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A note on rationally slice knots

Adam Simon Levine

ABSTRACT. Kawauchi proved that every strongly negative amphichiral knot $K \subset S^3$ bounds a smoothly embedded disk in some rational homology ball V_K, whose construction *a priori* depends on K. We show that V_K is independent of K up to diffeomorphism. Thus, a single 4-manifold, along with connected sums thereof, accounts for all known examples of knots that are rationally slice but not slice.

CONTENTS

1. Introduction

Let K be a knot in S^3. If X is a smooth, compact, oriented 4-manifold with boundary S^3, we say that K is *slice in* X if there exists a smoothly embedded disk D in X with boundary equal to K. Note that if K is slice in X, then so is any knot that is smoothly concordant to K.

For a commutative ring R with unit, we say that K is R-*slice* if it is slice in some 4-manifold X that is an R-homology 4-ball. We will focus on the cases of $R = \mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}_p$ (for p prime). Note that a $\mathbb{Z}_p$-homology 4-ball X is the same as a $\mathbb{Q}$-homology 4-ball with the additional property that $|H_1(X;\mathbb{Z})|$ is not divisible by p. We use *rationally slice* as a synonym for $\mathbb{Q}$-slice.[1]

By a slight abuse of notation, if Z is a closed 4-manifold and K is slice in $Z - B^4$, we also say that K is slice in Z. If $X = Z - B^4$, then X is an R-homology 4-ball if and only if Z is an R-homology 4-sphere.

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[1] Some authors, e.g. Kawauchi [Kaw09], impose an additional homological constraint in the definition of *rationally slice*, and use *weakly rationally slice* for the definition we are using. Our terminology agrees with that of other recent papers on the subject, e.g. [KW18, HKPS22], which use *strongly rationally slice* for Kawauchi's version.

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Let $\mathcal{C}$ denote the smooth concordance group, and let $\mathcal{K}_R$ denote the subgroup of $\mathcal{C}$ consisting of concordance classes of knots that are R-slice. In other words, $\mathcal{K}_R$ is the kernel of the forgetful map $\mathcal{C} \to \mathcal{C}_R$, where $\mathcal{C}_R$ is the group of knots in S^3 up to concordance in R-homology cobordisms.

It remains an open question whether there exist knots that are $\mathbb{Z}$-slice but not slice, i.e. whether $\mathcal{K}_{\mathbb{Z}} \neq 0$. In contrast, it is well-known that there exist knots that are $\mathbb{Q}$-slice but not slice (or even $\mathbb{Z}$-slice), such as the figure-eight knot. Specifically, a knot $K \subset S^3$ is called *strongly negative amphichiral* if there exists an orientation-reversing involution $\phi\colon\ S^3 \to S^3$ preserving K setwise and having exactly two fixed points, both lying on K. Following terminology of Keegan Boyle, we refer to a **s**trongly **n**egative **a**mphi**c**hiral **k**not as a SNACK. Note that every SNACK represents a class of order at most 2 in $\mathcal{C}$. Kawauchi [Kaw80, Kaw09] showed that every SNACK is $\mathbb{Q}$-slice; more precisely, he proved that every SNACK K is slice in a certain rational homology 4-ball V_K, whose construction *a priori* depends on K.

The main theorem of this note is that V_K is in fact independent of K up to diffeomorphism; that is, all SNACKs are slice in the same rational homology 4-ball. We may describe the manifold explicitly as follows. Let $\tau\colon\ S^2 \times S^2 \to S^2 \times S^2$ be the map $\tau(x, y) = (r(x), -y)$, where $r\colon\ S^2 \to S^2$ is a reflection. This map is an orientation-preserving involution with no fixed points, so the quotient $Z_0 = S^2 \times S^2/\tau$ is a closed, orientable manifold. Some elementary algebraic topology (see Lemma 2.3 below) shows that Z_0 is a rational homology 4-sphere with $\pi_1(Z_0) \cong H_1(Z_0) \cong H_2(Z_0) \cong \mathbb{Z}_2$. Thus, for every odd prime p, Z_0 is a $\mathbb{Z}_p$-homology sphere. Note that the map $(x, y) \mapsto (x, -y)$ induces an orientation-reversing involution on Z_0.

