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The eigensheaf of an operator

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Abstract. The eigensheaf $\mathcal{F}_T$ of an operator T on a Hilbert space H is the subsheaf of $\mathcal{O}_H$ defined by the prescription:

$$\mathcal{F}_T(U) = \{f \in \mathcal{O}_H(U) \mid T(f(w)) = wf(w) \text{ for all } w \in U\},$$

where U is open in $\mathbb{C}$ and $\mathcal{O}_H$ is the sheaf of H-valued holomorphic functions defined on $\mathbb{C}$. If T lies in the Cowen-Douglas class, then its eigensheaf is locally free, but not conversely. We obtain a model for operators whose eigensheaves are locally free. We describe the eigensheaves for certain coanalytic Toeplitz operators, we show that the map from an operator to its eigensheaf is a functor from the category of bounded linear operators on Hilbert space to the category of Hilbert space-valued analytic sheaves, and we discuss relation between the eigensheaf of an operator and the sheaf that Putinar associates to an operator.

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1. The Cowen-Douglas correspondence

This article is partly motivated by the celebrated work [9] of Cowen and Douglas. In that paper, they introduced and studied a class of bounded linear Hilbert space operators with point spectrum containing an open set (now well-known as the Cowen-Douglas class $\mathbf{B}_n(\Omega)$). One of the main results in [9] proves that the classification of operators in $\mathbf{B}_n(\Omega)$ is equivalent to the classification of a class of holomorphic vector bundles associated to the operators in $\mathbf{B}_n(\Omega)$. Moreover, the present paper is very much in the spirit of the Putinar's

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work [26, 27, 28] on a sheaf associated to a bounded linear operator. In particular, we associate the class of analytic eigensheaves to the class of bounded linear Hilbert space operators (see Definition 2.1 below). Needless to say, it is different from the sheaf constructions appearing in [4, 12, 13, 26, 27, 28]. Indeed, the sheaf of this paper is a kernel sheaf while the one introduced and studied in [26] is a cokernel sheaf (see Section 4).

All the vector spaces here are over the field $\mathbb{C}$ of complex numbers. If E is a complex vector space, then $\mathbf{1}_E$ denotes the identity operator on E, and for every complex number w, we denote the linear operator $w\mathbf{1}_E : E \to E$ also by w. Throughout this paper, Ω stands for a bounded connected open subset of $\mathbb{C}$ and Ω^* denotes the set of complex conjugates of elements in Ω. For a complex Hilbert space, the space of bounded linear operators on H is denoted by $\mathcal{L}(H)$. For $T \in \mathcal{L}(H)$, the kernel, the point spectrum, the approximate point spectrum and the essential spectrum of T are denoted by $\mathrm{Ker}(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_e(T)$, respectively.

We now recall the definition of the Cowen-Douglas class (the reader is referred to [23] for an up-to-date exposition on this class). For a connected open subset Ω of $\mathbb{C}$ and n a positive integer, let $\mathbf{B}_n(\Omega)$ denote the collection of all operators $T \in \mathcal{L}(H)$ satisfying the following conditions:

- for every point $w \in \Omega$, $(T - w)(H) = H$ and $\dim \mathrm{Ker}(T - w) = n$,
- the subspace $\bigvee_{w\in\Omega} \mathrm{Ker}(T - w)$ of H equals H, where $\bigvee$ stands for the closed linear span.

We call $\mathbf{B}_n(\Omega)$ the *Cowen-Douglas class of degree n on H with respect to* Ω, and its elements as the *Cowen-Douglas operators of degree n on H with respect to* Ω.

The following is essentially [9, Proposition 1.11 and Corollary 1.12].

Proposition 1.1. *Let n be a positive integer, and T be a Cowen-Douglas operator of degree n on H with respect to* Ω. *Let*

$$E_T := \{(w, x) \in \Omega \times H \mid Tx = wx\}, \tag{1}$$

and let $\pi_T : E_T \to \Omega$ *be the restriction of the canonical projection from* $\Omega \times H$ *to* Ω. *Then, there is a natural structure of a complex manifold on* E_T *such that the map* $\pi_T : E_T \to \Omega$ *is a holomorphic vector bundle of rank n. Moreover, the function* $h_T : E_T \times_\Omega E_T \to \mathbb{C}$ *given by*

$$h_T((w, x), (w, x')) = \langle x,\, x'\rangle, \quad w \in E_T, x, x' \in H,$$

is a smooth Hermitian metric on this holomorphic vector bundle.

In view of the last result, one can define a map from the category of Cowen-Douglas class over an open connected subset Ω of $\mathbb{C}$ to the category of holomorphic vector bundles over Ω (see [9, Theorem 1.14]). It has been recorded in [3, Pg 1] that this correspondence is not essentially surjective (see [8, Section 0.3]; see also Example 3.5 below). On the other hand, there are many coanalytic Toeplitz operators belonging to $\mathbf{B}_m(\Omega')$ for some m and for some component Ω' of $\Omega \setminus \sigma_e(T)$, Also, the degree m may vary for different components of

$\Omega \setminus \sigma_e(T)$. The structure of these operators can be revealed by analysing their eigensheaves, and in turn, providing another motivation for the present work.

Here is the plan of this paper. In Section 2, we discuss the idea of the eigensheaf $\mathcal{F}_T$ of an arbitrary bounded linear Hilbert space operator T and observe that the support of $\mathcal{F}_T$ is always contained in the closure of the point spectrum of T (see Proposition 2.5). We also discuss the relation between the interior of the point spectrum of T and the support of the sheaf $\mathcal{F}_T$ (see Example 2.7). We show that if T is hyponormal on a separable Hilbert space, then $\mathcal{F}_T$ is necessarily trivial (see Corollary 2.8). Further, we investigate the class of operators with locally free eigensheaves and obtain a model for these operators (see Theorem 2.9). In Section 3, we exhibit a class of locally free analytic eigensheaves arising from certain coanalytic Toeplitz operators on the Hardy space of the unit disc (see Theorem 3.3). In Section 4, we discuss the relation between the Putinar's sheaf and the one investigated in this paper (see Proposition 4.1).

