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Hilbert modules, rigged modules, and stable isomorphism

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Abstract. Rigged modules over an operator algebra are a generalization of Hilbert modules over a C^*-algebra. We characterize the rigged modules over an operator algebra $\mathcal{A}$ which are orthogonally complemented in $C_\infty(\mathcal{A})$, the space of infinite columns with entries in $\mathcal{A}$. We show that every such rigged module 'restricts' to a bimodule of Morita equivalence between appropriate stably isomorphic operator algebras.

Contents

1. Introduction

Let X, Y be operator spaces. We call them stably isomorphic if the spatial tensor products $X \otimes \mathcal{K}$, $Y \otimes \mathcal{K}$ are completely isometrically isomorphic, where $\mathcal{K}$ is the algebra of compact operators acting on an infinite dimensional Hilbert space. We also denote by $C_\infty(X)$ the operator space of infinite columns with entries in X. In the case where X is a right rigged module over an operator algebra, $\mathcal{A}$, so is $C_\infty(X)$.

The notion of a Hilbert C^*-module was introduced and developed in the early 1970s by Paschke and Rieffel, see [14, 18]. A Hilbert module over a C^*-algebra $\mathcal{A}$ is a right $\mathcal{A}$-module Y together with a map $\langle \cdot, \cdot \rangle : Y \times Y \to \mathcal{A}$ which is linear in the second variable, and which also satisfies the following conditions:

(1) $\langle y, y \rangle \geq 0$ for all $y \in Y$,
(2) $\langle y, y \rangle = 0 \Leftrightarrow y = 0$,
(3) $\langle y, za \rangle = \langle y, z \rangle a$, for all $y,\ z \in Y,\ a \in \mathcal{A}$,

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(4) $\langle y, z\rangle^* = \langle z, y\rangle$ for all $y,\ z \in Y$,
(5) Y is complete in the norm $\|y\| = \|\langle y, y\rangle\|^{1/2}$.

Observe that the space $I_{\mathcal{A}}(Y)$, which is the closure of the linear span of the set $\{\langle y, z\rangle \in \mathcal{A} \mid y,\ z \in Y\}$, is an ideal of $\mathcal{A}$.

Consider the C^*-algebra $\mathbb{K}_{\mathcal{A}}(Y)$ of the 'compact' adjointable operators from Y to Y. It is known that Y is a bimodule of Morita equivalence between $I_{\mathcal{A}}(Y)$ and $\mathbb{K}_{\mathcal{A}}(Y)$. But these C^*-algebras are not always stably isomorphic.

Let Y be a right Hilbert $\mathcal{A}$-module. In case there exists a sequence $(y_k)_{k\in\mathbb{N}} \subseteq Y$ such that

$$y = \sum_{k=1}^{\infty} y_k \langle y_k, y\rangle,\ \forall\, y \in Y$$

where the series converges in the norm of Y, we say that $(y_k)_{k\in\mathbb{N}}$ is a right quasibasis for Y. It follows by the Brown–Kasparov stabilization theorem, see [3, Corollary 8.20], that the spaces $I_{\mathcal{A}}(Y)$, $\mathbb{K}_{\mathcal{A}}(Y)$, Y are all stably isomorphic.

Let Y be a right Hilbert $\mathcal{A}$-module. We call it countably generated if there exists a sequence $(y_k)_{k\in\mathbb{N}} \subseteq Y$ such that

$$Y = \overline{\mathrm{span}}(\{y_k\, a \mid k \in \mathbb{N},\ a \in \mathcal{A}\}).$$

If Y has a right quasibasis, then Y is countably generated and conversely. Every countably generated Hilbert $\mathcal{A}$-module is isomorphic as a Hilbert $\mathcal{A}$-module with an orthogonally complemented bimodule of $C_\infty(\mathcal{A})$.

Blecher in [1] generalized the notion of Hilbert modules to the setting of non-selfadjoint operator algebras. He called these modules rigged modules, see the definition below. Hilbert modules over a C^*-algebra are rigged modules in terms of this definition. Using the notion of a ternary ring of operators, we introduce a new category of $\mathcal{A}$-rigged modules, where $\mathcal{A}$ is an operator algebra, the $\sigma\Delta$-$\mathcal{A}$-rigged modules. We prove that an $\mathcal{A}$-rigged module is a $\sigma\Delta$-$\mathcal{A}$-rigged module if and only if it is isomorphic with an orthogonally complemented module in $C_\infty(\mathcal{A})$. We also introduce a subcategory of the $\sigma\Delta$-$\mathcal{A}$-rigged modules, the doubly $\sigma\Delta$-$\mathcal{A}$-rigged modules. In the case of C^*-algebras, these two categories coincide. Every doubly $\sigma\Delta$-$\mathcal{A}$-rigged module implements a stable isomorphism between the corresponding operator algebras. Conversely, if $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic operator algebras, there exists a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module Y which is a bimodule of strong Morita equivalence (BMP-Morita equivalence) for $\mathcal{A}$ and $\mathcal{B}$ in the sense of Blecher, Muhly and Paulsen, [4]. Every $\sigma\Delta$-$\mathcal{A}$-rigged module has a 'restriction' which is a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module. Thus, every orthogonally complemented rigged module in $C_\infty(\mathcal{A})$, has a 'restriction' making it into a bimodule of BMP-Morita equivalence over some operator algebras $\mathcal{C}$, $\mathcal{D}$. Furthermore, $\mathcal{C}$ and $\mathcal{D}$ are stably isomorphic.

In Section 4, we will develop a theory of Morita equivalence for rigged modules. If $\mathcal{A}$, $\mathcal{B}$ are operator algebras, E is a right $\mathcal{B}$-rigged module, and F is a right $\mathcal{A}$-rigged module, we call E and F σ-Morita equivalent if there exists a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module Y such that

(i) $\mathcal{A} \cong \tilde{Y} \otimes^h_{\mathcal{B}} Y$,

(ii) $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} \tilde{Y}$,
(iii) $F \cong E \otimes^h_{\mathcal{B}} Y$,

where $\tilde{Y}$ is the counterpart bimodule of Y. In this case we write $E \sim_{\sigma M} F$. We will prove that if $E \sim_{\sigma M} F$, then E and F are stably isomorphic.

This paper has been written with an emphasis on the theory of non-selfadjoint operator algebras, but the conclusions for C^*-algebras follow easily.

At this point, we recall some definitions, notation and lemmas which will be useful for what follows.

Definition 1.1. [1]

Let $\mathcal{A}$ be an approximately unital operator algebra, i.e. an operator algebra with a contractive approximate identity, and let Y be a right $\mathcal{A}$-operator module. Suppose there is a net $(n(b))_{b\in B}$ of positive integers and right $\mathcal{A}$-module maps

$$\Phi_b : Y \to C_{n(b)}(\mathcal{A}),\ \Psi_b : C_{n(b)}(\mathcal{A}) \to Y,\ b \in B$$

such that

(i) the maps Φ_b, Ψ_b are completely contractive,
(ii) $\Psi_b \circ \Phi_b \to Id_Y$ strongly on Y,
(iii) the maps Ψ_b, $b \in B$ are right $\mathcal{A}$-essential maps (that is, $\Psi_b\, e_i \to \Psi_b$ for a bounded approximate identity $(e_i)_{i\in I}$ of $\mathcal{A}$),
(iv) $\Phi_c \circ \Psi_b \circ \Phi_b \to \Phi_c$, $\forall\, c \in B$ (uniformly in norm)

Then we say that Y is a right $\mathcal{A}$-rigged module.

We denote by $\mathbb{B}(H,K)$ the space of all linear and bounded operators from the Hilbert space H to the Hilbert space K. If $H = K$, we write $\mathbb{B}(H,H) = \mathbb{B}(H)$. If X is a subset of $\mathbb{B}(H,K)$ and Y is a subset of $\mathbb{B}(K,L)$, then we denote by $\overline{[YX]}$ the norm-closure of the linear span of the set

$$\{y\, x \in \mathbb{B}(H,L) \mid y \in Y,\ x \in X\}.$$

Similarly, if Z is a subset of $\mathbb{B}(L,R)$, we define the space $\overline{[Z\, YX]}$.

Definition 1.2. (i) A linear subspace $M \subseteq \mathbb{B}(H,K)$ is called a ternary ring of operators (TRO) if $M\, M^*\, M \subseteq M$.

(ii) A norm closed ternary ring of operators M is called a σ-TRO if there exist sequences $\{m_i \in M \mid i \in \mathbb{N}\}$ and $\{n_j \in M \mid j \in \mathbb{N}\}$ such that

$$\lim_n \sum_{i=1}^{n} m_i\, m_i^*\, m = m,\ \lim_t \sum_{j=1}^{t} m\, n_j^*\, n_j = m,\ \forall\, m \in M$$

and

$$\left\| \sum_{i=1}^{n} m_i\, m_i^* \right\| \leq 1,\ \left\| \sum_{j=1}^{t} n_j^*\, n_j \right\| \leq 1,\ \forall\, n,\ t \in \mathbb{N}.$$

A norm closed TRO M is a σ-TRO if and only if the C^*-algebras $\overline{[M^*\, M]}$ and $\overline{[M\, M^*]}$ have σ-units, [6].

