

## Split metacyclic actions on surfaces

Neeraj K. Dhanwani, Kashyap Rajeevsarathy  
and Apeksha Sanghi

ABSTRACT. Let  $\text{Mod}(S_g)$  be the mapping class group of the closed orientable surface  $S_g$  of genus  $g \geq 2$ . In this paper, we derive necessary and sufficient conditions under which two torsion elements in  $\text{Mod}(S_g)$  will have conjugates that generate a non-abelian finite split metacyclic subgroup of  $\text{Mod}(S_g)$ . As applications of the main result, we give a complete characterization of the finite dihedral and the generalized quaternionic subgroups of  $\text{Mod}(S_g)$  up to a certain equivalence that we will call weak conjugacy. Furthermore, we show that any finite-order mapping class whose corresponding orbifold is a sphere has a conjugate that lifts under certain finite-sheeted regular cyclic covers of  $S_g$ . Moreover, for  $g \geq 5$ , we show the existence of an infinite dihedral subgroup of  $\text{Mod}(S_g)$  that is generated by an involution and a root of a bounding pair map of degree 3. Finally, we provide a complete classification of the weak conjugacy classes of the non-abelian finite split metacyclic subgroups of  $\text{Mod}(S_3)$  and  $\text{Mod}(S_5)$ . We also describe nontrivial geometric realizations of some of these actions.

### CONTENTS

1. Introduction	617
2. Preliminaries	620
3. Main theorem	626
4. Applications	630
5. Hyperbolic structures realizing split metacyclic actions	642
6. Classification of the weak conjugacy classes in $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$	644
Acknowledgements	648
References	648

### 1. Introduction

Let  $S_g$  be the closed orientable surface of genus  $g \geq 0$ ,  $\text{Homeo}^+(S_g)$  be the group of orientation-preserving homeomorphisms of  $S_g$ , and let  $\text{Mod}(S_g)$  be the

---

Received October 4, 2021.

2020 *Mathematics Subject Classification*. Primary 57K20; Secondary 57M60.

*Key words and phrases*. surface; mapping class; finite order maps; metacyclic subgroups.

mapping class group of  $S_g$ . Given  $F, G \in \text{Mod}(S_g)$  of finite order, a pair of conjugates  $F', G'$  (of  $F, G$  resp.) may (or may not) generate a subgroup isomorphic to  $\langle F, G \rangle$ . For example, consider the periodic mapping classes  $F, G \in \text{Mod}(S_7)$  represented by homeomorphisms  $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_7)$  (see [23] for details), as shown in the first subfigure of Figure 1 below. From Figure 1, it is apparent that

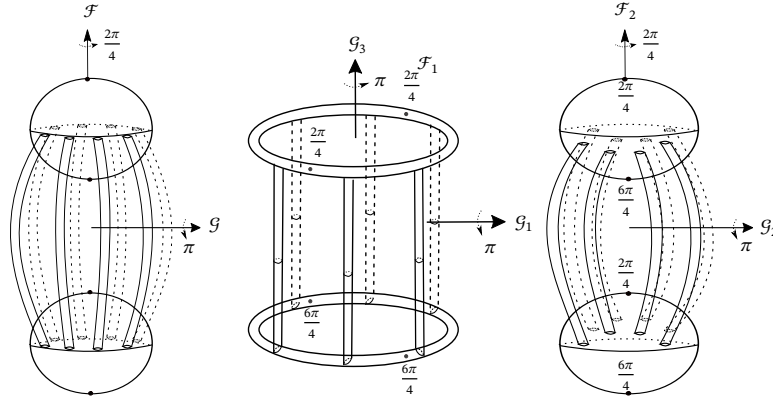


FIGURE 1. Split metacyclic subgroups of  $\text{Mod}(S_7)$  with conjugate generators.

$\langle F, G \rangle \cong D_8$  (i.e. the dihedral group of order 8). For  $1 \leq i \leq 3$ , we consider the conjugates  $G_i$  of  $G$ , represented by the  $\mathcal{G}_i \in \text{Homeo}^+(S_7)$  and for  $1 \leq j \leq 2$ , we consider the conjugates  $\mathcal{F}_j$  of  $\mathcal{F}$  indicated in the (second and third) subfigures. In the second subfigure, we have marked the fixed points of a conjugate  $\mathcal{F}_1$  of  $\mathcal{F}$  (with the same local rotation angles as  $\mathcal{F}$ ). Also, note that the third subfigure is different from the first (as an imbedding  $S_7 \hookrightarrow \mathbb{R}^3$ ), since it has four pairs of tubes connecting the spheres, where in each pair, the tubes are aligned one behind the other. As it turns out,  $\langle F_1, G_1 \rangle \cong \langle F_2, G_2 \rangle \cong D_8$ , but since  $\mathcal{F}_1$  and  $\mathcal{G}_3$  commute, we have  $\langle F_1, G_3 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ . Considering that the finite abelian subgroups of  $\text{Mod}(S_g)$  have been extensively studied [9, 11, 14, 20], this example motivates the following natural question: Given  $F', G' \in \text{Mod}(S_g)$  of orders  $n, m$  respectively, can one derive equivalent conditions under which there exist conjugates  $F, G$  (of  $F', G'$  resp.) such that  $\langle F, G \rangle$  is a finite non-abelian split metacyclic subgroup of order  $m \cdot n$  and twist factor  $k$  admitting the presentation

$$\langle F, G \mid F^n = G^m = 1, G^{-1}FG = F^k \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m?$$

The main result in this paper answers this question in the affirmative (see Theorem 3.3). This result is a generalization of an analogous result from [11] for two-generator finite abelian subgroups.

Given a finite split (non-abelian) metacyclic subgroup  $H = \langle F, G \rangle$  of  $\text{Mod}(S_g)$  as above, the Nielsen realization theorem [17, 22] asserts that we may also view  $H$  as a subgroup of  $\text{Homeo}^+(S_g)$  with an associated  $H$ -action on  $S_g$  inducing the branched cover  $S_g \rightarrow S_g/H$ . Consider representatives  $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$  of  $F, G \in \text{Mod}(S_g)$  (resp.) with the same orders. Given a branched cover  $S_g \rightarrow$

$S_g/\langle \mathcal{F} \rangle (= X)$  and a  $\bar{G} \in \text{Mod}(X)$  that lifts under this cover to a  $G \in \text{Mod}(S_g)$ , it follows from Birman-Hilden theory [2, 3, 4, 5] that there is an exact sequence:

$$1 \rightarrow \langle F \rangle \rightarrow \langle F, G \rangle \rightarrow \langle \bar{G} \rangle \rightarrow 1. \tag{†}$$

A key ingredient in the proof of the main result is the derivation of elementary number-theoretic conditions under which such a  $\bar{G}$  will have a conjugate that lifts so that the sequence (†) splits (see Section 3). The proof integrates ideas from the theory of group actions on surfaces [17, 19] with elements of Thurston’s orbifold theory [26, Chapter 13]. Another crucial aspect of the proof (of the main result) is the analysis of the geometric properties of the automorphism  $\bar{\mathcal{G}}$  induced by  $\mathcal{G}$  on  $S_g/\langle \mathcal{F} \rangle$ .

In Section 4, we provide several applications of our main theorem. The first application concerns the finite dihedral subgroups of  $\text{Mod}(S_g)$ . Let  $D_{2n} = \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_2$  be the dihedral group of order  $2n$ . We derive the following characterization of dihedral subgroups of  $\text{Mod}(S_g)$  in Subsection 4.1 (see Proposition 4.2).

**Proposition 1.** *Let  $F \in \text{Mod}(S_g)$  be of order  $n$ . Then there exists an involution  $G \in \text{Mod}(S_g)$  such that  $\langle F, G \rangle \cong D_{2n}$  if and only if  $F$  and  $F^{-1}$  are conjugate in  $\text{Mod}(S_g)$ .*

It is worth mentioning here that dihedral actions on Riemann surfaces have been classified in [10].

For  $n \geq 2$ , the generalized quaternion group  $Q_{2^{n+1}}$  is a metacyclic group of order  $2^{n+1}$  that admits the presentation

$$\langle F, G \mid F^{2^n} = G^4 = 1, F^{2^{n-1}} = G^2, G^{-1}FG = F^{-1} \rangle.$$

In Subsection 4.2, we obtain the following characterization of generalized quaternionic actions on  $S_g$  (see Proposition 4.7).

**Proposition 2.** *For  $g \geq 2$ , let  $F \in \text{Mod}(S_g)$  be of order  $2^n$ . Then there exists a  $G \in \text{Mod}(S_g)$  such that  $\langle F, G \rangle \cong Q_{2^{n+1}}$  if and only if the  $\langle \mathcal{F}, \mathcal{G} \rangle$ -action on  $S_g$  lifts to a  $(\langle \tilde{\mathcal{F}}, \tilde{\mathcal{G}} \rangle \cong \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4)$ -action on  $S_{2g-1}$  under the 2-sheeted regular cyclic cover  $S_{2g-1} \rightarrow S_g$  with deck transformation group  $\langle \tilde{\mathcal{G}}^2 \tilde{\mathcal{F}}^{2^{n-1}} \rangle$ .*

For a periodic mapping class  $F \in \text{Mod}(S_g)$ , the corresponding orbifold  $\mathcal{O}_{\langle \mathcal{F} \rangle} := S_g/\langle \mathcal{F} \rangle \approx S_{g_0,r}$ , where  $S_{g_0,r}$  is the surface of genus  $g_0 \geq 0$  with  $r \geq 0$  marked points. It is known [13] that  $F$  is irreducible if and only if  $\mathcal{O}_{\langle \mathcal{F} \rangle} \approx S_{0,3}$ . In Subsection 4.3, we provide a characterization of the split metacyclic subgroups  $\langle F, G \rangle$  of  $\text{Mod}(S_g)$  when  $F$  is irreducible (see Corollary 4.11).

Let  $\text{LMod}_p(S_g)$  be the liftable mapping class group of a finite  $n$ -sheeted regular cyclic cover  $p : S_{n(g-1)+1} \rightarrow S_g$  with deck transformation group  $\mathbb{Z}_n = \langle \mathcal{F} \rangle$ , and let  $\text{SMod}_p(S_{n(g-1)+1})$  be the symmetric mapping class group of  $p$  (see [21]), which in this case turns out to be the normalizer of  $\langle F \rangle$  in  $\text{Mod}(S_{n(g-1)+1})$ . In this context, we have the following result.

**Proposition 3.** *For  $g, n \geq 2$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Then any involution  $G' \in \text{Mod}(S_g)$  has a*

conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong D_{2n}$ .

Moreover, we provide sufficient conditions for the liftability of a periodic mapping class (under  $p$ ) whose corresponding orbifold is a sphere (see Propositions 4.15 - 4.16). As a consequence, we obtain the following corollary.

**Corollary 1.** *For  $g \geq 2$  and prime  $n$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular  $n$ -sheeted cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Let  $G' \in \text{Mod}(S_g)$  be of order  $m$  such that the genus of  $\mathcal{O}_{\langle G' \rangle}$  is zero. Then  $G'$  has a conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  if there exists  $k \in \mathbb{Z}_n^\times$  such that  $|k| = m$ .*

Consider an infinite metacyclic group [15] that admits a presentation of the form

$$\langle x, y \mid y^{2m} = 1, y^{-1}xy = x^{-1} \rangle.$$

When  $m = 1$ , we call such a group an *infinite dihedral group*. By a *root of a mapping class*  $F \in \text{Mod}(S_g)$  of degree  $n$ , we mean a  $G \in \text{Mod}(S_g)$  such that  $G^n = F$ . In Subsection 4.4, we use the theory developed in [24, 25] to construct roots of multitwists (i.e. products of powers of commuting Dehn twists) in  $\text{Mod}(S_g)$  which together with certain mapping classes of order  $2m$  generate infinite split metacyclic subgroups of  $\text{Mod}(S_g)$  (of the form described above) for  $g \geq 5$  (see Proposition 4.21). In particular, for  $m = 1$ , we have the following corollary.

**Corollary 2.** *For  $g \geq 5$ , there exists an infinite dihedral subgroup of  $\text{Mod}(S_g)$  that is generated by an involution and a root of a bounding pair map of degree 3.*

In Section 5, we classify the non-abelian finite split metacyclic subgroups of  $\text{Mod}(S_3)$  and  $\text{Mod}(S_5)$  up to a certain weaker notion of conjugacy that we call *weak conjugacy* (see Definition 2.10), which arises naturally in our setting. It may be noted that similar classifications for  $2 \leq g \leq 4$  can also be obtained through the techniques developed in [6, 8, 18]. Finally, we apply the results in [23] to provide an algorithm for determining the hyperbolic structures that realize split metacyclic subgroups as groups of isometries. We conclude the paper by giving nontrivial geometric realizations of some non-abelian finite split metacyclic subgroups of  $\text{Mod}(S_3)$  and  $\text{Mod}(S_5)$ .