2. The eigensheaf of an operator

Let M be a complex manifold and let X be a complex Banach space. We say that a map $f : M \to X$ is *X-valued holomorphic* if the function $\alpha \circ f : M \to \mathbb{C}$ is holomorphic for every continuous linear functional $\alpha : X \to \mathbb{C}$. Note that if H is a separable complex Hilbert space and $\{e_n\}_{n \geqslant 0}$ is an orthonormal basis for H, then $f : M \to H$ is holomorphic if and only if $\langle f(\cdot), e_n \rangle : M \to \mathbb{C}$ is holomorphic for every integer $n \geqslant 0$ (refer to [2] for the basic theory of vector-valued holomorphic functions). Consider the sheaf $\mathcal{O}_{M,H}$ of H-valued holomorphic functions on M. This sheaf $\mathcal{O}_{M,H}$ has a canonical structure of an $\mathcal{O}_M$-module, where $\mathcal{O}_M$ is the sheaf of $\mathbb{C}$-valued holomorphic functions on M. If $M = \mathbb{C}$, then $\mathcal{O}_{M,H}$ is denoted by $\mathcal{O}_H$. The restriction of a sheaf $\mathcal{F}$ to a non-empty open subset Ω is denoted by $\mathcal{F}|_\Omega$ (refer to [31, Chapter 2] for the definition of sheaf and related notions).

The idea of associating a sheaf with a bounded linear operator first appears in the work [26]. This provides a novel approach to the local spectral theory (see [12, 27, 28, 13]; refer also to [4, Section 1.3] for discussion on a coherent sheaf associated to the Cowen-Douglas operators of degree 1).

Definition 2.1. For $T \in \mathcal{L}(H)$, define an $\mathcal{O}_{\mathbb{C}}$-submodule $\mathcal{F}_T$ of $\mathcal{O}_H$ by

$$\mathcal{F}_T(U) = \{f \in \mathcal{O}_H(U) \mid T(f(w)) = wf(w) \text{ for all } w \in U\}$$

for every open subset U of $\mathbb{C}$. We will call $\mathcal{F}_T$ the *eigensheaf of T*.

Remark 2.2. For open subsets U, V of $\mathbb{C}$ such that $V \subseteq U$, the restriction morphism $\mathrm{res}_{U,V} : \mathcal{F}_T(U) \to \mathcal{F}_T(V)$ is given by

$$\mathrm{res}_{U,V}(f) = f|_V, \quad f \in \mathcal{F}_T(U).$$

Clearly, for every open set U of $\mathbb{C}$, $\mathrm{res}_{U,U}$ is the identity morphism on $\mathcal{F}_T(U)$. Also, for open subsets U, V, W of $\mathbb{C}$ such that $W \subseteq V \subseteq U$, $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$. Thus $\mathcal{F}_T$ is a presheaf. If $\{\mathcal{U}_i\}_{i \in I}$ is an open covering of an open set U, and if $f, g \in \mathcal{F}_T(U)$ have the property $f|_{U_i} = g|_{U_i}$ for each $i \in I$, then

$f = g$. Moreover, if $\{\mathcal{U}_i\}_{i\in I}$ is an open covering of an open set U, and if for each $i \in I$ and $f_i \in \mathcal{F}_T(U_i)$ is given such that for each pair U_i, U_j of the covering sets $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$, then $f = f_i$ on U_i, $i \in I$, defines a section $f \in \mathcal{F}_T(U)$. Thus $\mathcal{F}_T$ is a sheaf. Finally, note that one can define the eigensheaf of a continuous linear operator on any Frechet space (recall that if X is a Frechet space, then $\mathcal{O}_X$ is the projective tensor product $\mathcal{O}_{\mathbb{C}}\widehat{\otimes}X$; see [13, Appendix 1]).

Let $\mathcal{B}$ denote the category of bounded linear Hilbert space operators. If $T : H \to H$ and $S : K \to K$ are two bounded linear operators then

$$\mathrm{Hom}_{\mathcal{B}}(\mathrm{T}, \mathrm{S}) = \{\mathrm{u} : \mathrm{H} \to \mathrm{K} : \mathrm{u}\circ\mathrm{T} = \mathrm{S}\circ\mathrm{u}\}.$$

Let $\mathcal{O}_{\mathbb{C}}$-**mod** denote the category of $\mathcal{O}_{\mathbb{C}}$-module over $\mathbb{C}$ and $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}}}(\mathcal{F}_1, \mathcal{F}_2)$ denote the set of $\mathcal{O}_{\mathbb{C}}$-module homomorphisms between the $\mathcal{O}_{\mathbb{C}}$-modules $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proposition 2.3. *Consider the category $\mathcal{B}$ of bounded linear Hilbert space operators and the category $\mathcal{O}_{\mathbb{C}}$-**mod** of $\mathcal{O}_{\mathbb{C}}$-module over $\mathbb{C}$. Define $F(T) = \mathcal{F}_T$ for $T \in \mathcal{B}$. For $u \in \mathrm{Hom}_{\mathcal{B}}(\mathrm{T}, \mathrm{S})$, define $F(u)$ as the $\mathcal{O}_{\mathbb{C}}$-module homomorphism $F(u) : \mathcal{F}_T \to \mathcal{F}_S$ given by*

$$F(u)_U : \mathcal{F}_T(U) \to \mathcal{F}_S(U), \quad F(u)_U(f) = u\circ f$$

for every open subset $U \subseteq \mathbb{C}$. Then F defines a functor.