If X is an operator space, then the spatial tensor product $X \otimes \mathcal{K}$ is completely isometrically isomorphic with the space $K_\infty(X)$, which is the norm closure of the finitely supported matrices in $\mathbb{M}_\infty(X)$. Here, $\mathbb{M}_\infty(X)$ is the space of $\infty \times \infty$ matrices with entries in X which define bounded operators. Also, for another operator space Y, we denote by $X \otimes^h Y$ the Haagerup tensor product of X and Y. If $\mathcal{A}$ is an operator algebra, X is a right $\mathcal{A}$-module, and Y is a left $\mathcal{A}$-module, then we denote by $X \otimes^h_{\mathcal{A}} Y$ the balanced Haagerup tensor product of X and Y over $\mathcal{A}$, see [4]. We now give two basic definitions.

Definition 1.3. Let $X \subseteq \mathbb{B}(H,K)$, $Y \subseteq \mathbb{B}(L,R)$ be operator spaces. We call them σ-TRO equivalent if there exist σ-TROs $M_1 \subseteq \mathbb{B}(H,L)$, $M_2 \subseteq \mathbb{B}(K,R)$ such that

$$X = \overline{[M_2^* Y M_1]}, \ Y = \overline{[M_2 X M_1^*]}.$$

In this case we write $X \sim_{\sigma TRO} Y$.

Definition 1.4. Let X, Y be operator spaces. We call them $\sigma\Delta$ equivalent if there exist completely isometric maps $\phi : X \to \mathbb{B}(H,K)$, $\psi : Y \to \mathbb{B}(L,R)$ such that $\phi(X) \sim_{\sigma TRO} \psi(Y)$. We write $X \sim_{\sigma\Delta} Y$.

If $\mathcal{A}$, $\mathcal{B}$ are abstract or concrete operator algebras, we say that they are $\sigma\Delta$ equivalent and we write $\mathcal{A} \sim_{\sigma\Delta} \mathcal{B}$ if there exist completely isometric representations $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$, $\beta : \mathcal{B} \to \beta(\mathcal{B}) \subseteq \mathbb{B}(K)$ and a σ-TRO $M \subseteq \mathbb{B}(H,K)$ such that

$$a(\mathcal{A}) = \overline{[M^* \beta(\mathcal{B}) M]}, \ \beta(\mathcal{B}) = \overline{[M a(\mathcal{A}) M^*]}.$$

For further details about the notion of $\sigma\Delta$ equivalence of operator algebras and operator spaces, we refer the reader to [7, 8, 9, 10]. If X, Y are operator spaces, then $X \sim_{\sigma\Delta} Y$ if and only if X and Y are stably isomorphic, that is, $K_\infty(X) \cong K_\infty(Y)$ (similarly for operator algebras). We present a lemma which will be used in some of the proofs in the following sections.

Lemma 1.5. *Suppose that $\mathcal{A}$, $\mathcal{B}$ are operator algebras and $D \subseteq \mathcal{B}$ is a C^*-algebra such that $\overline{[D\mathcal{B}]} = \overline{[\mathcal{B}D]} = \mathcal{B}$. Let $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $\overline{[M^* M]} \cong D$ (as C^*-algebras) and assume that $\mathcal{A} \cong M \otimes^h_D \mathcal{B} \otimes^h_D M^\star$. Then, $\mathcal{A} \sim_{\sigma\Delta} \mathcal{B}$.*

A proof of this lemma can be found in [10, Lemma 2.2].

2. Orthogonally complemented modules and $\sigma\Delta$-rigged modules

Let $\mathcal{A}$ be an approximately unital operator algebra and $P : C_\infty(\mathcal{A}) \to C_\infty(\mathcal{A})$ be a left multiplier of $C_\infty(\mathcal{A})$ (that is, $P \in M_\ell(C_\infty(\mathcal{A}))$) such that P is contractive and $P^2 = P$. Then the space $W = P(C_\infty(\mathcal{A}))$ is said to be orthogonally complemented in $C_\infty(\mathcal{A})$. In this section we characterize the orthogonally complemented modules in the the terms of ternary rings of operators. A dual version of the results obtained here is in [2].

Definition 2.1. Let $\mathcal{A} \subseteq \mathbb{B}(H)$ be an approximately unital operator algebra and $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $M^* M \mathcal{A} \subseteq \mathcal{A}$. The operator space $Y_0 = \overline{[M \mathcal{A}]} \subseteq \mathbb{B}(H,K)$ is called a σ-TRO-$\mathcal{A}$-rigged module.

We recall that Y_0 is a right $\mathcal{A}$-operator module with action

$$(m\, a) \cdot x = m(a\, x),\ m \in M,\ a,\ x \in \mathcal{A}.$$

Definition 2.2. Let $\mathcal{A}$ be an abstract approximately unital operator algebra and let Y be an abstract right $\mathcal{A}$-module. We call Y a $\sigma\Delta$-$\mathcal{A}$-rigged module if there exists a completely isometric homomorphism $a : \mathcal{A} \to a(\mathcal{A})$ and there exist a σ-TRO-$a(\mathcal{A})$-rigged module Y_0 and a complete surjective isometry $\rho : Y \to Y_0$ which is also a right $\mathcal{A}$-module map. In case $\mathcal{A}$ is a C^*-algebra we call Y a $\sigma\Delta$-$\mathcal{A}$-Hilbert module.

Proposition 2.3. *Let $\mathcal{A}$ be an approximately unital operator algebra. Every $\sigma\Delta$-$\mathcal{A}$-rigged module is a right rigged module over $\mathcal{A}$ in the sense of Definition* 1.1.

Proof. Let Y be a right $\sigma\Delta$-$\mathcal{A}$-rigged module. Then there exist a completely isometric homomorphism $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$, a σ-TRO $M \subseteq \mathbb{B}(H,K)$ and a complete surjective isometry $\rho : Y \to Y_0 = \overline{[M\, a(\mathcal{A})]}$ which is also a right $\mathcal{A}$-module map. So, if we choose a $\{\Phi_b, \Psi_b \mid b \in B\}$ for the module Y_0, then we define for each $b \in B$ the maps $\Phi'_b = \Phi_b \circ \rho$, $\Psi'_b = \rho^{-1} \circ \Psi_b$ and we can see that the $\{\Phi'_b, \Psi'_b \mid b \in B\}$ satisfy the conditions of Definition 1.1. So, Y becomes a right $\mathcal{A}$-rigged module. Therefore, it suffices to prove the proposition when $Y = \overline{[M\, a(\mathcal{A})]} \subseteq \mathbb{B}(H,K)$. Since M is a σ-TRO, there exists a sequence $\{m_i \in M \mid i \in \mathbb{N}\}$ such that $||(m_i)_{i\in\mathbb{N}}|| \leq 1$ and

$$\sum_{i=1}^{\infty} m_i\, m_i^*\, m = m,\ \forall\, m \in M.$$

Since $Y = \overline{[M\, a(\mathcal{A})]}$, it follows that

$$\sum_{i=1}^{\infty} m_i\, m_i^*\, y = y,\ \forall\, y \in Y.$$

For $n \in \mathbb{N}$ we define

$$\Phi_n : Y \to C_n(\mathcal{A}),\ \Phi_n(y) = \begin{pmatrix} m_1^*\, y \\ \dots \\ m_n^*\, y \end{pmatrix},$$

which is linear and a completely contractive right $\mathcal{A}$-module map. We also define the linear, completely contractive and right $\mathcal{A}$-module map

$$\Psi_n : C_n(\mathcal{A}) \to Y,\ \Psi_n\left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) = \sum_{i=1}^{n} m_i\, a_i.$$

For all $y \in Y$, it holds that

$$\Psi_n \circ \Phi_n(y) = \Psi_n\left(\begin{pmatrix} m_1^* y \\ \dots \\ m_n^* y \end{pmatrix}\right) = \sum_{i=1}^{n} m_i\, m_i^*\, y \to y = Id_Y(y)$$

and we conclude that $\Psi_n \circ \Phi_n \to Id_Y$ strongly on Y. The next step is to prove that Ψ_n, $n \in \mathbb{N}$, is a right $\mathcal{A}$-essential map. To this end, let $(e_i)_{i \in I}$ be a contractive approximate identity of $\mathcal{A}$. We have that

$$\begin{aligned} \left\| \Psi_n e_i \left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) - \Psi_n\left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) \right\| &= \left\| \Psi_n\left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) e_i - \Psi_n\left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) \right\| \\ &= \left\| \sum_{j=1}^{n} (m_j\, a_j)\, e_i - \sum_{j=1}^{n} m_j\, a_j \right\| \\ &= \left\| \sum_{j=1}^{n} m_j\, (a_j\, e_i - a_j) \right\| \\ &\le \sum_{j=1}^{n} ||m_j||\; ||a_j\, e_i - a_j|| \end{aligned}$$

where

$$\lim_i ||a_j\, e_i - a_j|| = 0$$

for all $j = 1, \dots, n$, so

$$\lim_i \left\| \Psi_n e_i \left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) - \Psi_n\left(\begin{pmatrix} a_1 \\ \dots \\ a_n \end{pmatrix}\right) \right\| = 0.$$

Finally, let $r \in \mathbb{N}$. We shall show that

$$\lim_n ||\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r|| = 0.$$

We denote by s_n the operators

$$s_n = \sum_{i=1}^{n} m_i\, m_i^*,\; n \in \mathbb{N}.$$

Hence, if $y \in Y$, we have that

$$\begin{aligned} ||\Phi_r \circ \Psi_n \circ \Phi_n(y) - \Phi_r(y)|| &= ||\Phi_r\,(\Psi_n \circ \Phi_n(y) - y)\,|| \\ &= \left\| \begin{pmatrix} m_1^*\, s_n - m_1^* \\ \dots \\ m_r^*\, s_n - m_r^* \end{pmatrix} y \right\| \\ &\le \left\| \begin{pmatrix} m_1^*\, s_n - m_1^* \\ \dots \\ m_r^*\, s_n - m_r^* \end{pmatrix} \right\| \; ||y|| \end{aligned}$$

Therefore, $||\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r|| \leq \left\| \begin{pmatrix} m_1^* \, s_n - m_1^* \\ \dots \\ m_r^* \, s_n - m_r^* \end{pmatrix} \right\|$

Since

$$\lim_n ||m_i^* \, s_n - m_i^*|| = 0, \, \forall \, i = 1, \dots, r,$$

we have that

$$\lim_n ||\Phi_r \circ \Psi_n \circ \Phi_n - \Phi_r|| = 0.$$

We conclude that Y is a right $\mathcal{A}$-rigged module in the sense of Definition 1.1. □

Theorem 2.4. *Let $\mathcal{A}$ be an approximately unital operator algebra and Y be a right $\mathcal{A}$-operator module. Then the following are equivalent:*

(i) Y is a right $\sigma\Delta$-$\mathcal{A}$-rigged module.
(ii) Y is orthogonally complemented in $C_\infty(\mathcal{A})$.

