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L^p regularity of Szegö projections on quotient domains

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ABSTRACT. We introduce a family of Hardy spaces $\{\mathcal{H}_\varrho\}_{\varrho\in\widehat{G}_1}$ on the distinguished boundary of the quotient domain $\mathbb{D}^n/G$, where G is a finite pseudoreflection group acting on $\mathbb{D}^n$ and $\widehat{G}_1$ is the set of equivalence classes of one-dimensional representations of G. We establish a uniform platform to study L^p regularity properties of the generalized Szegö projections associated to Hardy spaces $\mathcal{H}_\varrho$ for every $\varrho \in \widehat{G}_1$.

CONTENTS

1. Introduction

The boundedness of projection operators on analytic function spaces such as Hardy space and Bergman space has a rich history and has been studied over several domains. The primary goal of this article is twofold:

- Let $\mathbb{D}^n$ denote the unit polydisc in $\mathbb{C}^n$ and G be a finite pseudoreflection group acting on $\mathbb{D}^n$. We first define an appropriate notion of Hardy space on a quotient domain $\mathbb{D}^n/G$. The notion of Hardy space over a quotient domain is not canonical in nature. We prescribe a unified approach to define a family of Hardy spaces on $\mathbb{D}^n/G$ indexed by the equivalence classes of one-dimensional representations of G. These spaces can be realized as subspaces of some weighted L^2 spaces on the distinguished boundary of the quotient domain where the weights are dictated by the associated representations.

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- Secondly, corresponding to each one-dimensional representation ϱ of G, one can naturally consider the orthogonal projection operator from the associated weighted L^2 space to the Hardy space on $\mathbb{D}^n/G$ associated to ϱ. These projection operators are analogues of the classical Szegö projections and without loss of generality we call them generalized Szegö projections. Finally, employing representation theoretic information, we obtain a range (a_ϱ, b_ϱ) with $1 < a_\varrho < 2 < b_\varrho < \infty$ such that these projection operators are L^p regular for $p \in (a_\varrho, b_\varrho)$. Moreover, the interval (a_ϱ, b_ϱ) is Hölder symmetric, that is, if $r \in (a_\varrho, b_\varrho)$ then $r' \in (a_\varrho, b_\varrho)$ where $r' := \frac{r}{r-1}$ is the Hölder conjugate of r.

The study of L^p regularity for singular integral operators is of immense interest in harmonic analysis and operator theory. Since its inception, the Szegö projection reflects crucially the geometry of the domain under study and is studied in various contexts. More precisely, the regularity of the Szegö projection depends on the smoothness of the boundary of the domain. To start with, we recall the fundamental result by Kerzman–Stein [13] where the authors exhibit a close connection between the Cauchy integral and the Szegö projection on bounded smooth domains in the complex plane $\mathbb{C}$, needless to mention that the two coincide when the domain is a disc in $\mathbb{C}$. Therefore, when the domain under consideration is a disc, the Szegö projection maps L^p to itself for $1 < p < \infty$ and is of weak-type $(1, 1)$. More generally, the same result is true if the domain is in C^1. For more results in this direction, we refer to [10, 18] and others.

However, while working on $\mathbb{C}^n$, the scenario is much more complicated and the main difficulty lies in the fact that the kernel for the Szegö projection is not known explicitly in most of the cases. A recent breakthrough in this direction was made by Lanzani and Stein [16] where the authors studied the L^p regularity of the Szegö projection on strongly pseudo-convex domains on $\mathbb{C}^n$ with C^2 boundary. Currently, there is a renewed interest in studying the Szegö projection on specialized domains due to the influential works [16, 18, 24], see also [28, 30] for some significant developments. Also, in a fundamental work [30] Wagner and Wick introduced an appropriate notion of Muckenhoupt weights suitable to the intrinsic quasi-metric of the boundary of a strongly pseudoconvex domain with C^2 boundary and proved weighted L^p regularity results for the Szegö projection (see Theorem 1.1 in [30]). In this direction, interesting end-point estimates for the Szegö projection are obtained in [28].

Hardy spaces are also considered on the distinguished boundary of domains and an early influential work in this direction is by Bekollé and Bonami [1] where they considered Hardy spaces on the distinguished boundary of tube domains over spherical cones. In [21], the authors have studied the Szegö projection on the unbounded model worm domain in this setting and obtained its sharp L^p regularity; subsequently, the analogous question was addressed on the Hartogs triangle in [20]. Both the works mentioned above reduce the study of the Szegö projection to some suitable Fourier multiplier operators and an application of Mihlin–Hörmander multiplier theorem concludes their proof.

We also refer to the article [23] where sharp L^p regularity is achieved for some weighted Szegö projection operators and the proof relies on the characterization of power weights belonging to Muckenhoupt A_p classes which is also a key ingredient in our proof.

When it comes to the quotient domains we have a very limited literature at hand. In [19], authors have defined a suitable notion of Hardy space on the symmetrized polydisc and very recently in [11], L^p regularity of the associated Szegö projection is studied. We are also motivated by the results in [2, 4] where Bergman projections on various quotient domains are studied. These can be thought of as our point of departure and in this work we explicitly prove the following regularity results for the Szegö projection on quotient domains $\mathbb{D}^n/G$, whenever G is a finite pseudoreflection group. A *pseudoreflection* on $\mathbb{C}^n$ is a linear homomorphism $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ such that σ has finite order in $GL(n, \mathbb{C})$ and the rank of $(I_n - \sigma)$ is 1. A group generated by pseudoreflections is called a pseudoreflection group. For example, any finite cyclic group, the permutation group $\mathfrak{S}_n$ on n symbols, and the dihedral groups are all finite pseudoreflection groups [17]. Suppose that G acts on $\mathbb{D}^n$ as in Equation (2) then $\mathbb{D}^n/G$ is not necessarily a domain. However, if G is a finite pseudoreflection group then $\mathbb{D}^n/G$ is biholomorphically equivalent to $\theta(\mathbb{D}^n)$, where $\theta : \mathbb{C}^n \to \mathbb{C}^n$ is a basic polynomial map associated to the group G [3, Subsection 3.1]. Therefore, we restrict our attention to finite pseudoreflection groups for the rest of the article. Note that if Ω is a domain such that there exists a proper holomorphic map $\boldsymbol{f} : \mathbb{D}^n \to \Omega$ with G as the group of deck transformations, then Ω is biholomorphic to $\mathbb{D}^n/G$ and θ is a representative of $\boldsymbol{f}$, that is, $\boldsymbol{f} = \theta \circ \boldsymbol{h}$ for some biholomorphism $\boldsymbol{h} : \theta(\mathbb{D}^n) \to \Omega$ [9, Proposition 2.2]. Therefore, we work with the domain $\theta(\mathbb{D}^n)$ instead of $\mathbb{D}^n/G$ and without loss of generality call them *quotient domains*. Also, the choice of a basic polynomial is not unique for the group G and our result is independent of the choice of basic polynomial θ. We now state our main result.

Let $\varrho \in \widehat{G}_1$ and $1 < p < \infty$. In Definition 2.7, we provide a notion of Hardy space on $\theta(\mathbb{D}^n)$ associated to the representation ϱ and denote it by $H^p_\varrho(\theta(\mathbb{D}^n))$. We show that each $H^2_\varrho(\theta(\mathbb{D}^n))$ (for simplicity, denoted by $\mathcal{H}_\varrho$ in the abstract) can be thought as a closed subspace of some weighted $L^2_\varrho(\theta(\mathbb{T}^n))$ (cf. Lemma 2.8). Let $\mathcal{S}_{\theta,\varrho} : L^2_\varrho(\theta(\mathbb{T}^n)) \to H^2_\varrho(\theta(\mathbb{D}^n))$ denote the corresponding orthogonal projection, which we call the generalized Szegö projection associated to the representation ϱ. Then the following holds:

Theorem 1.1. *Suppose that G is a finite pseudoreflection group acting on the unit polydisc $\mathbb{D}^n$ and θ is a basic polynomial associated to G.*

(1) *For the trivial representation of G,*

$$\mathcal{S}_{\theta,\text{trivial}} : L^p_{\text{trivial}}(\theta(\mathbb{T}^n)) \to H^p_{\text{trivial}}(\theta(\mathbb{D}^n))$$

is bounded for $p \in (1, \infty)$.

(2) *For a one-dimensional representation ϱ of G which is not equivalent to the trivial representation of G, there exists an interval (a_ϱ, b_ϱ), $1 < a_\varrho < 2 < b_\varrho < \infty$, such that the generalized Szegö projection $\mathcal{S}_{\theta,\varrho}$ is bounded from $L^p_\varrho(\theta(\mathbb{T}^n))$ to $H^p_\varrho(\theta(\mathbb{D}^n))$ if $p \in (a_\varrho, b_\varrho)$.*

Let us highlight some key features of Theorem 1.1.

- Let θ' be another basic polynomial associated to the group G. Then for every $\varrho \in \widehat{G}_1$, the generalized Szegö projection $\mathcal{S}_{\theta',\varrho}$ is bounded from $L^p_\varrho(\theta'(\mathbb{T}^n))$ to $H^p_\varrho(\theta'(\mathbb{D}^n))$ if $p \in (a_\varrho, b_\varrho)$. Therefore, the value of a_ϱ and b_ϱ is independent of the choice of basic polynomial of the group G.
- We ensure not only the existence of a_ϱ and b_ϱ but also provide an explicit expression of the range (a_ϱ, b_ϱ) depending solely on the representation ϱ, see Theorem 3.5 and Equation (20).
- Since we only need minimal knowledge about the representation ϱ to determine the values of a_ϱ and b_ϱ, we can avoid difficulties arising from the complexity of the boundary of the quotient domains. For example, the symmetrized polydisc (biholomorphic to $\mathbb{D}^n/\mathfrak{S}_n$) is a nonsmoothly bounded pseudoconvex domain without any strongly pseudoconvex boundary point and Theorem 1.1 is applicable for it also, see Subsection 3.3.