## 2. Preliminaries

**2.1. Fuchsian groups.** We let  $\text{Homeo}^+(S_g)$  denote the group of orientation-preserving homeomorphisms of  $S_g$ , and let  $H < \text{Homeo}^+(S_g)$  be a finite group. A faithful and properly discontinuous  $H$ -action on  $S_g$  induces a branched covering

$$S_g \rightarrow \mathcal{O}_H := S_g/H$$

with  $\ell$  cone points  $x_1, \dots, x_\ell$  on the quotient orbifold  $\mathcal{O}_H \approx S_{g_0}$  (which we will call the *corresponding orbifold*) of orders  $n_1, \dots, n_\ell$ , respectively. Then the orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{O}_H)$  of  $\mathcal{O}_H$  has a presentation given by

$$\left\langle \alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \dots, \xi_\ell \mid \xi_1^{n_1}, \dots, \xi_\ell^{n_\ell}, \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] \right\rangle. \quad (1)$$

In classical parlance,  $\pi_1^{\text{orb}}(\mathcal{O}_H)$  is also known as a *Fuchsian group* [16, 19] with signature

$$\Gamma(\mathcal{O}_H) := (g_0; n_1, \dots, n_\ell),$$

and the relation  $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i]$  appearing in its presentation is called the *long relation*. From Thurston’s orbifold theory [26, Chapter 13], we obtain an exact sequence

$$1 \rightarrow \pi_1(S_g) \rightarrow \pi_1^{\text{orb}}(\mathcal{O}_H) \xrightarrow{\phi_H} H \rightarrow 1. \quad (*)$$

In this context, we will require the following result due to Harvey [14].

**Lemma 2.1.** *A finite group  $H$  acts faithfully on  $S_g$  with  $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$  if and only if it satisfies the following two conditions:*

- (i)  $\frac{2g - 2}{|H|} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right)$ , and
- (ii) *there exists a surjective homomorphism  $\phi_H : \pi_1^{\text{orb}}(\mathcal{O}_H) \rightarrow H$  that preserves the orders of all torsion elements of  $\pi_1^{\text{orb}}(\mathcal{O}_H)$ .*

**2.2. Cyclic actions on surfaces.** For  $g \geq 1$ , let  $F \in \text{Mod}(S_g)$  be of order  $n$ . The Nielsen-Kerckhoff theorem [17, 22] asserts that  $F$  is represented by a *standard representative*  $\mathcal{F} \in \text{Homeo}^+(S_g)$  of the same order. We refer to both  $\mathcal{F}$  and the group it generates, interchangeably, as a  $\mathbb{Z}_n$ -*action on  $S_g$* . Each cone point  $x_i \in \mathcal{O}_{\langle \mathcal{F} \rangle}$  lifts to an orbit of size  $n/n_i$  on  $S_g$ , and the local rotation induced by  $\mathcal{F}$  around the points in each orbit is given by  $2\pi c_i^{-1}/n_i$ , where  $\text{gcd}(c_i, n_i) = 1$  and  $c_i c_i^{-1} \equiv 1 \pmod{n_i}$ . Further, it is known (see [14] and the references therein) that the exact sequence in (\*) takes the following form

$$1 \rightarrow \pi_1(S_g) \rightarrow \pi_1^{\text{orb}}(\mathcal{O}_{\langle \mathcal{F} \rangle}) \xrightarrow{\phi_{\langle \mathcal{F} \rangle}} \langle \mathcal{F} \rangle \rightarrow 1,$$

where  $\phi_{\langle \mathcal{F} \rangle}(\xi_i) = \mathcal{F}^{(n/n_i)c_i}$ , for  $1 \leq i \leq \ell$ . We will now introduce a tuple of integers that encodes the conjugacy class of a  $\mathbb{Z}_n$ -action on  $S_g$ .

**Definition 2.2.** A *data set of degree  $n$*  is a tuple

$$D = (n, g_0, r; (c_1, n_1), \dots, (c_\ell, n_\ell)),$$

where  $n \geq 2$ ,  $g_0 \geq 0$ , and  $0 \leq r \leq n - 1$  are integers, and each  $c_i \in \mathbb{Z}_{n_i}^\times$  such that:

- (i)  $r > 0$  if and only if  $\ell = 0$  and  $\text{gcd}(r, n) = 1$ , whenever  $r > 0$ ,
- (ii) each  $n_i \mid n$ ,

(iii)  $\text{lcm}(n_1, \dots, \hat{n}_i, \dots, n_\ell) = N$ , for  $1 \leq i \leq \ell$ , where  $N = n$  if  $g_0 = 0$ , and

(iv)  $\sum_{j=1}^{\ell} \frac{n}{n_j} c_j \equiv 0 \pmod{n}$ .

The number  $g$  determined by the Riemann-Hurwitz equation

$$\frac{2-2g}{n} = 2-2g_0 + \sum_{j=1}^{\ell} \left( \frac{1}{n_j} - 1 \right) \quad (2)$$

is called the *genus* of the data set, denoted by  $g(D)$ .

Note that quantity  $r$  (in Definition 2.2) will be non-zero if and only if  $D$  represents a free rotation of  $S_g$  by  $2\pi r/n$ , in which case,  $D$  will take the form  $(n, g_0, r; )$ . We will not include  $r$  in the notation of a data set, whenever  $r = 0$ .

By the Nielsen-Kerckhoff theorem, the canonical projection  $\text{Homeo}^+(S_g) \rightarrow \text{Mod}(S_g)$  induces a bijective correspondence between the conjugacy classes of finite-order maps in  $\text{Homeo}^+(S_g)$  and the conjugacy classes of finite-order mapping classes in  $\text{Mod}(S_g)$ . This leads us to the following lemma (that follows from [25, Theorem 3.8] and [14]), which allows us to use data sets to describe the conjugacy classes of cyclic actions on  $S_g$ .

**Lemma 2.3.** *For  $g \geq 1$  and  $n \geq 2$ , data sets of degree  $n$  and genus  $g$  correspond to conjugacy classes of  $\mathbb{Z}_n$ -actions on  $S_g$ .*

We will denote the data set corresponding to the conjugacy class of a periodic mapping class  $F$  by  $D_F$ . For compactness of notation, we also write a data set  $D$  (as in Definition 2.2) as

$$D = (n, g_0, r; ((d_1, m_1), \alpha_1), \dots, ((d_{\ell'}, m_{\ell'}), \alpha_{\ell'})),$$

where  $(d_i, m_i)$  are the distinct pairs in the multiset  $S = \{(c_1, n_1), \dots, (c_\ell, n_\ell)\}$ , and the  $\alpha_i$  denote the multiplicity of the pair  $(d_i, m_i)$  in the multiset  $S$ . Further, we note that every cone point  $[x] \in \mathcal{O}_{(\mathcal{F})}$  corresponds to a unique pair in the multiset  $S$  appearing in  $D_F$ , which we denote by  $\mathcal{P}_x := (c_x, n_x)$ .

Given  $u \in \mathbb{Z}_m^\times$  and  $\mathcal{G} \in H \leq \text{Homeo}^+(S_g)$  be of order  $m$ , let  $\mathbb{F}_{\mathcal{G}}(u, m)$  denote the set of fixed points of  $\mathcal{G}$  with induced rotation angle  $2\pi u/m$ . Let  $C_H(\mathcal{G})$  be the centralizer of  $\mathcal{G} \in H$  and  $\sim$  denote the conjugation relation between any two elements in  $H$ . We conclude this subsection by stating the following result from the theory of Riemann surfaces [7], which we will use in the proof of our main theorem.

**Lemma 2.4.** *Let  $H < \text{Homeo}^+(S_g)$  of finite order with  $\Gamma(\mathcal{O}_H) = (g_0; n_1, \dots, n_\ell)$ , and let  $\mathcal{G} \in H$  be of order  $m$ . Then for  $u \in \mathbb{Z}_m^\times$ , we have*

$$|\mathbb{F}_{\mathcal{G}}(u, m)| = |C_H(\mathcal{G})| \cdot \sum_{\substack{1 \leq i \leq \ell \\ m | n_i \\ \mathcal{G} \sim \phi_H(\xi_i)^{n_i u/m}}} \frac{1}{n_i}.$$

**2.3. Hyperbolic structures realizing cyclic actions.** For a finite subgroup  $H < \text{Mod}(S_g)$ , let  $\text{Fix}(H)$  denote the subspace of fixed points in the Teichmüller space  $\text{Teich}(S_g)$  under the action of  $H$ . When  $H$  is cyclic, a method for constructing the hyperbolic metrics representing the points in  $\text{Fix}(H)$  was described in [1] and [23], thereby yielding explicit solutions to the Nielsen realization problem [17, 22]. This method involved the construction of an arbitrary periodic element in  $\text{Mod}(S_g)$  (that is not realizable as a rotation of  $S_g$ ) by the “compatibilities” of irreducible periodic components, which are uniquely realized as rotations of certain special hyperbolic polygons with side-pairings.

A mapping class that is not reducible is called *irreducible*. Let  $F \in \text{Mod}(S_g)$  be of order  $n$ . Gilman [13] showed that  $F$  is irreducible if and only if  $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle})$  has the form  $(0; n_1, n_2, n_3)$  (i.e. the quotient orbifold  $\mathcal{O}_{\langle \mathcal{F} \rangle}$  is a sphere with three cone points.) Following the nomenclature in [1, 23],  $F$  is *rotational* if  $\mathcal{F}$  is either of order 2, or  $\mathcal{F}$  has at most 2 fixed points. A non-rotational  $F$  is said to be of *Type 1* if  $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle}) = (g_0; n_1, n_2, n)$ , otherwise, it is called a *Type 2* action. The following result describes the unique hyperbolic structure that realizes an irreducible Type 1 action.

**Theorem 2.5.** *For  $g \geq 2$ , consider a irreducible Type 1 action  $F \in \text{Mod}(S_g)$  with*

$$D_F = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n)).$$

*Then  $F$  can be realized explicitly as the rotation  $\theta_F = \frac{2\pi c_3^{-1}}{n}$  of a hyperbolic polygon  $\mathcal{P}_F$  with a suitable side-pairing  $W(\mathcal{P}_F)$ , where  $\mathcal{P}_F$  is a hyperbolic  $k(F)$ -gon with*

$$k(F) := \begin{cases} 2n, & \text{if } n_1, n_2 \neq 2, \text{ and} \\ n, & \text{otherwise,} \end{cases}$$

*and for  $0 \leq m \leq n - 1$ ,*

$$W(\mathcal{P}_F) = \begin{cases} \prod_{i=1}^n a_{2i-1} a_{2i} \text{ with } a_{2m+1}^{-1} \sim a_{2z}, & \text{if } k(F) = 2n, \text{ and} \\ \prod_{i=1}^n a_i \text{ with } a_{m+1}^{-1} \sim a_z, & \text{otherwise,} \end{cases}$$

*where  $z \equiv m + qj \pmod{n}$  with  $q = (n/n_2)c_3^{-1}$  and  $j = n_2 - c_2$ .*

Further, it was shown [23] that the process of realizing an arbitrary non-rotational action  $F$  of order  $n$  using these unique hyperbolic structures realizing irreducible Type 1 components involved two broad types of processes.

- (a) *k-compatibility.* In this process, for  $i = 1, 2$ , we take a pair of irreducible Type 1 mapping classes  $F_i \in \text{Mod}(S_{g_i})$  such that the  $\langle \mathcal{F}_i \rangle$ -action on  $S_{g_i}$  induces a pair of *compatible orbits* of size  $k$  (where the induced local rotation angles add upto 0 modulo  $2\pi$ ). We remove (cyclically permuted)  $\langle \mathcal{F}_i \rangle$ -invariant disks around points in the compatible orbits and then identify the resulting boundary components realizing a periodic mapping class



$F \in \text{Mod}(S_{g_1+g_2+k-1})$ . An analogous construction can also be performed using a pair of orbits induced by a single  $\langle \mathcal{F}' \rangle$ -action on  $S_g$  to realize a periodic mapping class  $F \in \text{Mod}(S_{g+k})$ .

- (b) *Permutation additions and deletions.* The *addition of a permutation component* involves the removal of (cyclically permuted) invariant disks around points in an orbit of size  $n$  induced by an  $\langle \mathcal{F} \rangle$ -action on  $S_g$  and then pasting  $n$  copies of  $S_{g'}^1$  (i.e.  $S_{g'}$  with one boundary component) to the resultant boundary components. This realizes an action on  $S_{g+ng'}$  with the same fixed point and orbit data as  $F$ . The reversal of this process is called a *permutation deletion*.

Thus, in summary, we have the following:

**Theorem 2.6.** [23, Theorem 2.24] *For  $g \geq 2$ , a non-rotational periodic mapping class in  $\text{Mod}(S_g)$  can be realized through finitely many  $k$ -compatibilities, permutation additions, and permutation deletions on the unique structures of type  $\mathcal{P}_F$  realizing irreducible Type 1 mapping classes.*

A final but yet vital ingredient in the realization of split metacyclic actions is the following elementary lemma, which is a direct generalization of [11, Lemma 6.1].

**Lemma 2.7.** *Let  $H = \langle F, G \rangle$  be a finite metacyclic subgroup of  $\text{Mod}(S_g)$ . Then*

$$\text{Fix}(H) = \text{Fix}(\langle F \rangle) \cap \text{Fix}(\langle G \rangle).$$

**2.4. Split metacyclic actions on surfaces.** Given integers  $m, n \geq 2$ , and  $k \in \mathbb{Z}_n^\times$  such that  $k^m \equiv 1 \pmod{n}$ , a *finite split metacyclic action of order  $mn$*  (written as  $m \cdot n$ ) on  $S_g$  is a tuple  $(H, (\mathcal{G}, \mathcal{F}))$ , where  $H < \text{Homeo}^+(S_g)$ , and

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^m = 1, \mathcal{G}^{-1} \mathcal{F} \mathcal{G} = \mathcal{F}^k \rangle.$$

The multiplicative class  $k$  will be called the *twist factor* of the split metacyclic action  $(H, (\mathcal{G}, \mathcal{F}))$ . As we are only interested in non-abelian split metacyclic subgroups, we will assume from here on that  $k \neq 1$ . Note that in classical notation  $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ . As  $\langle \mathcal{F} \rangle \triangleleft H$ , it is known [7, 27] that  $\mathcal{G}$  would induce a  $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_{\langle \mathcal{F} \rangle})$  that preserves the set of cone points in  $\mathcal{O}_{\langle \mathcal{F} \rangle}$  along with their orders. We will call  $\bar{\mathcal{G}}$ , the *induced automorphism on  $\mathcal{O}_{\langle \mathcal{F} \rangle}$*  by  $\mathcal{G}$ , and we formalize this notion in the following definition.