Proof. Note that $u\circ f \in \mathcal{F}_S(U)$ for every $u \in \mathrm{Hom}_{\mathcal{B}}(\mathrm{T}, \mathrm{S})$. Indeed, since $f \in \mathcal{F}_T(U)$, $T(f(w)) = wf(w)$ for all $w \in U$, and hence

$$S(u\circ f(w)) = (S\circ u)(f(w)) = u\circ T(f(w)) = u(w(f(w))) = w(u\circ f)(w).$$

It is easy to see that $F(u)$ is a $\mathcal{O}_{\mathbb{C}}$-module homomorphism. Since $F(\mathbf{1}_H) = \mathbf{1}_{\mathcal{F}_T}$ (note that $\mathbf{1}_H$ is the identity morphism of T in the category $\mathcal{B}$) and $F(u\circ v) = F(u)\circ F(v)$ for every $u \in \mathrm{Hom}_{\mathcal{B}}(\mathrm{S}, \mathrm{T})$ and $v \in \mathrm{Hom}_{\mathcal{B}}(\mathrm{T}, \mathrm{R})$, F is a functor. □

Since F is a functor, if u is a Hilbert space isomorphism then $F(u)$ is a $\mathcal{O}_{\mathbb{C}}$-module isomorphism. Let us see a particular instance in which the last proposition is applicable. Consider the subspace K of H given by

$$K = \sum_{w\in\mathbb{C}} \mathrm{Ker}(T - w).$$

Define $S : K \to K$ be $S = T|_K$. Clearly, S is a bounded linear operator on K. Further, $T\circ u = u\circ S$, where $u : K \hookrightarrow H$ denotes the inclusion map. This yields a natural $\mathcal{O}_{\mathbb{C}}$-module homomorphism $\Phi : \mathcal{F}_T \to \mathcal{F}_S$. Moreover, since j is injective, so is Φ. Note that the above observation is applicable to any closed T-invariant subspace K of H.

By the correspondence between vector bundles and locally free sheaves (see [31]), the eigensheaf of a Cowen-Douglas operator of degree n is locally free sheaf of rank n (see [24, Definition 2.1.35]). This yields an analogue of Proposition 1.1 formulated in the language of eigensheaves. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free* if for every point x in a complex manifold X, there exist an open neighborhood U of x and a set I_x such that $\mathcal{F}|_U$ is isomorphic to the direct sum

$\mathcal{O}_X^{(I_x)}|_U$ as an $\mathcal{O}_X|_U$-module. For a nonnegative integer r, we say $\mathcal{F}$ is *locally free of rank* r if for every $x \in X$, the index set I_x has the cardinality r.

Proposition 2.4. *Let Ω be a connected open subset of $\mathbb{C}$, n be a positive integer, and T be a Cowen-Douglas operator of degree n on H with respect to Ω. Then the eigensheaf $\mathcal{F}_T|_\Omega$ of T is locally free of rank n.*

Proof. Note that the $\mathcal{O}_\Omega$-module $\mathcal{F}_T|_\Omega$ is the analytic sheaf of holomorphic sections of the holomorphic vector bundle $\pi_T : E_T \to \Omega$ (see (1)), and hence by Proposition 1.1 and the preceding discussion, the eigensheaf $\mathcal{F}_T|_\Omega$ is locally free of rank n. □

Note that $\mathcal{F}_T$ is an $\mathcal{O}_\mathbb{C}$-submodule of $\mathcal{O}_H$ obtained by imposing the condition on a section f that w is an eigenvalue of T whenever its evaluation $f(w)$ at w is non-zero. This motivates one to explore the relation between the set of eigenvalues of T and the support of $\mathcal{F}_T$. Recall that if $\mathcal{F}$ denotes a sheaf of abelian groups on a topological space X, then the *support of* $\mathcal{F}$ is given by

$$\mathrm{Supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}_x \neq 0\},$$

where $\mathcal{F}_x$ stands for the stalk of $\mathcal{F}$ at x. The following result describes the support of $\mathcal{F}_T$ (cf. [26, Lemma 2.1] and Corollary 4.3 below).

Proposition 2.5. *For $T \in \mathcal{L}(H)$, let $\mathcal{F}_T$ be the eigensheaf of T. Then the following statements are valid:*

(*i*) $\mathrm{Supp}(\mathcal{F}_T)$ *is an open subset of* $\mathbb{C}$ *such that*

$$\mathrm{Supp}(\mathcal{F}_T) \subseteq \overline{\sigma_p(T)}, \tag{2}$$

(*ii*) *for every open neighbourhood V of $z_0 \in \mathrm{Supp}(\mathcal{F}_T)$, there exists an open subset W of V contained in $\sigma_p(T)$,*

(*iii*) *if* $\mathrm{Supp}(\mathcal{F}_T) \neq \emptyset$, *then $\sigma_p(T)$ has non-empty interior,*

(*iv*) *if there exists an open connected subset U of $\mathbb{C}$ such that $\mathcal{F}_T(U) \neq \{0\}$, then $U \subseteq \overline{\sigma_p(T)}$.*

In particular, the support of $\mathcal{F}_T$ is contained in the approximate point spectrum of T.

Proof. Let $z_0 \in \mathrm{Supp}(\mathcal{F}_T)$. Then, $(\mathcal{F}_T)_{z_0} \neq 0$, so there is a nonzero element γ of $(\mathcal{F}_T)_{z_0}$. Choose a connected open neighbourhood U of z_0 and a section f of $\mathcal{F}_T(U)$ such that $\gamma = f_{z_0}$. Then, $f_z \neq 0$ for all $z \in U$. For, if $f_z = 0$ for some $z \in U$, then f vanishes on an open neighbourhood of z in U, and hence by the vector-valued identity theorem (see [2, Proposition A.2]), $f = 0$, which is not possible since $f_{z_0} \neq 0$. In particular, $(\mathcal{F}_T)_z \neq 0$ for all $z \in U$, and hence $U \subseteq \mathrm{Supp}(\mathcal{F}_T)$. This also shows that if V is any neighbourhood of z_0, then there exist an open neighbourhood U of z_0 and a point $z' \in U$ such that $f(z') \neq 0$. By the continuity of f, f is non-vanishing in some neighborhood $W \subseteq V$ of z'. As $f \in \mathcal{F}_T(U)$, we have $(T - w')(f(w)) = 0$ for every $w \in W$. Thus, $W \subseteq \sigma_p(T)$. This yields (ii) and also shows that $z_0 \in \overline{\sigma_p(T)}$, completing the proof of (i).

