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On the orbits of plane automorphisms and their stabilizers

Alvaro Rittatore and Iván Pan

ABSTRACT. Let $\Bbbk$ be a perfect field with algebraic closure $\overline{\Bbbk}$. If H is a subgroup of plane automorphisms over $\Bbbk$ and $p \in \overline{\Bbbk}^2$ is a point, we describe the subgroup consisting of plane automorphisms which stabilize the orbit of p under H, when this orbit has irreducible closure in $\overline{\Bbbk}^2$. As an application, we treat the case where H is cyclic and the closure of the orbit of p is an arbitrary (non-necessarily irreducible) curve.

CONTENTS

1. Introduction

In [1], the authors study the group of automorphisms of the affine plane preserving some given curve over an arbitrary field $\Bbbk$; in particular, they describe when the mentioned group is algebraic. As an application of their main results, they propose a classification of the group of automorphisms stabilizing an arbitrary set $\Delta \subset \overline{\Bbbk}^2$ — here $\overline{\Bbbk}$ denotes the algebraic closure of $\Bbbk$. However, this classification (see [1, Prop. 3.11]) is incomplete, since the authors assume implicitly that Δ is such that the Zariski closure of Δ in $\overline{\Bbbk}^2$ coincides with the zero locus of $I_{\Bbbk}(\Delta)$, the ideal of the $\Bbbk$–polynomials vanishing at Δ. This classification can be rephrased as the following problem:

> Let $\Delta \subset \overline{\Bbbk}^2$ be a subset. Then Δ and its closure $\overline{\Delta}$ are stabilized by their respective groups of automorphisms $\mathrm{Aut}(\mathbb{A}^2, \Delta)$

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> and $\mathrm{Aut}(\mathbb{A}^2,\overline{\Delta})$, with $\mathrm{Aut}(\mathbb{A}^2,\Delta) \subset \mathrm{Aut}(\mathbb{A}^2,\overline{\Delta})$ (see Definition 2.3 below). Describe $\mathrm{Aut}(\mathbb{A}^2,\Delta)$; in particular, when the mentioned inclusion is proper.

Clearly, the problem above divides into three main cases: (1) Δ is finite; (2) $\overline{\Delta}$ is a curve; and (3) $\overline{\Delta} = \overline{\mathbb{k}}^2$.

Case (1) has been solved (see [1, §3.1]), whereas case (3) is very difficult to tackle. In this paper, we concentrate on the study of case (2). Notice that both Δ and $\overline{\Delta}$ are stable under the action of $\mathrm{Aut}(\mathbb{A}^2,\Delta)$. Hence, a particular case of (2) is when Δ is a $\mathrm{Aut}(\mathbb{A}^2,\Delta)$-orbit — recall that Δ is $\mathrm{Aut}(\mathbb{A}^2,\Delta)$-stable. This led us to consider the following problem, which is interesting in its own right.

> If H is a subgroup of plane automorphisms over a field $\mathbb{k}$ and $p \in \overline{\mathbb{k}}^2$, then describe the stabilizer of its orbit $\mathrm{Aut}\big(\mathbb{A}^2,\overline{\mathrm{O}_H(p)}\big)$, i.e. the set of plane automorphisms which leave invariant $\mathrm{O}_H(p) = \big\{h(p) : h \in H\big\}$.

More precisely, if we denote A and G the group of automorphisms of the plane stabilizing $\mathrm{O}_H(p)$ and $\overline{\mathrm{O}_H(p)}$ respectively, then it is clear that $H \subset A \subset G$. Moreover, it follows from [1, Theorem 2] that the irreducible curves classify into six disjoint families according to the stabilizer of the elements of each family (see Theorem 2.8 below for a concrete description); if $\overline{\mathrm{O}_H(p)}$ is not irreducible, then one can obtain partial information on G (see [1, Theorem 1]). Taking into account this classification, we want to describe the algebraic structure of the inclusion $H \subset A$ in terms of H, assuming that G is known.

In Theorem 3.4, assuming $\mathbb{k}$ perfect, we solve the above problem when $\overline{\mathrm{O}_H(p)}$ is an irreducible curve: if we classify the curves $\overline{\mathrm{O}_H(p)}$ in families according to the description of their group of automorphisms, then for all but one family (when $\overline{\mathrm{O}_H(p)}$ is, up to an automorphism, a straight line) $\overline{\mathrm{O}_H(p)}$ is either H or an extension of H by an element of G — one must notice that since $H \subset A$ and $\overline{\mathrm{O}_H(p)}$ is a curve, A cannot be finite. In particular, in this way we obtain infinite families of orbits of cyclic groups whose stabilizer is countable infinite, and therefore it is not an algebraic group (see Corollary 3.7).

If $\overline{\mathrm{O}_H(p)}$ is a non irreducible curve, or all the plane, then our problem is more difficult to tackle; in Section 4 we consider a particular case of this problem, namely the case where H is a cyclic group and the closure of $\mathrm{O}_H(p)$ is a (non-necessarily irreducible) curve, see Theorem 4.8: if $\overline{\mathrm{O}_H(p)}$ is not contained in a fence, then again A is either H of an extension by an element, if $\overline{\mathrm{O}_H(p)}$ is contained in a fence, a partial description is given.

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2. Preliminaries

In this section, we introduce the definition of the automorphisms group of a subset of the affine plane $\mathbb{A}^2$ and recall the description of the automorphisms group of a curve in $\mathbb{A}^2$.

Notation 2.1. The following notations will be kept through the paper.

(1) In what follows, $\mathbb{k}$ is an arbitrary perfect field and $\overline{\mathbb{k}}$ is an algebraic closure of $\mathbb{k}$.

(2) We let $\mathbb{A}^2 = \operatorname{Spec}(\mathbb{k}[x,y])$ and denote the set of $\mathbb{k}$-points and $\overline{\mathbb{k}}$-points in $\mathbb{A}^2$ by $\mathbb{A}^2(\mathbb{k})$ and $\mathbb{A}^2(\overline{\mathbb{k}})$ respectively. Recall that $\mathbb{A}^2(\overline{\mathbb{k}})$ identifies with the $\overline{\mathbb{k}}$-points of $\mathbb{A}^2_{\overline{\mathbb{k}}} = \operatorname{Spec}(\overline{\mathbb{k}}) \times_{\mathbb{k}} \mathbb{A}^2$.

(3) If $\Delta \subset \mathbb{A}^2(\overline{\mathbb{k}})$, we denote the *vanishing ideals* of Δ by $\mathcal{I}(\Delta) \subset \mathbb{k}[x,y]$ and $\mathcal{I}_{\overline{\mathbb{k}}}(\Delta) \subset \overline{\mathbb{k}}[x,y]$. If $J \subset \mathbb{k}[x,y]$ is an ideal, we denote $\mathcal{V}(J) = \operatorname{Spec}(\mathbb{k}[x,y]/J)$ viewed as a closed subscheme of $\mathbb{A}^2$, and $\mathcal{V}_{\overline{\mathbb{k}}}(J) = \mathcal{V}(J)(\overline{\mathbb{k}}) \subset \mathbb{A}^2_{\overline{\mathbb{k}}}(\overline{\mathbb{k}}) \cong \mathbb{A}^2(\overline{\mathbb{k}})$ the *zero set of* J.

(4) If $\Delta \subset \mathbb{A}^2(\overline{\mathbb{k}})$, we set $\widehat{\Delta} = \mathcal{V}_{\overline{\mathbb{k}}}(I(\Delta)) \subset \mathbb{A}^2(\overline{\mathbb{k}})$; the Zariski closure of Δ in $\mathbb{A}^2(\overline{\mathbb{k}})$ is denoted by $\overline{\Delta}$ — hence, $\overline{\Delta} \subset \widehat{\Delta} \subset \mathbb{A}^2(\overline{\mathbb{k}})$.

(5) The *automorphisms group* of $\mathbb{A}^2$ is denoted by $\operatorname{Aut}(\mathbb{A}^2)$ — recall that $\operatorname{Aut}(\mathbb{A}^2)$ is the (abstract) group of isomorphisms of $\mathbb{k}$-schemes $f : \mathbb{A}^2 \to \mathbb{A}^2$.

The following well known result, that gives further insight on the relationship between $\overline{\Delta}$ and $\widehat{\Delta}$, follows from Galois descent (see for example [5, Appendix A.j]); we include its proof for the sake of completeness.

Lemma 2.2. *Consider the canonical action of* $G = \operatorname{Gal}(\overline{\mathbb{k}}/\mathbb{k})$ *on* $\mathbb{A}^2(\overline{\mathbb{k}})$. *If* $\Delta \subset \mathbb{A}^2(\overline{\mathbb{k}})$*, then* $\widehat{\Delta} = G \cdot \overline{\Delta}$. *In particular, every irreducible component of* $\widehat{\Delta}$ *is the image of an irreducible component of* $\overline{\Delta}$ *under an element in* G.

Proof. It is clear that $\widehat{\Delta} \subset \mathbb{A}^2(\overline{\mathbb{k}})$ is a G–stable closet subset containing $\overline{\Delta}$. So, by Galois descent, there exists an unique closed subscheme $X \subset \operatorname{Spec}(\mathbb{k}[x,y])$ such that $\widehat{\Delta}$ identifies with the set of $\overline{\mathbb{k}}$-points of $\operatorname{Spec}(\overline{\mathbb{k}}) \times_{\operatorname{Spec}(\mathbb{k})} X$. Under the identification $\mathbb{A}^2(\mathbb{k}) \subset \mathbb{A}^2(\overline{\mathbb{k}})$, we have that $X(\mathbb{k})$ is the set of $\overline{\mathbb{k}}$-points of $\widehat{\Delta}$ fixed by the Galois action and $\widehat{\Delta} = G\cdot(X \cap \mathbb{A}^2(\mathbb{k}))$. It follows that $X(\mathbb{k}) = \overline{\Delta} \cap \mathbb{A}^2(\mathbb{k})$ and $G \cdot \overline{\Delta} = \widehat{\Delta}$. The last assertion is an easy consequence of Galois descent. □

We introduce now the main object of study of this work, namely the group of automorphisms that stabilize an arbitrary subset of $\overline{\mathbb{k}}$-points of the affine plane.