**Definition 2.8.** Let  $H < \text{Homeo}^+(S_g)$  be a finite cyclic group with  $|H| = n$ . We say a  $\bar{\mathcal{G}} \in \text{Homeo}^+(\mathcal{O}_H)$  is an *automorphism of  $\mathcal{O}_H$*  if for  $[x], [y] \in \mathcal{O}_H$ ,  $k \in \mathbb{Z}_n^\times$  and  $\bar{\mathcal{G}}([x]) = [y]$ , we have:

- (i)  $n_x = n_y$ , and
- (ii)  $c_x = kc_y$ .

We denote the group of automorphisms of  $\mathcal{O}_H$  by  $\text{Aut}_k(\mathcal{O}_H)$ .

We note that the concept of an induced orbifold automorphism in Definition 2.8 is more general than the one that was used in the abelian case ([11]), which



required a more rigid condition that  $c_x = c_y$ . The following lemma, which provides some basic properties of the induced map  $\bar{\mathcal{G}}$ , is a split metacyclic analog of [11, Lemma 3.1].

**Lemma 2.9.** *Let  $\mathcal{G}, \mathcal{F} \in \text{Homeo}^+(S_g)$  be maps of orders  $m, n$ , respectively, such that  $\mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k$ , and let  $H = \langle \mathcal{F} \rangle$ . Then:*

(i)  $\mathcal{G}$  induces a  $\bar{\mathcal{G}} \in \text{Aut}_k(\mathcal{O}_H)$  such that

$$\mathcal{O}_H / \langle \bar{\mathcal{G}} \rangle = S_g / \langle \mathcal{F}, \mathcal{G} \rangle,$$

(ii)  $|\bar{\mathcal{G}}|$  divides  $|\mathcal{G}|$ , and

(iii)  $|\bar{\mathcal{G}}| < m$  if and only if  $\mathcal{F}^l = \mathcal{G}^u$ , for some  $0 < l < n$  and  $0 < u < m$ .

We will now formalize the notion of weak conjugacy from Section 1.

**Definition 2.10.** We say that two finite split metacyclic actions  $(H_1, (\mathcal{G}_1, \mathcal{F}_1))$  and  $(H_2, (\mathcal{G}_2, \mathcal{F}_2))$  of order  $m \cdot n$  and twist factor  $k$  are *weakly conjugate* if there exists an isomorphism

$$\psi : \pi_1^{\text{orb}}(\mathcal{O}_{H_1}) \cong \pi_1^{\text{orb}}(\mathcal{O}_{H_2})$$

and an isomorphism  $\chi : H_1 \rightarrow H_2$  such that the following conditions hold.

(i)  $\chi((\mathcal{G}_1, \mathcal{F}_1)) = (\mathcal{G}_2, \mathcal{F}_2)$ .

(ii) For  $i = 1, 2$ , let  $\phi_{H_i} : \pi_1^{\text{orb}}(\mathcal{O}_{H_i}) \rightarrow H_i$  be the surface kernel (in the exact sequence (\*) in Section 2). Then  $(\chi \circ \phi_{H_1})(g) = (\phi_{H_2} \circ \psi)(g)$ , whenever  $g \in \pi_1^{\text{orb}}(\mathcal{O}_{H_1})$  is of finite order.

(iii) The pair  $(\mathcal{G}_1, \mathcal{F}_1)$  is conjugate (component-wise) to the pair  $(\mathcal{G}_2, \mathcal{F}_2)$  in  $\text{Homeo}^+(S_g)$ .

The notion of weak conjugacy defines an equivalence relation on split metacyclic actions on  $S_g$  and the equivalence classes thus obtained will be called *weak conjugacy classes*.

**Remark 2.11.** By virtue of the Nielsen-Kerckhoff theorem, the notion of weak conjugacy in Definition 2.10 naturally extends to an analogous notion in  $\text{Mod}(S_g)$  via the natural association

$$(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F})) \leftrightarrow (\langle F, G \rangle, (G, F)).$$

For simplicity, we will now introduce the following notation.

**Definition 2.12.** Let  $F, G \in \text{Mod}(S_g)$  be a finite order map of orders  $n, m$ , respectively. Then for some  $k \in \mathbb{Z}_n^\times \setminus \{1\}$ , we say (in symbols) that  $\llbracket F, G \rrbracket_k = 1$  if there exists conjugates  $F', G'$  (of  $F, G$  resp.) such that  $\langle F', G' \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  satisfying the relation  $(G')^{-1}F'G' = (F')^k$ .

We conclude this subsection with the following crucial remark.

**Remark 2.13.** Let  $H < \text{Mod}(S_g)$  be a finite split metacyclic subgroup, and let  $I(H)$  denote the isomorphism class of  $H$  (in  $\text{Mod}(S_g)$ ). By Remark 2.11, we have

$$I(H) = \{H' : H' \cong H \text{ and } (H', (G', F')) \text{ represents a weak conjugacy class}\}$$

for some  $F', G' \in H'$  such that  $H' = \langle F', G' \rangle$ .

Consequently, periodic mapping classes  $F, G \in \text{Mod}(S_g)$  satisfy  $\llbracket F, G \rrbracket_k = 1$  if and only if there exists conjugates  $F', G'$  (of  $F, G$  resp.) such that the triple  $(\langle F', G' \rangle, (G', F'))$  represents a weak conjugacy class associated with a finite split metacyclic subgroup (of twist factor  $k$ ) of  $\text{Mod}(S_g)$ .

### 3. Main theorem

In this section, we establish the main result of the paper by deriving equivalent conditions under which torsion elements  $F, G \in \text{Mod}(S_g)$  would satisfy  $\llbracket F, G \rrbracket_k = 1$ . We will introduce an abstract tuple of integers that will capture each weak conjugacy class associated with a finite split metacyclic subgroup of  $\text{Mod}(S_g)$ .

**Definition 3.1.** A split metacyclic data set of degree  $m \cdot n$ , twist factor  $k$ , and genus  $g \geq 2$  is a tuple

$$((m \cdot n, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

where  $m, n \geq 2$ , the  $n_{ij}$  are positive integers for  $1 \leq i \leq \ell$ ,  $1 \leq j \leq 2$ , and  $k \in \mathbb{Z}_n^\times$  such that  $k^m \equiv 1 \pmod{n}$ , satisfying the following conditions.

- (i)  $\frac{2g-2}{mn} = 2g_0 - 2 + \sum_{i=1}^{\ell} \left(1 - \frac{1}{n_i}\right)$ .
- (ii) (a) For each  $i, j$ ,  $n_{i1} \mid m$ ,  $n_{i2} \mid n$ , either  $\gcd(c_{ij}, n_{ij}) = 1$  or  $c_{ij} = 0$ , and  $c_{ij} = 0$  if and only if  $n_{ij} = 1$ .
- (b) For each  $i$ ,  $n_i = n_{i1} \cdot \beta_i$ , where  $\beta_i$  is least positive integer such that

$$c_{i2} \frac{n}{n_{i2}} \left( \sum_{i'=0}^{n_{i1}\beta_i-1} k^{c_{i1} \frac{m}{n_{i1}} i'} \right) \equiv 0 \pmod{n}.$$

(iii)  $\sum_{i=1}^{\ell} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m}$ .

(iv) Defining  $A := \sum_{i=1}^{\ell} c_{i2} \frac{n}{n_{i2}} \prod_{s=i+1}^{\ell} k^{c_{s1} \frac{m}{n_{s1}}}$  and  $d := \gcd(n, k-1)$ , we have

$$A \equiv \begin{cases} 0 \pmod{n}, & \text{if } g_0 = 0, \text{ and} \\ d\theta \pmod{n}, \text{ for } \theta \in \mathbb{Z}_n, & \text{if } g_0 \geq 1. \end{cases}$$

- (v) If  $g_0 = 0$ , there exists  $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$  and  $v \in \mathbb{N}$  such that the following conditions hold.

$$(a) \sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 1 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left( \sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left( \prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 0 \pmod{n}.$$

$$(b) \sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left( \sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left( \prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 1 \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell & \text{otherwise,} \end{cases}$$

$$t \equiv \begin{cases} t' \pmod{\ell}, & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell, & \text{otherwise.} \end{cases}$$

(vi) If  $g_0 = 1$ , there exists  $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in (\mathbb{N} \cup \{0\})^{\ell v}$  and  $m', n' \in \mathbb{Z}$ ,  $v \in \mathbb{N}$  such that  $m' \mid m$  and  $n' \mid n$ , satisfying the following conditions.

$$(a) \sum_{i'=1}^{\ell v} p_{i'} c_{i1} \frac{m}{n_{i1}} \equiv m' \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left( \sum_{s=1}^{p_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (p_{i'} - s)} \right) \left( \prod_{t'=i'+1}^{\ell v} k^{p_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv 0 \pmod{n}.$$

$$(b) \sum_{i'=1}^{\ell v} q_{i'} c_{i1} \frac{m}{n_{i1}} \equiv 0 \pmod{m} \text{ and}$$

$$\sum_{i'=1}^{\ell v} c_{i2} \frac{n}{n_{i2}} \left( \sum_{s=1}^{q_{i'}} k^{c_{i1} \frac{m}{n_{i1}} (q_{i'} - s)} \right) \left( \prod_{t'=i'+1}^{\ell v} k^{q_{t'} c_{t1} \frac{m}{n_{t1}}} \right) \equiv n' \pmod{n}, \text{ where}$$

$$i \equiv \begin{cases} i' \pmod{\ell}, & \text{if } i' \not\equiv 0 \pmod{\ell}, \\ \ell & \text{otherwise,} \end{cases}$$

$$t \equiv \begin{cases} t' \pmod{\ell}, & \text{if } t' \not\equiv 0 \pmod{\ell}, \text{ and} \\ \ell, & \text{otherwise.} \end{cases}$$

(c)  $A \equiv -\beta k^\alpha + \beta \pmod{n}$  for some non-negative integers  $\alpha, \beta$ , where

$$\text{lcm} \left( \frac{m}{m'}, \frac{m}{\text{gcd}(m, \alpha)} \right) = m \text{ and } \text{lcm} \left( \frac{n}{n'}, \frac{n}{\text{gcd}(n, \beta)} \right) = n.$$

Furthermore, we set  $\alpha = 1$ , when  $m' = 0$ , and  $\beta = 1$ , when  $n' = 0$ .

We will now show that the split metacyclic data sets of genus  $g$  are in one-to-one correspondence with the weak conjugacy classes of split metacyclic subgroups of  $\text{Mod}(S_g)$ .

**Proposition 3.2.** *For integers  $n, m, g \geq 2$ , the split metacyclic data sets of degree  $m \cdot n$  with twist factor  $k$  and genus  $g$  correspond to the weak conjugacy classes of  $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -actions on  $S_g$ .*

**Proof.** Let  $\mathcal{D}$  be a split metacyclic data set of degree  $m \cdot n$  with twist factor  $k$  and genus  $g$  (as in Definition 3.1 above). We need to show that  $\mathcal{D}$  corresponds to the weak conjugacy class of a  $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -action on  $S_g$  represented by  $(H, (\mathcal{G}, \mathcal{F}))$ . To this effect, we first establish the existence of an epimorphism  $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$  which preserves the order of torsion elements. Let the presentations of  $H$  and  $\pi_1^{orb}(\mathcal{O}_H)$  be given by

$$H = \langle \mathcal{F}, \mathcal{G} \mid \mathcal{F}^n = \mathcal{G}^m = 1, \mathcal{G}^{-1}\mathcal{F}\mathcal{G} = \mathcal{F}^k \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m \text{ and}$$

$$\langle \alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \xi_1, \dots, \xi_\ell \mid \xi_1^{n_1} = \dots = \xi_\ell^{n_\ell} = \prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1 \rangle,$$

respectively. We consider the map

$$\xi_i \xrightarrow{\phi_H} \mathcal{G}^{c_{i1} \frac{m}{n_{i1}}} \mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell.$$

As  $|\mathcal{G}^{c_{i1} \frac{m}{n_{i1}}}| = n_{i1}$  and  $|\mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}| = n_{i2}$ , condition (ii) of Definition 3.1 would imply that  $\phi_H$  is a map which preserves the order of torsion elements. For clarity, we break the argument for the surjectivity of  $\phi_H$  into three cases.

First, we consider the case when  $g_0 = 0$ . Conditions (iii) and (iv) show that  $\phi_H$  satisfies the long relation  $\prod_{i=1}^{\ell} \xi_i = 1$  and the surjectivity of  $\phi_H$  follows from condition (v).