We close this section by demonstrating our result for the one-dimensional representations of finite reflection groups. Pseudoreflections of order two are called as *reflections*. A finite group generated by reflections is called a reflection group. The permutation group on n symbols and dihedral group are examples of reflection groups. As an application of Theorem 1.1, we have the following result.

Corollary 1.2. *Let G be a finite reflection group acting on $\mathbb{D}^n$. Let $\varrho \in \widehat{G}_1$ be any representation which is not equivalent to the trivial representation. Then there exists a natural number M_ϱ such that the generalized Szegö projection $\mathcal{S}_{\theta,\varrho}$ is bounded from $L^p_\varrho(\theta(\mathbb{T}^n))$ to itself if $p \in \left(\frac{2M_\varrho+1}{M_\varrho+1}, \frac{2M_\varrho+1}{M_\varrho}\right)$.*

The quantity M_ϱ can be completely determined from the representation ϱ (cf. Corollary 3.7). For instance, when one considers the sign representation (sgn) for the permutation group and the dihedral group, we obtain the following results:

(1) For the permutation group: The symmetrization map $\boldsymbol{s} : \mathbb{D}^n \to \boldsymbol{s}(\mathbb{D}^n)$ is a basic polynomial associated to the permutation group $\mathfrak{S}_n$, see Equation (21). The domain $\mathbb{G}_n = \boldsymbol{s}(\mathbb{D}^n)$ is known as the symmetrized polydisc. In this case, $M_{\text{sgn}} = n-1$ and thus $a_{\text{sgn}} = 2 - \frac{1}{n}$ and $b_{\text{sgn}} = 2 + \frac{1}{n-1}$ for the domain $\mathbb{G}_n$ (cf. Proposition 3.9). This is the main result of [11].

(2) For the dihedral group: The polynomial map $\boldsymbol{\phi}(z_1, z_2) = (z_1^k + z_2^k, z_1 z_2) : \mathbb{D}^2 \to \boldsymbol{\phi}(\mathbb{D}^2)$ is a basic polynomial map associated to the dihedral group

D_{2k}. Denote $\mathscr{D}_{2k} = \boldsymbol{\phi}(\mathbb{D}^2)$. Here, $M_{\rm sgn} = k$ and thus $a_{\rm sgn} = 2 - \frac{1}{k+1}$ and $b_{\rm sgn} = 2 + \frac{1}{k}$ for the domain $\mathscr{D}_{2k}$, (cf. Proposition 3.10).

The article is organized as follows. In the next section, we define Hardy spaces on the distinguished boundary of quotient domains and prove several essential properties. The notion of generalized Szegö projections is defined in Subsection 2.3. In Section 3, we prove our main results on L^p regularity estimates. Throughout the article, C denotes an all purpose constant which may change from line to line.

2. Hardy space

A holomorphic function f on $\mathbb{D}^n$ is in the Hardy space $H^2(\mathbb{D}^n)$ on the unit polydisc $\mathbb{D}^n$ if and only if

$$\sup_{0<r<1} \int_{\mathbb{T}^n} |f(re^{i\Theta})|^2 d\Theta < \infty, \tag{1}$$

where $d\Theta$ is the normalized Lebesgue measure on $\mathbb{T}^n$. Let G be a finite pseudoreflection group which acts (right action) on $\mathbb{D}^n$ by

$$\sigma \cdot \boldsymbol{z} = \sigma^{-1}\boldsymbol{z}, \quad \text{for } \sigma \in G \text{ and } \boldsymbol{z} \in \mathbb{D}^n. \tag{2}$$

The group action extends to the set of all complex-valued functions on $\mathbb{D}^n$ by $\sigma(f)(\boldsymbol{z}) = f(\sigma^{-1} \cdot \boldsymbol{z})$ and a function f is said to be G-invariant if $\sigma(f) = f$ for all $\sigma \in G$. There is a system of G-invariant algebraically independent homogeneous polynomials $\{\theta_i\}_{i=1}^n$ associated to a pseudoreflection group G, called a homogeneous system of parameters (hsop) or basic polynomials associated to G. In fact, the Chevalley-Shephard-Todd theorem provides a characterization of finite pseudoreflection groups in terms of hsop. It states that a finite group G is generated by pseudoreflections if and only if G-invariant polynomials in n variables form a polynomial ring $\mathbb{C}[\theta_1, \dots, \theta_n]$ [25, p.282]. The polynomial map

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) : \mathbb{D}^n \to \boldsymbol{\theta}(\mathbb{D}^n)$$

is a proper holomorphic map and the domain $\boldsymbol{\theta}(\mathbb{D}^n)$ is biholomorphically equivalent to the quotient $\mathbb{D}^n/G$ [29, 3]. So we refer to the domains of the form $\boldsymbol{\theta}(\mathbb{D}^n)$ by *quotient domains.*

We define a family of weighted Hardy spaces on the domain $\boldsymbol{\theta}(\mathbb{D}^n)$ indexed by the one-dimensional representations of the group G. As mentioned earlier there are several notions of Hardy spaces depending on the boundary, however, in this article, we define the Hardy space on the Shilov boundary of $\boldsymbol{\theta}(\mathbb{D}^n)$.

Definition 2.1. [7] *The Shilov boundary $\partial\Omega$ of a bounded domain Ω is given by the closure of the set of its peak points and a point $\boldsymbol{w} \in \overline{\Omega}$ is said to be a peak point of Ω if there exists a function $f \in \mathcal{A}(\Omega)$ such that $|f(\boldsymbol{w})| > |f(\boldsymbol{z})|$ for all $\boldsymbol{z} \in \overline{\Omega} \setminus \{\boldsymbol{w}\}$, where $\mathcal{A}(\Omega)$ denotes the algebra of all functions holomorphic on Ω and continuous on $\overline{\Omega}$.*

Since the distinguished boundary of $\overline{\Omega}$ in $\mathbb{C}^n$ is the Shilov boundary of Ω, these two notions will be frequently used without any confusion. The proper holomorphic map $\theta : \mathbb{D}^n \to \theta(\mathbb{D}^n)$ can be extended to a proper holomorphic map of the same multiplicity from D' to $\theta(D)'$, where the open sets D' and $\theta(D)'$ contain $\overline{\mathbb{D}^n}$ and $\overline{\theta(\mathbb{D}^n)}$, respectively. Then [14, p. 100, Corollary 3.2] states that $\theta^{-1}(\partial\theta(\mathbb{D}^n)) = \partial\mathbb{D}^n = \mathbb{T}^n$. Thus

$$\partial\theta(\mathbb{D}^n) = \theta(\mathbb{T}^n). \tag{3}$$

2.1. One-dimensional representations. Since the one-dimensional representations of G play an important role in our discussion, we elaborate on some relevant results for the same. We denote the one-dimensional representations of G by $\widehat{G}_1$.

A hyperplane H in $\mathbb{C}^n$ is called reflecting if there exists a pseudoreflection in G acting trivially on H. For a pseudoreflection $\sigma \in G$, define $H_\sigma := \ker(\mathrm{id}-\sigma)$. By definition, the subspace H_σ has dimension $n-1$. Clearly, σ fixes the hyperplane H_σ pointwise. Hence each H_σ is a reflecting hyperplane. By definition, H_σ is the zero set of a non-zero homogeneous linear polynomial L_σ on $\mathbb{C}^n$, determined up to a non-zero constant multiple, that is, $H_\sigma = \{\boldsymbol{z} \in \mathbb{C}^n : L_\sigma(\boldsymbol{z}) = 0\}$. Moreover, the elements of G acting trivially on a reflecting hyperplane form a cyclic subgroup of G.

Let $H_1, \ldots, H_t$ denote the distinct reflecting hyperplanes associated to the group G and the corresponding cyclic subgroups are $G_1, \ldots, G_t$, respectively. Suppose $G_i = \langle a_i \rangle$ and the order of each a_i is m_i for $i = 1, \ldots, t$. For every one-dimensional representation ϱ of G, there exists a unique t-tuple of non-negative integers $(c_1, \ldots, c_t)$, where c_i's are the least non-negative integers that satisfy the following:

$$\varrho(a_i) = \big(\det(a_i)\big)^{c_i}, \ i = 1, \ldots, t. \tag{4}$$

The t-tuple $(c_1, \ldots, c_t)$ solely depends on the representation ϱ. The character of the one-dimensional representation ϱ, $\chi_\varrho : G \to \mathbb{C}^*$ coincides with the representation ϱ. The set of elements of $H^2(\mathbb{D}^n)$ relative to the one-dimensional representation ϱ is given by

$$R_\varrho^G(H^2(\mathbb{D}^n)) = \{f \in H^2(\mathbb{D}^n) : \sigma(f) = \chi_\varrho(\sigma)f, \text{ for all } \sigma \in G\}. \tag{5}$$

The elements of the subspace $R_\varrho^G(H^2(\mathbb{D}^n))$ are said to be ϱ-invariant functions.