Definition 2.3. Let $a : \operatorname{Aut}(\mathbb{A}^2) \times \mathbb{A}^2_{\overline{\mathbb{k}}}(\overline{\mathbb{k}}) \to \mathbb{A}^2_{\overline{\mathbb{k}}}(\overline{\mathbb{k}})$, $a(f,p) = f(\overline{\mathbb{k}})(p)$, be the canonical action of $\operatorname{Aut}(\mathbb{A}^2)$ on $\mathbb{A}^2(\overline{\mathbb{k}})$. If $\Delta \subset \mathbb{A}^2(\overline{\mathbb{k}})$, we denote the *stabilizer* of Δ under a by $\operatorname{Aut}(\mathbb{A}^2, \Delta)$.

Remark 2.4. (1) Recall that an automorphism $\varphi \in \operatorname{Aut}(\mathbb{A}^2)$ is given by a pair $(f, g) \in \Bbbk[x, y] \times \Bbbk[x, y]$, such that the corresponding endomorphism of $\mathbb{A}^2_{\Bbbk}$ is an automorphism, with inverse $(f, g)^{-1} = (h, j) \in \Bbbk[x, y]\times\Bbbk[x, y]$. Under this identification, $a\big((f, g), p\big) = \big(f(p), g(p)\big)$.

(2) Let $\Delta \subset \mathbb{A}^2(\overline{\Bbbk})$ and $\varphi \in \operatorname{Aut}(\mathbb{A}^2, \Delta)$. If $\varphi^* : \overline{\Bbbk}[x, y] \to \overline{\Bbbk}[x, y]$ denotes the induced automorphism (which is defined over $\Bbbk$), then φ^* stabilizes the ideals $\mathcal{I}(\Delta)$ and $\mathcal{I}_{\overline{\Bbbk}}(\Delta)$. It follows that φ stabilizes $\overline{\Delta}$ and $\widehat{\Delta}$; therefore φ^* also stabilizes the ideals $\mathcal{I}(\widehat{\Delta})$ and and $\mathcal{I}_{\overline{\Bbbk}}(\widehat{\Delta})$.

In other words:

$$\operatorname{Aut}(\mathbb{A}^2, \Delta) \subset \operatorname{Aut}\big(\mathbb{A}^2, \overline{\Delta}\big) \subset \operatorname{Aut}\big(\mathbb{A}^2, \widehat{\Delta}\big). \tag{1}$$

Notation 2.5. If $H \subset \operatorname{Aut}(\mathbb{A}^2)$ is a subgroup and $p \in \mathbb{A}^2(\overline{\Bbbk})$, we denote the *$H$-orbit* of p as $\mathrm{O}_H(p)$.

Example 2.6. It is well known that the inclusions (1) may be strict. For example if $\Bbbk = \mathbb{Q}$ and $\Delta = \big\{(\sqrt{2}, 0)\big\}$, then $\widehat{\Delta} = \big\{(\sqrt{2}, 0), (-\sqrt{2}, 0)\big\}$ and $\varphi = (-x, y + x^2 - 2) \in \operatorname{Aut}(\mathbb{A}^2, \widehat{\Delta}) \setminus \operatorname{Aut}(\mathbb{A}^2, \Delta)$.

The fact that the first inclusion may be strict is a key point in our study of the stabilizers of orbits.

In [1], the authors study the stabilizer of a closed subset of the affine plane. In particular, they give an explicit description of the geometrically irreducible curves $\mathcal{C}$ such that $\operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$ is an algebraic group. We briefly recall now these results — according to Remark 2.4, we present an element of $\operatorname{Aut}(\mathbb{A}^2)$ as a pair of polynomials $(f, g) \in \Bbbk[x, y]$.

Remark 2.7. If $\Delta \subset \mathbb{A}^2(\overline{\Bbbk})$ is finite, then $\operatorname{Aut}(\mathbb{A}^2, \Delta)$ is not an algebraic group (see [1, Proposition 3.11]) — notice that if $\Delta = \big\{(a_i, b_i) \in \mathbb{A}^2(\overline{\Bbbk}); i = 1, \dots, \ell\big\}$ and $m \gg 0$, then there exists a polynomial $P_m \in \Bbbk[x]$ of degree m such that $P(a_i) = 0$ for any $i = 1, \dots, \ell$. Therefore, $\operatorname{Aut}_{\Bbbk}(\mathbb{A}^2, \Delta)$ contains the automorphisms $\big(x, y + P_m(x)\big) \in \Bbbk[x, y]$ for all $m \gg 0$, and it follows from [4, Theorem 1.3] that $\operatorname{Aut}_{\Bbbk}(\mathbb{A}^2, \Delta)$ is not algebraic.

Theorem 2.8. *Let $\mathcal{C} \subset \mathbb{A}^2$ be a geometrically irreducible and reduced curve. Then, up to applying a plane automorphism, $\mathcal{C}$ is one of the curves of the following list:*

(1) $\mathcal{C} = \mathcal{V}\big(x^b - \lambda y^a\big)$, where $a, b > 1$ are coprime integers and $\lambda \in \Bbbk^*$. If this is the case, then

$$\operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big) = \big\{(t^a x, t^b y) : t \in \Bbbk^*\big\} \cong \Bbbk^*.$$

(2) $\mathcal{C} = \mathcal{V}(x^b y^a - \lambda)$*, where* $a, b \geq 1$ *are coprime integers, with* $ab \neq 1$*, and* $\lambda \in \mathbb{k}^*$*. If this is the case, then*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \{(t^a x, t^{-b} y) : t \in \mathbb{k}^*\} \cong \mathbb{k}^*.$$

(3) $\mathcal{C} = \mathcal{V}(xy - \lambda)$*, where* $\lambda \in \mathbb{k}^*$*. If this is the case, then*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \{(tx, t^{-1} y) : t \in \mathbb{k}^*\} \rtimes \{\mathrm{id}, \sigma\} \cong \mathbb{k}^* \rtimes \mathbb{Z}_2,$$

where $\sigma(x, y) = (y, x)$ *is the permutation of coordinates.*

(4) $\mathcal{C} = \mathcal{V}(\lambda x^2 + \nu y^2 - 1)$*, where* $\mathrm{char}(\mathbb{k}) \neq 2$*, and* $\lambda, \nu \in \mathbb{k}^*$ *are such that* $-\lambda\nu$ *is not a square in* $\mathbb{k}$*. If this is the case, then* $\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}}))$ *is the subgroup of* $\mathrm{GL}_2(\mathbb{k})$ *preserving the form* $\lambda x^2 + \nu y^2$ *:*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \left\{\left(\begin{smallmatrix} a & -\nu b \\ \lambda b & a \end{smallmatrix}\right) : (a, b) \in \mathbb{k} \times \mathbb{k},\ a^2 + \lambda\nu b^2 = 1\right\} \rtimes \{\mathrm{id}, \tau\} = T_{\lambda,\nu} \rtimes \mathbb{Z}_2,$$

where $\tau(x, y) = (x, -y)$*. Notice that* $T_{\lambda,\nu}$ *is a non-*$\mathbb{k}$*-split torus.*

(5) $\mathcal{C} = \mathcal{V}(x^2 + \mu xy + y^2 - 1)$*, where* $\mathrm{char}(\mathbb{k}) = 2$ *and* $\mu \in \mathbb{k}^*$ *is such that* $x^2 + \mu x + 1$ *has no root in* $\mathbb{k}$*. If this is the case, then* $\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}}))$ *is the subgroup of* $\mathrm{GL}_2(\mathbb{k})$ *preserving the form* $x^2 + \mu xy + y^2$*:*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \left\{\left(\begin{smallmatrix} a & b \\ b & a+\mu b \end{smallmatrix}\right) : a, b \in \mathbb{k}, a^2 + \mu ab + b^2 = 1\right\} \rtimes \{\mathrm{id}, \sigma_\mu\} = T_\mu \rtimes \mathbb{Z}_2,$$

where $\sigma_\mu(x, y) = (x + \mu y, y)$*. Notice that* T_μ *is a non-*$\mathbb{k}$*-split torus.*

(6) $\mathcal{C}$ *is the line of equation* $x = 0$*, in which case*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \Big\{(ax, by + P(x)) : a, b \in \mathbb{k}^*,\ P \in \mathbb{k}[x]\Big\}.$$

In particular, if $R : \mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) \to \mathrm{Aut}(\mathbb{A}^1) = \{y \mapsto by + c : b \in \mathbb{k}^*, c \in \mathbb{k}\}$ *denotes the restriction map, then* $\mathrm{Ker}(R) = \Big\{(x, y) \mapsto (ax, y + P(x)) : a \in \mathbb{k}^*,\ P \in \mathbb{k}[x],\ P(0) = 0\Big\}$ *and*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}})) = \mathrm{Ker}(R) \rtimes \mathrm{Aut}(\mathbb{A}^1).$$

(7) In any other case, the curve $\mathcal{C}$ *is such that* $\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\mathbb{k}}))$ *is finite.*

Proof. See [1, Theorem 2]. □

Definition 2.9. We will abuse notations and say that $D \subset \mathbb{A}^2(\overline{\mathbb{k}})$ is a *curve of type* (1) – (7), if $D = \mathcal{C}(\overline{\mathbb{k}})$, where $\mathcal{C}$ verifies the respective condition in Theorem 2.8. The *canonical form* of a curve of type (1) – (6) is the curve of equations given in the Theorem 2.8 for the corresponding type.