When  $g_0 \geq 2$ ,  $\pi_1^{orb}(\mathcal{O}_H)$  has additional hyperbolic generators (viewing them as isometries of the hyperbolic plane), namely the  $\alpha_i$  and the  $\beta_i$ . Extending  $\phi_H$  by mapping  $\alpha_1 \xrightarrow{\phi_H} \mathcal{G}, \beta_1 \xrightarrow{\phi_H} \mathcal{F}$  yields an epimorphism. Moreover, by carefully choosing the images of the  $\alpha_i$  and the  $\beta_i$  under  $\phi_H$ , for  $i \geq 2$ , conditions (iii) and (iv) would together ensure that the long relation  $\prod_{j=1}^{\ell} \xi_j \prod_{i=1}^{g_0} [\alpha_i, \beta_i] = 1$  is satisfied.

When  $g_0 = 1$ ,  $\pi_1^{orb}(\mathcal{O}_H)$  has two additional hyperbolic generators, namely the  $\alpha_1$  and the  $\beta_1$ . We extend  $\phi_H$  by defining  $\alpha_1 \xrightarrow{\phi_H} \mathcal{G}^\alpha$  and  $\beta_1 \xrightarrow{\phi_H} \mathcal{F}^\beta$ , and apply conditions (iii), (iv), and (vi) to obtain the desired epimorphism.

It remains to show that  $\mathcal{D}$  determines  $\mathcal{F}, \mathcal{G} \in \text{Homeo}^+(S_g)$  up to conjugacy (i.e. condition (iii) of Definition 2.10). Let  $D_{\bar{\mathcal{G}}} = (m, g_0; (c_{11}, n_{11}), \dots, (c_{\ell 1}, n_{\ell 1}))$  represent the conjugacy class of the action  $\bar{\mathcal{G}}$  induced on the orbifold  $\mathcal{O}_{\langle \mathcal{F} \rangle}$  by

the action  $\mathcal{G} \in \text{Homeo}^+(S_g)$ . We note that by Lemma 2.9,  $\Gamma(\mathcal{O}_{\langle \mathcal{F} \rangle})$  has the form

$$\left( g_1; \underbrace{\frac{n_1}{n_{11}}, \dots, \frac{n_1}{n_{11}}}_{\frac{m}{n_{11}} \text{ times}}, \dots, \underbrace{\frac{n_\ell}{n_{\ell 1}}, \dots, \frac{n_\ell}{n_{\ell 1}}}_{\frac{m}{n_{\ell 1}} \text{ times}} \right),$$

where if  $n_i/n_{i1} = 1$ , for some  $1 \leq i \leq \ell$ , then we exclude it from the signature, and  $g_1 = g(D_{\bar{G}})$  is determined by Equation (2) of Definition 2.2. So, we get

$$D_F = (n, g_1; (d_{11}, \frac{n_1}{n_{11}}), \dots, (d_{1 \frac{m}{n_{11}}}, \frac{n_1}{n_{11}}), \dots, (d_{\ell 1}, \frac{n_\ell}{n_{\ell 1}}), \dots, (d_{\ell \frac{m}{n_{\ell 1}}}, \frac{n_\ell}{n_{\ell 1}})),$$

where

$$d_{i1} n_{i1} \equiv c_{i2} \frac{n_i}{n_{i2}} \sum_{j'=1}^{n_{i1}} k^{c_{i1} \frac{m}{n_{i1}} (j'-1)} \pmod{n_i} \text{ and}$$

$$d_{ij_i} \equiv d_{i1} k^{(j_i-1)} \pmod{\frac{n_i}{n_{i1}}} \quad 1 \leq i \leq l, \quad 1 \leq j_i \leq \frac{m}{n_{i1}}.$$

Moreover, by applying Lemma 2.4, we see that

$$D_G = (m, g_2; ((u_{ij}^{-1}, m_i), \frac{m_i |\mathbb{f}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)|}{m})) : u_{ij} \in \mathbb{Z}_{m_i}^\times \text{ and } m_i \mid m),$$

where

$$|\mathbb{f}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)| = |\mathbb{F}_{\frac{m}{g^{m_i}}}(u_{ij}, m_i)| - \sum_{\substack{m_{i'} \in \mathbb{N} \\ m_{i'} \neq m_i \\ m_i \mid m_{i'} \mid m}} \sum_{\substack{(u_{i'j'}, m_{i'})=1 \\ u_{ij} \equiv u_{i'j'} \pmod{m_i}}} |\mathbb{f}_{\frac{m}{g^{m_{i'}}}}(u_{i'j'}, m_{i'})|$$

and  $g_2$  is determined by Equation (2) of Definition 2.2.

Conversely, consider the weak conjugacy class of  $\mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ -actions on  $S_g$  represented by  $(H, (\mathcal{G}, \mathcal{F}))$ , where  $H = \langle \mathcal{F}, \mathcal{G} \rangle$ . So, Lemma 2.1 would imply that there exists a surjective homomorphism  $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$  defined by

$$\xi_i \xrightarrow{\phi_H} \mathcal{G}^{c_{i1} \frac{m}{n_{i1}}} \mathcal{F}^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

which is order-preserving on the torsion elements. This yields a split metacyclic data set of degree  $m \cdot n$  with twist factor  $k$  and genus  $g$  as in Definition 3.1. By Lemma 2.1, this tuple satisfies condition (i) of Definition 3.1, while condition (ii) follows from the fact that  $\phi_H$  is order-preserving on torsion elements. Conditions (iii)-(iv) follow from the long relation satisfied by  $\pi_1^{orb}(\mathcal{O}_H)$ , and condition (v)-(vi) are implied by the surjectivity of  $\phi_H$ . Thus, we obtain the split metacyclic data set of degree  $m \cdot n$  with twist factor  $k$  and genus  $g$ , and the result follows.  $\square$

We denote the data sets  $D_F$  and  $D_G$  (representing the cyclic factors of  $H$ ) derived from the split metacyclic data set  $\mathcal{D}$  appearing in the proof of Proposition 3.2 by  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , respectively. Thus, our main theorem will now follow from Remark 2.13 and Proposition 3.2.

**Theorem 3.3** (Main theorem). *Let  $F, G \in \text{Mod}(S_g)$  be of orders  $n, m$ , respectively. Then  $\llbracket F, G \rrbracket_k = 1$  if and only if there exists a split metacyclic data set  $\mathcal{D}$  of degree  $m \cdot n$ , twist factor  $k$ , and genus  $g$  such that  $\mathcal{D}_1 = D_F$  and  $\mathcal{D}_2 = D_G$ .*

We conclude this section with an example of a split metacyclic action of order 16 on  $S_5$ .

**Example 3.4.** The split metacyclic data set  $\mathcal{D} = ((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$  encodes the weak conjugacy class of a  $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$ -action on  $S_5$  represented by  $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$ , where

$$D_F = (4, 1; (1, 2), (1, 2), (1, 2), (1, 2)) \text{ and } D_G = (4, 2, 1; ).$$

The geometric realization of this action is illustrated in Figure 2 below.

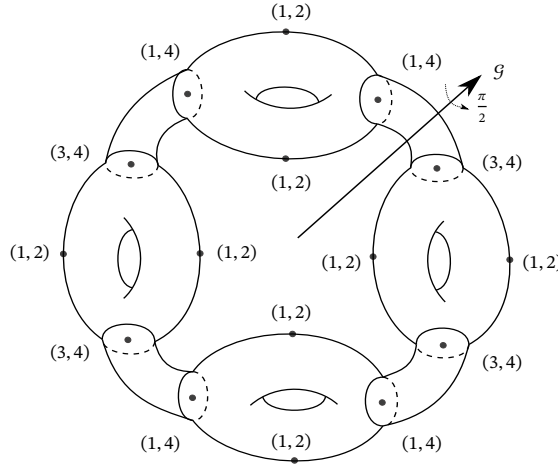


FIGURE 2. Realization of a  $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_4$ -action on  $S_5$ .

Note that the pairs of integers appearing in Figure 2 represent the compatible orbits involved in the realization of  $\mathcal{F}$ . Here, the action  $\mathcal{F}$  is realized via two 1-compatibilities between the action  $\mathcal{F}'$  on two copies of  $S_2$  with

$$D_{F'} = (4, 0; ((1, 2), 2), (1, 4), (3, 4)).$$

Furthermore, the action  $\mathcal{F}'$  is realized by a 1-compatibility between the actions  $\mathcal{F}''$  and  $(\mathcal{F}'')^3$  on two copies of  $S_1$  with

$$D_{F''} = (4, 0; (1, 2), (1, 4), (1, 4)).$$

### 4. Applications

**4.1. Dihedral groups.** Let  $D_{2n} = \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_2$  be the dihedral group of order  $2n$ . We will call a split metacyclic data set of degree  $2 \cdot n$  and twist factor  $-1$  a *dihedral data set*. A simple computation reveals that a dihedral data set

$$((2 \cdot n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

would have the property that  $(c_{j_1}, n_{j_1}) \in \{(0, 1), (1, 2)\}$ , for  $1 \leq j \leq \ell$ . The following is an immediate consequence of Proposition 3.2.

**Corollary 4.1.** *For  $g \geq 2$  and  $n \geq 3$ , dihedral data sets of degree  $2 \cdot n$  and genus  $g$  correspond to the weak conjugacy classes of  $D_{2n}$ -actions on  $S_g$ .*

The following proposition provides an alternative characterization of a  $D_{2n}$ -action in terms of the generator of its factor subgroup of order  $n$ .

**Proposition 4.2.** *Let  $F \in \text{Mod}(S_g)$  be of order  $n$ . Then there exists an involution  $G \in \text{Mod}(S_g)$  such that  $\langle F, G \rangle \cong D_{2n}$  if and only if  $D_F$  has the form*

$$(n, g_0, r; ((c_1, n_1), (-c_1, n_1), \dots, (c_s, n_s), (-c_s, n_s))). \tag{**}$$

**Proof.** Suppose that  $D_F$  has the form (\*\*). Then  $\mathcal{O}_{(\mathcal{F})}$  is an orbifold of genus  $g_0$  with  $2s$  cone points  $[x_1], [y_1], \dots, [x_s], [y_s]$ , where  $\mathcal{P}_{x_i} = (c_i, n_i)$  and  $\mathcal{P}_{y_i} = (-c_i, n_i)$ , for  $1 \leq i \leq s$ . Up to conjugacy, let  $\bar{\mathcal{G}} \in \text{Aut}_k(\mathcal{O}_{(\mathcal{F})})$  be the hyperelliptic involution so that  $\bar{\mathcal{G}}([x_i]) = [y_i]$ , for  $1 \leq i \leq s$ . To prove our assertion, it would suffice to show the existence of an involution  $\mathcal{G} \in \text{Homeo}^+(S_g)$  that induces  $\bar{\mathcal{G}}$ . This amounts to showing that there exists a split metacyclic data set  $\mathcal{D}$  of degree  $2 \cdot n$  with twist factor  $-1$  encoding the weak conjugacy class  $(H, (\mathcal{G}, \mathcal{F}))$  so that  $D_G$  has degree 2. Consider the tuple

$$\mathcal{D} = ((2 \cdot n, -1), 0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t-2 \text{ times}}, [(1, 2), (c_{(t-1)2}, n_{(t-1)2}), 2], [(1, 2), (c_{t_2}, n_{t_2}), 2], [(0, 1), (c_1, n_1), n_1], \dots, [(0, 1), (c_s, n_s), n_s]),$$

where  $t = 2g_0 + 2$ ,

$$(c_{(t-1)2}n/n_{(t-1)2}, c_{t_2}n/n_{t_2}) = \begin{cases} (0, -\sum_{i=1}^s c_i \frac{n}{n_i} \pmod{n}), & \text{if } g_0 = 0, \text{ and} \\ (1, 1 - \sum_{i=1}^s c_i \frac{n}{n_i} \pmod{n}), & \text{if } g_0 > 0. \end{cases}$$

It follows immediately that  $\mathcal{D}$  satisfies conditions (i)-(iv) of Definition 3.1. As  $t \geq 2$ , by taking  $v = 1$ , we may choose  $(p_1, \dots, p_{t+s}) = (1, 0, \dots, 0)$  to conclude that  $\mathcal{D}$  also satisfies condition (v)(a). Since  $t = 2 \iff g_0 = 0$ , and when  $g_0 = 0$ , we have that  $\text{lcm}(n_1, \dots, n_s) = n$ , from which condition (v)(b) follows. Finally, for the case when  $g_0 \neq 0$ , (v)(b) follows by choosing  $(q_1, \dots, q_{t-2}, q_{t-1}, \dots, q_{t+s}) = (0, \dots, 1, 1, \dots, 0)$ . Thus, it follows that  $\mathcal{D}$  is a split metacyclic data set. Further, a direct application of Theorem 3.3 would show that  $\mathcal{D}$  indeed encodes the weak conjugacy represented by  $(H, (\mathcal{G}, \mathcal{F}))$ , as desired.

The converse follows immediately from Remark 2.11 and Proposition 3.2. □

We now provide a couple of examples of dihedral actions along with their realizations.