Clearly, (iii) follows from (ii). To see (iv), let $f : U \to H$ be a non-zero element of $\mathcal{F}_T(U)$. If $Z(f)$ denotes the zero set of f, then $U\backslash Z(f) \subseteq \sigma_p(T)$, and by the identity theorem, $U\backslash Z(f)$ is dense in U. It follows that

$$U = \overline{U\backslash Z(f)}^U \subseteq \overline{U\backslash Z(f)} \subseteq \overline{\sigma_p(T)},$$

where $\overline{W}^U$ denotes the closure of the subset W of U in U. Finally, since $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma_{ap}(T)$ is closed in $\mathbb{C}$, the remaining part is immediate from (2). □

Remark 2.6. If $\mathcal{F}_T|_U$ is of finite type, then its support is closed in U (see [16, Chapter 0, Section 5.2.2], [24, Corollary 2.1.17]).

In general, the point spectrum of T need not be contained in the support of $\mathcal{F}_T$. For instance, if T is a diagonal operator with diagonal entries $\{1/n\}_{n\geqslant 1}$ then $\sigma_p(T) \neq \emptyset$. However, by Proposition 2.5(iii), $\mathcal{F}_T = \{0\}$. Although equality does not hold in (2) in general, the natural question arises is whether one can recover the support of $\mathcal{F}_T$ from the point spectrum $\sigma_p(T)$ of T? By Proposition 2.5(iii), if the interior $\sigma_p(T)^\circ$ of $\sigma_p(T)$ is empty then the support of $\mathcal{F}_T$ is also empty. So perhaps the correct guess would be the following inclusion:

$$\sigma_p(T)^\circ \subseteq \mathrm{Supp}(\mathcal{F}_T). \tag{3}$$

This certainly holds if eigenfunctions of T are holomorphic functions on the interior of point spectrum (for instance, this happens for operators T in the Cowen-Douglas class). Here is a slightly different example (cf. Theorem 3.3).

Example 2.7. Consider the positive definite kernel $\kappa(z, w)$ given by

$$\kappa(z, w) = \frac{(z^2 + \overline{z} + 2)(\overline{w}^2 + w + 2)}{1 - z\overline{w}}, \quad z, w \in \mathbb{D}.$$

By [25, Theorem 5.21], z is a multiplier of the reproducing kernel Hilbert space H_κ associated with κ. Let M_z denote the operator of multiplication by z on H_κ. Then any w in the open unit disc $\mathbb{D}$ is an eigenvalue of $T := M_z^*$ with corresponding eigenfunction $\kappa(\cdot, \overline{w}) \in H_\kappa$, which is clearly not holomorphic in w. Still, the holomorphic section $w \mapsto \frac{z^2+\overline{z}+2}{1-zw}$ belongs to $\mathcal{F}_T|_{\mathbb{D}}$, and hence the inclusion (3) holds. ◇

For the definition of the single-valued extension property (for short, SVEP), see [1, Definition 2.3]. The following is certainly well-known. Indeed, it is a consequence of [1, Theorem 3.96] and the discussion prior to it.

Corollary 2.8. *Let H be a separable Hilbert space. For $T \in \mathcal{L}(H)$, let $\mathcal{F}_T$ be the eigensheaf of T. If T is hyponormal, that is, $T^*T - TT^* \geq 0$, then $\mathcal{F}_T = \{0\}$. In particular, a hyponormal operator on a separable Hilbert space has SVEP.*

Proof. Recall the well-known fact about a hyponormal operator that its eigenvectors corresponding to distinct eigenvalues are orthogonal [21]. Since H is separable, T can have at most countable distinct eigenvalues. The desired conclusion now follows from Proposition 2.5(iii). The remaining part now follows from the fact that T has the SVEP if and only if $\mathcal{F}_T$ is trivial. □

In the remaining part of this section, we study operators with locally free eigensheaves. The following result sheds some light on the relation between Cowen-Douglas operators and operators with locally free eigensheaves over connected domains. The later one is generically an extension of the adjoint of the multiplication operator M_z on a reproducing kernel Hilbert space of vector-valued homomorphic functions. We make this statement precise in the following theorem. The proof presented below is an adaptation of the method of modelling Cowen-Douglas class to the present situation (see [9, Pg 194] for Cowen-Douglas class of degree 1, and [10, Theorem 4.12], [33, Theorem B], [14, Pg 279-280] for the general case).

Theorem 2.9. *Let Ω be a bounded open connected subset of $\mathbb{C}$ and let $T \in \mathcal{L}(H)$. If the eigensheaf $\mathcal{F}_T|_\Omega$ of T is locally free of rank n, then there exists a reproducing kernel Hilbert space $\mathcal{H}$ of holomorphic $\mathbb{C}^n$-valued functions on Ω^* and a surjective partial isometry $V : H \to \mathcal{H}$ such that $VT^* = M_zV$, where M_z denotes the operator of multiplication by the coordinate function z acting on $\mathcal{H}$.*

Proof. Suppose that $\mathcal{F}_T|_\Omega$ is locally free of rank n. By a theorem of Grauert, any holomorphic vector bundle over an open subset of $\mathbb{C}$ is holomorphically trivial, and hence it admits a global holomorphic frame ([15, Theorem 30.4]). Let $\{f_1, \dots, f_n\}$ denote a global holomorphic frame for the holomorphic vector bundle associated with $\mathcal{F}_T|_\Omega$. Define $V : H \to \mathcal{O}(\Omega^*, \mathbb{C}^n)$ by

$$(Vh)(w) = (\langle h, f_1(\overline{w})\rangle, \dots, \langle h, f_n(\overline{w})\rangle), \quad h \in H,\ w \in \Omega^*.$$

Note that $Vh = 0$ if and only if h is orthogonal to

$$K := \sum_{w\in\Omega} \mathrm{Ker}(T - w) = \sum\{f_j(w) : w \in \Omega,\ j = 1, \dots, n\}.$$