Following [1], we say that a curve $\mathcal{C}$ is of *fence type* if up to applying an automorphism of $\mathrm{Aut}(\mathbb{A}^2)$, $\mathcal{C}$ has equation $P(x) = 0$, with $P \in \mathbb{k}[x]$.

Corollary 2.10. *Assume that Δ is a proper closed subset of $\mathbb{A}^2(\overline{\Bbbk})$. Then one of the following assertions holds:*

(a) Δ is contained in a curve of fence type, in which case $\mathrm{Aut}(\mathbb{A}^2, \Delta)$ is not an algebraic group.

(b) The group $\mathrm{Aut}(\mathbb{A}^2, \Delta)$ is conjugate to an algebraic subgroup of either $\mathrm{Aff}(\mathbb{A}^2)$ or J_n for some $n \geq 2$ (notations as in [1]).

Proof. With the added hypothesis of the closedness of Δ, the proof of [1, Proposition 3.11] now applies without obstructions. □

3. The orbit of a group of automorphisms of the plane

Let $H \subset \mathrm{Aut}(\mathbb{A}^2)$ be a subgroup and $p \in \mathbb{A}^2(\overline{\Bbbk})$. We are interested now in the calculation of $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(p)\big)$. Notice that we have that

$$H \subset \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(p)\big) \subset \mathrm{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_H(p)}\big).$$

Lemma 3.1. *Let $H \subset \mathrm{Aut}(\mathbb{A}^2)$ be a subgroup and $p \in \mathbb{A}^2(\overline{\Bbbk})$. Then $\overline{\mathrm{O}_H(p)}$ is either:*

(1) a finite set;

(2) a curve $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$, with $\mathcal{C}_i \cong \mathcal{C}_j$ for all i, j, where $\mathcal{C}_1$ is a curve of type (1) – (6);

(3) $\mathbb{A}^2(\overline{\Bbbk})$.

Proof. Assume that $\overline{\mathrm{O}_H(p)}$ is neither finite nor all the plane; then $\dim \overline{\mathrm{O}_H(p)} = 1$ and therefore H is necessarily infinite. Let $\overline{\mathrm{O}_H(p)} = \bigcup_{i=1}^{s} \mathcal{C}_i$ be the decomposition in irreducible components, with $\dim \mathcal{C}_1 = 1$. By continuity and the irreducibility of $\mathcal{C}_1$, it follows that there exists $h_i \in H$ such that $h_i \cdot \mathcal{C}_1 = \mathcal{C}_i$. Hence, it remains to prove that $\mathcal{C}_1$ is a curve of type (1) – (6).

Let $H_i = \{h \in H \,:\, h \cdot \mathcal{C}_1 = \mathcal{C}_i\} \subset H$, $i = 1, \dots s$. Since H is infinite, there exists $i \in \{1, \dots, s\}$ such that $\# H_i = \infty$. Let $h_i \in H_i$ be a fixed element. Then $h_i^{-1} H_i \subset H_1 \subset \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)$, and it follows that $\#\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big) = \infty$. We deduce from Theorem 2.8 that $\mathcal{C}_1$ is a curve of type (1) – (6). □

We collect now some technical remarks that we need in order to calculate the stabilizers of an orbit.

Lemma 3.2. *Let $\mathcal{C}$ be a curve of type (1) – (6) given by its canonical form and $p = (x_p, y_p) \in \mathcal{C}(\overline{\Bbbk})$. If we keep the notations of Theorem 2.8, then the isotropy group $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)_p$ is trivial unless*

(i) $\mathcal{C} = \mathcal{V}(x^b - \lambda x^a)$, $a, b > 1$ coprime integers, and $p = (0,0)$, in which case $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)_{(0,0)} = \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$.

(ii) $\mathcal{C} = \mathcal{V}(xy-\lambda)$*, in which case* $\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk}))_p = \{\mathrm{Id}, \frac{x_p}{y_p}\sigma\}$*, where* $\sigma = (y, x)$ *is the permutation of coordinates.*

(iii) $\mathcal{C} = \mathcal{V}(\lambda x^2 + \nu y^2 - 1)$*, in which case*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk}))_p = \{\mathrm{Id}, t_{\lambda x_p^2 - \nu y_p^2, 2x_p y_p}\tau\} = \{\mathrm{Id}, t_{2\lambda x_p^2 - 1, 2x_p y_p}\tau\},$$

where $t_{a,b} = \left(\begin{smallmatrix} a & -\nu b \\ \lambda b & a \end{smallmatrix}\right) \in T_{\lambda,\nu}$ *and* $\tau = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$*.*

(iv) $\mathcal{C} = \mathcal{V}(x^2 + \mu xy + y^2 - 1)$*, in which case*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk}))_p = \{\mathrm{Id}, t_{\mu x_p y_p + 1, \mu y_p^2}\sigma_\mu\} = \{\mathrm{Id}, t_{x_p^2 + y_p^2, \mu y_p^2}\sigma_\mu\},$$

where $t_{a,b} = \left(\begin{smallmatrix} a & b \\ b & a+\mu b \end{smallmatrix}\right) \in T_\mu$ *and* $\sigma_\mu = \left(\begin{smallmatrix} 1 & \mu \\ 0 & -1 \end{smallmatrix}\right)$*.*

(v) $\mathcal{C}$ *is the line of equation* $x = 0$*, in which case*

$$\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk}))_p = \Big\{(ax, by + P(x)) \,:\, a, b \in \Bbbk^*,\ P \in \Bbbk[x], P(0) = -by_p\Big\} \cong \mathrm{Ker}(R) \rtimes \{(0, y) \mapsto (0, by + (1-b)y_p) \,:\ b \in \Bbbk^*\},$$

where $R : \mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk})) \to \mathrm{Aut}(\mathbb{A}^1)$ *is the restriction map (see Theorem* 2.8*).*

Proof. Let $G = \mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\bar{\Bbbk}))$. If $\mathcal{C}$ is a curve of type (1) or (2), given by its canonical form, then $(0,0)$ is the only fixed point of the action of G on $\mathbb{A}^2$. It follows that in these cases $G_p = \{\mathrm{Id}\}$ unless $\mathcal{C} = \mathcal{V}(x^b - \lambda y^a)$ with $b, a > 1$ coprimes and $p = (0,0)$.

(i) If $\mathcal{C} = \mathcal{V}(x^b - \lambda y^a)$, then clearly $G_{(0,0)} = G$.

(ii) Easy calculations show that if $p = (x_p, y_p)$ is a $\bar{\Bbbk}$-point of $\mathcal{V}(xy-\lambda)$, then $G_p = \{\mathrm{Id}, \frac{x_p}{y_p}\sigma\}$.

(iii) Consider $\mathcal{C} = \mathcal{V}(\lambda x^2 + \nu y^2 - 1)$ and let $p = (x_p, y_p) \in \mathcal{C}(\bar{\Bbbk})$. Since the closure of $\mathcal{C}$ in $\mathbb{P}^2(\bar{\Bbbk})$ is a smooth conic, it follows that if $\psi \in G_p$, then ψ induces an automorphism of $\mathrm{Aut}(\mathbb{P}^1(\bar{\Bbbk})) = \mathrm{PGL}_2(\bar{\Bbbk})$ that fixes p and stabilizes the set of (two) points at infinity. Since the subgroup of such automorphisms has at most two elements, solving the equation

$$t_{a,b}\tau(p) = \begin{pmatrix} a & \nu b \\ \lambda b & -a \end{pmatrix} \begin{pmatrix} x_p \\ y_p \end{pmatrix} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}$$

we deduce that $G_p = \{\mathrm{Id}, t_{(a,b)}\tau\}$, with $(a, b) = (\lambda x_p^2 - \nu y_p^2, 2x_p y_p) = (2\lambda x_p^2 - 1, 2x_p y_p)$.

(iv) If $\mathcal{C} = \mathcal{V}(x^2 + \mu xy + y^2 - 1)$ and $p = (x_p, y_p) \in \mathcal{C}(\bar{\Bbbk})$, we proceed as in the previous case and deduce that $G_p = \{\mathrm{Id}, t_{a,b}\sigma_\mu\}$, where $t_{(a,b)}\sigma_\mu = \left(\begin{smallmatrix} a & \mu a + b \\ b & a \end{smallmatrix}\right)$, with $(a, b) = (\mu x_p y_p + 1, \mu y_p^2) = (x_p^2 + y_p^2, \mu y_p^2)$.

(v) For the case $\mathcal{C} = \mathcal{V}(x)$, just recall that $\operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$ is the semi-direct product of $\operatorname{Ker}(R)$ and $\operatorname{Aut}(\mathbb{A}^1)$.

□

Remark 3.3. (1) Let $\mathcal{C}$ be a curve of type (3) – (5) and consider $G = \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$. Let $p \in \mathcal{C}$ be such that $\overline{\mathrm{O}_G(p)} = \mathcal{C}$ — it is clear that such a p exists. Then, it follows from Theorem 2.8 and Lemma 3.2 that

$$G = T \rtimes G_p = T \rtimes \{\mathrm{Id}, \tau_p\} = T \cup T\tau_p,$$

where T is a possibly non-split torus and τ_p is the involution given in Lemma 3.2.

Notice that $t\tau_p = t^{-1}\tau_p$ for all $t \in T$. In particular, any element of $T\tau_p$ is also an involution and $\operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big) = T \rtimes \{\mathrm{Id}, t\tau_p\}$ for all $t \in T$.