Proof. $(i) \implies (ii)$ Let $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$ be a completely isometric representation of $\mathcal{A}$ on H and assume there is a σ-TRO $M \subseteq \mathbb{B}(H, K)$ such that $M^* \, M \, a(\mathcal{A}) \subseteq a(\mathcal{A})$. Consider the $\sigma\Delta$-$\mathcal{A}$-rigged module $Y_0 = \overline{[M \, a(\mathcal{A})]} \subseteq \mathbb{B}(H, K)$ and a complete surjective isometry $\phi : Y \to Y_0$ which is also a right $\mathcal{A}$-module map. Let $\{m_i \in M \mid i \in \mathbb{N}\}$ be a sequence of elements of M having the property

$$\left\| \sum_{i=1}^{n} m_i \, m_i^* \right\| \leq 1, \, \forall \, n \in \mathbb{N}, \, \sum_{i=1}^{\infty} m_i \, m_i^* \, m = m, \, \forall \, m \in M.$$

It follows that

$$\sum_{i=1}^{\infty} m_i \, m_i^* \, y = y, \, \forall \, y \in Y_0.$$

We define the map $f : Y_0 \to C_\infty(\alpha(\mathcal{A}))$ by $f(y) = (m_i^* \, y)_{i \in \mathbb{N}}$, which is linear and a $\mathcal{A}$-module map. Also,

$$||f(y)||^2 = \left\| \sum_{i=1}^{\infty} (m_i^* \, y)^* \, m_i^* \, y \right\| = \left\| \sum_{i=1}^{\infty} y^* \, m_i \, m_i^* \, y \right\| = ||y^* \, y|| = ||y||^2,$$

so f is an isometry. We also define

$$g : C_\infty(\alpha(\mathcal{A})) \to Y_0, \, g((\alpha(x_i))_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i \alpha(x_i),$$

which is linear and a contractive $\mathcal{A}$-right module map. We see that

$$(g \circ f)(y) = g((m_i^* \, y)_{i \in \mathbb{N}}) = \sum_{i=1}^{\infty} m_i \, m_i^* \, y = y, \, \forall \, y \in Y_0,$$

that is, $g \circ f = Id_{Y_0}$. We now define $P = f \circ g : C_\infty(\alpha(\mathcal{A})) \to C_\infty(\alpha(\mathcal{A}))$. Clearly P is a contractive map satisfying $P^2 = P$. We shall prove that $P \in M_\ell(C_\infty(\mathcal{A}))$.

For all $x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \end{pmatrix} \in C_\infty(a(\mathcal{A}))$ we have that

$$P(x) = \left(m_i^* \sum_{j=1}^{\infty} m_j \, x_j \right)_{i \in \mathbb{N}} = s\, x,$$

where $s = \left(m_i^* \, m_j\right)_{i,j=1}^{\infty} \in \mathbb{M}_\infty(\mathbb{B}(H))$. Observe that $s = \begin{pmatrix} m_1^* \\ m_2^* \\ \dots \end{pmatrix} (m_1, m_2, \dots)$ and due to the fact that $||(m_1, m_2, \dots)|| \leq 1$ we get $||s|| \leq 1$. We define the map

$$\tau_P \, : \, C_2(C_\infty(a(\mathcal{A}))) \to C_2(C_\infty(a(\mathcal{A}))), \; \tau_P\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} P(x) \\ y \end{pmatrix} = \begin{pmatrix} s\, x \\ y \end{pmatrix}$$

and for all $\begin{pmatrix} x \\ y \end{pmatrix} \in C_2(C_\infty(a(\mathcal{A})))$ holds that

$$\left\| \tau_P\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) \right\| = \left\| \begin{pmatrix} s\, x \\ y \end{pmatrix} \right\| = \left\| \begin{pmatrix} s & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|,$$

so τ_P is a contraction. Similarly, we can prove that τ_P is completely contractive. Therefore by [3, Theorem 4.5.2], P is a left multiplier of $C_\infty(a(\mathcal{A}))$. It is easy to see that $f(Y_0) = P(C_\infty(\alpha(\mathcal{A})))$ and thus $Y \simeq P(C_\infty(\alpha(\mathcal{A})))$.

$(ii) \implies (i)$ Suppose that $\mathcal{A} \subseteq \mathcal{A}^{**} \subseteq \mathbb{B}(H)$. Let $P \, : \, C_\infty(\mathcal{A}) \to C_\infty(\mathcal{A})$ be a left multiplier of $C_\infty(\mathcal{A})$ which is a right $\mathcal{A}$-module map with $||P||_{cb} \leq 1$ and such that $P^2 = P$, $Y \cong P(C_\infty(\mathcal{A}))$. According to [5, Appendix B], there is an extension $\tilde{P} \, : \, C_\infty^w(\mathcal{A}^{**}) \to C_\infty^w(\mathcal{A}^{**})$ of P. The operator $\tilde{P}$ lies in the diagonal of $M_l(C_\infty^w(\mathcal{A}^{**}))$, which is contained in $\mathbb{M}_\infty(\mathcal{A}^{**})$. Therefore, $\tilde{P} = (\tilde{p_{i,j}})_{i,j \in \mathbb{N}}$ where $\tilde{p_{i,j}} \in \mathcal{A}^{**}$, $\forall \, i, \; j \in \mathbb{N}$. Thus,

$$\tilde{P}(u) = (\tilde{p_{i,j}})_{i,j \in \mathbb{N}} \cdot u, \;\; \forall \, u = \begin{pmatrix} u_1 \\ u_2 \\ \dots \end{pmatrix} \in C_\infty(\mathcal{A}).$$

In what follows we identify $\tilde{P}$ and $(\tilde{p_{i,j}})_{i,j}$. We have that $Y \cong P(C_\infty(\mathcal{A})) = \tilde{P}(C_\infty(\mathcal{A}))$ and $\tilde{P}^2 = \tilde{P} = \tilde{P}^*$. Let $N_2 = [\tilde{P}]$, D be the C^*-algebra generated by $\tilde{P}$ and $\mathcal{K}_\infty$ and let $N_1 = C_\infty$. By [8, Lemma 2.5], $M = \overline{[N_2 \, D \, N_1]} = \overline{[\tilde{P} \, D \, C_\infty]}$ is a σ-TRO. We claim that $D \, C_\infty(\mathcal{A}) \subseteq C_\infty(\mathcal{A})$. Indeed,

$$\tilde{P}(C_\infty(\mathcal{A})) = P(C_\infty(\mathcal{A})) \subseteq C_\infty(\mathcal{A}) \tag{2.1}$$

and $C_\infty \, R_\infty \, C_\infty(\mathcal{A}) \subseteq C_\infty(\mathcal{A})$. Due to the fact that $\mathcal{K}_\infty = C_\infty \, R_\infty$, we have that

$$\mathcal{K}_\infty \, C_\infty(\mathcal{A}) \subseteq C_\infty(\mathcal{A}). \tag{2.2}$$

But since D is generated by $\tilde{P}\,, \mathcal{K}_\infty$ by (2.1) and (2.2) we have that $D \, C_\infty(\mathcal{A}) \subseteq C_\infty(\mathcal{A})$. Now,

$$P(C_\infty(\mathcal{A})) \subseteq \tilde{P} \, D \, C_\infty(\mathcal{A}) = \overline{[M \, \mathcal{A}]}.$$

On the other hand,

$$\overline{[M\mathcal{A}]} = \overline{[\tilde{P}\, D\, C_\infty \cdot \mathcal{A}]} \subseteq \tilde{P}\,(C_\infty(\mathcal{A})) = P(C_\infty(\mathcal{A}))$$

so, $\overline{[M\mathcal{A}]} = P(C_\infty(\mathcal{A}))$.

Finally,

$$\begin{aligned} M^*\, M\, \mathcal{A} &\subseteq M^*\, P(C_\infty(\mathcal{A})) = R_\infty\, D\, \tilde{P}(C_\infty(\mathcal{A})) \\ &\subseteq R_\infty\, D\, C_\infty(\mathcal{A}) \subseteq R_\infty\, C_\infty(\mathcal{A}) \\ &= R_\infty\, C_\infty \cdot \mathcal{A} = \mathcal{A} \end{aligned}$$

so Y is a right $\sigma\Delta$-$\mathcal{A}$-rigged module. □

There is a category of rigged modules, the so-called countably column generated and approximately projective modules. We are going to examine whether there is a connection between them and the $\sigma\Delta$-rigged modules.

Definition 2.5. [1].

