

Examples of non-minimal open books with high fractional Dehn twist coefficient

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ABSTRACT. For a fixed surface S , one can ask if there are conditions on open books with pages S that imply maximality of the Euler characteristic of S among all pages of open books encoding the same 3-manifold (or at least imply maximality among those open books that encode the same 3-manifold and support the same contact structure). In this short note we propose an explicit variant of this question with a condition that involves the amount of twisting of the monodromy and the topological type of S , and we construct examples of open books for 3-manifolds that support our choice of condition. In particular, our examples show that the condition on twisting necessarily depends on the topological type of S . We find these examples of open books as the double branched covers of families of closed braids studied by Malyutin and Netsvetsev.

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1. Introduction

Denote by $S_{g,b}$ the compact oriented connected surface of genus $g \geq 0$ with $b \geq 1$ boundary components. Denote by $\text{MCG}(S_{g,b})$ the mapping class group of $S_{g,b}$. We recall (see e.g. [Etn06]) that every (conjugacy class of a) $\phi \in \text{MCG}(S_{g,b})$ determines a unique closed oriented connected 3-manifold M_ϕ together with an open book O_ϕ on M_ϕ with pages homeomorphic to $S_{g,b}$ and ϕ as its monodromy. We denote by $\omega(\phi) \in \mathbb{Q}$ the fractional Dehn twist coefficient of $\phi \in \text{MCG}(S_{g,1})$. Roughly, $\omega(\phi)$ measures the amount of twisting effected around the boundary of $S_{g,1}$ by ϕ .

Question 1.1. Fix an integer $g \geq 1$. Let ϕ in $\text{MCG}(S_{g,1})$ satisfy $|\omega(\phi)| > g - \frac{1}{2}$.

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- (i) Do the pages of open books for M_ϕ have Euler characteristic at most $1 - 2g$?
- (ii) Do the pages of open books for M_ϕ that support the same contact structure as O_ϕ have Euler characteristic at most $1 - 2g$?

Etnyre and Ozbagci defined the *norm* of a contact structure as the minimum of the negative Euler characteristic of the pages among all open books supporting the contact structure [EO08]. In their language, the second part of Question 1.1 can be rephrased as follows. If ϕ in $\text{MCG}(S_{g,1})$ satisfy $|\omega(\phi)| > g - \frac{1}{2}$, does the open book O_ϕ realize the norm of the contact structure on M_ϕ supported by O_ϕ ?

As a variant of Question 1.1, we ask: does there exist a family of constants $\{c_g\}_{g \in \mathbb{N}}$ such that there are positive answers to the above questions when replacing the assumption $|\omega(\phi)| > g - \frac{1}{2}$ with $|\omega(\phi)| > c_g$? A first naive hope could be that the variant of Question 1.1 can be answered in the positive for $c_g = 1$ or at least for $c_g = c$, where c is some universal constant. In other words, maybe there exists a $c > 0$ such that if $|\omega(\phi)| > c$, then the pages of the open book O_ϕ have the largest Euler characteristic among the pages among the open books for M_ϕ , or at least among the open books that support the same contact structure as O_ϕ . This hope is motivated by the fact that $|\omega(\phi)| > 1$ implies that the the open book O_ϕ on M_ϕ cannot be destabilized (destabilization is an operation that increases the Euler characteristic of pages without changing the manifold). However, this is wrong. In fact, the following theorem implies that any constants c_g as above have to grow at least linearly in g .

Theorem 1.2. *For every integer $k \geq 0$, there exists a Stein fillable contact 3-manifold supported by an open book with connected binding and pages of genus $k + 1$ whose monodromy has fractional Dehn twist equal to k such that its contact structure is also supported by an open book with connected binding and pages of genus k .*

We point to the end of Section 2 for (circumstantial) evidence that such constants c_g might exist and can be chosen to grow at most linearly in g ; in other words, evidence that Question 1.1 could have a positive answer. In addition to Theorem 1.2, this evidence is what brings us to propose the lower-bound $g - \frac{1}{2}$ in Question 1.1 as the potentially correct one (rather than any other affine linear expression in g).

