4. Proving Theorem 1.5. We now have the tools to prove the main result for medium complexity surfaces, namely that for any $c \in \mathcal{C}\Sigma$ and $r > 0$, we have that $S_r(c)$ is connected.

Proof of Theorem 1.5. Suppose $x, y \in S_{r+1}$ are arbitrary. By Lemma 4.6 we can connect x, y to $x', y' \in S^c_{r+1}$ respectively, so it suffices to find a path connecting x', y' inside S_{r+1}. By Lemma 4.5, $S^c_r \cup S^c_{r+1}$ is connected, so there exists a path $x' = x_0, x_1, ..., x_k = y'$ contained in $S^c_r \cup S^c_{r+1}$.

The path from x' to y' above can be taken to have no two consecutive vertices in S^c_r. This follows from [13, Lemma 5.4, part (2)] that for each $x_i, x_{i+1} \in S^c_r$, there exists a path $x_i = x^0_i, x^1_i, x^2_i = x_{i+1}$ such that $x^1_i \in S^c_{r+1}$.

Thus, for each vertex x_i in the path from x' to y', if $x_i \in S^c_r$, then both x_{i-1} and x_{i+1} must be in S^c_{r+1}. In particular, since x_{i-1}, x_i, x_{i+1} is a path, we have $x_{i-1}, x_{i+1} \in S_1(x_i) \cap S^c_{r+1}$.

Now applying Lemma 4.8, to x_i, there exists x'_{i-1} and x'_{i+1} in $\mathcal{O}(x_i)$ which can be connected to x_{i-1} and x_{i+1} respectively with paths contained in $S_1(x_i) \cap S^c_{r+1}$ such that $d_U(x'_{i-1}, c) \gg M$ and $d_U(x'_{i+1}, c) \gg M$.

Applying Lemma 4.9, we can connect x'_{i-1} and x'_{i+1} by a path entirely in S_{r+1}. Thus, for consecutive vertices x_{i-1}, x_i, x_{i+1} in the path from x' to y' where $x_{i-1}, x_{i+1} \in S^c_{r+1}$ and $x_i \in S^c_r$, we can remove x_i and connect x_{i-1} to x_{i+1} by a path contained in S_{r+1}. Since no two consecutive vertices in the path were in S^c_r, this construction eliminates all vertices in S_r and results in a path from x' to y' contained in S_{r+1}, as desired.

□

5. Structure of S_2 in low complexity

The main aim of this section is to prove Theorem 1.7. Throughout this section, we will work with the low complexity surface $\Sigma = \Sigma_{0,5}$. During the proof, we will also show that in $\mathcal{C}\Sigma_{0,5}$, the sphere $S'_2(c)$ has the structure of a $\mathbb{Z}$-bundle over $S_1(c)$.

5.1. Basic Definitions. We begin with two basic definitions.

Definition 5.1. Suppose $x \in S_1(c)$. Consider $S_1(x)$, which is a copy of the Farey graph, in which c is a vertex. Let E_x denote the subset of $S_1(x)$ that has Farey distance 1 from c. In other words, E_x consists of curves that are disjoint from x and have intersection number 2 with c.

Definition 5.2. Let $v \in S_2(c)$ be any vertex. Define $\beta(v)$ as the unique backtrack of v in $S_1(c)$. In other words, $\beta(v)$ is the unique vertex adjacent to both v and c.

Proposition 5.3. Suppose $v \in S_2(c)$. If v is a non-isolated vertex in $S_2(c)$, then $v \in E_{\beta(v)}$.

Proof. Since v is non-isolated in $S_2(c)$, v is adjacent to some $w \in S_2(c)$. Since $\mathcal{C}\Sigma_{0,5}$ has no triangles, $\beta(w) \neq \beta(v)$. So $c, \beta(v), v, w, \beta(w)$ is a cycle of length 5. Since all cycles of length 5 in the low complexity curve graph are pentagons ([1, Theorem 3.1]), $c, \beta(v), v, w, \beta(v)$ is a pentagon. Thus, v is Farey adjacent to c in $S_1(\beta(v))$, proving that $v \in E_{\beta(v)}$. □

Remark 5.4. Proposition 5.3 implies that $S_2'(c) = \bigsqcup_{x \in S_1(c)} E_x$. So the map $\beta|_{S_2'(c)} : S_2'(c) \to S_1(c)$ gives a fiber bundle. We will refer to the map $\beta|_{S_2'(c)}$ as just β.

We now give the above fiber bundle the additional structure of a $\mathbb{Z}$-bundle. We begin by recalling the notion of a half Dehn twist.

Notation 5.5. Let $v \in \mathcal{C}\Sigma_{0,5}$ be any vertex. Then we let τ_v denote the half (right) Dehn twist around v. Furthermore, we let H_v denote the infinite cyclic group generated by τ_v (viewed as a subgroup of the mapping class group of $\Sigma_{0,5}$).