Lemma 2.2. [9] *Suppose that the linear polynomial ℓ_i is a defining function of H_i for $i = 1, \ldots, t$ and $\ell_\varrho = \prod_{i=1}^t \ell_i^{c_i}$ is a homogeneous polynomial where c_i's are unique non-negative integers as described in Equation* (4)*. Any element $f \in R_\varrho^G(H^2(\mathbb{D}^n))$ can be written as $f = \ell_\varrho \widetilde{f} \circ \theta$ for a holomorphic function $\widetilde{f}$ on $\theta(\mathbb{D}^n)$.*

The *sign representation* of a finite pseudoreflection group G, $\mathrm{sgn} : G \to \mathbb{C}^*$, is defined by [26, p. 139, Remark (1)]

$$\mathrm{sgn}(\sigma) = (\det(\sigma))^{-1}, \quad \sigma \in G. \tag{6}$$

Additionally, we note that $\operatorname{sgn}(a_i) = (\det(a_i))^{-1} = \big(\det(a_i)\big)^{m_i-1}$, $i = 1, \dots, t$, which invokes the following result from Lemma 2.2.

Corollary 2.3. [27, p. 616, Lemma] *Let $H_1, \dots, H_t$ denote the distinct reflecting hyperplanes associated to the group G and let $m_1, \dots, m_t$ be the orders of the corresponding cyclic subgroups $G_1, \dots, G_t$, respectively. Then*

$$\ell_{\operatorname{sgn}}(\boldsymbol{z}) = J_\theta(\boldsymbol{z}) = c \prod_{i=1}^{t} \ell_i^{m_i-1}(\boldsymbol{z}),$$

where J_θ is the determinant of the complex Jacobian matrix of the basic polynomial map θ and c is a non-zero constant.

2.2. Hardy spaces associated to the representations. For the symmetrized polydisc, the notion of Hardy space was defined in [19] and our definition is partly motivated by that. Recall that $d\Theta$ is the normalized Lebesgue measure on $\mathbb{T}^n$.

Definition 2.4. *Let $\varrho \in \widehat{G}_1$. The function space consisting of holomorphic functions f on $\theta(\mathbb{D}^n)$ which satisfy*

$$\sup_{0<r<1} \int_{\mathbb{T}^n} |f \circ \theta(re^{i\Theta})|^2 |\ell_\varrho(re^{i\Theta})|^2 \, d\Theta < \infty$$

is said to be the Hardy space associated to the representation ϱ and is denoted by $H^2_\varrho(\theta(\mathbb{D}^n))$.

We list the following facts:

- Each $H^2_\varrho(\theta(\mathbb{D}^n))$ is a Hilbert space with the norm

$$||f||_\varrho^2 = \frac{1}{d} \sup_{0<r<1} \int_{\mathbb{T}^n} |f \circ \theta(re^{i\Theta})|^2 |\ell_\varrho(re^{i\Theta})|^2 d\Theta, \tag{7}$$

  where d is the order of the group G. Moreover, we will also show that $H^2_\varrho(\theta(\mathbb{D}^n))$ is a reproducing kernel Hilbert space.
- The Hardy space associated to the sign representation of G is said to be *the Hardy space* on $\theta(\mathbb{D}^n)$ and we denote it by $H^2(\theta(\mathbb{D}^n))$. In particular, $H^2(\mathbb{G}_n)$ coincides with the definition of the Hardy space on the symmetrized polydisc $\mathbb{G}_n$ defined in [19].

Proposition 2.5. *For each $\varrho \in \widehat{G}_1$, the Hilbert space $H^2_\varrho(\theta(\mathbb{D}^n))$ is a reproducing kernel Hilbert space.*

Before proving Proposition 2.5, we state some relevant results. Each $\varrho \in \widehat{G}_1$ induces an orthogonal projection $\mathbb{P}_\varrho : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ such that

$$\mathbb{P}_\varrho \phi = \frac{\deg \varrho}{d} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1}) \, \phi \circ \sigma^{-1}, \quad \phi \in L^2(\mathbb{T}^n), \tag{8}$$

where $\deg \varrho$ is the degree of the representation ϱ, d is the order of the group G and χ_ϱ is the character of ϱ. However, since we are only dealing with one-dimensional representations, $\deg \varrho = 1$ throughout this article. Each $\mathbb{P}_\varrho$ is well-defined because $\phi\circ\sigma^{-1}$ is in $L^2(\mathbb{T}^n)$ whenever $\phi \in L^2(\mathbb{T}^n)$. An application of Schur's Lemma implies that $\mathbb{P}_\varrho^2 = \mathbb{P}_\varrho$ [15, p. 24, Theorem 4.1]. We now show that $\mathbb{P}_\varrho$ is self-adjoint. Performing change of variables, we get that for all $\phi, \psi \in L^2(\mathbb{T}^n)$ and $\sigma \in G$,

$$\langle \sigma \cdot \phi, \sigma \cdot \psi \rangle = \langle \phi, \psi \rangle, \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{T}^n)$. For $\phi, \psi \in L^2(\mathbb{T}^n)$, we have

$$\begin{aligned} \langle \mathbb{P}_\varrho^* \phi, \psi \rangle = \langle \phi, \mathbb{P}_\varrho \psi \rangle &= \langle \phi, \frac{1}{d} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1})\, \psi\circ\sigma^{-1} \rangle \\ &= \frac{1}{d} \sum_{\sigma \in G} \chi_\varrho(\sigma) \langle \phi, \psi\circ\sigma^{-1} \rangle \\ &= \frac{1}{d} \sum_{\sigma \in G} \chi_\varrho(\sigma) \langle \phi\circ\sigma, \psi \rangle \\ &= \langle \mathbb{P}_\varrho \phi, \psi \rangle, \end{aligned}$$

where the penultimate equality follows from Equation (9). It is known that $\mathbb{P}_\varrho : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n)$ is the orthogonal projection onto the subspace $R_\varrho^G(H^2(\mathbb{D}^n))$ [9, Lemma 2.10]. Therefore,

$$\mathbb{P}_\varrho(H^2(\mathbb{D}^n)) = R_\varrho^G(H^2(\mathbb{D}^n)).$$

We use this identification in the following proof.

Proof of Proposition 2.5. Fix $\varrho \in \widehat{G}_1$ and consider the operator

$$\Gamma_\varrho : H_\varrho^2(\theta(\mathbb{D}^n)) \to \mathbb{P}_\varrho(H^2(\mathbb{D}^n))$$

defined by

$$\Gamma_\varrho f = \frac{1}{\sqrt{d}} \ell_\varrho f\circ\theta. \tag{10}$$

From Equation (7) it follows that the operator Γ_ϱ is an isometry. From the above argument and Lemma 2.2, we know that any element $\widetilde{f}$ in $\mathbb{P}_\varrho(H^2(\mathbb{D}^n))$ can be written as $\widetilde{f} = \ell_\varrho\, f\circ\theta$ and from Equation (7), it follows that $f \in H_\varrho^2(\theta(\mathbb{D}^n))$. Thus $\Gamma_\varrho(\sqrt{d} f) = \widetilde{f}$ and hence Γ_ϱ is unitary.

Let $S_{\mathbb{D}^n}$ denotes the reproducing kernel of $H^2(\mathbb{D}^n)$. The expression

$$\frac{1}{\ell_\varrho(\boldsymbol{z})\overline{\ell_\varrho(\boldsymbol{w})}} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1}) S_{\mathbb{D}^n}(\sigma^{-1} \cdot \boldsymbol{z}, \boldsymbol{w})$$

is G-invariant in both variables, separately. Using analytic version of Chevalley-Shephard-Todd theorem [3], we write

$$S_{\theta,\varrho}(\theta(\boldsymbol{z}),\theta(\boldsymbol{w})) = \frac{1}{\ell_\varrho(\boldsymbol{z})\overline{\ell_\varrho(\boldsymbol{w})}}\sum_{\sigma\in G}\chi_\varrho(\sigma^{-1})S_{\mathbb{D}^n}(\sigma^{-1}\cdot\boldsymbol{z},\boldsymbol{w}). \tag{11}$$

We now show that $S_{\theta,\varrho}(\theta(\boldsymbol{z}),\theta(\boldsymbol{w}))$ is the reproducing kernel of $H^2_\varrho(\theta(\mathbb{D}^n))$.

For a fixed $\boldsymbol{w}\in\mathbb{D}^n$, $\ell_\varrho(\cdot)S_{\theta,\varrho}(\theta(\cdot),\theta(\boldsymbol{w}))\in\mathbb{P}_\varrho(H^2(\mathbb{D}^n))$ since we have the following,

$$\begin{aligned}\ell_\varrho(\boldsymbol{z})S_{\theta,\varrho}(\theta(\boldsymbol{z}),\theta(\boldsymbol{w})) &= \frac{1}{\overline{\ell_\varrho(\boldsymbol{w})}}\sum_{\sigma\in G}\chi_\varrho(\sigma^{-1})S_{\mathbb{D}^n}(\sigma^{-1}\cdot\boldsymbol{z},\boldsymbol{w})\\ &= \frac{d}{\overline{\ell_\varrho(\boldsymbol{w})}}\mathbb{P}_\varrho S_{\mathbb{D}^n}(\boldsymbol{z},\boldsymbol{w}).\end{aligned}$$

Note that Γ_ϱ is unitary and $\Gamma_\varrho(S_{\theta,\varrho}(\cdot,\theta(\boldsymbol{w})))(\boldsymbol{z}) = \frac{1}{\sqrt{d}}\ell_\varrho(\boldsymbol{z})S_{\theta,\varrho}(\theta(\boldsymbol{z}),\theta(\boldsymbol{w}))$ for fixed $\boldsymbol{w}\in\mathbb{D}^n$. Therefore, for every $\boldsymbol{w}\in\mathbb{D}^n$, $S_{\theta,\varrho}(\cdot,\theta(\boldsymbol{w}))$ is in $H^2_\varrho(\theta(\mathbb{D}^n))$.