Thus $\mathrm{Ker}(V) = H \ominus K$. Moreover, for any $h \in H$ and $w \in \Omega^*$,

$$\begin{aligned}(VT^*h)(w) &= (\langle T^*h, f_1(\overline{w})\rangle, \dots, \langle T^*h, f_n(\overline{w})\rangle) \\ &= (\langle h, \overline{w} f_1(\overline{w})\rangle, \dots, \langle h, \overline{w} f_n(\overline{w})\rangle) \\ &= (M_zVh)(w),\end{aligned}$$

where $M_z : \mathcal{O}(\Omega^*, \mathbb{C}^n) \to \mathcal{O}(\Omega^*, \mathbb{C}^n)$ is given by $(M_zF)(w) = wF(w)$, $F \in \mathcal{O}(\Omega^*, \mathbb{C}^n)$. Consider the complex vector space $\mathcal{H} := \{Vh : h \in H\}$ endowed with the inner product

$$\langle Vh,\, Vh'\rangle_{\mathcal{H}} = \langle P_K h,\, P_K h'\rangle_H, \quad h, h' \in H, \tag{4}$$

where P_K denotes the orthogonal projection of H onto K. Thus $\tilde{V} : H \to \mathcal{H}$ given by $\tilde{V}h = Vh$, $h \in H$, is a partial isometry with the initial space K and the final space $\mathcal{H}$. Therefore $\mathcal{H}$ is a Hilbert space with the natural inner product induced by the norm $\|\cdot\|_{\mathcal{H}}$. The intertwining relation $VT^* = M_zV$ shows that M_z is a bounded linear operator on $\mathcal{H}$. It now suffices to check that $\mathcal{H}$ is a reproducing kernel Hilbert space. To see that, let $L(\mathbb{C}^n, K)$ denote the vector space of linear operators from $\mathbb{C}^n$ into K, and define $\gamma : \Omega^* \to L(\mathbb{C}^n, K)$ by

$$\gamma(w)(\alpha) = \sum_{j=1}^{n} \alpha_j f_j(\overline{w}), \quad \alpha = (\alpha_1, \dots, \alpha_n),\ w \in \Omega^*.$$

Consider the positive definite kernel $\kappa : \Omega^* \times \Omega^* \to L(\mathbb{C}^n)$ given by $\kappa(z, w) = \gamma(w)^*\gamma(w)$. It is easy to see that $\gamma(w)^*h = (Vh)(w)$ for every $w \in \Omega^*$ and $h \in K$. It now follows from (4) that for $h \in H$, $w \in \Omega^*$ and $\alpha \in \mathbb{C}^n$,

$$\begin{aligned} \langle Vh,\, \kappa(\cdot, w)(\alpha)\rangle_{\mathcal{H}} &= \langle Vh,\, \gamma(\cdot)^*\gamma(w)(\alpha)\rangle_{\mathcal{H}} \\ &= \langle Vh,\, V(\gamma(w)(\alpha))\rangle_{\mathcal{H}} \\ &= \langle P_K h,\, P_K \gamma(w)(\alpha)\rangle_H \\ &= \langle (Vh)(w),\, \alpha\rangle_{\mathbb{C}^n}. \end{aligned}$$

This completes the proof. □

We conclude the section with a brief discussion on the class $\mathbf{S}_{m,n}(\Omega_1, \Omega_2)$ of operators T for which $\mathcal{F}_T|_{\Omega_1}$ and $\mathcal{F}_{T^*}|_{\Omega_2}$ are locally free of ranks m and n respectively, where Ω_1, Ω_2 are bounded, open connected subsets of $\mathbb{C}$. This class appears to be closely related to the class $\mathbf{B}_{m,n}(\Omega_1, \Omega_2)$ as investigated by M. Cowen in [8]. To see this, let T be an operator on H in $\mathbf{S}_{m,n}(\Omega_1, \Omega_2)$, and consider closed subspaces K_1, K_2 of H given by

$$K_1 := \sum_{w_1 \in \Omega_1} \operatorname{Ker}(T - w_1), \quad K_2 := \sum_{w_2 \in \Omega_2} \operatorname{Ker}(T^* - w_2).$$

If $H = \sum(K_1 \cup K_2)$, then T admits the decomposition

$$T = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \text{ on } H = K_1 \oplus K_2,$$

where T_1, T_2 are bounded linear operators on K_1, K_2 such that $\mathcal{F}_{T_1}|_{\Omega_1}$, $\mathcal{F}_{T_2^*}|_{\Omega_2}$ are locally free of ranks m, n respectively, and X is a bounded linear transformation from K_1 into K_2 (cf. [8, Proposition 1.6]). The fact that K_1 and K_2 are mutually orthogonal follows from the following general fact.

Proposition 2.10. *For $T \in \mathcal{L}(H)$, let $\mathcal{F}_T$ be the eigensheaf of T. If U, V are open subsets of $\mathbb{C}$, then for every $f \in \mathcal{F}_T(U)$ and $g \in \mathcal{F}_{T^*}(V)$, the map $\phi : U \times V \to \mathbb{C}$ given by*

$$\phi(u, v) = \langle f(u), g(v)\rangle, \quad u \in U,\ v \in V,$$

is identically 0.

Proof. For $u \in U$ and $v \in V$, we have

$$u\langle f(u), g(v)\rangle = \langle Tf(u), g(v)\rangle = \langle f(u), T^*g(v)\rangle = \overline{v}\langle f(u), g(v)\rangle.$$

Thus $\langle f(u), g(v)\rangle = 0$ whenever $u \neq \overline{v}$. By continuity of f and g, we get the same conclusion for every $u \in U$ and $v \in V$. □

The classification of operators with locally free eigensheaves (in the spirit of [9, Theorem 1.14]) seems to be beyond reach at present. In this context, we conclude from Theorem 2.9 that any operator with locally free eigensheaf over a connected open set is necessarily an extension of the adjoint of multiplication operator M_z in a reproducing kernel Hilbert space.