We conclude the introduction by briefly comparing Question 1.1(ii) to an open question in contact geometry. Every overtwisted contact structure is supported by a planar open book, but the same is not true for tight contact structures: there exist such structures that require pages of genus one [Etn04]. It is an open question whether there exist any tight contact structures that are not supported by an open book whose pages have genus zero or one. By [HKM08] any contact structure that is supported by an open book O (with connected binding) whose fractional Dehn twist coefficient is greater than or equal to one

must be tight. A positive answer to Question 1.1(ii), when restricted just to positive fractional Dehn twist coefficients, would provide information about what types of open books for such tight contact structures are possible; but, it would not directly address the open question. One may also ask Question (1.1)(ii), when replacing *Euler characteristic at most* $1 - 2g$ with *genus at least* g . Namely, in the language of [EO08], if ϕ in $\text{MCG}(S_{g,1})$ satisfies $|\omega(\phi)| > g - \frac{1}{2}$, does the open book O_ϕ realize the support genus of the contact structure on M_ϕ supported by O_ϕ ?

Organization. In Section 2, we discuss the setup and motivation, in Section 3, we find the open books in Theorem 1.2 as the double branched covers of families of closed braids studied by Malyutin and Netsvetsev [MN03], and in Section 4, we discuss the distinctness of the underlying 3-manifolds of the open books we construct.

2. From braided links to open books via double branched covering

A general reference for open books and contact structures is [Etn06]. For a general reference for braids, we point to [Bir74, BB05].

Braids, braided links, and transverse links. Links are isotopy classes of smooth nonempty closed oriented 1-manifolds in S^3 . A *braiding* of a link L is a choice of representative of L that is transverse to the trivial open book of S^3 (the one with binding the unknot U). We consider braidings up to isotopy in the complement of U transverse to all pages. We refer to a link with a choice of braiding as a *braided link*. The number of transverse intersections of a braided link with a page is called *the number of strands*. The minimal number of strands among braidings of a link L is called the *braid index* of L . Braided links with $n \geq 1$ strands are canonically identified with conjugacy classes in *Artin’s braid group* on n -strands B_n [Art25]. Elements in B_n are called *braids*. Denoting by D_n the closed disc with n punctures, B_n can be defined as the mapping class group $\text{MCG}(D_n)$. We write $\widehat{\beta}$ for the braided link given by the conjugacy class of a braid β . Any two braidings of a given link are related by a sequence of so-called Markov stabilizations and destabilizations.

A braided link canonically determines a *transverse link*—a transverse isotopy class of a link transverse to the standard contact structure on S^3 (the one corresponding to the trivial open book). In fact, two braided links determine the same transverse link if and only if they are related by positive Markov stabilizations and destabilizations [Wri02, OS03].

Double branched covers. The double branched cover construction associates to a link L an oriented closed connected 3-manifold $\Sigma(L)$. This extends to braided links and open books:

$$\Sigma : \{\text{braided links}\} \rightarrow \{\text{open books on closed connected oriented 3-manifolds}\}.$$

Indeed, denoting by $\pi : \Sigma(L) \rightarrow S^3$ the double branched covering where a braiding of L is fixed, every page P of the trivial open book on S^3 has $\pi^{-1}(P)$ as the pages of an open book: π restricts to a double branched covering $\pi^{-1}(P) \rightarrow P$ of P along the intersection of P with the braiding. In particular, a braided link with n -strands yield an open book with pages $\Sigma(D_n)$, where $\Sigma(D_n)$ is $S_{\frac{n-1}{2},1}$ respectively $S_{\frac{n-2}{2},2}$ for n odd and even, respectively.

For another perspective, we note that Σ is induced by the Birman-Hilden embedding of groups

$$\text{BH} : B_n \hookrightarrow \text{MCG}(\Sigma(D_n)),$$

by considering the induced map that maps a conjugacy class $\widehat{\beta}$ of a braid β to the conjugacy class $[\text{BH}(\beta)]$ of $\text{BH}(\beta)$ and considering the corresponding open book $O_{\text{BH}(\beta)}$. This follows from the fact that BH is defined by taking the lift to the double branched cover; see e.g. [FM11, Chapter 5].

Recall that from an open book on a closed oriented 3-manifold one can construct others by so-called stabilizations and destabilizations (also known as Hopf plumbing and Hopf deplumbing). If two open books are related by so-called positive stabilization and destabilization then they support the same contact structure; and conversely, any two open books that support the same contact structure are related by a sequence of positive stabilizations and destabilizations by a result known as Giroux's correspondence [Gir03]. With this setup we note the following.

If a braided link $\widehat{\beta}'$ is obtained from a braided link $\widehat{\beta}$ by a positive (negative) Markov stabilization, then the open book $\Sigma(\widehat{\beta}')$ is obtained from $\Sigma(\widehat{\beta})$ by a positive (negative) stabilization. In particular, a transverse link (i.e. a class of braided links related by positive Markov stabilizations and destabilizations) give rise to a class of open books up to positive stabilizations and destabilizations, and hence a contact structure on the double branched cover of the link; see e.g. [Pla06].