In Section 2, we will prove:

Theorem 1.1. *For every strongly negative amphichiral knot $K \subset S^3$, Kawauchi's manifold V_K is diffeomorphic to $Z_0 - B^4$. Thus, every SNACK is slice in $Z_0 - B^4$.*

Remark 1.2. For another characterization of Z_0, consider the map $q\colon\ S^2 \times S^2 \to \mathbb{R}\mathrm{P}^2$ taking (x, y) to the class of y. Then $q \circ \tau = q$, so q descends to a map $\bar{q}\colon\ Z_0 \to \mathbb{R}\mathrm{P}^2$, which gives Z_0 the structure of an S^2-bundle over $\mathbb{R}\mathrm{P}^2$. If $x \in S^2$ is any fixed point of the reflection r, we obtain a section $\sigma_x\colon\ \mathbb{R}\mathrm{P}^2 \to Z_0$ by defining $\sigma_x([y]) = [(x, y)]$ for each $y \in S^2$. Since the fixed-point set of r is a circle, we in fact find a 1-dimensional family of nearby disjoint sections. The manifold Z_0 is thus characterized by being the unique S^2 bundle over $\mathbb{R}\mathrm{P}^2$ with orientable total space and a section of self-intersection 0. (See [Hil02, p. 237] for further discussion of S^2-bundles over $\mathbb{R}\mathrm{P}^2$.)

We claim that Z_0 is represented by the handle diagram in Figure 1 (using dotted 1-handle notation). As seen in [GS99, Figure 6.2], the 0-handle, 1-handle, and 2-framed 2-handle from the figure produce the D^2-bundle over $\mathbb{R}\mathrm{P}^2$ with orientable total space and Euler number 0. The double of that D^2-bundle is the S^2-bundle described above, which is Z_0. We obtain the double by adding a 0-framed 2-handle along the meridian of the first 2-handle, and then a 3-handle and 4-handle, which yields Figure 1.

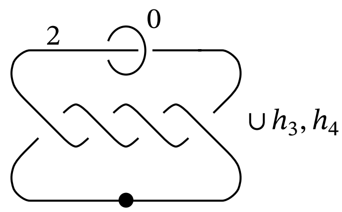

FIGURE 1. Kirby diagram for Z_0.

Before we turn to the proof of Theorem 1.1, we discuss its implications for the study of rationally slice knots, albeit with more questions than answers. Surprisingly, Kawauchi's construction actually accounts for all known examples of knots that are $\mathbb{Q}$-slice but not slice, that is, all known nontrivial elements of $\mathcal{K}_{\mathbb{Q}}$. We make this explicit as follows.

First, note that if K and K' are knots, and if K is slice in a 4-manifold X and K' is slice in X', then $K \# K'$ is slice in $X \natural X'$, and $-K$ is slice in $\overline{X}$ (i.e. X with reversed orientation). Let $\mathcal{S}$ denote the set of concordance classes of knots that are slice in $\natural n(Z_0 - B^4)$ for some $n \in \mathbb{N}$ (or equivalently in $\# nZ_0$). Because $Z_0 \cong \overline{Z_0}$ as oriented manifolds, we thus see that $\mathcal{S}$ is a subgroup of $\mathcal{C}$ and is contained in $\mathcal{K}_{\mathbb{Q}}$. Indeed, for every odd prime p, we have $\mathcal{S} \subset \mathcal{K}_{\mathbb{Z}_p}$.

For any knots $P \subset S^1 \times D^2$ and $K \subset S^3$, let $P(K)$ denote the satellite knot with pattern P and companion K (i.e. the image of P under the embedding $S^1 \times D^2 \to S^3$ determined by the 0-framing of K). The operation $K \mapsto P(K)$ descends to a function on $\mathcal{C}$. If $P(O)$ is slice (where O denotes the unknot[2]), we call P a *slice pattern*, and the operation $K \mapsto P(K)$ a *slice satellite operation*. If a knot K is slice in a particular 4-manifold X, then so is $P(K)$ for any slice pattern P; thus, the subgroups $\mathcal{K}_R$ (for any ring R) and $\mathcal{S}$ are closed under slice satellite operations.