Also, let $f\in H^2_\varrho(\theta(\mathbb{D}^n))$. Then

$$\begin{aligned}\langle f, S_{\theta,\varrho}(\cdot,\theta(\boldsymbol{w}))\rangle &= \langle\Gamma_\varrho f,\Gamma_\varrho S_{\theta,\varrho}(\cdot,\theta(\boldsymbol{w}))\rangle\\ &= \frac{1}{d}\langle\ell_\varrho\, f\circ\theta, \ell_\varrho S_{\theta,\varrho}(\theta(\cdot),\theta(\boldsymbol{w}))\rangle\\ &= \frac{1}{d}\langle\ell_\varrho\, f\circ\theta, \frac{d}{\overline{\ell_\varrho(\boldsymbol{w})}}\mathbb{P}_\varrho S_{\mathbb{D}^n}(\cdot,\boldsymbol{w})\rangle\\ &= \frac{1}{\ell_\varrho(\boldsymbol{w})}\langle\ell_\varrho\, f\circ\theta, S_{\mathbb{D}^n}(\cdot,\boldsymbol{w})\rangle = f(\theta(\boldsymbol{w})).\end{aligned}$$

□

Remark 2.6. *For $\varrho\in\widehat{G}_1$, $\mathbb{P}_\varrho(H^2(\mathbb{D}^n))$ is a closed subspace of $H^2(\mathbb{D}^n)$ and the reproducing kernel S_ϱ of $\mathbb{P}_\varrho(H^2(\mathbb{D}^n))$ is given by*

$$S_\varrho(\boldsymbol{z},\boldsymbol{w}) = \frac{1}{d}\sum_{\sigma\in G}\chi_\varrho(\sigma^{-1})S_{\mathbb{D}^n}(\sigma^{-1}\cdot\boldsymbol{z},\boldsymbol{w}).$$

For a fixed $\boldsymbol{w}$,

$$\overline{\ell_\varrho(\boldsymbol{w})}\Gamma_\varrho(S_{\theta,\varrho}(\cdot,\theta(\boldsymbol{w})))(\boldsymbol{z}) = \frac{1}{\sqrt{d}}\ell_\varrho(\boldsymbol{z})\overline{\ell_\varrho(\boldsymbol{w})}S_{\theta,\varrho}(\theta(\boldsymbol{z}),\theta(\boldsymbol{w})) = \sqrt{d}S_\varrho(\boldsymbol{z},\boldsymbol{w}).$$

Let us define the notion of Hardy spaces for $1<p<\infty$.

Definition 2.7. *Let $1<p<\infty$. The Hardy space $H^p_\varrho(\theta(\mathbb{D}^n))$ is the holomorphic function space on $\theta(\mathbb{D}^n)$ defined as following:*

$$H^p_\varrho(\theta(\mathbb{D}^n)) =$$

$$\{f:\theta(\mathbb{D}^n)\to\mathbb{C}\ \textit{holomorphic}: \sup_{0<r<1}\int_{\mathbb{T}^n}|f\circ\theta(re^{i\Theta})|^p|\ell_\varrho(re^{i\Theta})|^2d\Theta<\infty\}.$$

From the definition, it follows that the operator

$$\Gamma_\varrho : H^p_\varrho(\theta(\mathbb{D}^n)) \to H^p(\mathbb{D}^n, |\ell_\varrho|^{2-p}),$$

defined by $\Gamma_\varrho f = f\circ\theta\, \ell_\varrho$, is an isometry.

2.3. Szegö Projections. Let $d\Theta_{\varrho,\theta}$ be the measure supported on the Shilov boundary $\theta(\mathbb{T}^n)$ of $\theta(\mathbb{D}^n)$ obtained from the following:

$$\int_{\theta(\mathbb{T}^n)} f\, d\Theta_{\varrho,\theta} = \int_{\mathbb{T}^n} f\circ\theta\, |\ell_\varrho|^2 d\Theta, \tag{12}$$

where ℓ_ϱ is as defined in Lemma 2.2. For $1 \le p < \infty$, and $\varrho \in \widehat{G}_1$, the Lebesgue spaces on $\theta(\mathbb{T}^n)$ with respect to the measure $d\Theta_{\varrho,\theta}$ are defined by

$$L^p_\varrho(\theta(\mathbb{T}^n)) = \{f : \theta(\mathbb{T}^n) \to \mathbb{C} | \int_{\theta(\mathbb{T}^n)} |f|^p\, d\Theta_{\varrho,\theta} < \infty\}.$$

The operator $\Gamma_\varrho : L^p_\varrho(\theta(\mathbb{T}^n)) \to L^p(\mathbb{T}^n, |\ell_\varrho|^{2-p})$ defined by $\Gamma_\varrho f = f\circ\theta\, \ell_\varrho$ is an isometry. In particular, we observe that Γ_ϱ maps $L^2_\varrho(\theta(\mathbb{T}^n))$ onto the closed subspace $\mathbb{P}_\varrho(L^2(\mathbb{T}^n))$ of $L^2(\mathbb{T}^n)$. We use two elementary properties such as $\theta\circ\sigma^{-1} = \theta$ for every $\sigma \in G$ and $\mathbb{P}_\varrho(\ell_\varrho) = \ell_\varrho$ to prove that

$$\begin{aligned} \mathbb{P}_\varrho(\Gamma_\varrho f) &= \frac{1}{d}\sum_{\sigma\in G} \chi_\varrho(\sigma^{-1})\, (\Gamma_\varrho f)\circ\sigma^{-1} \\ &= (f\circ\theta)\left(\frac{1}{d}\sum_{\sigma\in G} \chi_\varrho(\sigma^{-1})\ell_\varrho\circ\sigma^{-1}\right) \\ &= f\circ\theta\, \ell_\varrho = \Gamma_\varrho f. \end{aligned}$$

Lemma 2.8. *For $\varrho \in \widehat{G}_1$, $H^2_\varrho(\theta(\mathbb{D}^n))$ is isometrically embedded in $L^2_\varrho(\theta(\mathbb{T}^n))$.*

Proof. Note that $\Gamma_\varrho : L^2_\varrho(\theta(\mathbb{T}^n)) \to \mathbb{P}_\varrho(L^2(\mathbb{T}^n))$, defined by $\Gamma_\varrho f = f\circ\theta\, \ell_\varrho$, is an isometry. So is $\Gamma_\varrho : H^2_\varrho(\theta(\mathbb{D}^n)) \to \mathbb{P}_\varrho(H^2(\mathbb{D}^n))$. Let $i_\varrho : \mathbb{P}_\varrho(H^2(\mathbb{D}^n)) \to \mathbb{P}_\varrho(L^2(\mathbb{T}^n))$ be the canonical isometric embedding. Observe that the following diagram commutes:

$$\begin{array}{ccc} H^2_\varrho(\theta(\mathbb{D}^n)) & \xrightarrow{\Gamma_\varrho^{-1}\circ i_\varrho\circ\Gamma_\varrho} & L^2_\varrho(\theta(\mathbb{T}^n)) \\ \Gamma_\varrho\downarrow & & \downarrow\Gamma_\varrho \\ \mathbb{P}_\varrho(H^2(\mathbb{D}^n)) & \xrightarrow{i_\varrho} & \mathbb{P}_\varrho(L^2(\mathbb{T}^n)) \end{array}$$

Thus, $H^2_\varrho(\theta(\mathbb{D}^n))$ is embedded into $L^2_\varrho(\theta(\mathbb{T}^n))$ by the isometry $\Gamma_\varrho^{-1}\circ i_\varrho\circ\Gamma_\varrho$. □

Therefore, one can realize $H^2_\varrho(\theta(\mathbb{D}^n))$ as a closed subspace of $L^2_\varrho(\theta(\mathbb{T}^n))$. Now we define generalized Szegö projections.

Definition 2.9. *Let $\varrho \in \widehat{G}_1$. The Szegö projection associated to the representation ϱ is defined to be the orthogonal projection*

$$\mathcal{S}_{\theta,\varrho} : L^2_\varrho(\theta(\mathbb{T}^n)) \to H^2_\varrho(\theta(\mathbb{D}^n)). \tag{13}$$

When ϱ is the sign representation, we simply denote the Szegö projection $\mathcal{S}_{\theta,\mathrm{sgn}}$ as $\mathcal{S}_\theta$.

In this article, our primary goal will be to obtain L^p boundedness of the generalized Szegö projections $\mathcal{S}_{\theta,\varrho}$. The following result connects $\mathcal{S}_{\theta,\varrho}$ with the classical Szegö projection $\mathcal{S}_{\mathbb{D}^n}$ on the polydisc.