Let $\mathcal{A}$ be an approximately unital operator algebra. A right $\mathcal{A}$ operator module Y is called countably column generated and approximately projective (CCGP for short) if there are completely contractive right $\mathcal{A}$-module maps $\phi : Y \to C_\infty(\mathcal{A})$ and $\psi : C_\infty(\mathcal{A}) \to Y$ with ψ finitely $\mathcal{A}$-essential (that is, for all $n \in \mathbb{N}$ the restriction map of ψ to $C_n(A) \subseteq C_\infty(A)$ is right $\mathcal{A}$-essential) and also $\psi\circ\phi = Id_Y$.

Remark 2.6. From [1, Theorem 8.3] and Theorem 2.4, it is obvious that a CCGP module is a $\sigma\Delta$-rigged module. The converse is not true. Indeed, by [1, Theorem 8.2], we have that the CCGP modules over C^*-algebras are precisely the countably generated right Hilbert modules, but there exist $\sigma\Delta$-Hilbert modules which are not countably generated. For example if $\mathcal{A}$ a C^*-algebra without σ-unit, since $\mathbb{C}\mathcal{A} = \mathcal{A}$ then $\mathcal{A}$ is a $\sigma\Delta$-Hilbert module over itself, but clearly is not countably generated, so it is not a CCGP module.

3. Doubly $\sigma\Delta$-rigged modules

In this section we introduce a subcategory of $\sigma\Delta$-rigged modules, the doubly $\sigma\Delta$-rigged modules and we prove that these modules implement stable isomorphism between the corresponding operator algebras.

Definition 3.1. Let Y be a right $\mathcal{A}$-operator module over the approximately unital operator algebra $\mathcal{A}$. We call Y a BMP equivalence bimodule if there exist an operator algebra $\mathcal{B}$ such that Y is a left $\mathcal{B}$-operator module and a $\mathcal{B}$-$\mathcal{A}$-operator module X such that

$$\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} X,\ \ \mathcal{A} \cong X \otimes^h_{\mathcal{B}} Y.$$

In this case we call X and Y bimodules of BMP-Morita equivalence.

We note that every $\mathcal{B}$-$\mathcal{A}$-bimodule of Morita equivalence is a right $\mathcal{A}$-rigged module. We now introduce the notion of a doubly $\sigma\Delta$-rigged module.

Definition 3.2. Let $\mathcal{A} \subseteq \mathbb{B}(H)$ be an approximately unital operator algebra and $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that

$$M^* M \mathcal{A} \subseteq \mathcal{A}, \ \overline{[M^* M \mathcal{A}]} = \overline{[\mathcal{A} M^* M]}.$$

We call the operator space $Y = \overline{[M \mathcal{A}]} \subseteq \mathbb{B}(H,K)$ a doubly σ-TRO-$\mathcal{A}$-rigged module.

We note that every doubly σ-TRO-$\mathcal{A}$-rigged module is also a σ-TRO-$\mathcal{A}$-rigged module in the sense of Definition 2.1.

Definition 3.3. Let $\mathcal{A}$ be an abstract approximately unital operator algebra and Y be an abstract right $\mathcal{A}$-module. We call Y a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module if there exists a completely isometric homomorphism $a : \mathcal{A} \to a(\mathcal{A})$ and also there exists a doubly σ-TRO-$a(\mathcal{A})$-rigged module Y_0 and a complete onto isometry $\phi : Y \to Y_0$ which is a right $\mathcal{A}$-module map.

Definition 3.4. Let $\mathcal{A}$ be an approximately unital operator algebra and Y be a $\sigma\Delta$-$\mathcal{A}$-rigged module. There exist $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$, a completely isometric representation of $\mathcal{A}$ on H and a σ-TRO $M \subseteq \mathbb{B}(H,K)$ such that $\overline{M^* M\, a(\mathcal{A})} \subseteq a(\mathcal{A})$ and $Y \cong Y_0 = \overline{[M\, a(\mathcal{A})]}$. Then the operator space $Z = \overline{[Y_0 M^* M]} \subseteq \mathbb{B}(H,K)$ is called the restriction of Y over $\mathcal{A}$. Observe that Z is a right module over the operator algebra $\overline{[a(\mathcal{A}) M^* M]}$.

In the following theorem we prove that the notions of $\sigma\Delta$ right Hilbert modules and of doubly $\sigma\Delta$ right Hilbert modules coincide:

Theorem 3.5. *Let $\mathcal{A}$ be a C^*-algebra and let Y be a right Hilbert module over $\mathcal{A}$. The following are equivalent:*

(i) Y is orthogonally complemented in $C_\infty(\mathcal{A})$.
(ii) Y is a $\sigma\Delta$ right Hilbert module over $\mathcal{A}$.
(iii) Y is a doubly $\sigma\Delta$ right Hilbert module over $\mathcal{A}$.

Proof. $(i) \implies (iii)$ Let $P : C_\infty(\mathcal{A}) \to C_\infty(\mathcal{A})$ be an adjointable map such that $P = P^2 = P^*$ and $Y \cong P(C_\infty(\mathcal{A}))$. Since $P \in M_l(C_\infty(\mathcal{A}))$, where $M_l(C_\infty(\mathcal{A}))$ is the left multiplier algebra of $C_\infty(\mathcal{A})$, P can be extended to a multiplier of $C_\infty^w(\mathcal{A}^{**})$. Here $\mathcal{A}^{**}$ is the second dual of $\mathcal{A}$ and $C_\infty^w(\mathcal{A}^{**})$ is the space of columns with entries in $\mathcal{A}^{**}$ which define bounded operators. The algebra of left multipliers of $C_\infty^w(\mathcal{A}^{**})$ is isomorphic to $\mathbb{M}_\infty(\mathcal{A}^{**})$ (we refer the reader to [5]). Therefore, we may assume that there exist $a_{ij} \in \mathcal{A}^{**}$, $i, j \in \mathbb{N}$ such that

$$P(u) = (a_{ij}) \cdot u, \ \forall\, u \in C_\infty(\mathcal{A}).$$

In what follows we identify P with the matrix (a_{ij}). We also may consider a Hilbert space K such that $\mathcal{A} \subseteq \mathcal{A}^{**} \subseteq \mathbb{B}(K)$ and also $I_K \in \mathcal{A}^{**}$.

Let N_2 be the linear span of the element P. Since $P^2 = P = P^*$ we get that N_2 is a σ-TRO. Let $\mathcal{A}^1 = \overline{[\mathcal{A} + \mathbb{C}\, I_K]}$ and $N_1 = C_\infty(\mathcal{A}^1)$. Clearly N_1 is a σ-TRO. If D is the C^*-algebra generated by P and $K_\infty(\mathcal{A}^1)$, then $M = \overline{[N_2\, D\, N_1]}$ is a σ-TRO, [8, Lemma 2.5].

We note that

$$\overline{[M^* M \mathcal{A}]} = \overline{[N_1^* D N_2^* N_2 D N_1 \mathcal{A}]} = \overline{[N_1^* D N_2^* C_\infty(\mathcal{A}^1)\mathcal{A}]} \subseteq \mathcal{A}.$$

If $Y_0 = \overline{[M \mathcal{A}]}$, then

$$Y_0 = \overline{[N_2 D N_1 \mathcal{A}]} = \overline{[N_2 D C_\infty(\mathcal{A})]} = \overline{[P D C_\infty(\mathcal{A})]} = P(C_\infty(\mathcal{A})).$$

We have that

$$\begin{aligned}\overline{[M^* M \mathcal{A}]} &= \overline{[M^* P(C_\infty(\mathcal{A}))]} = \overline{[N_1^* D N_2^* P C_\infty(\mathcal{A})]} \\ &= \overline{[N_1^* D P C_\infty(\mathcal{A})]} = \overline{[R_\infty(\mathcal{A}^1) P(C_\infty(\mathcal{A}))]} \\ &= \overline{[R_\infty(\mathcal{A}) P(C_\infty(\mathcal{A}))]}\end{aligned}$$

and therefore

$$(\overline{[M^* M \mathcal{A}]})^* = (\overline{[R_\infty(\mathcal{A}) P(C_\infty(\mathcal{A}))]})^*,$$

that is

$$\overline{[\mathcal{A} M^* M]} = \overline{[R_\infty(\mathcal{A}) P(C_\infty(\mathcal{A}))]} = \overline{[M^* M \mathcal{A}]},$$

which implies that

$$\overline{[M^* M \mathcal{A}]} = \overline{[\mathcal{A} M^* M]} \subseteq \mathcal{A}.$$

Since also

$$Y \cong P(C_\infty(\mathcal{A})) = Y_0 = \overline{[M \mathcal{A}]},$$

we conclude that Y is a doubly $\sigma\Delta$ Hilbert module.

$(iii) \implies (ii)$ This is obvious.

$(ii) \implies (i)$ This is a consequence of Theorem 2.4. □

At this point, we prove a Lemma which will be very useful for what follows.