3. Toeplitz operators with coanalytic symbols

The class of Toeplitz operators has been extensively studied in the literature, particularly, in the context of Cowen-Douglas theory (see [19, 20, 18, 32, 17]). It is worth noting that Cowen-Douglas class does not support Toeplitz operators with analytic symbols. Indeed, as a consequence of Corollary 2.8, one may conclude that the eigensheaf of the multiplication operator M_ϕ on Hardy space $H^2(\mathbb{D})$ of the unit disc with analytic symbol ϕ is trivial. Recall that $H^2(\mathbb{D})$ is the Hilbert space of holomorphic functions $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ on the open unit disc $\mathbb{D}$ endowed with the norm

$$\|f\| = \Big(\sum_{k=0}^{\infty} |\hat{f}(k)|^2\Big)^{1/2}, \quad f \in H^2(\mathbb{D}).$$

It turns out that $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel κ given by

$$\kappa(z,w) = \frac{1}{1-z\overline{w}}, \quad z, w \in \mathbb{D}. \tag{5}$$

This means that

$$f(w) = \langle f, \kappa(\cdot, w)\rangle, \qquad w \in \mathbb{D},\ f \in H^2(\mathbb{D}). \tag{6}$$

Recall further that for $\phi \in H^\infty(\mathbb{D})$ (the space of bounded holomorphic functions on $\mathbb{D}$), the multiplication (bounded linear) operator M_ϕ is defined by

$$(M_\phi f)(z) = \phi(z)f(z), \quad f \in H^2(\mathbb{D}),\ z \in \mathbb{D}.$$

The Hardy space $H^2(\mathbb{D})$ can be identified with the the closed subspace $H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ if } n < 0\}$ of $L^2(\mathbb{T})$ by associating every $f \in H^2(\mathbb{D})$ to its boundary function $\tilde{f}$. Here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. The *Toeplitz* operator T_ϕ on $H^2(\mathbb{T})$ with *symbol* $\phi \in C(\mathbb{T})$ is defined by $T_\phi = PN_\phi|_{H^2(\mathbb{T})}$, where N_ϕ denotes the operator of multiplication by ϕ in $L^2(\mathbb{T})$ and P is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It is easy to see that if $\phi \in H^\infty(\mathbb{D})$ then M_ϕ is unitarily equivalent to $T_{\tilde{\phi}}$ (refer to [22] for the basics of Toeplitz operators).

In this section, we describe the analytic sheaf $\mathcal{F}_T$ for a class of Toeplitz operators T with coanalytic symbols (cf. [6, Theorem 1]). We begin with a couple of lemmas. The first of which is well-known (see, for instance, [11, Theorem 7.26] and [30, Theorem 10]).

Lemma 3.1. *For $\phi \in H^\infty(\mathbb{D})$ with boundary function $\tilde{\phi}$ on $\mathbb{T}$, the multiplication operator M_ϕ (or equivalently M_ϕ^*) is Fredholm if and only if there exists $M > 0$*

such that $|\tilde{\phi}| > M$ almost everywhere on $\mathbb{T}$ and ϕ has finitely many zeros inside $\mathbb{D}$. In this case, the Fredholm index is given by

$$\operatorname{ind} M_\phi^* = \dim \operatorname{Ker}(M_\phi^*) = \textit{the number of zeros of } \phi \textit{ inside } \mathbb{D}.$$

The following lemma appears to be known (see [29, 9]). Since we could not locate it in the form we need in the sequel, we include it.

Lemma 3.2. *Let U be a bounded open subset of $\mathbb{C}$ and let $T \in \mathcal{L}(H)$. If the range of $T - w$ is closed for every $w \in U$, then $\mathcal{F}_T|_U$ is locally free of rank n if and only if*

$$\dim \operatorname{Ker}(T - w) = n \textit{ for every } w \in U. \tag{7}$$

Proof. Suppose that the range of $T - w$ is closed for every $w \in U$. If $\mathcal{F}_T|_U$ is locally free of rank n, then $U \subseteq \operatorname{Supp}(\mathcal{F}_T)$, and hence we obtain (7). To see the sufficiency part, assume that (7) holds. Note that $T - w$ is left Fredholm for every $w \in U$. One may now argue as in the proof of [9, Proposition 1.11] (the situation is similar here except that possibly $\operatorname{ind}(T - w)$ could be $-\infty$) to conclude that for every $w_0 \in U$, there exist H-valued holomorphic functions $e_1(w), \dots, e_n(w)$ defined on some neighborhood $\Delta \subseteq U$ of w_0 such that $\{e_1(w), \dots, e_n(w)\}$ forms a basis for $\operatorname{Ker}(T - w)$ for every $w \in \Delta$. In view of the correspondence between vector bundles and locally free sheaves, $\mathcal{F}_T|_U$ is locally free of rank n. □

We now state the main result of this section (cf. [26, Section 5]).

Theorem 3.3. *Let p be a non-constant polynomial in the complex variable z of degree n and let T be the bounded linear operator M_p^* on $H^2(\mathbb{D})$. Then there exist clopen (possibly empty) subsets $\Omega_1, \dots, \Omega_n$ of $p(\mathbb{D})$ such that*

$$p(\mathbb{D}) \setminus p(\mathbb{T}) = \bigsqcup_{k=1}^{n} \Omega_k \ \textit{(disjoint union)}.$$

Moreover, the following statements are valid:

(*i*) *for every $k = 1, \dots, n$, $\mathcal{F}_T|_{\Omega_k}$ is locally free of rank k provided Ω_k is non-empty,*

(*ii*) *if, in addition, p has real coefficients, then*

$$H^2(\mathbb{D}) = \sum_{\substack{w \in \Omega_k \\ k=1,\dots,n}} \operatorname{Ker}(T - w).$$

Proof. For $k = 1, \dots, n$, define Ω_k by

$$\Omega_k = \big\{\alpha \in p(\mathbb{D}) \setminus p(\mathbb{T}) : p(z) = \alpha \text{ has precisely } k \text{ roots in } \mathbb{D} \\ \text{(counted with multiplicities)}\big\}. \tag{8}$$

Let $\alpha \in \Omega_k$ and note that $\epsilon := \inf_{z \in \mathbb{T}} |p(z) - \alpha| > 0$. In particular, for $\beta \in \mathbb{C}$ such that $|\beta - \alpha| < \epsilon$,

$$|(p(z) - \alpha) - (p(z) - \beta)| < \epsilon \leqslant |p(z) - \alpha|, \quad z \in \mathbb{T}.$$