Lemma 2.10. *The following diagram commutes:*

$$\begin{array}{ccc} L^2_\varrho(\theta(\mathbb{T}^n)) & \xrightarrow{\mathcal{S}_{\theta,\varrho}} & H^2_\varrho(\theta(\mathbb{D}^n)) \\ \Gamma_\varrho \downarrow & & \downarrow \Gamma_\varrho \\ \mathbb{P}_\varrho(L^2(\mathbb{T}^n)) & \xrightarrow{\mathcal{S}_{\mathbb{D}^n}} & \mathbb{P}_\varrho(H^2(\mathbb{D}^n)). \end{array}$$

Proof. Note that for $f \in L^2_\varrho(\theta(\mathbb{T}^n))$, we have

$$\begin{aligned} (\Gamma_\varrho \mathcal{S}_{\theta,\varrho} f)(\boldsymbol{z}) &= \frac{1}{\sqrt{d}}(\mathcal{S}_{\theta,\varrho} f \circ \theta)(\boldsymbol{z})\ell_\varrho(\boldsymbol{z}) = \frac{1}{\sqrt{d}}\ell_\varrho(\boldsymbol{z})\langle f, S_{\theta,\varrho}(\cdot, \theta(\boldsymbol{z}))\rangle \\ &= \frac{1}{\sqrt{d}}\ell_\varrho(\boldsymbol{z})\langle \Gamma_\varrho f, \Gamma_\varrho\big(S_{\theta,\varrho}(\cdot, \theta(\boldsymbol{z}))\big)\rangle = \frac{1}{\sqrt{d}}\langle \Gamma_\varrho f, \sqrt{d} S_\varrho(\cdot, \boldsymbol{z})\rangle \\ &= (\mathcal{S}_{\mathbb{D}^n}\Gamma_\varrho f)(\boldsymbol{z}), \end{aligned} \tag{14}$$

where the penultimate equality follows from Remark 2.6. □

As a consequence of Lemma 2.10 along with previous discussion, we conclude the following which is crucial to prove our main result.

Lemma 2.11. *Under the assumption that $\mathcal{S}_{\mathbb{D}_n}$ is bounded on $L^p(\mathbb{T}^n, |\ell_\rho|^{2-p})$, the following diagram commutes:*

$$\begin{array}{ccc} L^p_\varrho(\theta(\mathbb{T}^n)) \cap L^2_\varrho(\theta(\mathbb{T}^n)) & \xrightarrow{\Gamma_\varrho} & L^p(\mathbb{T}^n, |\ell_\varrho|^{2-p}) \cap L^2(\mathbb{T}^n) \\ \mathcal{S}_{\theta,\varrho} \downarrow & & \downarrow \mathcal{S}_{\mathbb{D}^n} \\ H^p_\varrho(\theta(\mathbb{D}^n)) \cap H^2_\varrho(\theta(\mathbb{D}^n)) & \xrightarrow{\Gamma_\varrho} & H^p(\mathbb{D}^n, |\ell_\varrho|^{2-p}) \cap H^2(\mathbb{D}^n). \end{array}$$

3. L^p regularity of Szegö projections

3.1. Muckenhoupt weights. In this section, we recall the preliminaries related to the theory of Muckenhoupt weights on the circle, denoted by $A_p(\mathbb{T})$. A *weight* ω is a locally integrable function and $\omega(x) > 0$ almost everywhere. In 1972, Muckenhoupt [22] characterized the class of weights for which the Hardy-Littlewood maximal operator maps weighted L^p spaces to itself. Subsequently, Coifman and Fefferman [5] studied these weights in connection with

Calderón-Zygmund operators. Since in this article we confine ourselves to the Szegö projection, let us recall the definition of $A_p(\mathbb{T})$ weights.

Definition 3.1. *Let $1 < p < \infty$. We say $\omega \in A_p(\mathbb{T})$ if there exists a constant $C > 0$ such that*

$$[\omega]_{A_p(\mathbb{T})} := \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|} \int_I \omega \right) \left(\frac{1}{|I|} \int_I \omega^{-\frac{1}{p-1}} \right)^{p-1} \leq C < \infty, \tag{15}$$

where the supremum is over all intervals I in $\mathbb{T}$ and $|I|$ denotes its arc-length measure.

The following result characterizes $A_p(\mathbb{T})$ weights in terms of the Szegö projection and it will be heavily used in our proof of L^p regularity.

Theorem 3.2 ([8])**.** *Let $1 < p < \infty$. The Szegö projection $\mathcal{S}_{\mathbb{D}}$ maps $L^p(\omega, \mathbb{T})$ to itself if and only if $\omega \in A_p(\mathbb{T})$.*

We also need the following characterization of power weights. It is well known, however, to make the exposition complete, we supply a simple proof.

Lemma 3.3. *Let $\alpha, \beta, \gamma > 0$, $1 < p < \infty$ and $\kappa \in \mathbb{R}$. Then the weight $\omega(z) := |z - e^{i\kappa}|^{\frac{\beta - \gamma p}{\alpha}} \in A_p(\mathbb{T})$ if and only if $p \in \left(\frac{\beta+\alpha}{\alpha+\gamma}, \frac{\beta+\alpha}{\gamma} \right)$, with $[\omega]_{A_p(\mathbb{T})}$ independent of κ.*

Proof. We first prove that if $p \in \left(\frac{\beta+\alpha}{\alpha+\gamma}, \frac{\beta+\alpha}{\gamma} \right)$ then ω satisfies the condition (15) and $[\omega]_{A_p(\mathbb{T})}$ depends on α, β, γ, and p. Also, note that using change of variables, we may simply assume that $\kappa = 0$. Let $I(\vartheta, r) = \{e^{i\varsigma} : |e^{i\vartheta} - e^{i\varsigma}| < r\}$ be any interval on $\mathbb{T}$. Depending on the position of the interval I, we need to handle the following cases:

Case 1: $|e^{i\vartheta} - 1| \geq 5r$. Observe that in this case we have

$$|e^{i\varsigma} - 1| \geq |e^{i\vartheta} - 1| - |e^{i\vartheta} - e^{i\varsigma}| \geq \frac{4}{5} |e^{i\vartheta} - 1|,$$

for all $e^{i\varsigma} \in I(\vartheta, r)$. Similarly, $|e^{i\varsigma} - 1| \leq \frac{6}{5} |e^{i\vartheta} - 1|$. Therefore, in this case, we obtain

$$\begin{aligned} &\left(\frac{1}{|I(\vartheta, r)|} \int_{I(\vartheta, r)} \omega \right) \left(\frac{1}{|I(\vartheta, r)|} \int_{I(\vartheta, r)} \omega^{-\frac{1}{p-1}} \right)^{p-1} \\ &\simeq C |e^{i\vartheta} - 1|^{\frac{(\beta - \gamma p)}{\alpha}} \left(|e^{i\vartheta} - 1|^{-\frac{(\beta - \gamma p)}{\alpha(p-1)}} \right)^{p-1} \simeq C, \end{aligned}$$

where C is a fixed dimensional constant.

Case 2: $|e^{i\vartheta}-1|\le 5r$. Without loss of generality we may assume r is small since the weight ω (and $\omega^{-\frac{1}{p-1}}$) is only singular near 1. In this case $I(\vartheta,r)\subset I(0,7r)$. Therefore,

$$\begin{aligned}
&\left(\frac{1}{|I(\vartheta,r)|}\int_{I(\vartheta,r)}\omega\right)\left(\frac{1}{|I(\vartheta,r)|}\int_{I(\vartheta,r)}\omega^{-\frac{1}{p-1}}\right)^{p-1}\\
&\le\left(\frac{1}{r}\int_{I(0,7r)}|e^{i\varsigma}-1|^{\frac{(\beta-\gamma p)}{\alpha}}\,d\varsigma\right)\left(\frac{1}{r}\int_{I(0,7r)}|e^{i\varsigma}-1|^{-\frac{(\beta-\gamma p)}{\alpha(p-1)}}\,d\varsigma\right)^{p-1}\\
&\le C\left(\frac{1}{r}\int_0^{cr}\varsigma^{\frac{(\beta-\gamma p)}{\alpha}}\,d\varsigma\right)\left(\frac{1}{r}\int_0^{cr}\varsigma^{-\frac{(\beta-\gamma p)}{\alpha(p-1)}}\,d\varsigma\right)^{p-1}\\
&\le C_{\beta,\gamma,p,\alpha}\,\frac{r^{\frac{(\beta-\gamma p)}{\alpha}+1}}{r}\left(\frac{r^{1-\frac{(\beta-\gamma p)}{\alpha(p-1)}}}{r}\right)^{p-1}\le C_{\beta,\gamma,p,\alpha},
\end{aligned}$$

provided $\frac{(\beta-\gamma p)}{\alpha}+1>0$ and $1-\frac{(\beta-\gamma p)}{\alpha(p-1)}>0$. Rewriting the above inequalities, we obtain the specified range $\frac{\beta+\alpha}{\alpha+\gamma}<p<\frac{\beta+\alpha}{\gamma}$. The necessary part is easy to see since for $p\notin\left(\frac{\beta+\alpha}{\alpha+\gamma},\frac{\beta+\alpha}{\gamma}\right)$ either ω or $\omega^{-\frac{1}{p-1}}$ is not even locally integrable. □

The following is a simple consequence of Hölder's inequality and can be found in [8].