Lemma 3.6. *Let $\mathcal{A}$ be an operator algebra with cai $(a_k)_{k\in K}$ and $\mathcal{C}$ be a C^*-algebra with cai $(c_i)_{i\in I}$. Assume that $\mathcal{C}\,\mathcal{A} \subseteq \mathcal{A}$, $\mathcal{A}\,\mathcal{C} \subseteq \mathcal{A}$. We define $\mathcal{A}_0 = \overline{[\mathcal{C}\,\mathcal{A}\,\mathcal{C}]} \subseteq \mathcal{A}$. Then $\mathcal{A}_0$ is an operator algebra with a two-sided approximate identity*

$$x_{(i,k)} = c_i\, a_k\, c_i,\ i \in I,\ k \in K.$$

Proof. The space $\mathcal{A}_0$ is a closed subspace of $\mathcal{A}$ and is an algebra since

$$\mathcal{A}_0 \mathcal{A}_0 \subseteq \overline{[\mathcal{C}\,\mathcal{A}\,\mathcal{C}\,\mathcal{C}\,\mathcal{A}\,\mathcal{C}]} \subseteq \overline{[\mathcal{C}\,\mathcal{A}\,\mathcal{C}\,\mathcal{A}\,\mathcal{C}]} \subseteq \overline{[\mathcal{C}\,\mathcal{A}\ \mathcal{A}\,\mathcal{C}]} \subseteq \overline{[\mathcal{C}\,\mathcal{A}\,\mathcal{C}]} = \mathcal{A}_0.$$

It is obvious that $x_{(i,k)} = c_i\, a_k\, c_i \in \mathcal{A}_0$, $i \in I$, $k \in K$ and $\mathcal{A}_0 \subseteq \mathcal{A}$. Now, if $a \in \mathcal{A}_0$, then $c_i\, a \to a$ and $a_k\, a \to a$. For all $i \in I$, $k \in K$ we have that

$$\begin{aligned}||x_{(i,k)}\, a - a|| &= ||c_i\, a_k\, c_i\, a - a|| \\ &\leq ||c_i\, a_k\, c_i\, a - c_i\, a|| + ||c_i\, a - a|| \\ &\leq ||a_k\, c_i\, a - a|| + ||c_i\, a - a|| \\ &\leq ||a_k\, c_i\, a - a_k\, a|| + ||a_k\, a - a|| + ||c_i\, a - a|| \\ &\leq ||c_i\, a - a|| + ||a_k\, a - a|| + ||c_i\, a - a|| \\ &= 2\,||c_i\, a - a|| + ||a_k\, a - a||\end{aligned}$$

Thus,

$$\lim_{(i,k)} x_{(i,k)}\, a = a.$$

Similarly, we can prove that

$$\lim_{(i,k)} a\, x_{(i,k)} = a. \quad \square$$

Lemma 3.7. *Let $\mathcal{A} \subseteq \mathbb{B}(H)$ be an approximately unital operator algebra and $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $M^* M \mathcal{A} \subseteq \mathcal{A}$. We also assume that $\mathcal{A} M^* M \subseteq \mathcal{A}$. We define $\mathcal{B} = \overline{[M \mathcal{A} M^*]} \subseteq \mathbb{B}(K)$ and also $\mathcal{A}_0 = \overline{[M^* \mathcal{B} M]} \subseteq \mathbb{B}(H)$. Then $\mathcal{A}_0$ and $\mathcal{B}$ are approximately unital operator algebras and $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$.*

Proof. It suffices to prove that $\mathcal{A}_0$, $\mathcal{B}$ are closed under multiplication and that $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$. Indeed,

$$\mathcal{B}\,\mathcal{B} \subseteq \overline{[M \mathcal{A} M^* M \mathcal{A} M^*]} \subseteq \overline{[M \mathcal{A} \mathcal{A} M^*]} = \overline{[M \mathcal{A} M^*]} = \mathcal{B}$$

so $\mathcal{B}$ is an operator algebra. Now, we observe that $M M^* \mathcal{B} \subseteq \mathcal{B}$ and then

$$\mathcal{A}_0 \mathcal{A}_0 \subseteq \overline{[M^* \mathcal{B} M M^* \mathcal{B} M]} \subseteq \overline{[M^* \mathcal{B} \mathcal{B} M]} \subseteq \overline{[M^* \mathcal{B} M]} = \mathcal{A}_0$$

which means that $\mathcal{A}_0$ is an operator algebra. We have that $\mathcal{A}_0 = \overline{[M^* \mathcal{B} M]} = \overline{[M^* M \mathcal{A} M^* M]}$. If C is the C^*-algebra $\overline{[M^* M]}$, then $C \mathcal{A} \subseteq \mathcal{A}$, $\mathcal{A} C \subseteq \mathcal{A}$. By Lemma 3.6, the operator algebra $\mathcal{A}_0$ has a cai. Also, since $\mathcal{A}_0 = \overline{[M^* \mathcal{B} M]}$ and on the other hand

$$\overline{[M \mathcal{A}_0 M^*]} = \overline{[M M^* \mathcal{B} M M^*]} = \overline{[M M^* M \mathcal{A} M^* M M^*]} = \overline{[M \mathcal{A} M^*]} = \mathcal{B}$$

we deduce that $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$. Since $\mathcal{A}_0$ has a cai, we have that $\mathcal{B}$ has also a cai. $\square$

Theorem 3.8. *Let $\mathcal{A}$ be an approximately unital operator algebra and Y be a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module. Then, there exist operator algebras $\mathcal{A}_0$, $\mathcal{B}$ with cai's such that $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$ and also $\mathcal{B} \sim_{\sigma TRO} Y$. In case $\mathcal{A}$ is a C^*-algebra and Y is a $\sigma\Delta$-$\mathcal{A}$-Hilbert module then $\mathcal{A}_0 \simeq I_{\mathcal{A}}(Y)$, $\mathcal{B} \simeq K_{\mathcal{A}}(Y)$.*

Proof. Let H be a Hilbert space, $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$ be a completely isometric representation of $\mathcal{A}$ on H and let $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $M^* M\, a(\mathcal{A}) \subseteq a(\mathcal{A})$ and also

$$\overline{[M^* M\, a(\mathcal{A})]} = \overline{[a(\mathcal{A}) M^* M]} \tag{3.1}$$

Consider now a complete surjective isometry

$$\phi : Y \to Y_0 = \overline{[M\, a(\mathcal{A})]} \subseteq \mathbb{B}(H,K)$$

which is a right $\mathcal{A}$-module map. We define the spaces $\mathcal{B} = \overline{[M\, a(\mathcal{A}) M^*]} \subseteq \mathbb{B}(K)$ and $\mathcal{A}_0 = \overline{[M^* \mathcal{B} M]} \subseteq \mathbb{B}(H)$. Now by Lemma 3.7, $\mathcal{A}_0$, $\mathcal{B}$ are operator algebras with cai's such that $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$. It remains to prove that $\mathcal{B} \sim_{\sigma TRO} Y$.

Set $M_1 = M^* \subseteq \mathbb{B}(K,H)$ and $M_2 = \overline{[M\,M^*]} \subseteq \mathbb{B}(K)$. Then, M_1, M_2 are σ-TRO's and we have that

$$\overline{[M_2^*\,\phi(Y)\,M_1]} = \overline{[M\,M^*\,M\,a(\mathcal{A})\,M^*]} = \overline{[M\,a(\mathcal{A})\,M^*]} = \mathcal{B}$$

and

$$\overline{[M_2\,\mathcal{B}\,M_1^*]} = \overline{[M\,a(\mathcal{A})\,M^*\,M]} \overset{(3.1)}{=} \overline{[M\,M^*\,M\,a(\mathcal{A})]} = \overline{[M\,a(\mathcal{A})]} = \phi(Y).$$

From Definition 1.4 we get that $\mathcal{B} \sim_{\sigma TRO} Y$.

The other assertions follow easily. □

Theorem 3.9. *Let $\mathcal{A}$ be an approximately unital operator algebra, $a : \mathcal{A} \to \mathbb{B}(H)$ be a completely isometric homomorphism and $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $M^*\,M\,a(\mathcal{A}) \subseteq a(\mathcal{A})$. We define the $\sigma\Delta$-$\mathcal{A}$-rigged module $Y = \overline{[M\,a(\mathcal{A})]}$. Then there exist operator algebras $\mathcal{A}_0$, $\mathcal{B}$ with cai's and a restriction Z of Y such that Z is a doubly $\sigma\Delta$-$\mathcal{A}_0$-rigged module and $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B} \sim_{\sigma TRO} Z$.*

Proof. We define the restriction $Z = \overline{[Y\,M^*\,M]} = \overline{[M\,a(\mathcal{A})\,M^*\,M]}$ of Y. Let $\mathcal{A}_0 = \overline{[\,M^*\,M a(\mathcal{A})\,M^*\,M]}$. Then $\mathcal{A}_0$ is an operator algebra and

$$\overline{[M\,\mathcal{A}_0]} = \overline{[M\,M^*\,M\,a(\mathcal{A})\,M^*\,M]} = \overline{[M\,a(\mathcal{A})\,M^*\,M]} = Z$$

such that

$$\overline{[\,M^*M\,\mathcal{A}_0]} = \overline{[M^*\,M\,M^*\,M\,a(\mathcal{A})\,M^*\,M]} = \overline{[M^*\,M\,a(\mathcal{A})\,M^*\,M]} = \mathcal{A}_0$$

$$\overline{[\mathcal{A}_0\,M^*\,M]} = \overline{[M^*\,M\,a(\mathcal{A})\,M^*\,M\,M^*\,M]} = \overline{[M^*\,M\,a(\mathcal{A})\,M^*\,M]} = \mathcal{A}_0$$

which means that $\overline{[M^*\,M\,\mathcal{A}_0]} = \overline{[\mathcal{A}_0\,M^*\,M]}$, that is, $Z = \overline{[M\,\mathcal{A}_0]}$ is a doubly $\sigma\Delta$-$\mathcal{A}_0$-rigged module. If we define $\mathcal{B} = \overline{[M\,a(\mathcal{A})\,M^*]}$ then by Lemma 3.6, $\mathcal{A}_0$ and $\mathcal{B}$ have cai's and by Lemma 3.7, we get that $\mathcal{A}_0 \sim_{\sigma TRO} \mathcal{B}$.