By Rouché's Theorem [7], β must belong to Ω_k, and hence Ω_k is an open subset of $p(\mathbb{D}) \setminus p(\mathbb{T})$. We next check that each Ω_k is relatively closed in $p(\mathbb{D}) \setminus p(\mathbb{T})$. By Lemma 3.1 (applied to $\phi = p - \alpha$) and (8),

$$\Omega_k = \{\alpha \in p(\mathbb{D}) \setminus p(\mathbb{T}) : \dim \operatorname{Ker}(T - \overline{\alpha}) = k\}. \tag{9}$$

Now, if $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Omega_k$ is such that $\alpha_n \to \alpha$ for some $\alpha \in p(\mathbb{D}) \setminus p(\mathbb{T})$, then $T - \alpha_n$ converges to $T - \alpha$ in the operator norm, and hence by the continuity of the Fredholm index, $\operatorname{ind}(T - \overline{\alpha}) = k$. Hence, by (9), $\alpha \in \Omega_k$. Also, since the range of $T - \alpha$ is closed for every $\alpha \in p(\mathbb{D}) \setminus p(\mathbb{T})$ (see Lemma 3.1), the desired conclusion in (i) is now immediate from Lemma 3.2.

To see (ii), let $f \in H^2(\mathbb{D})$ be such that $\langle f, g \rangle = 0$ for every $g \in \operatorname{Ker}(T - w)$, $w \in \Omega_j$ and $k = 1, \dots, n$. Choose $k \in \{1, \dots, n\}$ such that Ω_k is non-empty (this is possible since p is non-constant). Let $\alpha \in \Omega_k$ be such that $p^{-1}\{\alpha\} \cap Z(p') = \emptyset$, where $Z(p')$ denotes the zero set of the derivative p' of p. We contend that for $j = 1, \dots, k$, there exist an open neighborhood V of α and a holomorphic function $g_j : V \to H^2(\mathbb{D})$ such that $\{g_j(w) : j = 1, \dots, k\}$ forms a basis $\operatorname{Ker}(T - w)$ for every $w \in V$. Note that p is a covering map from $\mathbb{D} \setminus Z(p')$ onto $p(\mathbb{D} \setminus Z(p'))$. Hence, one can choose a neighborhood V centered at α such that

$$p^{-1}(V) \cap Z(p') = \emptyset, \quad p^{-1}(V) = \sqcup_{j=1}^{k} W_j$$

such that $p|_{W_j}$ is a one-one holomorphic map onto V. Let $q_j : V \to W_j$ be the holomorphic inverse of $p|_{W_j}$, and define $g_j(w) = \kappa(\cdot, \overline{q_j(w)})$, $w \in V$ (see (5)). Assume now that p has real coefficients. Note that

$$\begin{aligned} Tg_j(w) &= M_p^* \kappa(\cdot, \overline{q_j(w)}) = \overline{p(\overline{q_j(w)})}\kappa(\cdot, \overline{q_j(w)}) \\ &= p(q_j(w))\kappa(\cdot, \overline{q_j(w)}) = w g_j(w), \quad w \in V, \ j = 1, \dots, k. \end{aligned}$$

Thus $g_j(w) \in \operatorname{Ker}(T - w)$, and hence by (6),

$$f \circ \overline{q_j(w)} = \langle f, g_j(w) \rangle = 0, \quad w \in V.$$

By the open mapping and the identity theorem (see [7]), f is identically zero. This completes the verification of part (ii). □

Remark 3.4. We make several remarks in order:

(1) The part (ii) says that the eigenvectors of M_p^* corresponding to the points in the complement of the essential spectrum $\sigma_e(T_p^*)$ of T_p^* are complete (see [6, Corollary 2], [32, Corollary 1] for variants).

(2) The argument given for part (ii) actually yields the following stronger assertion: If Ω_k is non-empty for some $k = 1, \dots, n$, then

$$H^2(\mathbb{D}) = \sum_{w \in \Omega_k} \operatorname{Ker}(T - w).$$

(3) The conclusions in (i) and (ii) hold true, with obvious modifications, if we replace the polynomial p by a bounded holomorphic function g satisfying $\overline{g(\overline{z})} = g(z)$, $z \in \mathbb{D}$.

(4) In the terminology of [19], p is an m-analytic cover of the connected component Ω_m of $p(\mathbb{D}) \setminus p(\mathbb{T})$ for every $m = 1, \dots, n$. In particular, M_p^* belongs to the Cowen-Douglas class $\mathbf{B}_m(\Omega_m)$ provided $\Omega_m \neq \emptyset$.

(5) It may be deduced from Oka's Coherence Theorem and Three Lemma for coherent sheaves (see [24, Corollary 2.1.31]) that $\mathcal{F}_T|_{p(\mathbb{D})\setminus p(\mathbb{T})}$ is a coherent sheaf (cf. [4, Proposition 1.4]).

In view of Remark 3.4(4), we certainly know that M_p^* is in $\mathbf{B}_m(U)$ for some positive integer m and some domain U. The essential point in Theorem 3.3 is that it captures spectral information about M_p^* beyond U. This is achieved after examining the sheaf $\mathcal{F}_{M_p^*}$.

Example 3.5. We discuss here two examples.

(1) If $p(z) = az^n + b$ for a non-zero scalar a, then $\Omega_j = \emptyset$ if and only if $j \neq n$, and hence $p(\mathbb{D}) \setminus p(\mathbb{T}) = \Omega_n$. In this case, $\mathcal{F}_T|_{\Omega_n}$ is a locally free sheaf of rank n.

(2) It may happen that Ω_j is non-empty for more than one value of positive integer j. For instance, if $p(z) = z(z^2 - \frac{z}{2} - 1)$, then $-\frac{1}{2} \in \Omega_1$ (the roots of $p(z) = -1/2$ are $\pm 1, \frac{1}{2}$) and $0 \in \Omega_2$ (the roots of $p(z) = 0$ are $0, \frac{1}{4} \pm \frac{\sqrt{17}}{4}$). ◇

4. Relation with the Putinar's sheaf

In this short section, we reveal the relation of the sheaf $\mathcal{F}_T$ with the cokernel sheaf introduced in [26]. Recall that the *cokernel sheaf* $\mathcal{G}_T$ associated to a bounded linear operator T on H is the sheaf associated with the presheaf

$$\mathcal{G}_T(U) = \mathcal{O}_H(U)/(T - w)\mathcal{O}_H(U), \quad U \subseteq \mathbb{C} \text{ open}.$$

This sheaf first appears in [26] in the context of local spectral theory. It turns out that if T has the Bishop's property (β) (see [1, Definition 6.14]), then $\mathcal{G}_T$ is a sheaf (see [26, Proposition 1.3], [13, P. 9]). To explain the precise relation between the cokernel sheaf and the eigensheaf, we need a definition. Following [9], we say that $w_0 \in \mathbb{C}$ is a *point of stability* of a bounded linear operator T on H if $T - w_0$ is Fredholm and $\dim \operatorname{Ker}(T - w)$ is constant in a neighbourhood of w_0.