(2) It follows that if $H \subset G = \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$ is a subgroup, then either $H \subset T$ or

$$H = H_0 \cup H_0\gamma = H_0 \rtimes \{\mathrm{Id}, \gamma\},$$

where $H_0 = H \cap T$ and $\gamma \in H \cap T\tau_p$ is a non trivial involution. Indeed, if $\gamma, \gamma' \in H \cap (T\gamma)$, then $\gamma' = h\gamma$, with $h \in H_0$.

(3) In particular, if a subgroup $H \subset \operatorname{Aut}(\mathbb{A}^2)$ and $p \in \mathbb{A}^2(\overline{\Bbbk})$ are such that $\overline{\mathrm{O}_H(p)}$ is a curve of type (3) – (5), it follows that either H is a subgroup of the corresponding torus $T \subset G = \operatorname{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_H(p)}\big)$ or

$$H = H_0 \rtimes \{\mathrm{Id}, t_0\tau_p\} = H_0 \cup H_0 t_0\tau_p,$$

where $H_0 = T \cap H$ and $t_0 \in T$.

Analogously, we have that $A = \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(p)\big)$ is either a subgroup of T or

$$A = A_0 \rtimes \{\mathrm{Id}, t_1\tau_p\} = A_0 \cup A_0 t_1\tau_p,$$

where $A_0 = T \cap A$ and $t_1 \in T$.

Notice in particular that $T \cap G_p = \{\mathrm{Id}\}$, and therefore $H_0 \cap G_p = A_0 \cap G_p = \{\mathrm{Id}\}$.

Theorem 3.4. *Let $H \subset \operatorname{Aut}(\mathbb{A}^2)$ be a subgroup, $p \in \mathbb{A}^2(\overline{\Bbbk})$ such that $\mathcal{C} = \overline{\mathrm{O}_H(p)}$ is an irreducible curve. Let $G = \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$, $A = \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(p)\big)$, $A_0 = A \cap T$, $H_0 = H \cap T$ and if $\#G_p = 2$, let $\tau_p \in G_p \setminus \{\mathrm{Id}\}$ (see Remark 3.3). Then:*

(a) If $\mathcal{C}$ is a curve of type (1) or (2) then $A = H$.

(b) If $\mathcal{C}$ is a curve of type (3), (4) or (5), then $G_p = \{\mathrm{Id}, \tau_p\} \cong \mathbb{Z}_2$ and:

(i) If $H = H_0 \subset T$ then $A = H_0 \cup H_0\tau_p = H_0 \rtimes G_p$.

(ii) If $H = H_0 \rtimes G_p = H_0 \cup H_0\tau_p$, then $A = H$.

(iii) If $H = H_0 \rtimes \{\mathrm{Id}, t_0\tau_p\} = H_0 \cup H_0 t_0\tau_p$, with $t_0 \in T \setminus H_0$, then $A = H$ unless $t_0^2 \in H_0$, in which case

$$A = \big\langle H_0 \cup \{t_0\}\big\rangle \cup \big\langle H_0 \cup \{t_0\}\big\rangle t_0\tau_p = \big\langle H_0 \cup \{t_0\}\big\rangle \rtimes \{\mathrm{Id}, t_0\tau_p\};$$

in particular, $A_0 = \langle H_0 \cup \{t_0\} \rangle$ *and* $A_0/H_0 \cong \mathbb{Z}_2$.

(c) If $\mathcal{C}$ *is a curve of type (6), then*

$$A = H(G_p \cap A).$$

In particular, $H\,\mathrm{Ker}(\mathrm{R}) \subset A$.

Proof. In what follows, we keep the notations of Theorem 2.8 and assume (without loss of generality) that $\mathcal{C}$ is given by its canonical form. It is clear that $H \subset A \subset G$ — the last inclusion follows by continuity of the action; then $\mathcal{C} = \overline{\mathrm{O}_H(p)} = \overline{\mathrm{O}_A(p)} = \overline{\mathrm{O}_G(p)}$.

(a) First, observe that if $\mathcal{C} = \overline{\mathrm{O}_H(p)}$ is of type (1), then $p \neq (0,0)$. Indeed, if this were the case then $G_{(0,0)} = G$, and therefore $\mathrm{O}_H\big((0,0)\big) \subset \mathrm{O}_G\big((0,0)\big) = \big\{(0,0)\big\}$ and we obtain a contradiction.

It follows from Lemma 3.2 that if $\mathcal{C}$ is of type (1) or (2), then $G_p = \{\mathrm{Id}\}$. If $\varphi \in A$ then $\varphi(p) \in \mathrm{O}_H(p)$ as $A = \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(p)\big)$, and therefore there exists $h \in H$ such that $\varphi(p) = hp$, so $h^{-1}\varphi \in G_p = \{\mathrm{Id}\}$. It follows that $A = H$.

(b) Assume now that $\mathcal{C}$ is of type (3) – (5) and given by its canonical form. Then, it follows from Lemma 3.2 and Remark 3.3 that $G = T \rtimes \{\mathrm{Id}, \tau_p\}$, where τ_p is a non trivial involution fixing p. Since $H \subset A \subset G$, it follows (again from Remark 3.3) that both H and A are determined by their intersections with T (H_0 and A_0 respectively), if they are not contained in T, an involution $t_0\tau_p$, for some t_0 belonging to H_0 or A_0 respectively — notice that if $H \neq H_0$, then $A \neq A_0$ but it may happen that $H = H_0$ and $A \neq A_0$. Hence, we may distinguish three cases:

(i) $H = H_0$. If $s \in A_0$ then $sp \in \mathrm{O}_H(p)$ and therefore $sp = hp$ for some $h \in H = H_0 \subset A_0$, so $h^{-1}s \in A_0 \cap G_p = \{\mathrm{Id}\}$ (see Remark 3.3). Thus, $s = h \in H \cap A_0 = H$. On the other hand $\tau_p(hp) = h^{-1}\tau_p p = h^{-1}p \in \mathrm{O}_H(p)$ for any $h \in H$. It follows that $A = H \rtimes \{\mathrm{Id}, \tau_p\}$.

(ii) $H = H_0 \cup H_0\tau_p = H_0 \rtimes G_p$. If $s \in A_0$ then $sp = hp$ for some $h \in H$. If $h \in H_0$ we deduce as in the previous case that $s = h \in H_0$. If $h = h_0\tau_p$, then $hp = h_0p$ and it follows that $s \in H_0$. Since $\tau_p \in H \subset A$, then $A = H_0 \rtimes \{\mathrm{Id}, \tau_p\} = H$.

(iii) $H = H_0 \cup H_0t_0\tau_p = H_0 \rtimes \{\mathrm{Id}, t_0\tau_p\}$, where $t_0 \in T \setminus H_0$. As in the previous cases, if $s \in A_0$, then $sp = hp$ for some $h \in H$, and therefore $h^{-1}s \in G_p$. If $h \in H_0$ we deduce as before that $s = h \in H_0$.

If $h = h_0t_0\tau_p$, with $h_0 \in H_0$, then $hp = h_0t_0p$ and, since $h_0t_0 \in G_0$, it follows that $s = h_0t_0$ (see Remark 3.3). Hence, $s \in A_0$ if and only if $t_0 \in A_0$ (because $h_0 \in H_0 \subset A_0$), and it remains to determine under which conditions we have that $t_0 \in A_0$.

Consider an arbitrary $h_1 \in H_0$; then $t_0(h_1 p) = h_1 t_0 p = h_1 t_0 \tau_p(p) \in \mathrm{O}_H(p)$ (since $t_0\tau_p \in H$) and $t_0\big(h_1 t_0 \tau_p(p)\big) = h_1 t_0^2 \tau_p(p) = h_1 t_0^2 p$. Since $H = H_0 \rtimes H_0 t_0 \tau_p$, it follows that $t_0 \in A_0$ if and only if $h_1 t_0^2 p \in \mathrm{O}_H(p)$, if and only if $t_0^2 p \in \mathrm{O}_H(p)$ (since $h_1 \in H$), and this happens if and only if there exists $h_2 \in H_0$ such that either $t_0^2 p = h_2 p$, or $t_0^2 p = h_2 t_0 \tau_p(p) = h_2 t_0 p$. In the first case it follows that $t_0^2 \in H_0$ (since $h_0^{-1} t_0^2 \in G_0 \cap G_p = \{\mathrm{Id}\}$), and in the last case the same argument shows that $t_0^2 = h_2 t_0$; therefore, $t_0 \in H_0$. We conclude that if $t_0 \in A_0$ then $t_0^2 \in H_0$. Conversely, it $t_0^2 \in H_0$, then $t_0^2 p \in \mathrm{O}_H(p)$ and $t_0 \in A_0$.

(c) If $\mathcal{C}$ is the line of equation $x = 0$ and $\varphi = \big(ax, by + P(x)\big) \in A$, then $\big(0, by_p + P(0)\big) = h \cdot p$ for some $h \in H$. It follows that $h^{-1}\varphi \in G_p$ and therefore $\varphi \in HG_p$; the result follows. □

If $\overline{\mathrm{O}_H(p)}$ is an irreducible curve of type (6), the inclusion $H\,\mathrm{Ker}(R) \subset A$ may be strict, as the following example shows.

Example 3.5. Assume that $\mathrm{char}(\Bbbk) = 0$ and let $\varphi = (x, y + c)$, $c \neq 0$. If $H = \langle \varphi \rangle = \{\varphi^n \,:\, n \in \mathbb{Z}\}$, then $\psi = (ax, ny)$ belongs to $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_H(0,0)\big)$ for all $n \in \mathbb{Z}$ – indeed $\varphi^n(0,0) = (0, nc)$ for all $n \in \mathbb{Z}$. Therefore, the inclusion $H\ker(R)$ in Theorem 3.4 (c) can be strict.

The following example gives a little insight on Theorem 3.4 (b-iii).