Finally, $\mathcal{B} \sim_{\sigma TRO} Z$. Indeed, if we consider the σ-TRO's $M_1 = M$ and $M_2 = \overline{[M\,M^*]}$, then

$$\overline{[M_2\,Z\,M_1^*]} = \overline{[M\,M^*\,M\,a(\mathcal{A})\,M^*\,M\,M^*]} = \overline{[M\,\mathcal{A}_0\,M^*]} = \mathcal{B}$$

$$\overline{[M_2^*\,\mathcal{B}\,M_1]} = \overline{[M\,M^*\,M\,a(\mathcal{A})\,M^*\,M]} = \overline{[M\,M^*\,\mathcal{B}\,M]} = \overline{[M\,\mathcal{A}_0]} = Z.$$

□

Corollary 3.10. Every $\sigma\Delta$-$\mathcal{A}$-rigged-module Y over an approximately unital operator algebra $\mathcal{A}$ has a restriction which is a bimodule of BMP equivalence, which actually implements a stable isomorphism over the operator algebras $\mathcal{A}_0$ and $\mathcal{B}$ defined as in Theorem 3.9.

Corollary 3.11. Every orthogonally complemented module over an approximately unital operator algebra $\mathcal{A}$ has a restriction which is a bimodule of BMP equivalence between operator algebras which are stably isomorphic.

Proof. If Y is an orthogonally complemented module over the operator algebra $\mathcal{A}$, then according to Theorem 2.4, Y is a $\sigma\Delta$-$\mathcal{A}$-rigged module and due to the previous corollary, Y has a restriction which is a bimodule of BMP equivalence between operator algebras which are stably isomorphic. □

Another interesting category of rigged modules is the category of column stable generator modules. We prove that the restriction of a $\sigma\Delta$-rigged module over $\mathcal{A}$ is a column stable generated module (maybe over another operator algebra than $\mathcal{A}$). We refer the reader to [1, Section 8] for facts about column stable generated modules.

Definition 3.12. [1].

A right $\mathcal{A}$-rigged module Y is called a column stable generator (CSG for short) if there exist completely contractive right $\mathcal{A}$-module maps $\sigma : \mathcal{A} \to C_\infty(Y)$ and $\tau : C_\infty(Y) \to \mathcal{A}$ such that $\tau\circ\sigma = Id_{\mathcal{A}}$.

Proposition 3.13. *Let $\mathcal{A}$ be an approximately unital operator algebra, $a : \mathcal{A} \to a(\mathcal{A}) \subseteq \mathbb{B}(H)$ be a completely isometric homomorphism and suppose there is a σ-TRO $M \subseteq \mathbb{B}(H,K)$ such that*

$$M^*\, M\, a(\mathcal{A}) \subseteq a(\mathcal{A}),\;\; a(\mathcal{A})\, M^*\, M \subseteq a(\mathcal{A}).$$

Consider the $\sigma\Delta$-$\mathcal{A}$-rigged module $Y = \overline{[M\, a(\mathcal{A})]}$. Then, there exist operator algebras $\mathcal{A}_0$ and $\mathcal{B}$ and a restriction Z of Y over $\mathcal{A}_0$ such that Z is a CSG module over $\mathcal{A}_0$.

Proof. Since M is a σ-TRO, we fix a sequence $\{m_i \in M \mid i \in \mathbb{N}\} \subseteq M$ such that

$$\left\|\sum_{i=1}^{n} m_i^*\, m_i\right\| \leq 1,\; \forall\, n \in \mathbb{N},\;\; \sum_{i=1}^{\infty} m_i^*\, m_i\, m^* = m^*,\; \forall\, m \in M. \tag{3.2}$$

We define the operator algebras $\mathcal{B} = \overline{[M\, a(\mathcal{A})\, M^*]} \subseteq \mathbb{B}(K)$, $\mathcal{A}_0 = \overline{[M^*\, \mathcal{B}\, M]} \subseteq \mathbb{B}(H)$ and also $Z = \overline{[Y\, M^*\, M]} = \overline{[\mathcal{B}\, M]}$, which is a restriction of Y, and is also a doubly $\sigma\Delta$-$\mathcal{A}_0$-rigged module (Theorem 3.9). Since

$$\overline{[M\, \mathcal{A}_0]} = \overline{[M\, M^*\, M\, a(\mathcal{A})\, M^*\, M]} = \overline{[M\, a(\mathcal{A})\, M^*\, M]} = \overline{[\mathcal{B}\, M]} = Z$$

and $\overline{[M^*\, Z]} = \overline{[M^*\, \mathcal{B}\, M]} = \mathcal{A}_0$, the maps

$$\sigma : \mathcal{A}_0 \to C_\infty(Z),\; \sigma(a) = (m_i\, a)_{i\in\mathbb{N}}$$

and

$$\tau : C_\infty(Z) \to \mathcal{A}_0,\; \tau((z_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} m_i^*\, z_i$$

are well defined and also completely contractive right $\mathcal{A}_0$-module maps. For all $m^*\, b\, n \in M^*\, \mathcal{B}\, M \subseteq \mathcal{A}_0$ we have that

$$(\tau\circ\sigma)(m^*\, b\, n) = \tau((m_i\, m^*\, b\, n)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} m_i^*\, m_i\, m^*\, b\, n \overset{(3.2)}{=} m^*\, b\, n = Id_{\mathcal{A}_0}(m^*\, b\, n).$$

It follows that $(\tau\circ\sigma)(a) = Id_{\mathcal{A}_0}(a),\; \forall\, a \in \mathcal{A}_0 \implies \tau\circ\sigma = Id_{\mathcal{A}_0}$. □

Theorem 3.14. *Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras such that $\mathcal{A}$, $\mathcal{B}$ are stably isomorphic. Then, there exists a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module Y which is also an $\mathcal{A}$-$\mathcal{B}$-operator module and there exists a $\mathcal{B}$-$\mathcal{A}$-operator module X such that $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} X$ and $\mathcal{A} \cong X \otimes^h_{\mathcal{B}} Y$. Furthermore, $\mathcal{A}$, $\mathcal{B}$, X, Y are all stably isomorphic.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic, we have that they are also $\sigma\Delta$ equivalent, that is, $\mathcal{A} \sim_{\sigma\Delta} \mathcal{B}$, [8, Theorem 3.3]. So, there exist Hilbert spaces H, K and completely isometric homomorphisms $a : \mathcal{A} \to \mathbb{B}(H)$ and $\beta : \mathcal{B} \to \mathbb{B}(K)$ and also a σ-TRO $M \subseteq \mathbb{B}(H,K)$ such that $a(\mathcal{A}) = \overline{[M^* \beta(\mathcal{B}) M]}$, $\beta(\mathcal{B}) = \overline{[M\, a(\mathcal{A}) M^*]}$. We have that

$$\overline{[a(\mathcal{A}) M^* M]} = a(\mathcal{A}) = \overline{[M^* M\, a(\mathcal{A})]}$$

and so $Y = \overline{[M\, a(\mathcal{A})]} \subseteq \mathbb{B}(H,K)$ is a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module which is also a left $\mathcal{B}$-operator module since

$$\beta(\mathcal{B})\, Y \subseteq \overline{[M\, a(\mathcal{A}) M^* M\, a(\mathcal{A})]} \subseteq \overline{[M\, a(\mathcal{A})\, a(\mathcal{A})]} \subseteq \overline{[M\, a(\mathcal{A})]} = Y.$$

We also define $X = \overline{[a(\mathcal{A}) M^*]} \subseteq \mathbb{B}(K,H)$ which is a left $\mathcal{A}$-operator module via the module action

$$a(x) \cdot (a(y)\, m^*) = a(x\, y)\, m^*,\ x,\ y \in \mathcal{A},\ m \in M.$$

Furthermore X is a right $\mathcal{B}$-operator module since

$$X\, \beta(\mathcal{B}) \subseteq \overline{[a(\mathcal{A}) M^* M\, a(\mathcal{A}) M^*]} = \overline{[a(\mathcal{A})\, a(\mathcal{A}) M^* M M^*]} \subseteq \overline{[a(\mathcal{A}) M^*]} = X.$$

By Lemma 1.5, if $D_1 = \overline{[M^* M]}$, then

$$\begin{aligned} Y \otimes^h_{a(\mathcal{A})} X &= \overline{[M\, a(\mathcal{A})]} \otimes^h_{a(\mathcal{A})} \overline{[a(\mathcal{A}) M^*]} \\ &\cong \left(M \otimes^h_{D_1} a(\mathcal{A})\right) \otimes^h_{a(\mathcal{A})} \left(a(\mathcal{A}) \otimes^h_{D_1} M^*\right) \\ &\cong M \otimes^h_{D_1} a(\mathcal{A}) \otimes^h_{D_1} M^* \\ &\overset{(1.5)}{\cong} \overline{[M\, a(\mathcal{A}) M^*]} = \beta(\mathcal{B}) \end{aligned}$$

and also, due to the fact that $Y = \overline{[a(\mathcal{A})M^*]} = \overline{[M^*\beta(\mathcal{B})]}$, if $D_2 = \overline{[MM^*]}$ we have that

$$\begin{aligned}
X \otimes^h_{\beta(\mathcal{B})} Y &= \overline{[M^*\beta(\mathcal{B})]} \otimes^h_{\beta(\mathcal{B})} \overline{[MM^*\beta(\mathcal{B})M]} \\
&\cong \left(M^* \otimes^h_{D_2} \beta(\mathcal{B})\right) \otimes^h_{\beta(\mathcal{B})} \overline{[M\,a(\mathcal{A})]} \\
&\cong M^* \otimes^h_{D_2} \left(\beta(\mathcal{B}) \otimes^h_{\beta(\mathcal{B})} \overline{[M\,a(\mathcal{A})]}\right) \\
&\cong M^* \otimes^h_{D_2} \overline{[M\,a(\mathcal{A})]} \\
&\cong M^* \otimes^h_{D_2} \left(M \otimes^h_{D_1} a(\mathcal{A})\right) \\
&\cong \left(M^* \otimes^h_{D_2} M\right) \otimes^h_{D_1} a(\mathcal{A}) \\
&\cong \overline{[M^*M]} \otimes^h_{D_1} a(\mathcal{A}) \\
&\cong \overline{[M^*M\,a(\mathcal{A})]} \\
&= \overline{[M^*MM^*\beta(\mathcal{B})M]} \\
&= \overline{[M^*\beta(\mathcal{B})M]} = a(\mathcal{A}).
\end{aligned}$$