Lemma 3.4. *Let $\omega_1,\omega_2,\dots,\omega_k\in A_p(\mathbb{T})$. For any collection of scalars $\{\alpha_i\}$ with $\sum_{i=1}^{k}\alpha_i=1$, we have $\omega=\prod_{i=1}^{k}\omega_i^{\alpha_i}\in A_p(\mathbb{T})$.*

3.2. L^p regularity. The following is our main result regarding the L^p regularity of generalized Szegö projections on quotient domains. Recall the linear form $\ell_\varrho=\prod_{j=1}^{t}\ell_j^{c_j}$, where t and the c_j's are described in Lemma 2.2. Observe that if ϱ is the trivial representation $\mathrm{tr}:\mathrm{G}\to\mathbb{C}^*$, defined by $\mathrm{tr}(\sigma)=1$ and $\sigma\in G$, then $\ell_{\mathrm{tr}}=1$, and in this case the generalized Szegö projection $\mathcal{S}_{\theta,\mathrm{tr}}$ is trivially bounded from $L^p_{\mathrm{tr}}(\theta(\mathbb{T}^n))\to H^p_{\mathrm{tr}}(\theta(\mathbb{D}^n))$ for $1<p<\infty$ since $\mathcal{S}_{\mathbb{D}^n}$ maps $L^p(\mathbb{T}^n)$ to itself for $1<p<\infty$ (see Equation (18)). Therefore, from here onwards we only consider representations which are not equivalent to the trivial representation. Without loss of generality, we may assume that $\ell_\varrho=\prod_{\substack{j=1\\c_j\neq0}}^{t}\ell_j^{c_j}$. Let us set $\ell_j(\boldsymbol{z})=\sum_{k=1}^{n}a_{jk}z_k$ for all $1\le j\le t$. In the sequel, Z_i, $1\le i\le n$, denotes the following set

$$Z_i=\{j:1\le j\le t\text{ and }a_{ji}\neq0\}. \tag{16}$$

Theorem 3.5. *Let G be a finite pseudoreflection group that acts on $\mathbb{D}^n$ and θ be a basic polynomial associated to G. For $\varrho \in \widehat{G}_1$, there exists an interval (a_ϱ, b_ϱ), $1 < a_\varrho < 2 < b_\varrho < \infty$, such that the generalized Szegö projection $\mathcal{S}_{\theta,\varrho}$ is bounded from $L^p_\varrho(\theta(\mathbb{T}^n))$ to $H^p_\varrho(\theta(\mathbb{D}^n))$ if $p \in (a_\varrho, b_\varrho)$, where*

$$(a_\varrho, b_\varrho) := \bigcap_{i=1}^{n} \bigcap_{j \in Z_i} \left(\frac{2c_j + \alpha_j^i}{c_j + \alpha_j^i}, \frac{2c_j + \alpha_j^i}{c_j} \right), \tag{17}$$

and for each $1 \le i \le n$, $\{\alpha_j^i\}_{j \in Z_i}$ are positive numbers such that $\sum_{j \in Z_i} \alpha_j^i = 1$.

Proof. L^p regularity follows from the following chain of arguments. Note that

$$\begin{aligned}
&\|\mathcal{S}_{\theta,\varrho} f\|_{L^p_\varrho(\theta(\mathbb{T}^n))} \lesssim \|f\|_{L^p_\varrho(\theta(\mathbb{T}^n))} \\
\iff &\int_{\theta(\mathbb{T}^n)} |\mathcal{S}_{\theta,\varrho} f|^p d\Theta_{\varrho,\theta} \le C \int_{\theta(\mathbb{T}^n)} |f|^p d\Theta_{\varrho,\theta} \\
\iff &\int_{\mathbb{T}^n} |\mathcal{S}_{\theta,\varrho} f \circ \theta|^p \ |\ell_\varrho|^2 d\Theta \le C \int_{\mathbb{T}^n} |f \circ \theta|^p \ |\ell_\varrho|^2 d\Theta \\
\iff &\int_{\mathbb{T}^n} |\mathcal{S}_{\mathbb{D}^n}(\ell_\varrho f \circ \theta)|^p \ |\ell_\varrho|^{2-p} d\Theta \le C \int_{\mathbb{T}^n} |\ell_\varrho f \circ \theta|^p \ |\ell_\varrho|^{2-p} d\Theta.
\end{aligned} \tag{18}$$

Therefore, the exponents p for which $\mathcal{S}_{\theta,\varrho}$ maps $L^p_\varrho(\partial\theta(\mathbb{D}^n))$ to itself will be dictated by the weighted boundedness of $\mathcal{S}_{\mathbb{D}^n}$. Since we are only concerned with the distinguished boundary, we have $\mathcal{S}_{\mathbb{D}^n} = \bigotimes_{i=1}^n \mathcal{S}_{\mathbb{D}}$. This allows us to consider the weight $|\ell_\varrho|^{2-p}$ coordinatewise and it is enough to assure the boundedness in each coordinate uniformly. More precisely,

$$\begin{aligned}
&\int_{\mathbb{T}^n} |\mathcal{S}_{\mathbb{D}^n}(\ell_\varrho f \circ \theta)|^p \ |\ell_\varrho|^{2-p} d\Theta \\
&= \int_{\mathbb{T}^n} |\bigotimes_{i=1}^{n} \mathcal{S}_{\mathbb{D}}(\ell_\varrho f \circ \theta)|^p \ |\ell_\varrho|^{2-p} d\Theta \\
&= \int_{\mathbb{T}^{n-1}} |\bigotimes_{i=1}^{n-1} \mathcal{S}_{\mathbb{D}}(\ell_\varrho f \circ \theta)|^p \prod_{i=2}^{n} dz_i \\
&\int_{\mathbb{T}} |\mathcal{S}_{\mathbb{D}}(\ell_\varrho f \circ \theta)|^p \ \prod_{j \in Z_1} |\ell_j|^{c_j(2-p)} dz_1 \prod_{j \notin Z_1} |\ell_j|^{c_j(2-p)} \\
&\le \int_{\mathbb{T}^{n-1}} |\bigotimes_{i=1}^{n-1} \mathcal{S}_{\mathbb{D}}(\ell_\varrho f \circ \theta)|^p \prod_{i=2}^{n} dz_i \\
&\int_{\mathbb{T}} |\mathcal{S}_{\mathbb{D}}(\ell_\varrho f \circ \theta)|^p \ \prod_{j \in Z_1} |a_{j1}|^{c_j(2-p)} |z_1 + \phi_j(z_2, \cdots, z_n)|^{\frac{c_j(2-p)}{\alpha_j^1}\alpha_j^1} \prod_{j \notin Z_1} |\ell_j|^{c_j(2-p)} dz_1
\end{aligned}$$

,

where $\{\alpha_j^1\}_{j\in Z_1}$ are such that $\sum_{j\in Z_1}\alpha_j^1 = 1$ and since each $\ell_j = \sum_{k=1}^n a_{jk}z_k, j \in Z_1$, we write it as $\ell_j = a_{j1}(z_1+\phi_j(z_2,\cdots,z_n))$ with $\phi_j(z_2,\cdots,z_n) := \sum_{k=2}^n \frac{a_{jk}}{a_{j1}}z_k$. At this point, invoking Theorem 3.2, Lemma 3.3 and Lemma 3.4, if

$$p \in \bigcap_{j\in Z_1}\left(\frac{2c_j+\alpha_j^1}{c_j+\alpha_j^1}, \frac{2c_j+\alpha_j^1}{c_j}\right),$$

we have

$$\begin{aligned}\int_{\mathbb{T}^n} &|\mathcal{S}_{\mathbb{D}^n}(\ell_\varrho f\circ\theta)|^p\ |\ell_\varrho|^{2-p}d\Theta \\ &\le C\int_{\mathbb{T}^{n-1}}|\bigotimes_{i=1}^{n-1}\mathcal{S}_{\mathbb{D}}(\ell_\varrho f\circ\theta)|^p\int_{\mathbb{T}}|\ell_\varrho f\circ\theta)|^p\prod_{j\in Z_1}|a_{j1}|^{c_j(2-p)} \qquad (19)\\ &\qquad |z_1+\phi_j(z_2,\cdots,z_n)|^{\frac{c_j(2-p)}{\alpha_j^1}\alpha_j^1}dz_1\prod_{j\notin Z_1}|\ell_j|^{c_j(2-p)}\prod_{i=2}^n dz_i.\end{aligned}$$

Now recursively applying the above procedure for each coordinate, we ensure the following

$$\int_{\mathbb{T}^n}|\mathcal{S}_{\mathbb{D}^n}(\ell_\varrho f\circ\theta)|^p\ |\ell_\varrho|^{2-p}d\Theta \le_{\alpha_j^i,p,c_i} C\int_{\mathbb{T}^n}|\ell_\varrho f\circ\theta|^p\ |\ell_\varrho|^{2-p}d\Theta$$

holds true for

$$p \in \bigcap_{i=1}^n\bigcap_{j\in Z_i}\left(\frac{2c_j+\alpha_j^i}{c_j+\alpha_j^i}, \frac{2c_j+\alpha_j^i}{c_j}\right),$$

where for each $1 \le i \le n$, $\{\alpha_j^i\}_{j\in Z_i}$ are positive numbers such that $\sum_{j\in Z_i}\alpha_j^i = 1$.