**Example 4.3.** Consider the  $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_2$ -action  $\langle \mathcal{F}, \mathcal{G} \rangle$  on  $S_3$  illustrated in Figure 3 below, where

$$D_F = (3, 1; (1, 3), (2, 3)) \text{ and } D_G = (2, 1; (1, 2), (1, 2), (1, 2), (1, 2)).$$



The weak conjugacy class of the action  $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$  is encoded by

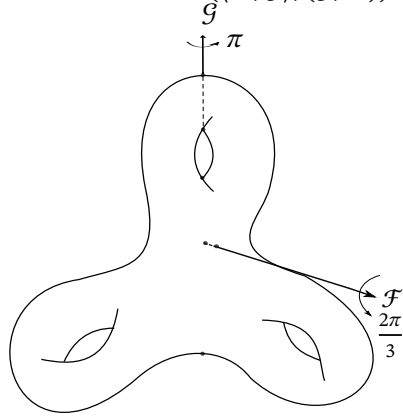


FIGURE 3. Realization of a  $D_6$ -action on  $S_3$ .

$$\mathcal{D} = ((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(1, 2), (1, 3), 2], [(0, 1), (2, 3), 3]).$$

**Example 4.4.** Consider the  $\mathbb{Z}_4 \rtimes_{-1} \mathbb{Z}_2$ -actions  $\langle \mathcal{F}, \mathcal{G} \rangle$  and  $\langle \mathcal{F}', \mathcal{G}' \rangle$  on  $S_3$  illustrated in Figure 4 below, where  $D_{\mathcal{F}} = (4, 0; (1, 4), (3, 4), (1, 4), (3, 4))$ ,  $D_{\mathcal{G}} = (2, 1; (1, 2), (1, 2), (1, 2), (1, 2))$ , and  $D_{\mathcal{G}'} = (2, 2, 1; )$ .

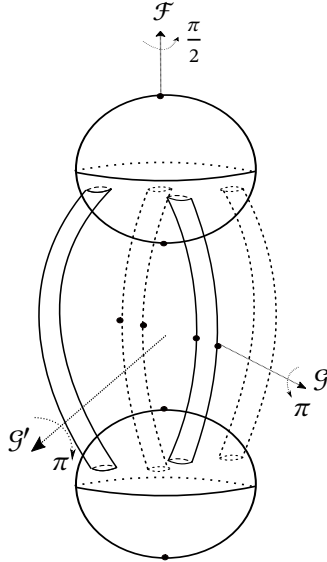


FIGURE 4. Realization of a  $D_8$ -action on  $S_3$ .

The weak conjugacy classes  $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$  and  $(\langle \mathcal{F}', \mathcal{G}' \rangle, (\mathcal{G}', \mathcal{F}'))$  are encoded by

$$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2], [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$$

and

$((2 \cdot 4, -1), 0; [(1, 2), (1, 4), 2], [(1, 2), (1, 4), 2], [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$ ,  
 respectively.

**4.2. Generalized quaternions.** For  $n \geq 2$ , the generalized quaternion group  $Q_{2^{n+1}}$  is a metacyclic group of order  $2^{n+1}$  that admits the presentation

$$\langle x, y \mid x^{2^n} = y^4 = 1, x^{2^{n-1}} = y^2, y^{-1}xy = x^{-1} \rangle.$$

**Remark 4.5.** Let  $\mathcal{D}$  be a split metacyclic data set of genus  $g$ , degree  $4 \cdot 2^n$  and twist factor  $-1$  (as in Definition 3.1) encoding a weak conjugacy class represented by  $(H, (\mathcal{G}, \mathcal{F}))$ . Suppose that  $\mathcal{D}$  has the property that

$$[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] = [(1, 2), (1, 2), 2]$$

for some  $1 \leq j \leq \ell$ . Then it follows from the proof of Proposition 3.2 that under the epimorphism  $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$  which preserves the order of torsion elements, the tuple  $[(1, 2), (1, 2), 2]$  would correspond to an involution  $\mathcal{G}^2 \mathcal{F}^{2^{n-1}} \in H$  which defines a non-free action on  $S_g$ .

Remark 4.5 motivates the following definition.

**Definition 4.6.** A *quaternionic data set* is a split metacyclic data set of degree  $4 \cdot 2^n$  that has the form

$$\mathcal{D} = ((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell]),$$

such that  $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j] \neq [(1, 2), (1, 2), 2]$ , for  $1 \leq j \leq \ell$ .

**Proposition 4.7.** For  $g, n \geq 2$ , quaternionic data sets of genus  $2g - 1$  and degree  $4 \cdot 2^n$  correspond to  $Q_{2^{n+1}}$ -actions on  $S_g$ .

**Proof.** Suppose that there exists an action of  $H = Q_{2^{n+1}}$  on  $S_g$ . By Lemma 2.1, there exists an epimorphism  $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$

$$\xi_i \xrightarrow{\phi_H} y^{c_{i1} \frac{m}{n_{i1}}} x^{c_{i2} \frac{n}{n_{i2}}}, \text{ for } 1 \leq i \leq \ell,$$

that is order-preserving on torsion elements. Let  $H' = \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4$ . Since the canonical projection  $q : H' \rightarrow H (\cong H'/\mathbb{Z}_2)$  preserves the order of torsion elements on  $H' \setminus \ker q$ , the map  $\phi_H$  naturally factors via  $q$ . Thus, as there are exactly two possible choices for  $\phi_H|_{\{\xi_i : 1 \leq i \leq \ell\}}$  that preserves the order, at least one of which yields an action  $H'$  on  $S_{g'}$  (for some  $g' > g$ ). A weak conjugacy class associated with this action is encoded by a split metacyclic data set of genus  $g'$  and degree  $2^{n+2} = 4 \cdot 2^n$ , which has one of the following forms

$$((4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_\ell])$$

or

$$(4 \cdot 2^n, -1), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n_\ell]),$$

where  $c'_{\ell_1} \frac{4}{n'_{\ell_1}} \equiv c_{\ell_1} \frac{4}{n_{\ell_1}} + 2 \pmod{4}$  and  $c'_{\ell_2} \frac{2^n}{n'_{\ell_2}} \equiv c_{\ell_2} \frac{2^n}{n_{\ell_2}} + 2^{n-1} \pmod{2^n}$ . Further, since  $\ker q \cong \mathbb{Z}_2$  and  $q$  preserves the orders of all  $x \in H' \setminus \ker q$ , it follows that  $\ker q$  acts freely on  $S_{g'}$ . Hence, it follows that  $g' = 2g - 1$  and further by Remark 4.5, both (possible) tuples cannot contain a triple of the type  $[(1, 2), (1, 2), 2]$ .

Conversely, if there exists a quaternionic data set  $\mathcal{D}$  of genus  $g' = 2g - 1$  as in Definition 4.6. Then we obtain an epimorphism  $\phi_{H'} : \pi_1^{orb}(\mathcal{O}_{H'}) \rightarrow H'$  which preserves the order of torsion elements, when composed with canonical projection  $q : H' \rightarrow H$ , yields an epimorphism  $\phi_H : \pi_1^{orb}(\mathcal{O}_H) \rightarrow H$  which preserves the order of torsion elements, where  $\pi_1^{orb}(\mathcal{O}_{H'}) = \pi_1^{orb}(\mathcal{O}_H)$ . Further, as  $\mathcal{D}$  does not contain a triple of type  $[(1, 2), (1, 2), 2]$ ,  $\ker q$  acts freely on  $S_{g'}$ , thereby yielding an action of  $Q_{2^{n+1}}$  on  $S_g$ , where  $g' = 2g - 1$ .  $\square$

**Remark 4.8.** A crucial step in the proof (of Proposition 4.7) is the establishment of the fact that the canonical projection  $q : \mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4 \rightarrow Q_{2^{n+1}}$  is order-preserving on  $(\mathbb{Z}_{2^n} \rtimes_{-1} \mathbb{Z}_4) \setminus \ker q$ . However, it is interesting to note that this fact does not generalize to arbitrary metacyclic groups [15] arising as quotients of split metacyclic groups. This motivates the study of finite non-split metacyclic actions on surfaces, which we plan to undertake in future works.

**Example 4.9.** The split metacyclic data set in Example 3.4 is quaternionic. Hence, this represents the weak conjugacy class of an induced  $Q_8$ -action on  $S_3$ .

**4.3. Lifting cyclic subgroups of mapping classes to split metacyclic groups.** For  $n, g \geq 2$ , let  $p : S_{\tilde{g}} \rightarrow S_g$  be a covering map (that is possibly branched) with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Let  $\text{LMod}_p(S_g)$  (resp.  $\text{SMod}_p(S_{\tilde{g}})$ ) denote the liftable (resp. symmetric) mapping class groups of  $S_g$  (resp.  $S_{\tilde{g}}$ ) under  $p$ .

**Remark 4.10.** From Birman-Hilden theory [2], we have the exact sequence

$$1 \rightarrow \langle \mathcal{F} \rangle \rightarrow \text{SMod}_p(S_{\tilde{g}}) \rightarrow \text{LMod}_p(S_g) \rightarrow 1. \quad (\text{B})$$

Let  $G \in \text{Mod}(S_g)$  be of finite order. Then  $G \in \text{LMod}_p(S_g)$  if and only if  $G$  has a lift  $\tilde{G} \in \text{SMod}_p(S_{\tilde{g}})$  of finite order so that the sequence (B) yields a sequence of the form

$$1 \rightarrow \langle \mathcal{F} \rangle \rightarrow \langle \mathcal{F}, \tilde{G} \rangle \rightarrow \langle G \rangle \rightarrow 1.$$

Thus,  $G \in \text{LMod}_p(S_g)$  if and only if for any lift  $\tilde{G}$  of  $G$ ,  $\langle G \rangle$  lifts under  $p$  to a metacyclic group  $\langle \mathcal{F}, \tilde{G} \rangle$ .

In the following corollary, we characterize the finite cyclic subgroups in  $\text{Mod}(S_{0,3})$  that lift to finite split metacyclic groups under branched covers induced by irreducible cyclic actions.

**Corollary 4.11.** For  $g, n \geq 2$ , let  $p : S_g \rightarrow S_{0,3}$  be a cover with deck transformation group  $\langle \mathcal{F} \rangle$  with  $D_{\mathcal{F}} = (n, 0; (c_1, n_1), (c_2, n_2), (c_3, n_3))$ . Then a  $G' \in \text{Mod}(S_{0,3})$

of order  $m$  has a conjugate  $G \in \text{LMod}_p(S_{0,3})$  with a lift  $\tilde{G} \in \text{SMod}_p(S_g)$  such that  $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  if and only if one of the following conditions hold.

- (a)  $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2k, n))$  for some  $k \in \mathbb{Z}_n^\times$  such that  $k^2 \equiv 1 \pmod{n}$ .
- (b)  $D_F = (n, 0; (c_1, n), (c_1k, n), (c_1k^2, n))$  for some  $k \in \mathbb{Z}_n^\times$  such that  $k^3 \equiv 1 \pmod{n}$ .

**Proof.** Suppose that  $G' \in \text{Mod}(S_{0,3})$  has a conjugate  $G \in \text{LMod}_p(S_{0,3})$  with a lift  $\tilde{G} \in \text{SMod}_p(S_g)$  such that  $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$ . First, we claim that the  $n_i$ , for  $1 \leq i \leq 3$ , are not distinct. Suppose that we assume on the contrary that the  $n_i$ , for  $1 \leq i \leq 3$ , are indeed distinct. Since  $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$  and  $|\mathcal{G}'| > 1$ , it would have to fix all three cone points of  $\mathcal{O}_{\langle \mathcal{F} \rangle}$ , which contradicts the fact that any nontrivial automorphism of the sphere has exactly two fixed points. Thus, the following two cases arise.

*Case 1:*  $n_2 = n_3 = n \neq n_1$ . In this case,  $\mathcal{G}'$  fixes the cone point, say of order  $n_1$ , and should permute the remaining 2 cone points of orders  $n_2$  and  $n_3$ . This implies that  $D_F$  takes the form in condition (a) in our hypothesis (by Definition 2.8), and hence  $H = \langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_2$ .

*Case 2:*  $n_i = n, 1 \leq i \leq 3$ . In this case, if  $\mathcal{G}'$  permutes all the three cone points cyclically, then  $D_F$  takes the form in condition (b) in our hypothesis, and hence  $H \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_3$ . Alternatively,  $\mathcal{G}'$  could also fix a cone point of order  $n$  and permute the remaining 2 cone points, in which case,  $D_F$  will take the form in condition (a).

Conversely, let  $D_F = (n, 0; (c_1, n_1), (c_2, n), (c_2k, n))$  for some  $k \in \mathbb{Z}_n^\times$  such that  $k^2 \equiv 1 \pmod{n}$ . Up to conjugacy, let  $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$  be an involution so that  $\mathcal{G}'$  maps the cone point represented by  $(c_2, n)$  to the cone point represented by  $(c_2k, n)$ . To prove our assertion, it would suffice to show the existence of an involution  $\mathcal{G} \in \text{Homeo}^+(S_g)$  that induces  $\mathcal{G}'$ . This amounts to showing that there exists a split metacyclic data set  $\mathcal{D}$  of degree  $2 \cdot n$  with twist factor  $k$  encoding the weak conjugacy class  $(H, (\mathcal{G}, \mathcal{F}))$  so that  $D_G$  has degree 2. Consider the tuple  $((2 \cdot n, k), 0; [(1, 2), (0, 1), 2], [(1, 2), (n - c_2, n), 2n_1], [(0, 1), (c_2, n), n])$ . A simple computation would reveal that conditions (i) - (iv) of Definition 3.1 hold true. Condition (v) is true by taking  $v = 1, (p_1, p_2, p_3) = (1, 0, 0)$  and  $(q_1, q_2, q_3) = (0, 0, w)$  such that  $wc_2 \equiv 1 \pmod{n}$ , which proves our claim.