To the author's knowledge, the only known concordance classes of knots that are rationally slice but not slice (that is, nontrivial elements of $\mathcal{K}_{\mathbb{Q}}$) arise from Kawauchi's construction, together with taking iterated slice satellite operations and/or connected sums, and thus they lie in $\mathcal{S}$. We note several such constructions in the literature:

- Cha [Cha07, Theorem 4.16] exhibited an family of SNACKs that generate a $\mathbb{Z}_2^\infty$ subgroup of $\mathcal{K}_{\mathbb{Q}}$. These knots can be distinguished up to concordance by their classes in the algebraic concordance group [Lev69]. Subsequently, Hedden, Kim, and Livingston [HKL16] found another such family of SNACKs with the additional property of being topologically slice (and hence algebraically slice). The proof that these knots fail to be slice relies on the Heegaard Floer d invariants [OS03a] of the knots' branched double covers.

[2] Many authors use U to denote the unknot, but we prefer O because of the obvious graphical similarity.

- For a knot K and relatively prime integers m, n, let $K_{m,n}$ denote the (m, n) cable of K (where m denotes the winding number in the longitudinal direction and n in the meridional direction). Then for any $m \in \mathbb{Z}$, the operation $K \mapsto K_{m,1}$ is a slice satellite operation. Let F denote the figure-eight knot, which is strongly negative amphichiral and hence slice in Z_0. Hom, Kang, Park, and Stoffregen [HKPS22] proved that the set of knots

$$\{F_{2n-1,1} \mid n \geq 2\}$$

is linearly independent in $\mathcal{C}$, and thus generates a $\mathbb{Z}^\infty$ subgroup of $\mathcal{C}$ contained in $\mathcal{K}_\mathbb{Q}$ (and indeed in $\mathcal{S}$).[3] This result provided the first known non-torsion elements of $\mathcal{K}_\mathbb{Q}$. For linear combinations consisting of more than one summand, the resulting knot is slice in some connected sum of copies of Z_0, but *a priori* not necessarily slice in Z_0 itself. The proof makes use of concordance invariants coming from involutive knot Floer homology [HM17].

  More recently, Dai, Kang, Mallick, Park, and Stoffregen [DKM+22], answering a long-standing question of Kawauchi [Kaw80], proved that $F_{2,1}$ is not slice (and indeed generates a $\mathbb{Z}$ subgroup of $\mathcal{S}$). This proof relies on using the involutive structure of the Heegaard Floer homology of the branched double cover and the action of the deck transformation.
- Kawauchi's result applies only to strongly negative amphichiral knots, but not necessarily to knots that are merely negative amphichiral (isotopic to their mirror reverses). However, Kim and Wu [KW18] proved that if K is a fibered, negative amphichiral knot whose Alexander polynomial is irreducible, then K is necessarily obtained from a SNACK by iterated slice satellite operations, and hence is rationally slice by Kawauchi's result. Again, any such knot must lie in $\mathcal{S}$.

In some sense, Theorem 1.1 illustrates how little is known about rational concordance: a single 4-manifold (along with connected sums of copies thereof) accounts for all known examples of knots that are rationally slice but not slice. That is, the following question is open:

Question 1.3. Is $\mathcal{K}_\mathbb{Q} = \mathcal{S}$? That is, is every rationally slice knot slice in a connected sum of copies of Z_0?

To try to answer Question 1.3 in the negative, it is instructive to consider not only $\mathbb{Q}$-concordance but also $\mathbb{Z}_p$-concordance. By the above discussion, all known elements of $\mathcal{K}_\mathbb{Q}$ are contained in $\mathcal{K}_{\mathbb{Z}_p}$ for every odd prime p. In contrast, the following question remains open:

Question 1.4. Is $\mathcal{K}_{\mathbb{Z}_2} \neq 0$? That is, does there exist a knot $K \subset S^3$ that is slice in a $\mathbb{Z}_2$-homology ball but not slice?