Fractional Dehn twist coefficient. For a compact connected surface of finite type with boundary $S_{g,b,p}$ (genus $g \geq 0$, $b \geq 1$ boundary components, $p \geq 0$ punctures), one considers the so-called fractional Dehn twist coefficient, a homogeneous quasimorphism

$$\omega : \text{MCG}(S_{g,b,p}) \rightarrow \mathbb{Q}$$

with respect to a fixed boundary component; see [HKM07, HKM08] and compare also, for instance, to [HM18, IK18]. To avoid dependence on the choice of boundary, we restrict to the case $b = 1$.

Under the Birman-Hilden map, the fractional Dehn twist coefficient behaves simply: for all odd $n \geq 1$,

$$\omega(\text{BH}(\beta)) = \frac{\omega(\beta)}{2} \quad \text{for all } \beta \in B_n; \quad (1)$$

see [IK18, Theorem 4.2]. (The assumption that n is odd is to assure that $\Sigma(D_n)$ has one boundary component.) A key observation to see (1) is the following. For the full-twist $\Delta^2 = \delta^n \in B_n$ —the positive Dehn twist along the boundary of D_n —we have that $\text{BH}((\Delta^2)^2)$ is the Dehn twist along the boundary of $\Sigma(D_n)$.

Analogue of Question 1.1 for braidings of links. Before we use the above setup to discuss our examples in Section 3, we discuss why we dare to hope that Question 1.1 has a positive answer. Considering the double branched cover construction, the following is an analogue of Question 1.1.

Question 2.1. Fix an integer $n \geq 3$. Let β in B_n satisfy $|\omega(\beta)| > n - 2$ and denote by L and L^{trans} the link isotopy class and transverse link isotopy class obtained as the closure of β , respectively.

- Does every braid with closure L have at least n strands? [FH19, Quest. 7.3]
- Does every braid with transverse closure L^{trans} have at least n strands?

And indeed, taking n odd, and noting that

$$\omega(\text{BH}(\beta)) \stackrel{(1)}{=} \frac{\omega(\beta)}{2} \quad \text{and} \quad g := \text{genus}(\Sigma(D_n)) = \frac{n-1}{2},$$

we see that the assumptions of the two questions correspond:

$$|\omega(\beta)| > n - 2 \quad \text{if and only if} \quad |\omega(\text{BH}(\beta))| > g - \frac{1}{2}.$$

We were able to answer Question 2.1 in the positive for $n = 3$ and, for general n , with the stronger assumption $|\omega(\beta)| > n - 1$; see [FH19]. Hence, we feel justified to ask Question 1.1.

3. Construction of the examples

We describe our examples for the proof of Theorem 1.2 as double branched covers of braided links. To describe braids, we use the standard Artin generators $\sigma_1, \dots, \sigma_{n-1}$ for the braid group on n -strands [Art25].

For any pair of integers $n, m \geq 1$, define the braid

$$\beta_{n,m} := (\delta \delta^\Delta)^{m-1} \delta,$$

where

$$\delta := \sigma_1 \sigma_2 \cdots \sigma_{n-1} \quad \text{and} \quad \delta^\Delta := \sigma_{n-1} \sigma_{n-2} \cdots \sigma_1.$$

Malyutin and Netsvetsev observed that the closures of $\beta_{n,m}$ and $\beta_{m,n}$ are isotopic as links in S^3 ; see [MN03, Figure 2]. In fact, we can say more: Etnyre and Van Horn-Morris showed that any two positive braids representing the same link L are related by positive Markov stabilizations and destabilizations and braid isotopy [EVHM11, Corollary 1.13]. In particular, since both $\beta_{n,m}$ and $\beta_{m,n}$ are positive braids, we have the first two items of the following.