Example 3.6. Let $\Bbbk$ be an infinite field, $\mathrm{char}(\Bbbk) = 0$. Consider the automorphisms of the plane $h_0, t_0 \in \mathrm{Aut}(\mathbb{A}^2)$, $h_0(x,y) = (4x, \frac{y}{4})$, $t_0(x,y) = (2x, \frac{y}{2})$, and let $H = \langle h_0 \rangle \rtimes \{\mathrm{Id}, t_0\sigma\} = \langle h_0 \rangle \bigcup \langle h_0 \rangle t_0 \sigma$, where σ is the transposition of coordinates. Then $t_0 \notin H_0$, $t_0^2 = h_0 \in H_0$ and

$$\mathrm{O}_H\big\{(1,1)\big\} = \big\{(2^{2n}, 2^{-2n}), (2^{2n-1}, 2^{-2n+1}) \,:\, n \in \mathbb{Z}\big\}$$

Clearly, $\overline{\mathrm{O}_H\big((1,1)\big)} = \mathcal{V}(xy-1)$ and, since $A \subset \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{V}(xy-1)\big) = \big\{(x,y) \mapsto (tx, t^{-1}y) \,:\, t \in \Bbbk^*\big\} \rtimes \{\mathrm{Id}, \sigma\}$, it follows that $A = \langle t_0 \rangle \rtimes \langle t_0 \rangle \sigma$.

If we consider now $h_1 = t_0^3$, and $L = \langle h_1 \rangle \rtimes \{\mathrm{Id}, t_0\sigma\}$ then

$$\mathrm{O}_L(1,1) = \big\{(2^{3n}, 2^{-3n}), (2^{3n-1}, 2^{-3n+1}) \,:\, n \in \mathbb{Z}\big\}$$

and $A = H$.

Corollary 3.7. *Let $H \subset \mathrm{Aut}(\mathbb{A}^2)$ be an infinite countable subgroup of plane automorphisms and $p \in \mathbb{A}^2(\overline{\Bbbk})$ a point such that $\mathcal{C} = \overline{\mathrm{O}_H(p)}$ is an irreducible curve. Then the stabilizer of $\mathrm{O}_H(p)$ is not an algebraic group.*

Proof. It follows from Theorem 3.4 that if $\mathcal{C}$ is of type (1)–(5) then A is infinite countable and therefore is not algebraic. If $\mathcal{C}$ is of type (6) then A contains $\mathrm{Ker}(R)$ and therefore A contains automorphisms of arbitrary degree; it follows from [4, Theorem 1.3] that A is not algebraic. □

4. The orbit of an automorphism of the plane

Let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and $p \in \mathbb{A}^2(\overline{\Bbbk})$. In this section, as an application of Theorem 3.4, we study the geometry of the *orbit of p by φ*. More precisely, we consider $H = \langle\varphi\rangle$ and describe $\mathcal{C} = \overline{\mathrm{O}_H(p)}$, $\mathrm{Aut}(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk}))$ and $\mathrm{Aut}(\mathbb{A}^2, \mathrm{O}_H(p))$. Along this section, we keep the previous notations and set $\mathrm{O}_\varphi(p) = \mathrm{O}_H(p)$.

As Lemma 3.1 indicates, the closure of the orbit $\mathrm{O}_\varphi(p)$ is either finite, a curve or the whole plane. We begin by showing that under mild conditions, there exist pairs $p \in \mathbb{A}^2(\overline{\Bbbk})$ and $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ such that $\mathrm{O}_\varphi(p)$ is infinite and $\mathcal{C} = \overline{\mathrm{O}_\varphi(p)}$ is of type (1)–(6) — in particular, $\mathrm{Aut}(\mathbb{A}^2, \mathrm{O}_H(p))$ is not algebraic (see Corollary 3.7).

Example 4.1. (a) Assume that $\Bbbk$ is such that it contains a non zero element t_0 that is not a root of unity. Then:

(i) If $p = (x_p, y_p)$, $x_p y_p \neq 0$ and $\varphi(x, y) = (t_0^a x, t_0^b y)$, with $a, b > 1$ coprimes, then $\mathcal{C} = \mathcal{V}(x^b - \frac{x_p^b}{y_p^a} y^a)$.

(ii) If $p = (x_p, y_p)$, $x_p y_p \neq 0$ and $\varphi(x, y) = (t_0^a x, t_0^{-b} y)$, with $a, b \geq 1$ coprimes, then $\mathcal{C} = \mathcal{V}(x^b y^a - x_p^b y_p^a)$. Notice that if $ab \neq 1$ then $\mathcal{C}$ is of type (2), whereas if $a = b = 1$, then $\mathcal{C}$ is of type (3).

(iii) If $p = (x_p, y_p) \neq (0, 0)$, and $\varphi(x, y) = t_0 \cdot (x, y) = (t_0 x, t_0 y)$, then $\mathcal{C}$ is a line.

(b) We want now to construct examples of orbits of automorphisms with curves of types (4) and (5) as closures. We assume that $\Bbbk$ is uncountable and keep the notations of Theorem 2.8.

(i) If $\mathrm{char}(\Bbbk) \neq 2$ and there exist $\lambda, \nu \in \Bbbk^*$ are such that $-\lambda\nu$ is not a square in $\Bbbk$, consider $T_{\lambda,\nu} = \left\{\left(\begin{smallmatrix} a & -\nu b \\ \lambda b & a \end{smallmatrix}\right) : (a, b) \in \Bbbk \times \Bbbk,\ a^2 + \lambda\nu b^2 = 1\right\}$. If $\varphi = \left(\begin{smallmatrix} a & -\nu b \\ \lambda b & a \end{smallmatrix}\right) \in T_{\lambda,\nu}$, then φ has eigenvalues $a \pm \sqrt{-\lambda\mu} b$. Since $\Bbbk$ is uncountable, we can choose $(a, b) \in \Bbbk^2$ such that $a \pm \sqrt{-\lambda\mu} b$ is not a root of unity, and therefore $(0, 0)$ is the unique periodic point of φ. It follows that if $p = (x_p, y_p) \in \mathcal{C} = \mathcal{V}(\lambda x^2 + \nu y^2 - 1)$, then $\mathrm{O}_\varphi(p)$ is infinite, with $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}$; in particular, $\overline{\mathrm{O}_\varphi(p)}$ is of type (4).

(ii) If $\mathrm{char}(\Bbbk) = 2$ and $\mu \in \Bbbk^*$ is such that $x^2 + \mu x + 1$ has no root in $\Bbbk$, let ζ be a root of $x^2 + \mu x + 1$, and consider

$$T_\mu = \left\{\left(\begin{smallmatrix} a & b \\ b & a+\mu b \end{smallmatrix}\right) : a, b \in \Bbbk, a^2 + \mu ab + b^2 = 1\right\}.$$

Then $\varphi = \left(\begin{smallmatrix} a & b \\ b & a+\mu b \end{smallmatrix}\right) \in T_\mu$ has $a + \zeta b$, $a + \zeta^{-1} b$ as eigenvalues and, since $\Bbbk$ is uncountable, it follows as before that we can choose a, b such that $(0, 0)$ is the unique periodic point of φ. Thus, for such a φ if $p \in \mathcal{C} =$

$\mathcal{V}(x^2+\mu xy+y^2-1)$ then $\mathrm{O}_\varphi(p)$ is infinite, with $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}$; in particular, $\overline{\mathrm{O}_\varphi(p)}$ is of type (5).

If $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$, let $\mathcal{S}$ be the union of (not necessarily irreducible) stable curves of φ and $\mathcal{P}$ be the set of periodic points of φ. Then it is clear that there exists $p \in \mathbb{A}^2(\overline{\Bbbk})$ such that $\overline{\mathrm{O}_\varphi(p)} = \mathbb{A}^2$ if and only if $\mathcal{S}\bigcup\mathcal{P} \neq \mathbb{A}^2$. This simple remark allows us to show that there exist automorphisms of the plane with orbits which closure is all the plane as follows.

Example 4.2. Let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ be such that $\varphi^n \neq \mathrm{Id}$ for all n; for $n \geq 1$, we denote by F_n the (closed) subset of fixed points of φ^n. Clearly, $\mathcal{P} = \bigcup F_n$. We affirm that F_n is contained in a curve for each n. Indeed, otherwise $\overline{F_n} = \mathbb{A}^2$ and therefore $\varphi^n = \mathrm{Id}$.

Recall that the (first) *dynamical degree* of φ is defined as

$$\lambda_1(\varphi) = \lim_{m\to\infty} \sqrt[m]{\deg \varphi^m}.$$

Notice that $\lambda_1(\varphi) > 1$ implies $\lambda_1(\varphi^n) > 1$ for any $n \geq 1$; for more details see [3], where the complex case is treated. It follows from [2, Thm. B] and [6, Thm. 5.5] that if $\lambda_1(\varphi) > 1$, then $\mathcal{S}$ is a curve. Therefore, $\mathcal{S} \cup \mathcal{P}$ is contained in a countable union of curves. Hence, if $\overline{\Bbbk}$ is uncountable and $\lambda_1(\varphi) > 1$, there exists $p \in \mathbb{A}^2(\overline{\Bbbk})$ such that $\overline{\mathrm{O}_\varphi(p)} = \mathbb{A}^2$.

One particular example of this situation concerns the so-called Hénon maps, that is morphisms of the form $\varphi = \big(y, -\delta x + P(y)\big)$, where $\delta \in \Bbbk^*$ and $P \in \Bbbk[y]$ is a polynomial of degree greater or equal that 2, since in this case $\lambda_1(\varphi) = \deg P$.