For the case when  $D_F = (n, 0; (c_1, n), (c_1k, n), (c_1k^2, n))$  for some  $k \in \mathbb{Z}_n^\times$  such that  $k^3 \equiv 1 \pmod{n}$ , let  $\mathcal{G}' \in \text{Aut}_k(\mathcal{O}_{\langle \mathcal{F} \rangle})$  be of order 3 so that for  $1 \leq i \leq 2, \mathcal{G}'^i$  maps the cone point represented by  $(c_1, n)$  to the cone point represented by  $(c_1k^{3-i}, n)$ . By similar argument as above, we can show that the tuple  $((3 \cdot n, k), 0; [(1, 3), (0, 1), 3], [(2, 3), (n - c_1, n), 3], [(0, 1), (c_1, n), n])$  forms a split metacyclic data set of degree  $3 \cdot n$  with twist factor  $k$ . □

**Example 4.12.** For  $i = 1, 2$ , consider the branched cover  $p : S_3 \rightarrow \mathcal{O}_{\langle \mathcal{F}_i \rangle} (\approx S_{0,3})$ , where  $D_{F_1} = (8, 0; (1, 4), (1, 8), (5, 8))$  and  $D_{F_2} = (8, 0; (3, 4), (3, 8), (7, 8))$ . Then (up to conjugacy) the order-2 mapping class  $G \in \text{LMod}_p(S_{0,3})$  represented by an automorphism  $\mathcal{G} \in \text{Aut}_5(S_{0,3})$ , that permutes two cone points

of order 8 and fixes order 4 cone point, lifts to a  $\tilde{G} \in \text{SMod}_p(S_3)$  with  $D_{\tilde{G}} = (2, 1; ((1, 2), 4))$  such that  $\langle F_i, \tilde{G} \rangle \cong \mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$ . Moreover, the weak conjugacy class of  $(\langle \mathcal{F}_i, \tilde{G} \rangle, (\tilde{G}, \mathcal{F}_i))$ , for  $i = 1, 2$ , is encoded by

$$\begin{aligned} &((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 8), 8], [(0, 1), (1, 8), 8]) \text{ and} \\ &((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 8), 8], [(0, 1), (7, 8), 8]), \end{aligned}$$

respectively. The geometric realization of these actions is illustrated in Figure 5 below, where for each  $i$ , the action  $\mathcal{F}_i$  is realized by the rotation of a polygon of type  $\mathcal{P}_{F_i}$  described in Theorem 2.5.

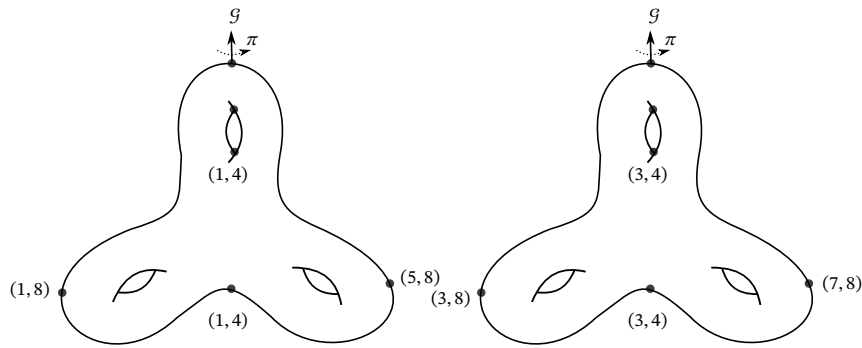


FIGURE 5. The realizations of two distinct  $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$ -actions on  $S_3$ .

**Proposition 4.13.** *For  $g, n \geq 2$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Then any involution  $G' \in \text{Mod}(S_g)$  has a conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong D_{2n}$ .*

**Proof.** Let  $G' \in \text{Mod}(S_g)$  be an involution. When  $\mathcal{G}'$  generates a free action on  $S_g$ , it is easy to see that  $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$  represents a weak conjugacy class in  $\text{Mod}(S_{n(g-1)+1})$  with  $\langle F, \tilde{G} \rangle \cong D_{2n}$ . Now, we assume that  $\mathcal{G}'$  generates a non-free action with  $D_{G'} = (2, g_0; ((1, 2), t))$ . By Theorem 3.3 and Remark 4.10, it suffices to show that there exists a dihedral data set  $\mathcal{D}$  of degree  $2 \cdot n$  and genus  $n(g - 1) + 1$  representing the weak conjugacy class of  $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$ . When  $g_0 \geq 1$ , we take  $\mathcal{D}$  to be the tuple

$$((2 \cdot n, -1), g_0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t \text{ times}}),$$

and when  $g_0 = 0$ ,  $t \geq 4$ , and so we take  $\mathcal{D}$  to be the tuple

$$\begin{aligned} &((2 \cdot n, -1), 0; \underbrace{[(1, 2), (0, 1), 2], \dots, [(1, 2), (0, 1), 2]}_{t-2 \text{ times}}, \\ &[(1, 2), (1, n), 2], [(1, 2), (1, n), 2]). \end{aligned}$$

It is an easy computation to check that  $\mathcal{D}$  satisfies conditions (i)-(iv) of Definition 3.1 in both cases. When  $g_0 = 0$ , taking  $\nu = 1$ ,

$$(p_1, \dots, p_t) = (1, 0, \dots, 0), \text{ and } (q_1, \dots, q_t) = (0, \dots, 0, 1, 1, 0)$$

we obtain condition (v). Moreover, when  $g_0 = 1$ , we take  $\nu = 1$ ,

$$(p_1, \dots, p_t) = (1, 0, \dots, 0), \text{ and } (q_1, \dots, q_t) = (0, \dots, 0),$$

thereby verifying condition (vi). Thus, we have shown that  $\mathcal{D}$  is a dihedral data set as desired. Finally, it follows from Theorem 3.3 that  $\mathcal{D}$  encodes the weak conjugacy class of  $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$ .  $\square$

Note that the same  $\mathbb{Z}_2$ -action can lift to multiple non-isomorphic groups under a regular cyclic cover. We illustrate this phenomenon in the following example.

**Example 4.14.** Let  $p : S_5 \rightarrow S_2$  be a regular 4-sheeted cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_4$  as illustrated in Figure 6 below.

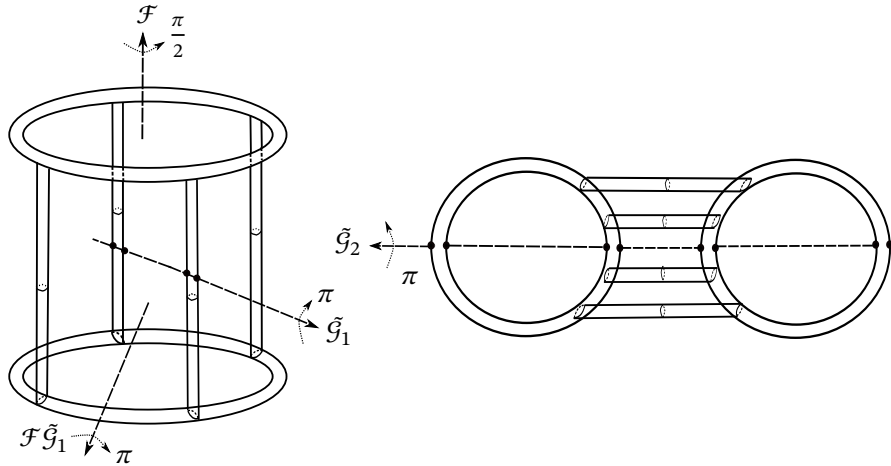


FIGURE 6. Two distinct lifts  $\tilde{G}_1, \tilde{G}_2 \in \text{SMod}(S_5)$  of an involution  $G \in \text{Mod}(S_2)$ . Note that  $\tilde{G}_1$  has four fixed points, while  $\tilde{G}_2$  has eight fixed points.

The involution  $G \in \text{Mod}(S_2)$  with  $D_G = (2, 1; (1, 2), (1, 2))$  has two distinct lifts  $\tilde{G}_1, \tilde{G}_2 \in \text{SMod}_p(S_5)$  (as indicated) such that  $\langle F, \tilde{G}_1 \rangle \cong D_8$  and  $\langle F, \tilde{G}_2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ . Note that the weak conjugacy class of  $(\langle \mathcal{F}, \tilde{G}_1 \rangle, (\tilde{G}_1, \mathcal{F}))$  is represented by  $((2 \cdot 4, -1), 1; [(1, 2), (0, 1), 2], [(1, 2), (0, 1), 2])$ .

The following proposition provides a sufficient condition for the liftability of  $\mathbb{Z}_m$ -actions whose corresponding orbifolds are spheres with a cone point of order  $m$ .

**Proposition 4.15.** For  $g, n \geq 2$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular  $n$ -sheeted cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Let  $G' \in \text{Mod}(S_g)$  be of order  $m$  such that  $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$  with  $m_\ell = m$  (say). Then  $G'$

has a conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  if the following conditions hold.

(a) There exists  $a_1, \dots, a_{\ell-1} \in \mathbb{Z}$ , and  $k \in \mathbb{Z}_n^\times$ ,  $k^m \equiv 1 \pmod{n}$  such that

$$\sum_{i=1}^{\ell-1} a_i (k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell-1} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod{n}.$$

(b) For  $1 \leq i \leq \ell - 1$ , there exists  $d_i, n_i \in \mathbb{Z}$  such that  $\gcd(d_i, n_i) = 1$ ,  $n_i \mid n$ ,

$$d_i \frac{n}{n_i} \equiv a_i (k^{c_i \frac{m}{m_i}} - 1) \pmod{n}, \text{ and}$$

$$\text{lcm}(n_1, n_2, \dots, n_{\ell-1}) = n.$$

**Proof.** By Theorem 3.3 and Remark 4.10, it suffices to show that the tuple

$$\mathcal{D} = ((m \cdot n, k), 0; [(c_1, m_1), (d_1, n_1), m_1], \dots,$$

$$[(c_{\ell-1}, m_{\ell-1}), (d_{\ell-1}, n_{\ell-1}), m_{\ell-1}], [(c_\ell, m_\ell), (0, 1), m_\ell])$$

forms a split metacyclic data set of genus  $n(g-1)+1$  that represents the weak conjugacy class of  $(\langle F, \tilde{G} \rangle, (\tilde{G}, F))$  for some lift  $\tilde{G}$  of  $G$  under  $p$ . It can be verified easily that  $\mathcal{D}$  satisfies conditions (i)-(iii) of Definition 3.1, and further, condition (iv) follows from condition (a) in our hypothesis. By taking  $v = 1$ ,  $(p_1, \dots, p_\ell) = (0, \dots, 0, w)$  such that  $wc_\ell \equiv 1 \pmod{m}$ , we see that condition (v)(a) holds. Finally, condition (v)(b) follows from condition (b) in our hypothesis, and our assertion follows.  $\square$

Using similar arguments, we can show the following.

**Proposition 4.16.** For  $g, n \geq 2$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular  $n$ -sheeted cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Let  $G' \in \text{Mod}(S_g)$  be of order  $m$  such that  $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$  with  $m_i \neq m$ , for  $1 \leq i \leq \ell$ . Then  $G'$  has a conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  if the following conditions hold.

(i) There exists  $a_1, \dots, a_\ell \in \mathbb{Z}$ , and  $k \in \mathbb{Z}_n^\times$ ,  $k^m \equiv 1 \pmod{n}$  such that

$$\sum_{i=1}^{\ell} a_i (k^{c_i \frac{m}{m_i}} - 1) \prod_{s=i+1}^{\ell} k^{c_s \frac{m}{m_s}} \equiv 0 \pmod{n}.$$

(ii) There exists  $(p_1, \dots, p_{\ell v}), (q_1, \dots, q_{\ell v}) \in \mathbb{Z}^{\ell v}$  and  $v \in \mathbb{N}$  such that condition (v)(b) of Definition 3.1 holds, where for  $1 \leq i \leq \ell$ , we have

$$c_{i1} \frac{m}{n_{i1}} \equiv c_i \frac{m}{m_i} \pmod{m} \text{ and } c_{i2} \frac{n}{n_{i2}} \equiv a_i (k^{c_i \frac{m}{m_i}} - 1) \pmod{n}.$$

A consequence of Propositions 4.15-4.16 is the following.

**Corollary 4.17.** For  $g \geq 2$  and prime  $n$ , let  $p : S_{n(g-1)+1} \rightarrow S_g$  be a regular  $n$ -sheeted cover with deck transformation group  $\langle \mathcal{F} \rangle \cong \mathbb{Z}_n$ . Let  $G' \in \text{Mod}(S_g)$  be of order  $m$  such that the genus of  $\mathcal{O}_{\langle G' \rangle}$  is zero. Then  $G'$  has a conjugate  $G \in \text{LMod}_p(S_g)$  with a lift  $\tilde{G} \in \text{SMod}_p(S_{n(g-1)+1})$  such that  $\langle F, \tilde{G} \rangle \cong \mathbb{Z}_n \rtimes_k \mathbb{Z}_m$  if there exists  $k \in \mathbb{Z}_n^\times$  such that  $|k| = m$ .



**Proof.** Let  $D_{G'} = (m, 0; (c_1, m_1), \dots, (c_\ell, m_\ell))$ . First, let us assume (without loss of generality) that  $m_\ell = m$ . By choosing

$$(a_1, \dots, a_{\ell-1}) = (0, \dots, 0, 1, -(k^{c_{\ell-2} \frac{m}{m_{\ell-2}}} - 1) \cdot k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} \cdot (k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} - 1)^{-1}),$$

we see that condition (i) of Proposition 4.15 holds true. Moreover, since  $|k| = m$ , we have  $\gcd((k^{c_{\ell-2} \frac{m}{m_{\ell-2}}} - 1), n) = 1$ , and so condition (ii) also holds, and our assertion follows.