[3] The result is stated in [HKPS22] for $F_{2n-1,-1}$, but note that $F_{2n-1,-1}$ is the mirror of $F_{2n-1,1}$.

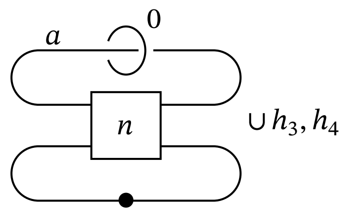

FIGURE 2. Kirby diagram for $X_{n,a}$. The box indicates n full positive twists.

In some sense, this question is nearly as difficult as that of the better-known problem of finding nontrivial elements of $\mathcal{K}_{\mathbb{Z}}$. A large number of knot invariants, including Heegaard Floer invariants such as τ [OS03b] and Υ [OSS17], necessarily vanish for all rationally slice knots. Most crucially, even the invariants used in the above-mentioned results, which can detect some nontrivial elements of $\mathcal{K}_{\mathbb{Q}}$, are unable to obstruct a knot from being slice in a $\mathbb{Z}_2$-homology 4-ball. Namely, if a knot K is $\mathbb{Z}_2$-slice, then:

- it is algebraically slice [CLR08, Theorem 3];
- the slice obstructions from involutive knot Floer homology vanish [HKPS22, Remark 1.8]; and
- the branched double cover of K bounds an equivariant $\mathbb{Z}_2$-homology ball, and hence the obstructions from d invariants and involutive Floer homology vanish [DKM+22, Remark 5.4].

It remains unknown whether Rasmussen's s invariant [Ras10] (or any of its generalizations) vanishes for all rationally slice knots.

Nevertheless, here is one potential approach to Questions 1.3 and 1.4. First, recall that if a knot K is slice in a $\mathbb{Z}_p$-homology ball X, then for any power p^k, the p^k-fold cyclic branched cover of X branched over the slice disk is again a $\mathbb{Z}_p$-homology ball whose boundary is $\Sigma_p(K)$. (On the other hand, if $H_1(X;\mathbb{Z}_p) \neq 0$, then this covering may not be a rational homology ball.) Thus, suppose one can find a knot K that is slice in a $\mathbb{Z}_2$-homology 4-ball X that is not an integer homology ball, and choose any odd prime p dividing $|H_1(X;\mathbb{Z})|$. If one can show that $\Sigma_{p^k}(K)$ does not bound any rational homology ball (using, say, d invariants), it then follows that K cannot be $\mathbb{Z}_p$-slice, and in particular it cannot be in $\mathcal{S}$. This would thus resolve both Question 1.3 (in the negative) and Question 1.4 (in the affirmative).

Klug and Ruppik [KR21, Corollary 2.5] proved that if X is a closed 4-manifold whose universal cover $\tilde{X}$ is $\mathbb{R}^4$ or S^4, then any knot that is slice in X is slice. However, this is not an issue if X is a rational homology 4-sphere with finite (nontrivial) fundamental group; a simple Euler characteristic argument shows that the universal cover must have nontrivial H_2. For instance, for any $n \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $X_{n,a}$ denote the closed 4-manifold indicated by the handle diagram in Figure 2, generalizing $Z_0 = X_{2,0}$. It is easy to verify that $X_{n,a}$ is a rational homology 4-sphere with $\pi_1(X_{n,a}) \cong H_1(X_{n,a}) \cong \mathbb{Z}/n$, essentially the simplest construction of a manifold with those properties. The diffeomorphism type of $X_{n,a}$ depends only on n and the parity of a. Thus, it is natural to ask a more concrete version of Question 1.4:

Question 1.5. For $n > 2$ and $a \in \{0, 1\}$, does there exist a non-slice knot $K \subset S^3$ that is slice in $X_{n,a} - B^4$?