A priori, the closure of an orbit $\mathrm{O}_\varphi(p)$ is not necessarily an irreducible curve. In what follows, we exploit Lemma 2.2 in order to describe $\widehat{\mathrm{O}_\varphi(p)}$. If $\mathrm{O}_\varphi(p)$ is finite then $\widehat{\mathrm{O}_\varphi(p)} = \mathrm{Gal}(\overline{\Bbbk}/\Bbbk) \cdot \mathrm{O}_\varphi(p)$. On the other hand, if $\overline{\mathrm{O}_\varphi(p)} = \mathbb{A}^2(\overline{\Bbbk})$, then $\widehat{\mathrm{O}_\varphi(p)}$ is the whole plane. Next proposition deal with the case when $\overline{\mathrm{O}_\varphi(p)}$ is a curve.

Proposition 4.3. *Let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and $p \in \mathbb{A}^2_{\overline{\Bbbk}}$ be such that $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}$ is a curve, and let $\mathcal{C} = \bigcup_{i=1}^{\ell} \mathcal{C}_i$ and $\widehat{\mathrm{O}_\varphi(p)} = \bigcup_{i=1}^{s} \mathcal{C}_i$ be the decomposition in irreducible components of $\mathcal{C}$ and $\widehat{\mathrm{O}_\varphi(p)}$ respectively (see Lemma 2.2). Then $s = k\ell$ and, up to reordering the indexes, we have that:*

(1) The morphism φ restricts to isomorphisms $\varphi|_{\mathcal{C}_{i+j\ell}} : \mathcal{C}_{i+j\ell} \to \mathcal{C}_{i+j\ell+1}$ and $\varphi|_{\mathcal{C}_{j\ell+\ell}} : \mathcal{C}_{j\ell+\ell} \to \mathcal{C}_{j\ell+1}$, for $i = 1, \dots, \ell-1$, $j = 0, \dots, k-1$.

(2) $\mathcal{C}_{i+j\ell} = \overline{\mathrm{O}_{\varphi^\ell}\big(\varphi^{i+j\ell-1}(p)\big)}$ for $i = 1, \dots, \ell$, $j = 0, \dots, k-1$.

(3) The curves $\mathcal{C}_i$ are isomorphic — in particular, the curves have the same type.

Proof. Without loss of generality we assume that $p \in \mathcal{C}_1$. We reorder $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ in such a way that $\mathcal{C}_i = \varphi^{i-1}(\mathcal{C}_1)$ for $i = 1, \dots, \ell$ (see Lemma 3.1); then $\mathcal{C}_1 =$

$\varphi(\mathcal{C}_\ell)$. It follows from Galois descent that

$$\widehat{\mathrm{O}_\varphi(p)} = \mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C} = \bigcup_{i=1}^{\ell}\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C}_i,$$

where if $i\neq j$, then $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C}_i\cap\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C}_j$ does not contain a curve. Since φ is fixed by the Galois action, it follows from Galois descent that if $i=1,\dots,\ell-1$, then $\varphi|_{\mathcal{C}_i}:\mathcal{C}_i\to\mathcal{C}_{i+1}$ induces an $\Bbbk$-isomorphism $\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C}_i\to\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)\cdot\mathcal{C}_{i+1}$. Assertion (1) follows; assertion (2) is clear.

In order to prove assertion (3), we first observe that $\mathcal{C}_{j\ell+1},\dots,\mathcal{C}_{j\ell+\ell}$ being isomorphic for $j=0,\dots k-1$, they have the same type. Up to reordering, we can assume that there exists $g_j\in\mathrm{Gal}(\overline{\Bbbk}/\Bbbk)$, $j=1,\dots,k-1$, such that $\mathcal{C}_{i+j\ell}=g_j\cdot\mathcal{C}_i$. Since $\psi\in\mathrm{Aut}\big(\mathbb{A}^2,g_j\mathcal{C}_1(\overline{\Bbbk})\big)$ if only if $\psi=g^{-1}\psi g\in\mathrm{Aut}\big(\mathbb{A}^2,\mathcal{C}_1(\overline{\Bbbk})\big)$, the result follows. □

Example 4.4. Suppose that $\Delta\subset\mathbb{A}^2(\overline{\Bbbk})$ is such that $\widehat{\Delta}$ is a fence of equation $F(x)=0$, with $F\in\Bbbk[x]$ — then it follows from [1, Theorem 1] that

$\mathrm{Aut}\big(\mathbb{A}^2,\widehat{\Delta}\big)=\big\{\big(\alpha x+\beta,\gamma y+P(x)\big) : \alpha,\beta,\gamma\in\Bbbk,\ F(\alpha x+\beta)/F(x)\in\Bbbk^*,\ P\in\Bbbk[x]\big\}.$

(a) If $F=x-1$, choose $\gamma\in\Bbbk^*$ which is not a root of the unity and let $P\in\Bbbk[x]$ be such that $P(1)=0$. Then $\varphi=\big(x,\gamma y+P(x)\big)\in\mathrm{Aut}\big(\mathbb{A}^2,\widehat{\Delta}\big)$ and if $p=(1,1)$, we have that $\mathrm{O}_\varphi(p)=\big\{(1,\gamma^n)\ n\in\mathbb{Z}\big\}\subset\Delta$ is infinite. It follows that $\overline{\mathrm{O}_\varphi(p)}=\widehat{\mathrm{O}_\varphi(p)}_{\overline{\Bbbk}}=\widehat{\Delta}$. Thus, in this case $\widehat{\mathrm{O}_\varphi(p)}_{\overline{\Bbbk}}$ is an irreducible curve, stable by φ — in the notations of Proposition 4.3, $\ell=s=1$. Recall that in this case the group $\mathrm{Aut}\big(\mathbb{A}^2,\mathrm{O}_\varphi(p)\big)$ is not algebraic.

(b) If $F=x^2-1$, then $\varphi=\big(\alpha x+\beta,\gamma y+P(x)\big)\in\mathrm{Aut}\big(\mathbb{A}^2,\widehat{\Delta}\big)$ if and only if $\alpha=\pm1$ and $\beta=0$. Choose $\gamma\in\Bbbk^*$ which is not a root of the unity, $d\geq1$, and consider the automorphism $\varphi_d=\big(-x,\gamma y+(x^2-1)^d\big)$. If $p=(1,y_0)$, with $y_0\neq0$, then $\mathrm{O}_{\varphi_d}(p)=\big\{\big((-1)^n,\gamma^n y_0\big);n\in\mathbb{Z}\big\}$ is dense in the fence $F(x)=0$. It follows that $\widehat{\mathrm{O}_\varphi(p)}_{\overline{\Bbbk}}=\overline{\mathrm{O}_\varphi(p)}=\widehat{\Delta}$, and therefore it is a curve with two irreducible components. Moreover, in the notations of Proposition 4.3, $\ell=s=2$ and once again $\mathrm{Aut}\big(\mathbb{A}^2,\mathrm{O}_\varphi(p)\big)$ is not algebraic because the orbit does not depend on $d=\deg\varphi_d$.

The following example, very similar to the ones given above shows that is not sufficient for Δ to be an orbit in order to guarantee $\overline{\Delta}$ and $\widehat{\Delta}$ to be equals.

Example 4.5. Assume that $\Bbbk\neq\overline{\Bbbk}$ and let $\xi\in\overline{\Bbbk}\setminus\Bbbk$. Denote by $P\in\Bbbk[x]$ the minimal polynomial of ξ over $\Bbbk$ and consider $\varphi=\big(x,\gamma y+P(x)\big)\in\mathrm{Aut}(\mathbb{A}^2)$, where $\gamma\in\Bbbk^*$ is not a root of unity. If $p=(\xi,1)$, then $\Delta=\mathrm{O}_\varphi(p)=\big\{(\xi,\gamma^n);n\in\mathbb{Z}\big\}$ is infinite. Clearly $\overline{\Delta}=\mathcal{V}_{\overline{\Bbbk}}(x-\xi)$ and $\widehat{\Delta}=\mathcal{V}_{\overline{\Bbbk}}(P)$ so $\Delta\subsetneq\overline{\Delta}\subsetneq\widehat{\Delta}$.

Notice that the automorphisms $(x,y)\mapsto\big(x,y+P(x)^d\big)$, $d\geq1$, fix every point in Δ and therefore they belong to $\mathrm{Aut}(\mathbb{A}^2,\Delta)$.

Remark 4.6. Let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and $p = (x_p, y_p) \in \mathbb{A}^2(\overline{\Bbbk})$ be such that $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_\ell$ is a curve. Then, by Proposition 4.3, up to reordering we can assume that $\mathcal{C}_i = \varphi^{i-1}\mathcal{C}_1$, for $i = 0, \dots, \ell - 1$, with $\varphi^\ell(\mathcal{C}_1) = \mathcal{C}_1$. Therefore

$$\mathcal{C}_1 = \overline{\mathrm{O}_{\varphi^\ell}(p)}.$$

Moreover, each curve $\mathcal{C}_i$ is of the same type and, up to conjugation by an automorphism of $\mathbb{A}^2$, we can assume that $\mathcal{C}_1$ is one of the list given in Theorem 2.8.

Since any automorphism $\psi \in \mathrm{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_\varphi(p)}\big)$ induces a permutation of the irreducible curves $\mathcal{C}_i$, we have the following exact sequence of abstract groups:

$$1 \longrightarrow \bigcap_{i=1}^{\ell} \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_i(\overline{\Bbbk})\big) \longrightarrow \mathrm{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_\varphi(p)}\big) \longrightarrow \mathfrak{S}_\ell$$

where $\mathfrak{S}_\ell$ is the group of permutations of ℓ elements. Since

$$\varphi^{n\ell} \in \mathrm{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_{\varphi^\ell}(\varphi^i(p))}\big),$$

if follows that $\bigcap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_i(\overline{\Bbbk})\big)$ is an infinite subgroup of $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)$.