Note that we can use (19) recursively since at the i-th iteration the constants only depend on α_j^i, c_j, p, where $1 \le i \le t$. Moreover, as each of the intervals $((2c_j+\alpha_j^i)/(c_j+\alpha_j^i), (2c_j+\alpha_j^i)/c_j), 1 \le i \le n, j \in Z_i$ is Hölder symmetric, it is easy to see that (a_ϱ, b_ϱ) is also Hölder symmetric. □

Remark 3.6. *We specialize Theorem 3.5 for some particular representations.*

- ***Sign representation:*** *Recall that for the sign representation we have the following*

$$\ell_{\text{sgn}}(\boldsymbol{z}) = c\prod_{i=1}^t \ell_i^{m_i-1}(\boldsymbol{z}),$$

where m_i is the order of the cyclic subgroup G_i for $1 \le i \le t$ (see Corollary 2.3). Therefore, $c_j = m_j - 1$, hence invoking Theorem 3.5, we obtain *that the Szegö projection $\mathcal{S}_\theta$ maps $L^p_{\mathrm{sgn}}(\theta(\mathbb{T}^n))$ to itself if*

$$p \in \bigcap_{i=1}^{n} \bigcap_{j\in Z_i} \left(\frac{2(m_j-1)+\alpha^i_j}{m_j-1+\alpha^i_j}, \frac{2(m_j-1)+\alpha^i_j}{m_j-1} \right).$$

- *Another very interesting case appears when c_i's are equal, that is, $c_i = \varkappa \neq 0$ for all $1 \le i \le t$. In that case, using homogeneity of ℓ_ϱ, we can conclude that $|Z_i| = |Z_j|$ for all $1 \le i, j \le n$. Let $|Z_i| = M$. Then an easy computation reveals that we can obtain the maximum range for L^p regularity provided we choose $|\alpha^i_j| = \frac{1}{|Z_i|} = \frac{1}{M}$ for all $1 \le i \le n$ and $j \in Z_i$, which in turn implies the L^p regularity for the range*

$$\left(\frac{2\varkappa M+1}{\varkappa M+1}, \frac{2\varkappa M+1}{\varkappa M} \right). \tag{20}$$

A more intrinsic description of the interval (a_ϱ, b_ϱ) can be given for one-dimensional representations of a reflection group. Recall that a reflection is a pseudoreflection of order 2 and a finite group generated by reflections is called a reflection group. The permutation group on n symbols and the dihedral group are examples of reflection groups. For a reflection group G, $m_j = 2$, for every $1 \le j \le t$. Thus $J_\theta = \prod_{i=1}^t \ell_i$. Recall that for a one-dimensional representation ϱ of G, the generating polynomial $\ell_\varrho = \prod_{i=1}^t \ell_i^{c_i}$ divides the complex Jacobian J_θ, hence the corresponding c_j's are either 1 or 0. Let

$$Z_{i,\varrho} = \{j : c_j \neq 0\} \cap Z_i, \quad i = 1, \dots, n,$$

where Z_i is as described in Equation (16). Using homogeneity of ℓ_ϱ, we conclude that there exists a natural number M_ϱ such that $|Z_{i,\varrho}| = M_\varrho$ for every $i = 1, \dots, n$. We specialize to the following corollary for the one-dimensional representations of reflection groups.

Corollary 3.7. *Let G be a finite reflection group that acts on $\mathbb{D}^n$ and θ be a basic polynomial associated to G. Let $\varrho \in \widehat{G}_1$ be a one-dimensional representation which is not equivalent to the trivial representation. Then there exists a natural number M_ϱ such that the generalized Szegö projection $\mathcal{S}_{\theta,\varrho}$ is bounded from $L^p_\varrho(\theta(\mathbb{T}^n))$ to itself if $p \in \left(\frac{2M_\varrho+1}{M_\varrho+1}, \frac{2M_\varrho+1}{M_\varrho}\right)$.*

In particular, the Szegö projection $\mathcal{S}_\theta$ is bounded from $L^p_{\mathrm{sgn}}(\theta(\mathbb{T}^n))$ to itself if $p \in \left(\frac{2M+1}{M+1}, \frac{2M+1}{M}\right)$, where $M = |Z_j|$ for all $j = 1, \dots, n$ and Z_j is as described in Equation (16).

Remark 3.8. *The reader should note that different choices of positive numbers $\{\alpha^i_j\}_{j\in Z_i}$ with $\sum_{j\in Z_i} \alpha^i_j = 1$, $1 \le i \le n$, lead to different intervals (a_ϱ, b_ϱ) of the form (17) for which L^p regularity holds in Theorem 3.5. Therefore, it is an interesting problem to optimize the choice of α^i_j so that one obtains the sharp range of L^p regularity for the Szegö projection. Very recently, the sharp range for L^p regularity for the Bergman projection on the symmetrized polydisc is obtained in* [12], *however, here we are more concerned with providing a framework which works for general quotient domains and we will take up issues regarding sharpness in future work.*

Now we are at a position to describe our applications of Theorem 3.5.

3.3. Symmetrized polydisc. As an application of the last Corollary, we obtain regularity properties of the Szegö projection on the symmetrized polydisc. The permutation group on n symbols is denoted by $\mathfrak{S}_n$. The group $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by permuting its coordinates, that is,

$$\sigma \cdot (z_1, \dots, z_n) = (z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)}),$$

for $\sigma \in \mathfrak{S}_n$ and $(z_1, \dots, z_n) \in \mathbb{C}^n$. Clearly, the open unit polydisc $\mathbb{D}^n$ is invariant under the action of the group $\mathfrak{S}_n$.

Let s_k denote the elementary symmetric polynomials of degree k in n variables, for $k = 1, \dots, n$. The symmetrization map

$$\boldsymbol{s} := (s_1, \dots, s_n) : \mathbb{C}^n \to \mathbb{C}^n \tag{21}$$

is a basic polynomial map associated to the pseudoreflection group $\mathfrak{S}_n$. The domain $\mathbb{G}_n := \boldsymbol{s}(\mathbb{D}^n)$ is known as the symmetrized polydisc. It is well-known that the symmetric group $\mathfrak{S}_n$ has only two one-dimensional representations in $\widehat{\mathfrak{S}}_n$, the sign representation and the trivial representation of $\mathfrak{S}_n$. For the trivial representation, we have already seen that the Szegö projection $\mathcal{S}_{\boldsymbol{s},\mathrm{tr}}$ maps $L^p_{\mathrm{tr}}(\boldsymbol{s}(\mathbb{T}^n))$ to $H^p_{\mathrm{tr}}(\mathbb{G}_n)$ for all $1 < p < \infty$.

- **Sign representation:** In [6, p. 370, Lemma 10] the generating polynomial for the sign representation is computed and is given by the following:

  $$\ell_{\mathrm{sgn}}(\boldsymbol{z}) = \prod_{i<j} (z_i - z_j).$$

  The following is our result in this setting. Recall that $\mathcal{S}_{\boldsymbol{s}}$ is the shorthand for $\mathcal{S}_{\boldsymbol{s},\mathrm{sgn}}$.

  Proposition 3.9. *Let $p \in (2 - \frac{1}{n}, 2 + \frac{1}{n-1})$ then the Szegö projection $\mathcal{S}_{\boldsymbol{s}}$ maps $L^p_{\mathrm{sgn}}(\boldsymbol{s}(\mathbb{T}^n))$ to $H^p_{\mathrm{sgn}}(\mathbb{G}_n)$.*

  Proof. The proof trivially follows from Corollary 3.7 since in this case $M = (n - 1)$. □

Finally, as a consequence of Theorem 3.5, we show L^p regularity of generalized Szegö projections on the quotient group $\mathbb{D}^2/D_{2k}$.

3.4. The domain $\mathscr{D}_{2k}$. Consider the polynomial map $\boldsymbol{\phi}(z_1, z_2) = (z_1^k + z_2^k, z_1 z_2)$ on $\mathbb{D}^2$. It is a proper holomorphic map of multiplicity $2k$ and let us denote the domain $\phi(\mathbb{D}^2)$ by $\mathscr{D}_{2k}$. The domain $\mathscr{D}_{2k}$ is biholomorphically equivalent to the quotient domain $\mathbb{D}^2/D_{2k}$ where D_{2k} is the dihedral group of order $2k$, that is,

$$D_{2k} = \langle \delta, \sigma : \delta^k = \sigma^2 = \mathrm{id}, \sigma\delta\sigma^{-1} = \delta^{-1} \rangle.$$

See subsection 3.1.1 in [3] and page 19 in [9] for more details. Clearly, $J_{\boldsymbol{\phi}}(z_1, z_2) = k(z_1^k - z_2^k)$. The number of one-dimensional representations of the dihedral group D_{2k} in $\widehat{D}_{2k}$ is 2 if k is odd and 4 if k is even. Clearly, for every $k \in \mathbb{N}$ the trivial representation of D_{2k} and the sign representation of D_{2k} are in $\widehat{D}_{2k}$. Since for the trivial representation $\ell_{\text{tr}} = 1$, we automatically obtain that the Szegö projection $\mathcal{S}_{\boldsymbol{\phi},\text{tr}}$ maps $L^p_{\text{tr}}(\boldsymbol{\phi}(\mathbb{T}^n))$ to $H^p_{\text{tr}}(\mathscr{D}_{2k})$ for all $1 < p < \infty$.

- **Sign representation:** For the sign representation we have the following:

$$\ell_{\text{sgn}}(\boldsymbol{z}) = k(z_1^k - z_2^k). \tag{22}$$

Now we state the L^p regularity in this setting.