We invite the reader to find a knot with the needed properties.

2. Proof of Theorem 1.1

Throughout this section, let $K \subset S^3$ be a SNACK. Up to equivariant isotopy, we may assume that K is fixed setwise by the map $\phi\colon S^3 \to S^3$ that is the restriction to S^3 of the linear involution $\Phi\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $\Phi(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, -x_4)$.

Let $r\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection $r(x_1, x_2) = (x_1, -x_2)$; we also denote its restrictions to D^2 and S^1 by the same symbol. Let $\psi_K\colon S^1 \to S^3$ denote the inclusion of K, chosen to be equivariant with respect to the involutions r on S^1 and ϕ on S^3. (In particular, ψ_K takes $(\pm 1, 0)$ to the two fixed points of ϕ.) By the equivariant tubular neighborhood theorem (see, e.g., [Kan07, Theorem 4.4]), we may extend ψ_K to an embedding $\Psi_K\colon S^1 \times D^2 \hookrightarrow S^3$ that parametrizes an equivariant closed tubular neighborhood of K, with the following properties:

- Ψ_K restricts to ψ_K on $S^1 \times \{\vec{0}\}$.
- For any $(x, y) \in S^1 \times D^2$, we have $\phi \circ \Psi_K(x, y) = \Psi_K(r(x), -y)$.
- Ψ_K determines the 0-framing of K; that is, for any nonzero $y \in D^2$, $\psi_K(S^1 \times \{y\})$ has linking number 0 with K.

Let X_K denote the 0-trace of K, obtained by attaching a 0-framed 2-handle $D^2 \times D^2$ to D^4 using the attaching map Ψ_K. This manifold acquires an orientation from that of D^4. The boundary of X_K is the 0-surgery $S^3_0(K)$. The involution $\Phi|_{D^4}$ extends to an orientation-reversing involution $\Phi_K\colon X_K \to X_K$, defined on the 2-handle $D^2 \times D^2$ by $\Phi_K(x, y) = (r(x), -y)$. The fixed point set of Φ_K is a circle, consisting of the arcs $[-1, 1] \times \{\vec{0}\} \subset D^4$ and $([-1, 1] \times \{0\}) \times \{\vec{0}\} \subset D^2 \times D^2$. In particular, observe that Φ_K restricts to a fixed-point-free, orientation-reversing involution of $S^3_0(K)$, which we denote by ϕ_K.

We now describe Kawauchi's construction (in slightly different terms). Let Z_K denote the quotient $X_K/\sim$, where for all $x \in S^3_0(K)$, we set $x \sim \phi_K(x)$. Let $\pi\colon X_K \to Z_K$ denote the quotient map. That is, we obtain Z_K by a self-gluing of the boundary of X_K. Because ϕ_K has no fixed points, Z_K is a smooth, closed 4-manifold, and because ϕ_K is orientation-reversing, Z_K naturally acquires an orientation from that of X_K.

Lemma 2.1. *The knot K is slice in Z_K.*

Proof. Let X'_K be the union of X_K with an exterior collar $S^3_0(K) \times [0, 1]$, attached along $S^3_0(K) \times \{1\}$, and let Z'_K be the quotient of X'_K by self-gluing by ϕ_K along $S^3_0(K) \times \{0\}$. Then clearly $X'_K \cong X_K$ and $Z'_K \cong Z_K$. Let $B \subset Z'_K$ denote the 0-handle of X_K, which is still an embedded closed 4-ball even after the gluing thanks to the collar. Let $V_K = Z'_K - \mathrm{int}(B)$; there is a natural identification $\partial V_K = S^3$. Then K bounds an embedded disk in V_K, namely the core of the 2-handle of X_K. Thus, K is slice in Z_K. □

Remark 2.2. In the paper [Kaw09], Kawauchi considers the more general case of a strongly negative amphichiral knot K in an arbitrary rational homology sphere Y, not just in S^3. He first considers $Y_0(K) \times [0,1]/\sim$, where $(x,0) \sim (\phi_K(x), 0)$, and proves that this is a rational homology $S^1 \times D^3$ bounded by $Y_0(K)$. Adding a 2-handle along the meridian of K then produces a rational homology ball bounded by Y, in which K is slice. In the case where $Y = S^3$, this agrees with the description of V_K in the previous paragraph.