Moreover, we have

$$\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_i(\overline{\Bbbk})\big) = \varphi^{i-1}\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\varphi^{1-i}$$

and if $\psi \in \mathrm{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_\varphi(p)}\big)$, then there exists $i \in \{0, \dots \ell - 1\}$ such that $\varphi^i\psi \in \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)$. It follows that $\psi \in \varphi^{-i}\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)$.

Notation 4.7. If G is a group and $A, B \subset G$ arbitrary subsets, we denote $AB = \{ab \,:\, a \in A\,,\ b \in B\}$.

Theorem 4.8. *Let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and $p = (x_p, y_p) \in \mathbb{A}^2(\overline{\Bbbk})$ be such that $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_\ell$ is a curve. Then up to applying an automorphism of $\mathbb{A}^2$ one of the following assertions holds:*

(a) $\mathcal{C}_1$ is of type (1) or (2) and $p \neq (0,0)$. In this case, $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) = \langle\varphi\rangle$.

(b) $\mathcal{C}_1$ is of type (3), (4) or (5). If we denote $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)_p = \{\mathrm{Id}, \tau_p\}$ (see Lemma 3.2), we have two possibilities:

(i) If $\tau_p\varphi = \varphi^i\tau_p$ for some $i \in \mathbb{Z}$, then

$$\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) = \langle\varphi\rangle \rtimes \{\mathrm{Id}, \tau_p\} = \langle\varphi\rangle \cup \langle\varphi\rangle\tau_p$$

(ii) In other case, $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) = \langle\varphi\rangle$.

(c) $\mathcal{C}_1$ is of type (6). In this case

$$\begin{aligned}\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) &= \bigcap_{i=0}^{\ell-1} \langle\varphi\rangle\Big(\varphi^i\Big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\Big)\varphi^{-i}\Big)\\ &= \bigcap_{i=0}^{\ell-1} \langle\varphi\rangle\Big(\Big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\Big)\varphi^{-i}\Big)\\ &= \bigcap_{i=0}^{\ell-1} \langle\varphi\rangle\Big(\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\varphi^{-i}\Big),\end{aligned}$$

where $G = \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)$.

Moreover, if $\mathbb{k}$ is uncountable, there exist infinitely many automorphisms $\varphi \in G$ such that

$$\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) = \bigcap_{i=0}^{\ell-1} \langle\varphi\rangle\big(\varphi^i \operatorname{Ker}(R)\varphi^{-i}\big) = \bigcap_{i=0}^{\ell-1} \langle\varphi\rangle\big(\operatorname{Ker}(R)\varphi^{-i}\big),$$

where $R : \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big) \to \operatorname{Aut}\big(\mathcal{C}_1(\overline{\mathbb{k}})\big)$ *is the restriction morphism (see* 2.8*).*

Proof. We use the notations of Remark 3.3 and Theorem 3.4. By Proposition 4.3, we have that $\{\varphi^n\}_{n\in\mathbb{Z}}$ is infinite and $\mathcal{C}_1$ is of type (1) – (6). Hence, it remains to describe $\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$ in each case. We assume that $\mathcal{C}_i = \varphi^{i-1}(\mathcal{C}_1)$, and $\varphi^\ell \in \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)$ (see Remark 4.6). Since φ^ℓ cannot be an involution we know it belongs to the torus $T \subset \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)$.

(a) If $\mathcal{C}_1 = \mathcal{V}(x^b - y^a)$ then, since $\varphi^\ell \in \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)$, we have that $\varphi^\ell(0,0) = (0,0)$ and therefore $p \neq (0,0)$. If $\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$, then by Remark 4.6 $\psi = \varphi^j t$ for some $t \in \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)$ and some $j \in \mathbb{Z}$. But if $n \in \mathbb{Z}$, then there exists s_n such that $\varphi^j t \varphi^{n\ell}(p) = \varphi^{s_n}(p)$. It follows that $\varphi^{s_n - j}(p) = t\varphi^{n\ell}(p) \in \mathcal{C}_1$, and therefore $s_n - j$ is a multiple of ℓ and $t \in \operatorname{Aut}(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p))$. By Theorem 3.4, $\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big) = \langle\varphi^\ell\rangle$ and therefore $\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) = \langle\varphi\rangle$.

If $\mathcal{C}_1$ is of type (2) an analogous argument applies.

(b) We proceed as in the previous case and we apply Theorem 3.4 to $H = \langle\varphi^\ell\rangle$; it follows that $\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big) = \langle\varphi^\ell\rangle \rtimes \{\mathrm{Id}, \tau_p\}$.

If $\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$ then there exist $j \in \mathbb{Z}$ and $t \in \langle\varphi^\ell\rangle \rtimes \{\mathrm{Id}, \tau_p\}$ such that $\psi = \varphi^i t$. Now, if $\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) \setminus \langle\varphi\rangle$, then there exists $n \in \mathbb{Z}$ such that $t = \varphi^{n\ell}\tau_p$, so $\tau_p \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) \subset \operatorname{Aut}\big(\mathbb{A}^2, \overline{\mathrm{O}_\varphi(p)}\big)$. Since an automorphism of $\overline{\mathrm{O}_\varphi(p)}$ sends irreducible components into irreducible components, then $\tau_p(\mathcal{C}_1) = \mathcal{C}_j$ for some $j \in Z$. It follows that there exists $i \in$

$\mathbb{Z}$ such that $\tau_p\big(\varphi(p)\big) = \varphi^{i\ell}(p)$, and therefore $\varphi^{-i\ell}\tau_p\varphi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\mathbb{k}})\big)_p = \{\operatorname{Id}, \tau_p\}$ (see Lemma 3.2). But if $\varphi^{-i\ell}\tau_p\varphi = \operatorname{Id}$ then φ is an involution, so the assertion is proved.

(c) Assume now that $\mathcal{C}_1 = \mathcal{V}(x)$. Since $\mathcal{C}_1 \cap \mathcal{C}_i$ is finite and φ^ℓ-stable for all $i = 2, \dots, \ell$, it follows that $p \notin \mathcal{C}_1 \cap \mathcal{C}_i$, because $\mathrm{O}_{\varphi^\ell}(p)$ is infinite — more in general, $\mathcal{C}_i \cap \mathcal{C}_j \cap \mathrm{O}_\varphi(p) = \emptyset$. Moreover, we deduce from Theorem 2.8 that $\varphi^\ell = \big(ax, by + P(x)\big)$, where $a, b \in \mathbb{k}^*$ and $P \in \mathbb{k}[x]$ — notice that b is not a root of unity different from 1, since $\varphi^{n\ell}(p) = \big(0, b^n y_p + (1 + \cdots + b^{n-1})P(0)\big)$, and that if $\operatorname{char}(\mathbb{k}) \neq 0$, then also $b \neq 1$.

If $\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$, then $\psi(p) = \varphi^r(p)$ for some r, and therefore $\varphi^{-r}\psi(p) = p \in \mathcal{C}_1$. Since $\varphi^{-r}\psi(\mathcal{C}) = \mathcal{C}$, it follows that $\varphi^{-r}\psi(\mathcal{C}_1) = \mathcal{C}_1$. Hence, in the notations of Theorem 2.8 and Lemma 3.2,

$$\begin{aligned}\varphi^{-r}\psi \in G_p = \Big\{\big(a'x, cy + S(x)\big) : a', c \in \mathbb{k}^*, \ S \in \mathbb{k}[x], S(0) = -cy_p\Big\} \cong \\ \operatorname{Ker}(R) \rtimes \big\{(0, y) \mapsto (0, cy + (1-c)y_p) : c \in \mathbb{k}^*\big\}\end{aligned}$$

Moreover, since $\mathrm{O}_{\varphi^\ell}(p) = \mathrm{O}_\varphi(p) \cap \mathcal{C}_1$, we have $\varphi^{-r}\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)$ and therefore

$$\psi \in \langle\varphi\rangle\big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\big) = \big\{\varphi^j\gamma : j \in \mathbb{Z}, \ \gamma \in G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\big\}.$$

Analogously, since ψ stabilizes $\mathrm{O}_\varphi(p)$, given $i = 1, \dots, \ell - 1$, there exists $j_i \in \mathbb{Z}$ such that $\psi\big(\varphi^i(p)\big) = \varphi^{j_i}(p)$. Hence, $\varphi^{i-j_i}\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_i(\overline{\mathbb{k}})\big)_{\varphi^i(p)}$, and

$$\begin{aligned}\psi \in &\langle\varphi\rangle\Big(\operatorname{Aut}\big(\mathbb{A}^2, \mathcal{C}_{i+1}(\overline{\mathbb{k}})\big)_{\varphi^i(p)} \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(\varphi^i(p))\big)\Big) = \\ &\langle\varphi\rangle\Big(\big(\varphi^i G \varphi^{-i}\big)_{\varphi^i(p)} \cap \big(\varphi^i \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\varphi^{-i}\big)\Big) = \\ &\langle\varphi\rangle\Big(\varphi^i G_p \varphi^{-i} \cap \big(\varphi^i \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\varphi^{-i}\big)\Big) = \\ &\langle\varphi\rangle\Big(\varphi^i\Big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\Big)\varphi^{-i}\Big).\end{aligned}$$

All in all, we have that

$$\begin{aligned}\psi &\in \bigcap_{i=0}^{\ell-1}\langle\varphi\rangle\Big(\varphi^i\Big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\Big)\varphi^{-i}\Big) \\ &= \bigcap_{i=0}^{\ell-1}\langle\varphi\rangle\Big(\Big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\Big)\varphi^{-i}\Big).\end{aligned}$$

Let $\psi \in \bigcap_{i=0}^{\ell-1}\langle\varphi\rangle\big(\varphi^i\big(G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)\big)\varphi^{-i}\big)$; in order to prove the first assertion of (c) we need to prove that $\psi \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$. If $j \in \mathbb{Z}$, let

$j = q\ell + r$ with $0 \le r < \ell$ and consider the decomposition $\psi = \varphi^r \gamma \varphi^{-r}$, with $\gamma \in G_p \cap \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)$. Then

$$\psi(\varphi^j(p)) = \varphi^r \gamma \varphi^{-r+j}(p) = \varphi^r \gamma \varphi^{q\ell}(p).$$

Since $\gamma \in \operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big)$, the assertion follows.