Proposition 3.10. *Let $p \in (\frac{2k+1}{k+1}, \frac{2k+1}{k})$ then the Szegö projection $\mathcal{S}_{\boldsymbol{\phi}}$ maps $L^p_{\text{sgn}}(\boldsymbol{\phi}(\mathbb{T}^n))$ to $H^p_{\text{sgn}}(\mathscr{D}_{2k})$.*

Proof. In view of (22), we obtain that ℓ_{sgn} can be factored into k many degree one polynomials in z_1, that is $\prod_{i=1}^k \ell_i$, thus $c_i = 1$ and the cardinality of the set Z_1 and Z_2 is k. Now a direct application of Corollary 3.7 implies that for $p \in (\frac{2k+1}{k+1}, \frac{2k+1}{k})$, the Szegö projection $\mathcal{S}_{\boldsymbol{\phi}}$ maps $L^p_{\text{sgn}}(\boldsymbol{\phi}(\mathbb{T}^n))$ to $H^p_{\text{sgn}}(\mathscr{D}_{2k})$. □

We now elaborate the L^p regularity for the Szegö projection associated to the additional two representations in the case when k is even. Let $k = 2j$ for some $j \in \mathbb{N}$.

(1) Let us consider the representation ϱ_1 defined as

$$\varrho_1(\delta) = -1 \quad \text{and} \quad \varrho_1(\tau) = 1 \text{ for } \tau \in \langle \delta^2, \sigma \rangle.$$

It is known that (see [9]) $\ell_{\varrho_1}(\boldsymbol{z}) = z_1^j + z_2^j$ and therefore as a consequence of Theorem 3.5 and Remark 3.6, the Szegö projection $\mathcal{S}_{\boldsymbol{\phi},\varrho_1}$ maps $L^p_{\varrho_1}(\boldsymbol{\phi}(\mathbb{T}^n))$ to $H^p_{\varrho_1}(\mathscr{D}_{2k})$ for $p \in (\frac{2j+1}{j+1}, \frac{2j+1}{j})$.

(2) The representation ϱ_2 is defined as following:

$$\varrho_2(\delta) = -1 \quad \text{and} \quad \varrho_2(\tau) = 1 \text{ for } \tau \in \langle \delta^2, \delta\sigma \rangle.$$

In this case $\ell_{\varrho_2}(\boldsymbol{z}) = z_1^j - z_2^j$, therefore, arguing similarly we obtain that the Szegö projection $\mathcal{S}_{\boldsymbol{\phi},\varrho_2}$ maps $L^p_{\varrho_2}(\boldsymbol{\phi}(\mathbb{T}^n))$ to $H^p_{\varrho_2}(\mathscr{D}_{2k})$ for $p \in (\frac{2j+1}{j+1}, \frac{2j+1}{j})$.

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References

[1] BÉKOLLÉ, D.; BONAMI, A. Estimates for the Bergman and Szegö projections in two symmetric domains of $\mathbf{C}^n$. *Colloq. Math.* **68** (1995), no. 1, 81–100. MR1311766, Zbl 0863.47018. 912

[2] BENDER, C; CHAKRABARTI, D; EDHOLM, L; MAINKAR, M. L^p-regularity of the Bergman projection on quotient domains. *Canad. J. Math.* **74** (2022), no. 3, 732–772. MR4430928, Zbl 1495.32019. 913

[3] BISWAS, S; DATTA, S; GHOSH, G; SHYAM ROY, S. Reducing submodules of Hilbert modules and Chevalley-Shephard-Todd theorem. *Adv. Math.* **403** (2022), Paper No. 108366, 54 pp. MR4404033, Zbl 7534687. 913, 915, 919, 928

[4] CHEN, L; KRANTZ, S. G; YUAN, Y. L^p regularity of the Bergman projection on domains covered by the polydisc. *J. Funct. Anal.* **279** (2020), no. 2, 20 pp. MR4088498, Zbl 1457.32003. 913

[5] COIFMAN, R.R; FEFFERMAN, C. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–250. MR358205, Zbl 0291.44007. 921

[6] EDIGARIAN, A; ZWONEK, W. Geometry of the symmetrized polydisc. *Arch. Math. (Basel).* **84** (2005), no. 4, 364-–374. MR2135687, Zbl 1068.32012. 927

[7] GAMELIN, T. W. Uniform algebras. *Prentice-Hall, Inc., Englewood Cliffs, N.J.* 1969. xiii+257 pp. MR0410387, Zbl 0213.40401. 915

[8] GARNETT, J. B. Bounded analytic functions. Graduate Texts in Mathematics, vol. 236, *Springer, New York.* 2007. xiv+459 pp. MR2261424, Zbl 0469.30024. 922, 923

[9] GHOSH, G. The weighted Bergman spaces and pseudoreflection groups. arXiv:2104.14162. 913, 916, 918, 928

[10] GRELLIER, S; PELOSO, M. M. Decomposition theorems for Hardy spaces on convex domains of finite type. *Illinois J. Math.* **46** (2002), no. 46, 207–232. MR1936086, Zbl 1016.32005. 912

[11] HAN, K; TANG, Y. L^p regularity of the Szegö projection on the symmetrized polydisc. *Bull. Aust. Math. Soc.* **106** (2022), no. 3, 481–490. MR4510140, Zbl 1505.32006. 913, 914

[12] HUO, Z; WICK, B. D. L^p regularity of the Bergman projection on the symmetrized polydisc. arXiv:2303.10002. 926

[13] KERZMAN, N; STEIN, E. M. The Cauchy kernel, the Szegö kernel, and the Riemann mapping function. *Math. Ann.* **236** (1978), no. 1, 85–93. MR486468, Zbl 0419.30012. 912

[14] KOSIŃSKI, L; ZWONEK, W. Proper holomorphic mappings vs. peak points and Shilov boundary. *Ann. Polon. Math.* **107** (2013), no. 1, 97–108. MR3001625, Zbl 1271.32036. 916

[15] KOSMANN-SCHWARZBACH, Y. Groups and symmetries. Universitext, *Springer, New York.* 2010. xvi+194 pp. MR2553682, Zbl 7471582. 918

[16] LANZANI, L; STEIN, E. M. The Cauchy-Szegö projection for domains in $\mathbb{C}^n$ with minimal smoothness. *Duke Math. J.* **166** (2017), no. 1, 125–176. MR3592690, Zbl 1367.32005. 912

[17] LEHRER, G. I.; TAYLOR, D. E. Unitary reflection groups. Australian Mathematical Society Lecture Series, vol. 20, *Cambridge University Press, Cambridge.* 2009. viii+294 pp. MR2542964, Zbl 1189.20001. 913

[18] MCNEAL, J. D.; STEIN, E. M. The Szegö projection on convex domains, *Math. Z.,* **224** (1997), no. 4, 519–553. MR1452048, Zbl 0948.32004. 912

[19] MISRA, G; SHYAM ROY, S; ZHANG, G. Reproducing kernel for a class of weighted Bergman spaces on the symmetrized polydisc, *Proc. Amer. Math. Soc.* **141** (2013), no. 7, 2361–2370. MR3043017, Zbl 1273.32011. 913, 917

[20] MONGUZZI, A. Holomorphic function spaces on the Hartogs triangle. Math. Nachr. **294** (2021), no. 11, 2209–2231. MR4371295. 912

[21] MONGUZZI, A; PELOSO, M. M. Sharp estimates for the Szegö projection on the distinguished boundary of model worm domains. *Integral Equations Operator Theory.* **89** (2017), no. 3 ,315–344. MR3722049, Zbl 1393.32015. 912

[22] MUCKENHOUPT, B. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* 165 (1972), 207–226. MR293384, Zbl 0236.26016. 921

[23] MUNASINGHE, S; ZEYTUNCU, Y. E. L^p regularity of weighted Szegö projections on the unit disc. *Pacific J. Math.* **276** (2015), no. 2, 449–458. MR3374068, Zbl 1331.30049. 913
[24] PHONG, D. H; STEIN, E. M. Estimates for the Bergman and Szegö projections on strongly pseudo-convex domains. *Duke Math. J.* **44** (1977), no. 3, 695–704. MR450623, Zbl 0392.32014. 912
[25] SHEPHARD, G. C; TODD, J. A. Finite unitary reflection groups. *Canad. J. Math.* **6** (1954), 274–304. MR59914, Zbl 0055.14305. 915
[26] STANLEY, R. P. Relative invariants of finite groups generated by pseudoreflections, *J. Algebra.* **49** (1977), no. 1, 134–148. MR460484, Zbl 0383.20029. 916
[27] STEINBERG, R. Invariants of finite reflection groups. *Canadian J. Math.* **12** (1960), 616–618. MR117285, Zbl 0099.36802. 917
[28] STOCKDALE, C. B.; WAGNER, N. A. Weighted endpoint bounds for the Bergman and Cauchy–Szegö projections on domains with near minimal smoothness, *Indiana Univ. Math. J.* **71** (2022), no. 5, 2099-2125. MR4509829, Zbl 1505.32007. 912
[29] TRYBULA, M. Proper holomorphic mappings, Bell's formula, and the Lu Qi-Keng problem on the tetrablock. *Arch. Math. (Basel).* **101** (2013), no. 6, 549–558. MR3133729, Zbl 1286.32002. 915
[30] WAGNER, N. A.; WICK, B. D. Weighted L^p estimates for the Bergman and Szegö projections on strongly pseudoconvex domains with near minimal smoothness. *Adv. Math.* **384** (2021), Paper No. 107745, 45 pp. MR4246099, Zbl 1482.32005. 912

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