Fact 5.6. Suppose $x \in S_1(c)$. Then H_c acts on E_x simply transitively. Indeed, the set of vertices adjacent to x, which includes c and E_x, can be naturally identified with the Farey graph, and H_c acts simply transitively on the set of vertices adjacent to c in this Farey graph.

This fact implies that the H_c-action makes the $\beta : S_2'(c) \to S_1(c)$ into a $\mathbb{Z}$-bundle, as we now make explicit.

Remark 5.7. For all $y \in S_1(c)$, we fix for the rest of this section some arbitrary $\overline{y} \in E_y$. Then there is an explicit bijection from $\mathbb{Z}$ to E_y given by $n \mapsto \tau_c^n(\overline{y})$. Let ζ_y denote the inverse of this bijection (so ζ_y maps E_y to $\mathbb{Z}$).

5.2. Perfect pairing between some of the fibers.

Definition 5.8. Suppose $x_1, x_2 \in S_1(c)$ and $i(x_1, x_2) = 2$ (i.e. x_1 and x_2 are adjacent if we interpret $S_1(c)$ as a copy of the Farey graph). Then we say that x_1, x_2 are *Farey connected*.

In this subsection, we show that if $x_1, x_2 \in S_1(c)$ and x_1, x_2 are Farey connected, then E_{x_1}, E_{x_2} have a "perfect pairing," which we will make precise below. We first introduce a piece of notation.

Definition 5.9. Suppose x_1, x_2 are in $\in S_1(c)$ and x_1, x_2 are Farey connected. Then let $\mathcal{E}(x_1, x_2)$ denote the set of all edges in $\mathcal{C}\Sigma_{0,5}$ with one vertex in E_{x_1} and another vertex in E_{x_2}.

The following proposition explains how $\mathcal{E}(x_1, x_2)$ gives a "perfect pairing" between E_{x_1} and E_{x_2}.

Proposition 5.10. Suppose $x_1, x_2 \in S_1(c)$ and x_1 is Farey connected to x_2. Then there exists a bijection $\psi : E_{x_1} \to E_{x_2}$ such that

$$\mathcal{E}(x_1, x_2) = \{\{v, \psi(v)\} : v \in E_{x_1}\}. \tag{8}$$

In other words, the proposition says that every vertex of E_{x_1} is joined by an edge to a unique vertex of E_{x_2}, and vice versa. The bijection is such that for all $v \in E_{x_1}$, $\psi(v)$ is the unique element of E_{x_2} joined to v by an edge.

Proof. Since x_1 is Farey connected to x_2, by [13, Lemma 6.6], there exists $s_1, s_2 \in S_2'(c)$ such that c, x_1, s_1, s_2, x_2 is a pentagon. By definition of a pentagon, $s_1 \in E_{x_1}$ and $s_2 \in E_{x_2}$. Applying all integer powers of the half twist τ_c to the edge $\{s_1, s_2\}$, we get a collection of edges

$$\{\{\tau_c^n(s_1), \tau_c^n(s_2)\} : n \in \mathbb{Z}\}.$$

Call the collection Ω.

By Fact 5.6, if $e_1, e_2 \in \Omega$, then e_1, e_2 share no vertices. Also by Fact 5.6, each vertex in E_{x_1} is contained in an edge Ω and likewise each vertex in E_{x_2} is contained in an edge Ω. These two facts guarantee the existence of a bijection $\psi : E_{x_1} \to E_{x_2}$ such that $\Omega = \{\{v, \psi(v)\} : v \in E_{x_1}\}$.

Now it remains to verify that $\Omega = \mathcal{E}(x_1, x_2)$. It is clear that $\Omega \subset \mathcal{E}(x_1, x_2)$. To prove the converse, first observe that any edge $e \in \mathcal{E}(x_1, x_2)$ forms a pentagon with the vertices x_1, x_2, c. We know that all the pentagons containing x_1, x_2, c are obtained from our initial pentagon $\{c, x_1, s_1, s_2, x_2\}$ by applying a power of τ_c (because given any two pentagons, there is a mapping class taking one to the other, and if this mapping class fixes x_1, x_2, c, it must be a power of τ_c). Hence, e is obtained by applying a power of τ_c to the edge $\{s_1, s_2\}$, and so $e \in \mathcal{E}(x_1, x_2)$. This shows that $\mathcal{E}(x_1, x_2) \subset \Omega$, and hence proves the proposition. □

This "perfect pairing" between E_{x_1} and E_{x_2} (for all Farey connected $x_1, x_2 \in S_1(c)$) that we just found is compatible with the action of H_c on E_{x_1} and E_{x_2}. More precisely, we have the following.