Proposition 4.1. *For $T \in \mathcal{L}(H)$, let $\mathcal{F}_T$ be the eigensheaf of T. If w_0 is a point of stability for T, then there exists a neighbourhood N_{w_0} of w_0 such that the following hold*:

(*i*) *if T^t denotes the real transpose of T, then $\mathcal{G}_{T^t}|_{N_{w_0}}$ is a sheaf,*

(*ii*) *the sheaves $\mathcal{G}_{T^t}|_{N_{w_0}}$ and $\mathcal{F}_T|_{N_{w_0}}$ are isomorphic.*

If (ii) holds, then $w_0 \in \operatorname{Supp}(\mathcal{G}_{T^t})$ if and only if $w_0 \in \operatorname{Supp}(\mathcal{F}_T)$.

Proof. By the proof of [9, Proposition 1.11], there exist a neighbourhood N_{w_0} of w_0 outside the essential spectrum of T and a holomorphic function $P : N_w \to \mathcal{L}(H)$ such that

$$\mathrm{Ker}(T - w) = P(w)H, \quad w \in N_{w_0}. \tag{10}$$

By the closed range theorem, the range of $T^t - w$ is closed for every $w \in N_{w_0}$. For any $f \in \mathcal{O}_H(N_{w_0})$, we now obtain the orthogonal decomposition of H when considered over the real field $\mathbb{R}$:

$$f(w) = f_1(w) + (T^t - w)f_2(w), \quad w \in N_{w_0}, \tag{11}$$

where $f_1(w) \in \mathrm{Ker}(T - w)$ and $f_2(w) \in \mathrm{Ker}(T^t - w)^{\perp}$. By (10), $w \mapsto f_1(w) \in \mathcal{O}_H(N_{w_0})$. It follows that $w \mapsto (T^t - w)f_2(w)$ is holomorphic on N_{w_0}. Thus, by the Chain rule and the continuity of T, for every $w \in N_{w_0}$ and $h \in H$,

$$\begin{aligned}\frac{\partial}{\partial \overline{w}}\Big\langle (T^t - w)f_2(w), h\Big\rangle &= \frac{\partial}{\partial \overline{w}}\Big\langle f_2(w), (T^t)^* h\Big\rangle - w\frac{\partial}{\partial \overline{w}}\Big\langle f_2(w), h\Big\rangle \\ &= \Big\langle (T^t - w)\frac{\partial}{\partial \overline{w}} f_2(w), h\Big\rangle,\end{aligned}$$

and hence by the Cauchy-Riemann equations, $\frac{\partial}{\partial \overline{w}} f_2(w) \in \mathrm{Ker}(T^t - w)$. Since $f_2(w) \in \mathrm{Ker}(T^t - w)^{\perp}$, by a routine limit argument, $\frac{\partial f_2(w)}{\partial \overline{w}} \in \mathrm{Ker}(T^t - w)^{\perp}$. This implies that $\frac{\partial f_2(w)}{\partial \overline{w}} = 0$, or equivalently

$$f_2 \in \mathcal{O}_H(N_{w_0}). \tag{12}$$

For any non-empty open subset U of N_{w_0}, define an $\mathcal{O}_{\mathbb{C}}(U)$-module morphism $\Phi(U) : \mathcal{G}_{T^t}|_U \to \mathcal{F}_T|_U$ by setting

$$\Phi(U)(f + (T^t - w)\mathcal{O}_H(U)) = f_1(w), \quad f \in \mathcal{O}_H(U).$$

Clearly, $\Phi(U)$ is surjective. If $f_1 = 0$, then by (11) and (12), f belongs to $(T^t - w)\mathcal{O}_H(U)$, and hence $\Phi(U)$ is injective. It is evident that this morphism is compatible with restrictions. Since isomorphic image of a sheaf is a sheaf, this yields (i) and (ii). The remaining part is now clear. □

Remark 4.2. If T is a hyponormal operator on a separable Hilbert space, then by Corollary 2.8, $\mathcal{F}_T$ is trivial, and hence by Proposition 4.1, $\mathcal{G}_{T^t}|_{N_{w_0}}$ is trivial.

Proposition 4.1 helps us in revealing the structure of the support of the sheaf $\mathcal{F}_T$ under the assumption of finite cyclicity of the adjoint T^* of T.

Corollary 4.3. *For $T \in \mathcal{L}(H)$, let $\mathcal{F}_T$ be the eigensheaf of T. Suppose that there exist finitely many vectors $h_1, \dots, h_k \in H$ such that $H = \vee\{T^{*n}h_j : n \geqslant 0, j = 1, \dots, k\}$. Then*

$$\mathrm{Supp}(\mathcal{F}_T) \setminus \sigma_r(T) = \sigma(T) \setminus \sigma_r(T),$$

where $\sigma_r(T)$ denotes the right spectrum of T.

Proof. Since $\sigma_{ap}(T^*) = \sigma_r(T)^*$, applying [5, Lemma 5.3] to T^*, we conclude that λ is a point of stability of T for every $\lambda \notin \sigma_r(T)$. By Proposition 4.1,

$$\mathrm{Supp}(\mathcal{F}_T) \setminus \sigma_r(T) = \mathrm{Supp}(\mathcal{G}_{T^t}) \setminus \sigma_r(T).$$

Since $\mathrm{Supp}(\mathcal{G}_{T^t}) = \sigma(T^t) = \sigma(T)$ (see [26, Lemma 2.1]), the desired conclusion is now immediate. □

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