Let DX_K denote the double of X_K: $DX_K = X_K \sqcup \overline{X_K}/\sim$, where the two copies are identified by the identity map of $S^3_0(K)$. This manifold acquires an orientation from that of X_K. Since DX_K is the union of two simply-connected spaces along a connected intersection, it is simply-connected. Indeed, because X_K is built with only a 0- and 2-handle (with even framing), it is well-known that $DX_K \cong S^2 \times S^2$, irrespective of K. (See, e.g., [GS99, Corollary 5.1.6].)

Let $\Pi_K : DX_K \to Z_K$ be defined by π on X_K and by $\pi \circ \phi_K$ on $\overline{X_K}$. It is easy to see that Π_K is a $2:1$ covering map, and hence it is the universal cover of Z_K. There is a nontrivial deck transformation $\tau_K : DX_K \to DX_K$ that interchanges the two copies X_K and $\overline{X_K}$ using Φ_K. Using this covering map, we can deduce the algebraic topology of Z_K, as follows.

Lemma 2.3. *The manifold Z_K is a rational homology 4-sphere and has $\pi_1(Z_K) \cong H_1(Z_K) \cong H_2(Z_K) \cong \mathbb{Z}_2$.*

Proof. Since the universal cover of Z_K is two-sheeted, we deduce that $\pi_1(Z_K) \cong H_1(Z_K) \cong \mathbb{Z}_2$ and hence $b_1(Z_K) = 0$. The nontrivial element of $\pi_1(Z_K)$ can be given by any arc connecting two points in $S^3_0(K)$ that are exchanged by ϕ_K.

To see that Z_K is a rational homology sphere, we first note that $\chi(DX_K) = 2\chi(X_K) - \chi(S^3_0(K)) = 4$, and then $\chi(Z_K) = \chi(DX_K)/2 = 2$. Since $\chi(Z_K) = 2 - 2b_1(Z_K) + b_2(Z_K)$, we have $b_2(Z_K) = 0$. Universal coefficients and Poincaré duality then imply that $H_2(Z_K) \cong H^3(Z_K) \cong H_1(Z_K) \cong \mathbb{Z}_2$, as required. □

Example 2.4. Let O denote the unknot; then $X_O \cong S^2 \times D^2$. To be explicit, let us identify D^4 with $D^2 \times D^2$, where the involution Φ is still given in coordinates by $\Phi(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, -x_4)$, and take O to be $S^1 \times \{0\}$. The framing Ψ_O is then just the inclusion of $S^1 \times D^2$. Then $X_O = (D^2 \times D^2) \cup (D^2 \times D^2)$, glued by the identity map of $S^1 \times D^2$. This is naturally identified as $(D^2 \cup_{S^1} D^2) \times D^2 = S^2 \times D^2$, and $S^3_0(O)$ is identified as $S^2 \times S^1$. By construction, Φ_O acts on each copy of $D^2 \times D^2$ by a reflection in the first factor and negation in the second. Thus, it acts on $S^2 \times D^2$ in the same fashion: a reflection $r : S^2 \to S^2$ in the first factor and negation in the second factor.

Taking the double, we have $DX_O = S^2 \times (D^2 \cup_{S^1} D^2) = S^2 \times S^2$. The deck transformation τ_O acts by the reflection r in the first factor, while interchanging the two copies of D^2 and negating in the second factor. That is, for $(x, y) \in$

$S^2 \times S^2$, we have $\tau_O(x,y) = (r(x), -y)$. We thus see that Z_O agrees with the construction of Z_0 in the introduction.