Corollary 5.11. Suppose $x_1, x_2 \in S_1(c)$ and x_1, x_2 are Farey connected. Let $\psi : E_{x_1} \to E_{x_2}$ constructed in 5.10. Then H_c acts on E_{x_1}, E_{x_2} ψ-equivariantly, i.e. for all $g \in H_c$ and all $v \in E_{x_1}$, we have

$$\psi(gv) = g\psi(v). \tag{9}$$

Proof. Define the set Ω as in the proof of Proposition 5.10.

Suppose $g = \tau_c^m$ and $v \in E_{x_1}$. We know that $\{v, \psi(v)\} \in \Omega$. By construction of Ω, we have $\{\tau_c^m(v), \tau_c^m(\psi(v))\} \in \Omega$ as well. This implies that $\psi(\tau_c^m(v)) = \tau_c^m(\psi(v))$. This proves the desired ψ-equivariance. □

5.3. Monodromy Number. In this subsection, we define the monodromy number associated to a Farey path in $S_1(c)$.

Suppose $x_1, \ldots, x_l$ all belong to $S_1(c)$ and that they form a Farey path. Let $\psi_{i,i+1}$, $1 \leq i \leq l-1$, be the bijections (between E_{x_i} and $E_{x_{i+1}}$) obtained in Proposition 5.10. Choose any $v \in E_{x_1}$. By Proposition 5.10, we obtain a path in $S_2'(c)$

$$v, \psi_{12}(v), \psi_{23}\psi_{12}(v), \ldots, \psi_{(l-1)l} \cdots \psi_{12}(v).$$

Using the identification of the two sets E_{x_1}, E_{x_l} with $\mathbb{Z}$ given by Remark 5.7, we compute an integer $\zeta_{x_l}(\psi_{(l-1)l} \cdots \psi_{12}(v)) - \zeta_{x_1}(v)$.

Proposition 5.12. For a fixed Farey path γ as above, the number

$$\zeta_{x_l}(\psi_{(l-1)l}\cdots\psi_{12}(v)) - \zeta_{x_1}(v)$$

is independent of the choice of $v \in E_{x_1}$.

Remark 5.13. If $x_l \neq x_1$ (i.e. our Farey path is not a Farey cycle), then the number $\zeta_{x_l}(\psi_{(l-1)l}\cdots\psi_{12}(v)) - \zeta_{x_1}(v)$ *does* depend on the choices of $\overline{x}_l \in E_{x_l}$ and $\overline{x}_1 \in E_{x_1}$ that we made in Remark 5.7 when we defined the bijections ζ_l and ζ_1 .

However, in the case $x_l = x_1$, then changing our choice of $\bar{x}_1$ would change $\zeta_{x_1}(\psi_{(l-1)1}\cdots\psi_{12}(v))$ and $\zeta_{x_1}(v)$ by the same integer. Hence the number

$$\zeta_{x_1}(\psi_{(l-1)1}\cdots\psi_{12}(v)) - \zeta_{x_1}(v)$$

is independent of the choice of $\overline{x}_1$ that we made in Remark 5.7.

Proof of Proposition 5.12. If the path γ has length 1, i.e. $l = 2$, then the proposition follows from equivariance (Corollary 5.11). The general case follows from the case $l = 1$. □

Definition 5.14. Suppose $\gamma = x_1, \dots, x_l$ is a Farey path in $S_1(c)$. We call the number $\zeta_{x_l}(\psi_{(l-1)1}\cdots\psi_{12}(v)) - \zeta_{x_1}(v)$ for some choice of v the "monodromy number" associated to γ. Proposition 5.12 shows that the monodromy number is independent of the choice of v. When γ is a Farey cycle, by Remark 5.13, the monodromy number is also independent of the choices made in Remark 5.7.

5.4. Monodromy Number for a Triangle. In this subsection, we explicitly construct a Farey triangle in $S_1(c)$ and calculate its monodromy number.

Remark 5.15. For the rest of Section 5, we fix two conventions for how we will pictorially represent $\Sigma_{0,5}$ and curves on it. First, we will label the five punctures on $\Sigma_{0,5}$ with elements of the set $\{1, 2, 3, 4, 5\}$, as shown in Fig. 1 and Fig. 2. Second, in these figures, we will represent an element $v \in \mathcal{C}\Sigma_{0,5}$ by an arc such that v is the boundary of an ε-neighborhood of the arc.

Construction 5.16. Let c be the loop around punctures $1, 2$ shown in Fig. 1 (note that Remark 5.15 is now in effect). We now construct a Farey cycle in $S_1(c)$. Let x_1 (resp. x_2, x_3) be the loops around punctures $3, 4$ (resp. punctures $3, 5$, punctures $4, 5$) also shown in Fig. 1. It is clear that x_1, x_2, x_3, x_1 is a Farey cycle of length 3 in $S_1(c)$. For the rest of Section 5.4, we call this Farey cycle the "fundamental triangle" and denote it by $\mathcal{T}$.