Similarly, for the case when each  $m_i < m$  for  $1 \leq i \leq \ell$ , the result follows by taking

$$(a_1, \dots, a_\ell) = (0, \dots, 0, 1, -(k^{c_{\ell-1} \frac{m}{m_{\ell-1}}} - 1) \cdot k^{c_\ell \frac{m}{m_\ell}} \cdot (k^{c_\ell \frac{m}{m_\ell}} - 1)^{-1}),$$

and applying Proposition 4.16. □

**4.4. Infinite split metacyclic subgroups of  $\text{Mod}(S_g)$ .** An infinite split metacyclic group that is isomorphic to  $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$  admits a presentation of the form

$$\langle x, y \mid y^{2m} = 1, y^{-1}xy = x^{-1} \rangle. \tag{3}$$

In this subsection, we give an explicit construction of an infinite metacyclic subgroup isomorphic to  $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$  of  $\text{Mod}(S_g)$ . Let  $T_c \in \text{Mod}(S_g)$  denote the left-handed Dehn twist about a simple closed curve  $c$  in  $S_g$ . A root of  $T_c$  of degree  $s$  is an  $F \in \text{Mod}(S_g)$  such that  $F^s = T_c$ . In the following lemma, by using some basic properties of Dehn twists [12, Chapter 3], we show that a root of Dehn twist cannot generate an infinite split metacyclic group that admits a presentation as in (3).

**Lemma 4.18.** *For  $g \geq 2$ , no root of  $T_c$  is a generator of any infinite split metacyclic subgroup of  $\text{Mod}(S_g)$  that is isomorphic to  $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ .*

**Proof.** Let  $F$  be a root of  $T_c$  of degree  $s$ . Suppose we assume on the contrary that for some  $g \geq 2$ , there exists an infinite split metacyclic subgroup  $H \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$  of  $\text{Mod}(S_g)$  that admits the presentation

$$H = \langle F, G \mid G^{2m} = 1, G^{-1}FG = F^{-1} \rangle.$$

First, we consider the case when  $s = 1$ , that is,  $F = T_c$ . Then we have that

$$G^{-1}T_cG = T_c^{-1} \implies T_{G^{-1}(c)} = T_c^{-1},$$

which is impossible. Thus, we have that  $H \neq \langle G, T_c \rangle$ , which contradicts our assumption.

For  $s > 1$ , suppose that  $H = \langle F, G \rangle$ . Then the subgroup  $\langle F^s, G \rangle$  of  $H$  would also be a split metacyclic group. Since  $F^s = T_c$ , this would contradict our conclusion in the previous case, and so our assertion follows. □

By a *multitwist* in  $\text{Mod}(S_g)$ , we mean a finite product of powers of commuting Dehn twists. In view of Lemma 4.18, a natural question that arises is whether a multitwist in  $\text{Mod}(S_g)$  can generate an infinite split metacyclic group. In the following examples, we answer this question in the affirmative.

**Example 4.19.** Let  $F' \in \text{Mod}(S_2)$  be of order 3 with

$$D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2)).$$

First, we note that  $\mathcal{F}'$  has four fixed points on  $S_2$ . Further, it induces a local rotation angle of  $2\pi/3$  around two of these points (corresponding to the two  $(1, 3)$  pairs in  $D_{F'}$ ) and rotation angle of  $4\pi/3$  around the remaining two points (corresponding to the two  $(2, 3)$  pairs in  $D_{F'}$ ), as indicated in Figure 7. Considering this action on two distinct copies of  $S_2$ , we remove invariant disks around a distinguished  $(1, 3)$ -type fixed point and a distinguished  $(2, 3)$ -type fixed point in each of the two copies. We now attach two annuli connecting the resulting boundary components across the two surfaces so that:

- (a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 7 below, and further,
- (b) the annulus connecting the boundary components with rotation  $4\pi/3$  (with the nonseparating curve  $c$ ) has a  $1/3^{rd}$  twist, while the other (with the nonseparating curve  $d$ ) has a  $-1/3^{rd}$  twist.

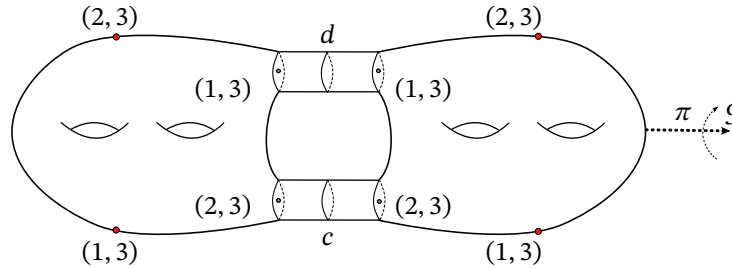


FIGURE 7. Realization of an infinite dihedral subgroup of  $\text{Mod}(S_5)$ .

Thus, by applying the theory developed in [25], we obtain an  $F \in \text{Mod}(S_5)$ , which is a root of the bounding pair map  $T_c T_d^{-1}$  of degree 3. Now, we consider the hyperelliptic involution  $G \in \text{Mod}(S_5)$  with  $D_G = (2, 0; ((1, 2), 12))$  (also indicated in Figure 7). By our construction, it follows that  $GFG^{-1} = F^{-1}$ , and so we have  $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_2$ .

**Example 4.20.** Let  $F' \in \text{Mod}(S_5)$  be of order 3 with

$$D_{F'} = (3, 1; ((1, 3), 2), ((2, 3), 2)).$$

First, we note that  $\mathcal{F}'$  has four fixed points on  $S_5$ . Furthermore, it induces a local rotation angle of  $2\pi/3$  around two of these points (corresponding to the two  $(1, 3)$  pairs in  $D_{F'}$ ) and rotation angle of  $4\pi/3$  around the remaining two points (corresponding to the two  $(2, 3)$  pairs in  $D_{F'}$ ), as indicated in Figure 8. Considering this action on two distinct copies of  $S_5$ , we remove invariant disks around all fixed point in each of the two copies. We now attach four annuli connecting the resulting boundary components across the two surfaces so that:

- (a) each annulus connects a pair of boundary components where the induced rotation angle is the same, as shown in Figure 8 below, and further,

- (b) the annulus connecting the boundary components with rotation  $4\pi/3$  (with the nonseparating curve  $c_1$  and  $c_3$ ) has a  $1/3^{rd}$  twist, while the other (with the nonseparating curve  $c_2$  and  $c_4$ ) has a  $-1/3^{rd}$  twist.

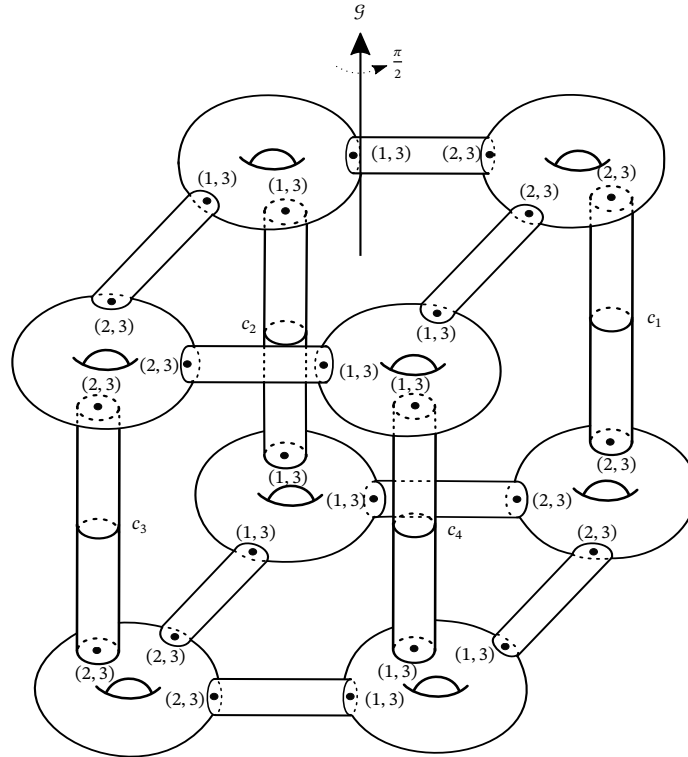


FIGURE 8. Realization of an infinite metacyclic subgroup of  $\text{Mod}(S_{13})$ .

Thus, by applying the theory developed in [25], we obtain an  $F \in \text{Mod}(S_{13})$ , which is a root of the multitwist  $T_{c_1} T_{c_2}^{-1} T_{c_3} T_{c_4}^{-1}$  of degree 3. Now, we consider a  $G \in \text{Mod}(S_{13})$  with  $D_G = (4, 4, 1;)$  (also indicated in Figure 8). By our construction, as  $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4 \cong \langle F', G' \rangle \leq \text{Mod}(S_5)$ , where  $D_{G'} = (4, 2, 1;)$ , it follows that  $GFG^{-1} = F^{-1}$ , and so we have  $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_4$ .

Generalizing the above all constructions in Example 4.19 and Example 4.20, we have the following.

**Proposition 4.21.** *For  $i = 1, 2$ , let  $H_i = \langle F_i, G_i \rangle \leq \text{Mod}(S_{g_i})$  with  $H_i \cong \mathbb{Z}_n \rtimes_{-1} \mathbb{Z}_{2m}$ , such that the weak conjugacy class  $(H_i, (G_i, F_i))$  is represented by a split metacyclic data set  $\mathcal{D}_{H_i}$  containing a tuple  $[(0, 1), (a_i, n), n]$ . Then there exists an infinite metacyclic subgroup of  $\text{Mod}(S_{g_1+g_2+2m-1})$  isomorphic to  $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$  that is generated by a periodic mapping class of order  $2m$  and a root of a multitwist of degree  $n$ .*

**Proof.** As  $\mathcal{D}_{H_i}$  contains a tuple  $[(0, 1), (a_i, n), n]$ , by Proposition 3.2, we have

$$D_{F_1} = (n, g_0; (c_1, n_1), \dots, (c_s, n_s), \underbrace{(a_1, n), (n - a_1, n), \dots, (a_1, n), (n - a_1, n)}_{m \text{ times}})$$

and

$$D_{F_2} = (n, g'_0; (c'_1, n'_1), \dots, (c'_t, n'_t), \underbrace{(a_2, n), (n - a_2, n), \dots, (a_2, n), (n - a_2, n)}_{m \text{ times}}).$$

Taking inspiration from the theory developed in [24, 25] and Examples 4.19-4.20, we glue  $2m$  annuli connecting the boundary components resulting from removing invariant disks around the orbit points corresponding to the pairs  $(a_1, n)$  and

$$\begin{cases} (a_2, n), & \text{if } a_2 \neq n - a_1, \text{ or} \\ (n - a_2, n), & \text{if } a_2 = n - a_1. \end{cases}$$

This yields a degree- $n$  root  $F$  of a multitwist of the form

$$\begin{cases} \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1} + a_2^{-1})}, & \text{if } a_2 \neq n - a_1, \text{ or} \\ \prod_{i=1}^{2m} T_{c_i}^{(-1)^{i+1}(a_1^{-1} + (n - a_2)^{-1})}, & \text{if } a_2 = n - a_1, \end{cases}$$

where  $a_i a_i^{-1} \equiv 1 \pmod{n}$  and  $a_1^{-1} + a_2^{-1} \in \mathbb{Z}_n$ . By considering the action  $\mathcal{G}$  obtained by performing a  $2m$ -compatibility on  $\mathcal{G}_1$  and  $\mathcal{G}_2$  (see Section 2), we see that  $\langle F, G \rangle \cong \mathbb{Z} \rtimes_{-1} \mathbb{Z}_{2m}$ , as desired.  $\square$

The group for  $m = 1$  in the presentation of the infinite split metacyclic group of the type in the Equation (3) is known as the *infinite dihedral group*. Here is the corollary, which directly follows from Proposition 4.21.

**Corollary 4.22.** *For  $g \geq 5$ , there exists an infinite dihedral subgroup of  $\text{Mod}(S_g)$  that is generated by an involution and a root of a bounding pair map of degree 3.*

## 5. Hyperbolic structures realizing split metacyclic actions

We begin this section by providing an algorithm for obtaining the hyperbolic structures that realize finite split metacyclic subgroups of  $\text{Mod}(S_g)$  (up to weak conjugacy) as groups of isometries.

*Step 1.* Consider a weak conjugacy class represented by  $(H, (\mathcal{G}, \mathcal{F}))$ .

*Step 2.* Use Theorem 3.3 to determine the conjugacy classes  $D_F$  (resp.  $D_G$ ) of the generators  $F$  (resp.  $G$ ).

*Step 3.* We apply Lemma 2.7, and Theorems 2.5-2.6, to obtain the hyperbolic structures that realize  $H$  as a group of isometries.

We now describe the geometric realizations of some split metacyclic actions on  $S_3$  and  $S_5$  represented by the split metacyclic data sets listed in Tables 1 and 2 in Section 6.