Proposition 3.1. Fix integers $n, m \geq 1$ and denote by L the closure of $\beta_{n,m}$ and $\beta_{m,n}$ as links in S^3 . Set

$$O_{n,m} := O_{\text{BH}(\beta_{n,m})} = \Sigma(\widehat{\beta_{n,m}}) \quad \text{and} \quad O_{m,n} := \Sigma(\widehat{\beta_{m,n}})$$

for the open books on $\Sigma(L)$ corresponding to $\beta_{n,m}$ and $\beta_{m,n}$, respectively. We have that

- i) the closures of $\beta_{n,m}$ and $\beta_{m,n}$ as transverse links in S^3 with the standard contact structure are transversely isotopic,
- ii) the two open books $O_{n,m}$ and $O_{m,n}$ correspond to the same contact structure $\xi_{\{n,m\}}$ on $\Sigma(L)$,
- iii) the contact manifold $(\Sigma(L), \xi_{\{n,m\}})$ is Stein fillable (and therefore tight)
- iv) for n odd, the fractional Dehn twist coefficients of $\text{BH}(\beta_{n,m})$ in $\text{MCG}(\Sigma(D_n))$, namely, the monodromy of the open book $O_{n,m}$, is $\omega(\text{BH}(\beta_{n,m})) = (m-1)/2$.

Proof. i) holds since $\beta_{n,m}$ and $\beta_{m,n}$ are related by positive Markov stabilizations and destabilizations. ii) holds since the corresponding open books $O_{n,m}$ and $O_{m,n}$ are related by corresponding positive stabilizations and destabilizations.

We discuss iii). It is a theorem of Giroux ([Gir03]; see also Etnyre [Etn06, Theorem 5.11]) that a contact manifold is Stein fillable if and only if there is an open book for it whose monodromy can be written as a composition of positive Dehn twists. The monodromy of $O_{n,m}$ is $[\text{BH}(\beta_{n,m})]$, and $\text{BH}(\beta_{n,m})$ is a composition of positive Dehn twists since $\beta_{n,m}$ is a composition of positive braid generators.

For iv), we use $\omega(\beta_{n,m}) = m-1$ (see [FH19, Example 6.1]) and (1). \square

Proof of Theorem 1.2. Fix a non-negative integer k . We consider the braid $\beta_{2k+3,2k+1}$ and denote by K_k its closure as a knot in S^3 . The fractional Dehn twist coefficient of the monodromy of the corresponding open book $O_{2k+3,2k+1}$ on $\Sigma(K_k)$ is k . The pages of $O_{2k+3,2k+1}$ have genus $k+1$. As the proposed contact manifold, we take the contact manifold $(\Sigma(K_k), \xi_{\{2k+3,2k+1\}})$ corresponding to the open book decomposition $O_{2k+3,2k+1}$. By Proposition 3.1, the contact manifold $(\Sigma(K_k), \xi_{\{2k+3,2k+1\}})$ is the same as to one corresponding to the open book $O_{2k+1,2k+3}$. We conclude the proof by noting that the pages of $O_{2k+1,2k+3}$ have genus k and that the contact manifold $(\Sigma(K_k), \xi_{\{2k+3,2k+1\}})$ is Stein fillable by Proposition 3.1. \square

4. Distinctness

The examples of contact manifolds we provided in our proof of Theorem 1.2 are pairwise distinct. In fact, the underlying manifolds are all pairwise non-homeomorphic.

Proposition 4.1. *For every integer $k \geq 1$, denote by K_k the closure of $\beta_{2k+1,2k+3}$ as a knot in S^3 . We have*

$$|H_1(\Sigma(K_k); \mathbb{Z})| = 4k^2 + 4k - 1.$$

Remark 4.2. Even without Proposition 4.1, it is clear that for every k , there exists an l_0 such that $\Sigma(K_l)$ is not homeomorphic to $\Sigma(K_k)$ for $l \geq l_0$. Indeed, for every closed oriented 3-manifold M , there exists a constant c_M such that all monodromies ϕ of open books with connected binding on M satisfy $|\omega(\phi)| \leq c_M$; see [HM18].

We establish Proposition 4.1 using that

$$|H_1(\Sigma(K); \mathbb{Z})| = |\det(K)| = |\Delta_K(-1)|$$

for all knots K , where $\det(K)$ and $\Delta(K)$ denote the knot determinant and the Alexander polynomial of K , respectively, and using the following connection to the Burau representation. For integers $n \geq 1$, we have

$$|\Delta_K(-1)| = |\det(I_{n-1} - f_*(\beta))| \text{ for all } \beta \in B_n,$$

where I_{n-1} denotes the identity matrix in $GL_{n-1}(\mathbb{Z})$ and $f_* : B_n \rightarrow GL_{n-1}(\mathbb{Z})$ denotes the reduced Burau representation evaluated at $t = -1$; see [Bir74] or [BB05, Equation (15)].