Proposition 5.17. The monodromy number of $\mathcal{T}$ is 1.

Proof. Let ψ_{12} be the bijection between E_{x_1} and E_{x_2} constructed in Proposition 5.10. Define ψ_{23} and ψ_{31} similarly.

Let v be the loop around punctures $2, 5$ as shown in the Fig. 2. Then the loops $\psi_{12}(v), \psi_{23}\psi_{12}(v), \psi_{31}\psi_{23}\psi_{12}(v)$ must be the ones shown in the same figure. We see that $\psi_{31}\psi_{23}\psi_{12}(v) = \tau_c(v)$. As a result, we have

$$\zeta_{x_1}(\psi_{31}\psi_{23}\psi_{12}(v)) - \zeta_{x_1}(v) = 1. \tag{10}$$

□

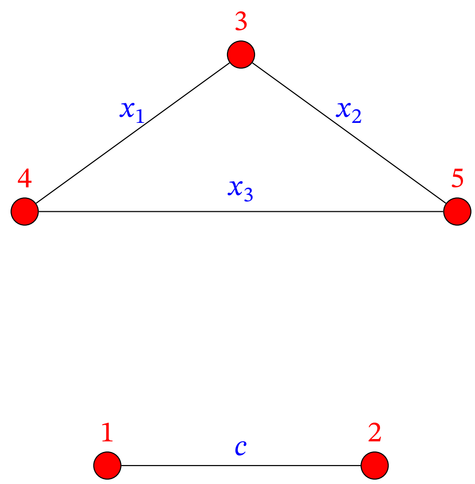

FIGURE 1. Fundamental Triangle

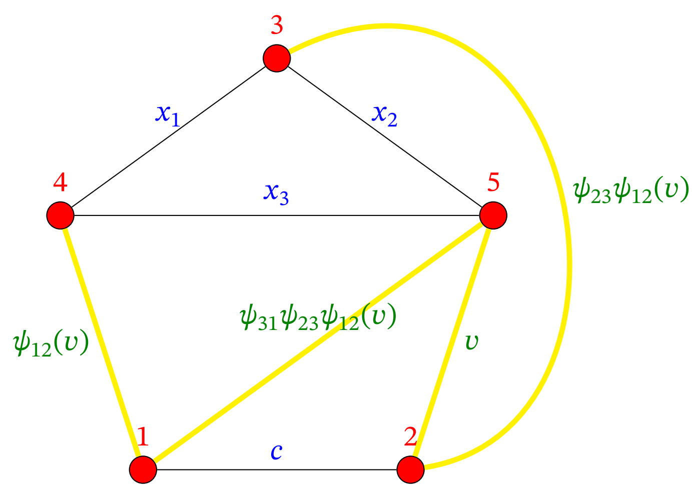

FIGURE 2. Monodromy Number of the Fundamental Triangle

Corollary 5.18. Suppose $v, w \in E_{x_1}$, where x_1 is still the vertex defined in Construction 5.16. Then v can be connected to w by a path in $S_2'(c)$.

Proof. We assume without loss of generality that $\zeta_{x_1}(w) - \zeta_{x_1}(v) = a > 0$. By Proposition 5.17, if $a = 1$, then v can be connected to w by a path in $S_2'(c)$.

Now we pass to the general case. Consider the vertices $v, \tau_c(v), \cdots, \tau_c^a(v)$. By definition of ζ_{x_1} (Remark 5.7), we know that $\tau_c^a(v) = w$ and for all $1 \le i \le a$, we have $\zeta_{x_1}(\tau_c^i(v)) - \zeta_{x_1}(\tau_c^{i-1}(v)) = 1$. So by the case of $a = 1$, for all $1 \leq i \leq a$, we obtain a path in $S_2'(c)$ that connects $\tau_c^{i-1}(v)$ to $\tau_c^i(v)$. Joining these paths together, we obtain a path in $S_2'(c)$ that connects v to w. This proves the corollary. □

5.5. Proving Theorem 1.7. We will see that Theorem 1.7 follows easily from Proposition 5.10 and Corollary 5.18.

Proof of Theorem 1.7. Fix some $v \in E_{x_1}$. Let $z \in S_1(c)$ and $s \in E_z$. It suffices to find a path in $S_2'(c)$ between v and s.

We first choose a Farey path $z = z_0, z_1, \dots, z_l = x_1$ contained in $S_1(c)$. By applying Proposition 5.10 l times, we see s is connected to some $s' \in E_{x_1}$ by some path in $S_2'(c)$. By Corollary 5.18, s' is connected to v by some path in $S_2'(c)$. This proves the theorem. □

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