To prove Theorem 1.1, we will use a 5-dimensional argument (inspired by one of Mazur [Maz61]) to show that the diffeomorphism $DX_K \cong S^2 \times S^2 = DX_O$ can be constructed equivariantly with respect to the deck transformations τ_K and τ_O. Let $Q_K = X_K \times [-1,1]$. This is a 5-manifold whose boundary is

$$(X_K \times \{1\}) \cup (S^3_0(K) \times [-1,1]) \cup (X_K \times \{-1\}).$$

Then $\partial Q(K)$ is naturally identified, after smoothing corners, with DX_K (or, more precisely, with DX'_K because of the collar). Define $\tilde{\tau}_K : Q_K \to Q_K$ by $\tilde{\tau}_K(x,t) = (\Phi_K(x), -t)$. This is an involution of Q_K, and it restricts to τ_K on ∂Q_K. Continuing with the above example, we may identify Q_O with $S^2 \times D^3$, where $\tilde{\tau}_O(x,y) = (r(x), -y)$.

Proposition 2.5. *For any SNACK K, the pairs $(Q_K, \tilde{\tau}_K)$ and $(Q_O, \tilde{\tau}_O)$ are equivariantly diffeomorphic.*

Proof. Note that Q_K has a 5-dimensional handle structure consisting of one 0-handle and one 2-handle, each of which is the product of the corresponding handle of X_K with an interval, and the involution $\tilde{\tau}_K$ preserves this handle structure. After smoothing corners, we may identify the 0-handle of Q_K with D^5, and the 2-handle with $D^2 \times D^3$, so that the involution $\tilde{\tau}_K$ is given on D^5 by

$$\tilde{\tau}_K|_{D^5}(x_1, x_2, x_3, x_4, x_5) = (x_1, -x_2, -x_3, -x_4, -x_5).$$

Since this is independent of K, we will omit the K subscript denote this map by $\tilde{\tau}$. The attaching circle for the 2-handle is $K \times \{0\}$, where we identify S^3 with $\partial D^5 \cap \{x_5 = 0\}$. The gluing map is an inclusion of $S^1 \times D^3$ into ∂D^5, parametrizing a $\tilde{\tau}$-invariant neighborhood of $K \times \{0\}$. We may likewise view the attaching circle for the 2-handle of Q_O, $O \times \{0\}$, as living in this same manifold.

By a theorem of Boyle and Chen [BC23, Proposition 3.12], there is a homotopy from K to O, equivariant with respect to our original involution $\phi : S^3 \to S^3$, which is an isotopy except for finitely many pairs of simultaneous crossing changes. By slightly perturbing this in the x_5 direction, we may promote this to a $\tilde{\tau}$-equivariant isotopy taking $K \times \{0\}$ to $O \times \{0\}$ in S^4.

By the equivariant isotopy extension theorem (see, e.g., [Kan07, Theorem 8.6]), we may then find an equivariant ambient isotopy of S^4 taking $K \times \{0\}$ to $O \times \{0\}$. Under this isotopy, the framing of $K \times \{0\}$ used to define Q_K induces a framing of $O \times \{0\}$, which *a priori* may or may not agree with the framing of $O \times \{0\}$ used to define Q_O. However, note that a circle in S^4 only has two framings, which are distinguished by their surgeries: one framing yields $S^2 \times S^2$, while the other framing yields $S^2 \tilde{\times} S^2 = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. Since we have already established that both Q_K and Q_O have boundary diffeomorphic to $S^2 \times S^2$, we deduce that the isotopy does indeed take the preferred framing of $K \times \{0\}$ to that of $O \times \{0\}$. Thus, the isotopy extends to an equivariant diffeomorphism from Q_K to Q_O, as required. ☐

Proof of Theorem 1.1. Restricting the diffeomorphism from Proposition 2.5 to the boundary gives an equivariant diffeomorphism $(DX_K, \tau_K) \cong (DX_O, \tau_O)$, and hence a diffeomorphism between the quotients, $Z_K \cong Z_O$. Thus, K is slice in Z_O. □

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(Adam Simon Levine) DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NC 27708, USA
alevine@math.duke.edu

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