□

By the same arguments, we obtain the following corollary:

Corollary 3.15. Let $\mathcal{A}$, $\mathcal{B}$ be stably isomorphic C^*-algebras. There exists a $\sigma\Delta$-Hilbert module Y over a C^*-algebra $\mathcal{D}$ such that

$$\mathcal{A} \simeq K_{\mathcal{D}}(Y),\ \mathcal{B} \simeq I_{\mathcal{D}}(Y).$$

Furthermore $\mathcal{A}$, $\mathcal{B}$ and Y are all stably isomorphic.

4. Morita equivalence of rigged modules

Definition 4.1. [1]. Let $\mathcal{A}$ be an approximately unital operator algebra and let Y be a right $\mathcal{A}$-rigged module. If $\{\Phi_b, \Psi_b \mid b \in B\}$ is a choice for Y as in Definition 1.1, then we write E_b for the map $E_b = \Psi_b \circ \Phi_b : Y \to Y,\ b \in B$. We define

$$\tilde{Y} = \{f \in CB_{\mathcal{A}}(Y, \mathcal{A}) \mid f \circ E_b \to f \text{ uniformly}\}$$

and $\mathbb{K}(Y)$ to be the closure in $CB_{\mathcal{A}}(Y, Y)$ of the set of finite rank operators

$$T_{y,f} : Y \to Y,\ T_{y,f}(y') = y\,f(y')$$

where $y \in Y,\ f \in \tilde{Y}$.

For further details we refer the reader to [1, Section 3]. We also note that $\mathbb{K}(Y)$ and $\tilde{Y}$ are actually independent of the particular directed set and nets $\{\Phi_b, \Psi_b \mid b \in B\}$. In the following lemma we use the notion of a complete quotient map. For further details we refer the reader to [4].

Lemma 4.2. *Let $\mathcal{A} \subseteq \mathbb{B}(H)$ be an approximately unital operator algebra, $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO and $Y = \overline{[M\mathcal{A}]} \subseteq \mathbb{B}(H,K)$. Assume that $M^* M \mathcal{A} \subseteq \mathcal{A}$, $\mathcal{A} M^* M \subseteq \mathcal{A}$ (thus Y is a $\sigma\Delta$-$\mathcal{A}$-rigged module). Then $\tilde{Y} \cong \overline{[\mathcal{A} M^*]}$ and $\mathbb{K}(Y) \cong \overline{[M\mathcal{A}M^*]}$.*

Proof. We define $\mathcal{B} = \overline{[M\mathcal{A}M^*]}$. Clearly, $\mathcal{B}$ is an operator algebra. By Lemma 3.6, the algebra $\mathcal{A}_0 = \overline{[M^* M\mathcal{A}M^* M]}$ has a cai. By Lemma 3.7, the algebra $\mathcal{B}$ has also cai and obviously the algebras $\mathcal{A}_0$ and $\mathcal{B}$ are σ-TRO equivalent. If $X = \overline{[\mathcal{A}M^*]}$, then we define the completely contractive maps

$$(\cdot,\cdot) : X \times Y \to \mathcal{A},\ (x,y) \mapsto (x,y) = xy$$

$$[\cdot,\cdot] : Y \times X \to \mathcal{B},\ (y,x) \mapsto [y,x] = yx.$$

These maps satisfy

$$(x,y)\,x' = x\,[y,x'],\ \ y\,(x,y') = [y,x]\,y',\ \forall\, x,\ x' \in X,\ y,\ y' \in Y.$$

The map $[\cdot,\cdot]$ induces a complete quotient map $Y \otimes^h X \to \mathcal{B}$, $y \otimes x \to yx$. Indeed, by making the same calculations as those of the proof of Theorem 3.14, we have that $Y \otimes^h_{\mathcal{A}} X \cong [M\mathcal{A}M^*] = \mathcal{B}$. Futhermore, the map ϕ : $Y \otimes^h X \to Y \otimes^h_{\mathcal{A}} X$, $y \otimes x \mapsto y \otimes_{\mathcal{A}} x$ is a complete quotient since the map $\hat{\phi} : \left(Y \otimes^h X\right)/\mathrm{Ker}(\phi) \to \mathrm{Y} \otimes^{\mathrm{h}}_{\mathcal{A}} \mathrm{X}$ is a complete surjective isometry. From [1, Theorem 5.1] it follows that $\tilde{Y} \cong \overline{[\mathcal{A}M^*]}$ and $\mathbb{K}(Y) \cong \mathcal{B} = \overline{[M\mathcal{A}M^*]}$. □

Theorem 4.3. *If $\mathcal{A}$ is an approximately unital operator algebra and Y is a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module, then there exist approximately unital operator algebras $\mathcal{A}_0 \subseteq \mathcal{A}$ and $\mathcal{B}$ such that*

(i) $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}_0} \tilde{Y}$

(ii) $\mathcal{A}_0 \cong \tilde{Y} \otimes^h_{\mathcal{B}} Y$

(iii) $\mathcal{A}_0 \sim_{\sigma\Delta} \mathcal{B}$, $\mathcal{A}_0 \sim_{\sigma\Delta} Y$, $Y \sim_{\sigma\Delta} \tilde{Y}$.

Proof. It suffices to prove the above assertions for the case of a doubly σ-TRO-$\mathcal{A}$-module $Y = \overline{[M\mathcal{A}]}$ where $\mathcal{A} \subseteq \mathbb{B}(H)$, $M \subseteq \mathbb{B}(H,K)$ is a σ-TRO such that $M^* M\mathcal{A} \subseteq \mathcal{A}$ and

$$\overline{[M^* M\mathcal{A}]} = \overline{[\mathcal{A}M^* M]}. \tag{4.1}$$

We set $\mathcal{A}_0 = \overline{[\mathcal{A}M^* M]} \subseteq \mathcal{A}$. Clearly $\mathcal{A}_0$ is an approximately unital operator algebra.

(i) By Lemma 4.2, $\tilde{Y} \cong \overline{[\mathcal{A}M^*]}$, and so

$$\overline{[\mathcal{A}_0 M^*]} = \overline{[\mathcal{A}M^* M M^*]} = \overline{[\mathcal{A}M^*]} = \tilde{Y}.$$

On the other hand

$$\overline{[M\mathcal{A}_0]} = \overline{[M\mathcal{A}M^* M]} \overset{(4.1)}{=} \overline{[M M^* M\mathcal{A}]} = \overline{[M\mathcal{A}]} = Y.$$

Using Lemma 1.5 and making the same calculations as in the proof of Theorem 3.14 we have that $Y \otimes^h_{\mathcal{A}_0} \tilde{Y} \cong \overline{[M\mathcal{A}M^*]}$. If we define $\mathcal{B} = \overline{[M\mathcal{A}M^*]}$, then $\mathcal{B}$ is an approximately unital operator algebra such that $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}_0} \tilde{Y}$.

(ii) It is true that $\tilde{Y} \cong \overline{[\mathcal{A} M^*]} = \overline{[M^* \mathcal{B}]}$, so if $D_1 = \overline{[M^* M]}$ and $D_2 = \overline{[M M^*]}$, it follows that

$$\begin{aligned}
\tilde{Y} \otimes^h_{\mathcal{B}} Y &= \overline{[M^* \mathcal{B}]} \otimes^h_{\mathcal{B}} \overline{[M \mathcal{A}]} \\
&\cong \left(M^* \otimes^h_{D_2} \mathcal{B}\right) \otimes^h_{\mathcal{B}} Y \\
&\cong M^* \otimes^h_{D_2} \left[\mathcal{B} \otimes^h_{\mathcal{B}} \left(M \otimes^h_{D_1} \mathcal{A}\right)\right] \\
&\cong M^* \otimes^h_{D_2} M \otimes^h_{D_1} \mathcal{A} \\
&\cong \overline{[M^* M]} \otimes^h_{D_1} \mathcal{A} \\
&\cong \overline{[M^* M \mathcal{A}]} \\
&= \overline{[\mathcal{A} M^* M]} = \mathcal{A}_0.
\end{aligned}$$

(iii) Consider the σ-TROs $M_1 = M^* \subseteq \mathbb{B}(K, H)$ and $M_2 = M \subseteq \mathbb{B}(H, K)$. Then

$$\overline{[M_2^* Y M_1]} = \overline{[M^* M \mathcal{A} M^*]} = \overline{[\mathcal{A} M^* M M^*]} = \overline{[\mathcal{A} M^*]} = \tilde{Y}$$

and

$$\overline{[M_2 \tilde{Y} M_1^*]} = \overline{[M \mathcal{A} M^* M]} = \overline{[M M^* M \mathcal{A}]} = \overline{[M \mathcal{A}]} = Y$$

so $Y \sim_{\sigma TRO} \tilde{Y}$. By Theorem 3.8, we also have that $\mathcal{B} \sim_{\sigma TRO} Y$ and $\mathcal{B} \sim_{\sigma\Delta} \mathcal{A}_0$. □

Definition 4.4. Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras, E be a right $\mathcal{B}$-rigged module and F be a right $\mathcal{A}$-rigged module. We call E and F Morita equivalent if there exists a right $\mathcal{A}$-rigged module Y such that

(i) $\mathcal{A} \cong \tilde{Y} \otimes^h_{\mathcal{B}} Y$ as operator algebras,
(ii) $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} \tilde{Y}$ as operator algebras,
(iii) $F \cong E \otimes^h_{\mathcal{B}} Y$ as right $\mathcal{A}$-rigged modules.