Proof of Proposition 4.1. For integers $i \geq 1, n \geq 1$, we set $\sigma_{i,n} := f_*(\sigma_i)$ for $\sigma_i \in B_n$. Using the following explicit matrices for $f_*(\sigma_i)$ for $n \geq 3$.

$$f_*(\sigma_i) = \left[\begin{array}{c|ccc|c} I_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n-i-2} \end{array} \right] \text{ for } 1 < i < n - 1,$$

$$f_*(\sigma_1) = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-3} \end{array} \right], \text{ and } f_*(\sigma_{n-1}) = \left[\begin{array}{c|cc} I_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -1 & 1 \end{array} \right].$$

Note that this agrees with the matrices used in [Bir74, BB05] up to replacing matrices by their inverse transpose.

With these explicit matrices one finds, for $n \geq 3$:

$$f_*(\delta) = \sigma_{1,n}\sigma_{2,n} \cdots \sigma_{n-1,n} = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline & & -I_{n-2} & 1 \\ & & & \vdots \\ & & & 1 \end{array} \right] \text{ and} \quad (2)$$

$$f_*(\delta^\Delta) = \sigma_{n-1,n}\sigma_{k-2,n} \cdots \sigma_{1,n} = \left[\begin{array}{c|cc} 1 & & \\ -1 & & I_{n-2} \\ \vdots & & \\ (-1)^{n-1} & & \\ \hline (-1)^n & 0 & \cdots & 0 \end{array} \right]. \quad (3)$$

Indeed, (2) and (3) are easily checked for $n = 3$, and, for $n \geq 4$, (2) and (3) follow inductively using that we have

$$\begin{aligned} \sigma_{1,n}\sigma_{2,n} \cdots \sigma_{n-2,n} &= \left[\begin{array}{ccc|c} \sigma_{1,n-1}\sigma_{2,n-1} \cdots \sigma_{n-2,n-1} & & & 1 \\ & & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right] \text{ and} \\ \sigma_{n-2,n}\sigma_{n-3,n} \cdots \sigma_{1,n} &= \left[\begin{array}{ccc|c} \sigma_{n-2,n-1}\sigma_{n-3,n-1} \cdots \sigma_{1,n-1} & & & 1 \\ & & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & 1 \end{array} \right]. \end{aligned}$$

So, we see that, for $n \geq 3$ odd, we have

$$(f_*(\delta\delta^\Delta))^2 = (f_*(\delta)f_*(\delta^\Delta))^2 \stackrel{(2),(3)}{=} \left[\begin{array}{ccc|c} -1 & 0 & \cdots & 0 \\ \hline -2 & & & \\ 0 & & & \\ -2 & & -I_{n-2} & \\ \vdots & & & \\ 0 & & & \\ -2 & & & \end{array} \right]^2 = \left[\begin{array}{ccc|c} 1 & 0 & \cdots & 0 \\ \hline 4 & & & \\ 0 & & & \\ 4 & & I_{n-2} & \\ \vdots & & & \\ 0 & & & \\ 4 & & & \end{array} \right],$$

and hence, for all integers $l \geq 1$, we have that $f_*(\delta\delta^\Delta)^{2l}f_*(\delta)$ equals

$$\left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 4 & \\ 0 & \\ 4 & I_{n-2} \\ \vdots & \\ 0 & \\ 4 & \end{array} \right]^l f_*(\delta) = \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 4l & \\ 0 & \\ 4l & I_{n-2} \\ \vdots & \\ 0 & \\ 4l & \end{array} \right] \left[\begin{array}{c|c} 0 \cdots 0 & 1 \\ \hline & 1 \\ & 1 \\ -I_{n-2} & 1 \\ & \vdots \\ & 1 \\ & 1 \end{array} \right] = \left[\begin{array}{c|c} 0 \cdots 0 & 1 \\ \hline & 4l + 1 \\ & 1 \\ -I_{n-2} & 4l + 1 \\ & \vdots \\ & 1 \\ & 4l + 1 \end{array} \right].$$

We conclude that the determinant

$$\det(I_{2k} - f_*(\beta_{2k+3,2k+1})) = \det(I_{2k} - f_*(\delta\delta^\Delta)^{2(k+1)}f_*(\delta))$$

equals

$$\det \left(\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & -1 \\ \hline 1 & 1 & 0 & \cdots & 0 & -4(k+1) - 1 \\ 0 & 1 & 1 & & & -1 \\ & & \ddots & & & \vdots \\ & & & 1 & 1 & -1 \\ & & & & 1 & -4(k+1) \end{array} \right] \right) = -4k^2 - 4k + 1,$$

where the last equality follows for example by developing the determinant using the last column. □

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