In order to prove the last assertion, we first observe that, up to conjugate with the translation of vector $(0, y_p)$, we can assume that $p = (0,0)$. Let $\psi = \big(a'x, b'y + Q(x)\big) \in \bigcap_{i=0}^{\ell-1} \varphi^i G_p \varphi^{-i} \setminus \operatorname{Ker}(R)$. Then $Q(0) = 0$ and, since $\psi \in \operatorname{Aut}(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p))$, there exists $j \in \mathbb{Z}$ such that

$$\big(0, b'P(0)\big) = \psi\big(0, P(0)\big) = \psi\big(\varphi^\ell(p)\big) = \varphi^{j\ell}(p).$$

If $j = 1$, then $\psi = \big(a'x, y + Q(x)\big) \in \operatorname{Ker}(R)$. Thus, up to work with ψ^{-1} if necessary, we can assume that $j > 1$. We deduce that $b' = 1 + \cdots + b^{j-1}$; in particular φ is such that $1 \neq 1 + \cdots + b^{j-1}$ — that is, $b \neq 0$ is not a $(j-1)$-root of unity.

Analogously, we have that $\psi\big(0, (1+b)P(0)\big) = \psi\big(\varphi^{2\ell}(p)\big) = \varphi^{m\ell}(p)$, with $m \neq 2$. Since $\varphi^{-\ell} = \big(a^{-1}x, b^{-1}y - b^{-1}P(x)\big)$, we deduce that

$$\begin{aligned}\big(0, (1 + \cdots + b^{j-1})(1+b)P(0)\big) &= \psi\big(\varphi^{2\ell}(p)\big)\\ &= \begin{cases}\big(0, (1 + \cdots + b^{m-1})P(0)\big), & \text{if } m > 2\\ \big(0, P(0)\big), & \text{if } m = 1\\ (0,0), & \text{if } m = 0\\ (0, -b^{-m}(1 + \cdots + b^{m})P(0)), & \text{if } m < 0\end{cases}\end{aligned}$$

For all four cases above we deduce a polynomial condition on b. Since there are countably many such conditions, each of them with finite solutions, we deduce that if $\Bbbk$ is uncountable there exist infinite values of b for which no such a ψ can be found. □

Remark 4.9. Notice that in the proof of the last assertion of Theorem 4.8(c), we omitted additional conditions on b that must be satisfied in order to allow the existence of $\psi \in \bigcap_{i=0}^{\ell-1} \varphi^i G_p \varphi^{-1} \setminus \operatorname{Ker}(R)$; namely, the conditions arising from the equations $\psi\big(\varphi^{j\ell}(p)\big) \in \mathrm{O}_{\varphi^\ell}(p)$, for $j = 3, \dots, \ell - 1$. However, Example 3.5 shows that there exists an automorphism $\varphi \in \operatorname{Aut}\big(\mathbb{A}^2, \{x = 0\}\big)$ such that $\operatorname{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) \neq \langle \varphi \rangle \big(\bigcap_{i=0}^{\ell-1} \varphi^i \operatorname{Ker}(R) \varphi^{-i}\big)$, therefore, these conditions may be satisfied in particular examples.

We finish by giving some examples that illustrate Theorem 4.8(c), and by proving that in this case $\operatorname{Aut}\big(\mathrm{O}_\varphi(p)\big)$ is not algebraic when $\Bbbk$ is uncountable — the countable case remains open.

Example 4.10. (1) If φ and p are such that $\mathcal{C} = \overline{\mathrm{O}_\varphi(p)}$ is a fence, then we can assume that $p = (0,0)$ and $\mathcal{C}$ is contained in the curve $F(x) = 0$ for some $F \in \overline{\Bbbk}[x]$. Therefore,

$$\big\{(ax, y + Q(x)F(x)), a \in \Bbbk^*, Q \in \Bbbk[x]\big\} \subset \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$$

and we deduce that A is not algebraic.

(2) Let $\Bbbk$ be an infinite field and $a \in \Bbbk$ such that a is not a root of unity. Consider $\varphi = (ay, ax)$ and let $p = (0,1)$. Then

$$\mathrm{O}_\varphi(p) = \big\{(a^{2n+1}, 0), (0, a^{2n}) \,:\, n \in \mathbb{Z}\big\}, \text{ and } \overline{\mathrm{O}_\varphi(p)} = \{xy = 0\}.$$

Notice that $\mathcal{C} = \{xy = 0\}$ is a union of curves of type (6) but it is not a fence. It is easy to see that $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big) = \big\{(ax, by), (ay, bx) \,:\, a, b \in \Bbbk^*\big\} \cong (\Bbbk^*)^2 \rtimes \mathbb{Z}_2$ and that

$$\begin{aligned}\mathrm{Aut}\big(\mathrm{O}_\varphi(0,1)\big) &= \big\{(a^{2n}x, a^{2m}y), (a^{2n}y, a^{2m}x) \,:\, n, m \in \mathbb{Z}\big\}\\ &\cong \mathbb{Z}^2 \rtimes \mathbb{Z}_2 \subsetneq \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big).\end{aligned}$$

Hence, $\mathrm{Aut}\big(\mathrm{O}_\varphi(0,1)\big)$ strictly contains $\langle\varphi\rangle$ and clearly is not an algebraic group.

(3) Let $\Bbbk = \mathbb{C}$ and consider $\varphi = \frac{a}{\sqrt{2}}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)$, the rotation of angle $\pi/4$ followed by a homothety of ratio a, where a is not a root of unity.

If $p = (0,1)$, then

$$\mathrm{O}_\varphi(p) = \left\{\begin{matrix}a^{8n}(0,1), \frac{1}{\sqrt{2}}a^{8n+1}(-1,1), a^{8n+2}(-1,0), a^{8n+3}\frac{1}{\sqrt{2}}(-1,-1),\\ a^{8n+4}(0,-1), a^{8n+5}\frac{1}{\sqrt{2}}(1,-1), a^{8n+6}(1,0), a^{8n+7}\frac{1}{\sqrt{2}}(1,1)\end{matrix} \,:\, n \in \mathbb{Z}\right\},$$

and $\mathcal{C} = \overline{\mathrm{O}_\varphi(p)} = \{xy(x+y)(x-y) = 0\}$. An easy calculation shows that

$$\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big) = \left\langle \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \,:\, a, b \in \Bbbk^* \right\rangle.$$

We deduce that if $\psi \in \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(0,1)\big)$, then ψ is a power of φ and $\mathrm{Aut}\big(\mathrm{O}_\varphi(p)\big) = \langle\varphi\rangle$.

Proposition 4.11. *Assume that $\Bbbk$ is uncountable and let $\varphi \in \mathrm{Aut}(\mathbb{A}^2)$ and $p = (x_p, y_p) \in \mathbb{A}^2(\overline{\Bbbk})$ be such that $\overline{\mathrm{O}_\varphi(p)} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_\ell$ is a curve, with $\mathcal{C}_i$ of type (6) (that is φ and p verify assertion (c) of Theorem* 4.8*). Then $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$ is not algebraic.*

Proof. If $\mathcal{C}$ is, up to an automorphism, a fence, the result is proved in Example 4.10(1).

Hence, we may assume that $\mathcal{C}$ is not a fence, and therefore $\mathrm{Aut}\big(\mathbb{A}^1, \mathcal{C}(\overline{\Bbbk})\big)$ is algebraic by [1, Theorem 1]. Up to conjugation, we may assume that $\mathcal{C}_1 = \{x = 0\}$ and $p = (0,1)$. Then $\varphi^\ell = \big(\alpha x, \beta y + Q(x)\big)$ for some $Q \in \Bbbk[x]$, and it follows from Theorem 3.4 that

$$A := \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_{\varphi^\ell}(p)\big) = \langle\varphi^\ell\rangle(G_p \cap A) \subset \langle\varphi^\ell\rangle\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big).$$

Since φ^ℓ does not fix p, we deduce that $\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big) \bigcap \varphi^\ell\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big) = \emptyset$. Analogously, we deduce that if $1 < i < \ell$, then $\varphi^i\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big) \cap \langle\varphi^\ell\rangle\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big) = \emptyset$.

Since $G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)$ is closed in $\mathrm{Aut}(\mathbb{A}^2)$ and $\mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)$ is algebraic, we have that

$$\begin{aligned} K &:= \langle\varphi\rangle\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big)\big) \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big) \\ &= \bigcup_{j\in\mathbb{Z}} \varphi^j\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big) \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)\big) \end{aligned}$$

is a countable disjoint union of closed subsets of the algebraic group $\mathrm{Aut}\big(\mathbb{A}??, \mathcal{C}(\overline{\Bbbk})\big)$.

On the other hand, we deduce from Theorem 4.8 that $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$ is contained in K and, since $\varphi^j \in \mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big) \cap \varphi^j\big(G_p \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}_1(\overline{\Bbbk})\big) \cap \mathrm{Aut}\big(\mathbb{A}^2, \mathcal{C}(\overline{\Bbbk})\big)\big)$ for any j, we conclude that $\mathrm{Aut}\big(\mathbb{A}^2, \mathrm{O}_\varphi(p)\big)$ has at least a countable number of irreducible components, and therefore it is not algebraic. □

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(Alvaro Rittatore) Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, Montevideo, Uruguay
alvaro@cmat.edu.uy

(Iván Pan) Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, Montevideo, Uruguay
ivan@cmat.edu.uy

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