In this case we write $E \sim_M F$.

Remark 4.5. If $\mathcal{A}$, $\mathcal{B}$, E and F are as above, then by [1, Theorem 6.1] we get that

$$\mathbb{K}(F) \cong \mathbb{K}\left(E \otimes^h_{\mathcal{B}} Y\right) \cong \mathbb{K}(E).$$

Definition 4.6. Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras, E be a right $\mathcal{B}$-rigged module and F be a right $\mathcal{A}$-rigged module. We call E and F σ-Morita equivalent if there exists a doubly $\sigma\Delta$-$\mathcal{A}$-rigged module Y such that

(i) $\mathcal{A} \cong \tilde{Y} \otimes^h_{\mathcal{B}} Y$ as operator algebras,
(ii) $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} \tilde{Y}$ as operator algebras,
(iii) $F \cong E \otimes^h_{\mathcal{B}} Y$ as right $\mathcal{A}$-rigged modules.

In this case we write $E \sim_{\sigma M} F$.

Remark 4.7. Other notions of Morita equivalence for the subcategory of Hilbert modules exist in [11, 17].

Proposition 4.8. *If $E \sim_{\sigma M} F$, then $\mathbb{K}(E) \cong \mathbb{K}(F)$.*

Proof. If $E \sim_{\sigma M} F$, then $E \sim_M F$ and the conclusion comes from Remark 4.5. □

Lemma 4.9. *Let M be a σ-TRO, $D_1 = \overline{[M M^*]}$, $D_2 = \overline{[M^* M]}$, E be a right D_1-module and F be a right D_2-module such that $F \cong E \otimes^h_{D_1} M$. Then $E \sim_{\sigma\Delta} F$.*

Proof. By [10, Theorem 3.8], it suffices to prove that E and F are stably isomorphic. We may assume that $F = E \otimes^h_{D_1} M$. Hence,

$$\begin{aligned} F \otimes^h_{D_2} M^* &= \left(E \otimes^h_{D_1} M\right) \otimes^h_{D_2} M^* \\ &\cong E \otimes^h_{D_1} \left(M \otimes^h_{D_2} M^*\right) \\ &\cong E \otimes^h_{D_1} D_1 \\ &\cong E. \end{aligned}$$

We can also assume that there exists a complete onto isometry

$$a : F \otimes^h_{D_2} M^* \to E$$

such that

$$a((e \otimes_{D_1} m) \otimes_{D_2} n^*) = e\, m\, n^*,\ \forall\, e \in E,\ m,\ n \in M. \tag{4.2}$$

There exists a sequence $\{m_i \in M \mid i \in \mathbb{N}\}$ such that

$$\left\|\sum_{i=1}^{n} m_i^*\, m_i\right\| \leq 1,\ \forall\, n \in \mathbb{N}$$

and also

$$\sum_{i=1}^{\infty} m\, m_i^*\, m_i = m,\ \forall\, m \in M.$$

We observe that for all $e \in E$ and $m \in M$ we have that

$$\begin{aligned} \sum_{i=1}^{\infty} a((e \otimes_{D_1} m) \otimes_{D_2} m_i^*) \otimes_{D_1} m_i &\overset{(4.2)}{=} \sum_{i=1}^{\infty} e\, m\, m_i^* \otimes_{D_1} m_i \\ &\overset{m\, m_i^* \in D_1}{=} \sum_{i=1}^{\infty} e \otimes_{D_1} m\, m_i^*\, m_i = e \otimes_{D_1} m. \end{aligned}$$

Thus,

$$\sum_{i=1}^{\infty} a(f \otimes_{D_2} m_i^*) \otimes_{D_1} m_i = f,\ \forall\, f \in F. \tag{4.3}$$

We define the completely contractive maps

$$\Phi : F \to R_\infty(E),\ \Phi(f) = (a(f \otimes_{D_2} m_i^*))_{i \in \mathbb{N}}$$

$$\Psi : R_\infty(E) \to F,\ \Psi((e_i)_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} e_i \otimes_{D_1} m_i.$$

Using (4.3) we have that

$$(\Psi\circ\Phi)(f) = \sum_{i=1}^{\infty} a(f \otimes_{D_2} m_i^*) \otimes_{D_1} m_i = f,\ \forall f \in F.$$

So, Φ is a complete isometry and $P = \Phi\circ\Psi : R_\infty(E) \to R_\infty(E)$ is a projection such that $\Phi(F) = \mathrm{Ran(P)}$. Now we employ the usual arguments, see for example the proof of [3, Corollary 8.2.6]:

$$R_\infty(E) \cong \mathrm{Ran(P)} \oplus_{\mathrm{r}} \mathrm{Ran(I-P)} \cong \Phi(\mathrm{F}) \oplus_{\mathrm{r}} \mathrm{Ran(I-P)} \cong \mathrm{F} \oplus_{\mathrm{r}} \mathrm{Ran(I-P)}$$

where $I = I_{R_\infty(E)}$. Thus,

$$\begin{aligned} R_\infty(E) &\cong R_\infty(R_\infty(E)) \\ &\cong (F \oplus_r \mathrm{Ran(I-P)}) \oplus_{\mathrm{r}} (\mathrm{F} \oplus_{\mathrm{r}} \mathrm{Ran(I-P)}) \oplus_{\mathrm{r}} \ldots \\ &\cong F \oplus_r (\mathrm{Ran(I-P)} \oplus_{\mathrm{r}} \mathrm{F}) \oplus_{\mathrm{r}} (\mathrm{Ran(I-P)} \oplus_{\mathrm{r}} \mathrm{F}) \oplus_{\mathrm{r}} \ldots \\ &\cong F \oplus_r R_\infty(E). \end{aligned}$$

Therefore, $R_\infty(E) \cong R_\infty(R_\infty(E)) \cong R_\infty(F) \oplus_r R_\infty(E)$. By symmetry, $R_\infty(F) \cong R_\infty(E) \oplus_r R_\infty(F)$, so $R_\infty(E) \cong R_\infty(F)$ which implies that $K_\infty(E) \cong K_\infty(F)$. □

Theorem 4.10. *Let $\mathcal{A}$, $\mathcal{B}$ be approximately unital operator algebras, E be a right $\mathcal{B}$-rigged module and F be a right $\mathcal{A}$-rigged module such that $E \sim_{\sigma M} F$. Then $E \sim_{\sigma\Delta} F$.*

Proof. Let $a : \mathcal{A} \to \mathbb{B}(H)$ be a completely-isometric representation of $\mathcal{A}$ on H and $M \subseteq \mathbb{B}(H,K)$ be a σ-TRO such that $M^* M\, a(\mathcal{A}) \subseteq a(\mathcal{A})$ and also $\overline{[M^* M\, a(\mathcal{A})]} = \overline{[a(\mathcal{A})\, M^* M]}$. Consider also the doubly $\sigma\Delta$-$\mathcal{A}$-rigged module $Y = \overline{[M\, a(\mathcal{A})]}$ such that $a(\mathcal{A}) \cong \tilde{Y} \otimes^h_{\mathcal{B}} Y$, $\mathcal{B} \cong Y \otimes^h_{\mathcal{A}} \tilde{Y} \cong \overline{[M\, a(\mathcal{A})\, M^*]}$ and also $F \cong E \otimes^h_{\mathcal{B}} Y$. We define $D_1 = \overline{[M\, M^*]}$ and we have that $\mathcal{B}\, M\, M^* \subseteq \mathcal{B}$. So

$$E = \overline{[E\,\mathcal{B}]} \supseteq \overline{[E\,\mathcal{B}\, M\, M^*]} = \overline{[E\, M\, M^*]}$$

which means that E is a right D_1-module. Therefore, since $Y = \overline{[M\, a(\mathcal{A})]} = \overline{[\mathcal{B}\, M]}$, it holds that

$$F \cong E \otimes^h_{\mathcal{B}} Y = E \otimes^h_{\mathcal{B}} \overline{[\mathcal{B}\, M]} \cong E \otimes^h_{\mathcal{B}} \left(\mathcal{B} \otimes^h_{D_1} M\right) \cong \left(E \otimes^h_{\mathcal{B}} \mathcal{B}\right) \otimes^h_{D_1} M \cong E \otimes^h_{D_1} M.$$

Observe that if $D_2 = \overline{[M^* M]}$, then $F = \overline{[F\,\mathcal{A}]} \supseteq \overline{[F\,\mathcal{A}\, M^* M]} = \overline{[F\, M^* M]}$ which means that F is a right D_2-module. From Lemma 4.9, we get that $E \sim_{\sigma\Delta} F$. □

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