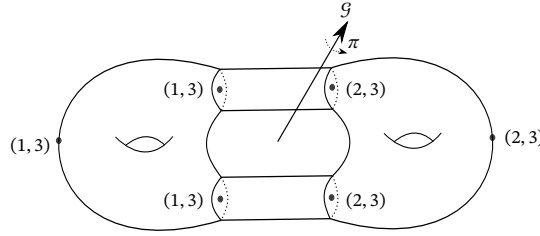


FIGURE 9. A realization of a  $D_6$ -action  $\langle \mathcal{F}, \mathcal{G} \rangle$  on  $S_3$ , where  $D_G = (2, 2, 1; )$  and  $D_F = (3, 1; (1, 3), (2, 3))$ . The action  $\mathcal{F}$  is realized through two 1-compatibilities between two actions  $\mathcal{F}'$  and  $\mathcal{F}''$  on  $S_1$  with  $D_{F'} = (3, 0; ((1, 3), 3))$  and  $D_{F''} = (3, 0; ((2, 3), 3))$ . The weak conjugacy class of  $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$  is encoded by the first split metacyclic data set in Table 1.

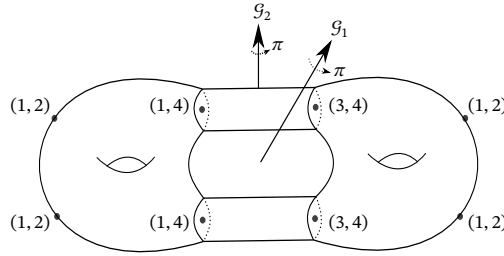


FIGURE 10. The realizations of two distinct  $D_8$ -actions  $\langle \mathcal{F}, \mathcal{G}_1 \rangle$  and  $\langle \mathcal{F}, \mathcal{G}_2 \rangle$  on  $S_3$ , where  $D_F = (4, 1; ((1, 2), 2))$ ,  $D_{G_1} = (2, 2, 1; )$ , and  $D_{G_2} = (2, 1; ((1, 2), 4))$ . The action  $\mathcal{F}$  is realized via two 1-compatibilities between two actions  $\mathcal{F}'$  and  $\mathcal{F}''$  on  $S_1$ , where  $D_{F'} = (4, 0; ((1, 4), 2), (1, 2))$  and  $D_{F''} = (4, 0; ((3, 4), 2), (1, 2))$ . The weak conjugacy classes of  $(\langle \mathcal{F}, \mathcal{G}_1 \rangle, (\mathcal{G}_1, \mathcal{F}))$  and  $(\langle \mathcal{F}, \mathcal{G}_2 \rangle, (\mathcal{G}_2, \mathcal{F}))$  are encoded by split metacyclic data sets nos. 3 and 6, respectively, in Table 1.

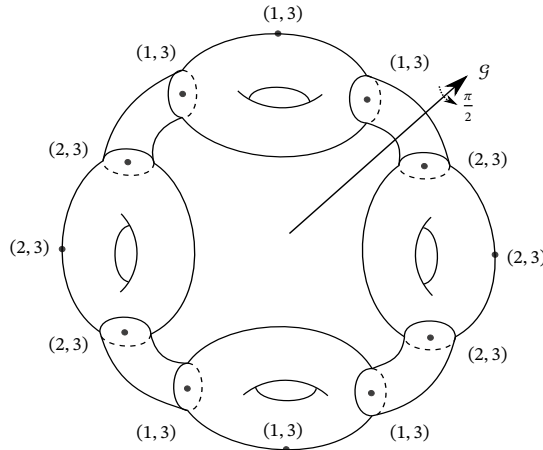


FIGURE 11. A realization of a  $\mathbb{Z}_3 \rtimes_{-1} \mathbb{Z}_4$ -action  $\langle \mathcal{F}, \mathcal{G} \rangle$  on  $S_5$ , where  $D_G = (4, 2, 1; )$  and  $D_F = (3, 1; ((1, 3), 2), ((2, 3), 2))$ . The action  $\mathcal{F}$  is realized via two 1-compatibilities between the action  $\mathcal{F}'$  on two copies of  $S_2$  with  $D_{F'} = (3, 0; ((1, 3), 2), ((2, 3), 2))$ . Furthermore, the action  $\mathcal{F}'$  is realized by a 1-compatibility between the actions  $\mathcal{F}''$  and  $\mathcal{F}'''$  on  $S_1$ , where  $D_{F''} = (3, 0; ((1, 3), 3))$  and  $D_{F'''} = (3, 0; ((2, 3), 3))$ . The weak conjugacy class of  $(\langle \mathcal{F}, \mathcal{G} \rangle, (\mathcal{G}, \mathcal{F}))$  is encoded by the split metacyclic data set no. 14 in Table 2.

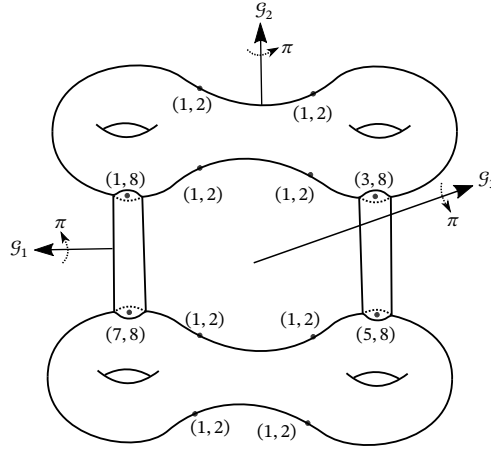


FIGURE 12. Realization of  $\mathbb{Z}_8 \rtimes_{-1} \mathbb{Z}_2$ -action  $\langle \mathcal{F}, \mathcal{G}_1 \rangle$ ,  $\mathbb{Z}_8 \rtimes_3 \mathbb{Z}_2$ -action  $\langle \mathcal{F}, \mathcal{G}_2 \rangle$  and  $\mathbb{Z}_8 \rtimes_5 \mathbb{Z}_2$ -action  $\langle \mathcal{F}, \mathcal{G}_3 \rangle$  on  $S_5$ , where  $D_{G_1} = D_{G_2} = (2, 2; ((1, 2), 4))$ ,  $D_{G_3} = (2, 3, 1; )$  and  $D_{\mathcal{F}} = (8, 1; ((1, 2), 2))$ . The action  $\mathcal{F}$  is realized via two 1-compatibilities between two actions  $\mathcal{F}'$  and  $\mathcal{F}''$  on  $S_2$  where  $D_{\mathcal{F}' } = (8, 0; (1, 2), (1, 8), (3, 8))$  and  $D_{\mathcal{F}''} = (8, 0; (1, 2), (5, 8), (7, 8))$ . The weak conjugacy class of  $(\langle \mathcal{F}, \mathcal{G}_i \rangle, (\mathcal{G}_i, \mathcal{F}))$   $1 \leq i \leq 3$  is encoded by the split metacyclic data set nos. 26, 25, and 22, respectively, in Table 2.

### 6. Classification of the weak conjugacy classes in $\text{Mod}(S_3)$ and $\text{Mod}(S_5)$

In this section, we will use Theorem 3.3 to classify the weak conjugacy classes in  $\text{Mod}(S_3)$  and  $\text{Mod}(S_5)$ . For brevity, we will further assume the following equivalence of the split metacyclic data sets (i.e. the weak conjugacy classes).

**Definition 6.1.** Two split metacyclic data sets

$$\mathcal{D} = ((m \cdot n, k), g_0; [(c_{11}, n_{11}), (c_{12}, n_{12}), n_1], \dots, [(c_{\ell 1}, n_{\ell 1}), (c_{\ell 2}, n_{\ell 2}), n_{\ell}])$$

$$\mathcal{D}' = ((m \cdot n, k), g_0; [(c'_{11}, n'_{11}), (c'_{12}, n'_{12}), n'_1], \dots, [(c'_{\ell 1}, n'_{\ell 1}), (c'_{\ell 2}, n'_{\ell 2}), n'_{\ell}])$$

are said to be *equivalent* if for each tuple  $[(c'_{i1}, n'_{i1}), (c'_{i2}, n'_{i2}), n'_i]$ , there exists a unique tuple  $[(c_{j1}, n_{j1}), (c_{j2}, n_{j2}), n_j]$  satisfying the following conditions:

- (i)  $(c'_{i1}, n'_{i1}) = (c_{j1}, n_{j1})$ ,
- (ii)  $n'_i = n_j$ , and
- (iii)  $c'_{i2} \frac{n}{n'_{i2}} \equiv c_{j2} \frac{n}{n_{j2}} k^{a_i} + b_i (k^{c_{j1} \frac{m}{n_{j1}}} - 1) \pmod{n}$  for some  $a_i, b_i \in \mathbb{Z}$ .

Note that equivalent data sets  $\mathcal{D}$  and  $\mathcal{D}'$  as in Definition 6.1 satisfy  $\mathcal{D}'_i = \mathcal{D}_i$ , for  $i = 1, 2$ . We will now provide a classification of the weak conjugacy classes of finite split metacyclic subgroups of  $\text{Mod}(S_3)$  and  $\text{Mod}(S_5)$  (up to this equivalence) in Tables 1 and 2, respectively.

Group	Weak conjugacy classes in $\text{Mod}(S_3)$	Cyclic factors $[D_G; D_F]$
$Z_3 \rtimes_{-1} Z_2$	$((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2]_3, [(0, 1), (1, 3), 2], [(0, 1), (2, 3), 3])^*$	$[(2, 2, 1); (3, 1; (1, 3), (2, 3))]$
$Z_4 \rtimes_{-1} Z_2$	$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 2), 2])$	$[(2, 2, 1); (4, 1; (1, 2), 2)]$
$Z_3 \rtimes_{-1} Z_4$	$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 4), 4], [(0, 1), (3, 4), 4])$	$[(2, 1; ((1, 2), 4); (4, 0; (1, 4), 2), (3, 4), 2))]$
$Z_6 \rtimes_{-1} Z_2$	$((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 4), 2], [(1, 2), (3, 4), 2], [(0, 1), (1, 2), 2])$	$[(2, 2, 1); (4, 0; ((1, 4), 2), (3, 4), 2))]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 3, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 3), 4], [(1, 2), (2, 3), 6])$	$[(4, 0; (1, 4), 2), ((1, 2), 3); (3, 1; (1, 3), (2, 3))]$
$Z_8 \rtimes_{-1} Z_2$	$((4 \cdot 3, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (1, 3), 4], [(1, 2), (2, 3), 6])$	$[(4, 0; (3, 4), 2), ((1, 2), 3); (3, 1; (1, 3), (2, 3))]$
$Z_4 \rtimes_{-1} Z_4$	$((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 3), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 6), 6])$	$[(2, 1; ((1, 2), 4); (6, 0; (1, 2), 2), (1, 6), (5, 6))]$
$Z_4 \rtimes_{-1} Z_4$	$((2 \cdot 6, -1), 0; [(1, 2), (1, 6), 2], [(1, 2), (1, 2), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 6), 6])$	$[(2, 2, 1); (6, 0; (1, 2), 2), (1, 6), (5, 6))]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 4), 4], [(3, 4), (1, 4), 4])^\dagger$	$[(4, 0; ((1, 4), 2), (1, 2), 3); (4, 0; (1, 4), 2), ((3, 4), 2))]$
$Z_4 \rtimes_{-1} Z_4$	$((4 \cdot 4, -1), 0; [(3, 4), (0, 1), 4], [(0, 1), (1, 4), 4], [(1, 4), (1, 4), 4])^\dagger$	$[(4, 0; ((3, 4), 2), (1, 2), 3); (4, 0; (1, 4), 2), ((3, 4), 2))]$
$Z_8 \rtimes_5 Z_2$	$((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 8), 8], [(0, 1), (1, 8), 8])$	$[(4, 0; ((3, 4), 2), (1, 2), 3); (4, 1; (1, 2), 2))]$
$Z_7 \rtimes_2 Z_3$	$((2 \cdot 8, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 8), 8], [(0, 1), (7, 8), 8])$	$[(2, 1; ((1, 2), 4); (8, 0; (1, 4), (1, 8), (5, 8))]$
$Z_{12} \rtimes_5 Z_2$	$((3 \cdot 7, 2), 0; [(1, 3), (0, 1), 3], [(2, 3), (6, 7), 3], [(0, 1), (1, 7), 7])$	$[(2, 1; (1, 3), (2, 3)); (7, 0; (1, 7), (2, 7), (4, 7))]$
$Z_{12} \rtimes_5 Z_2$	$((3 \cdot 7, 2), 0; [(1, 3), (0, 1), 3], [(2, 3), (1, 7), 3], [(0, 1), (6, 7), 7])$	$[(3, 1; (1, 3), (2, 3)); (7, 0; (3, 7), (6, 7), (5, 7))]$
$Z_{12} \rtimes_5 Z_2$	$((2 \cdot 12, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (11, 12), 4], [(0, 1), (1, 12), 12])$	$[(2, 1; (1, 2), 4); (12, 0; (1, 12), (5, 12), (1, 2))]$
$Z_{12} \rtimes_5 Z_2$	$((2 \cdot 12, 5), 0; [(1, 2), (0, 1), 2], [(1, 2), (5, 12), 4], [(0, 1), (7, 12), 12])$	$[(2, 1; ((1, 2), 4); (12, 0; (7, 12), (11, 12), (1, 2)))]$
$Z_{12} \rtimes_5 Z_2$	$((2 \cdot 12, 5), 0; [(1, 2), (1, 6), 2], [(1, 2), (1, 12), 4], [(0, 1), (1, 12), 12])$	$[(2, 2, 1); (12, 0; (1, 12), (5, 12), (1, 2))]$
$Z_{12} \rtimes_5 Z_2$	$((2 \cdot 12, 5), 0; [(1, 2), (1, 6), 2], [(1, 2), (7, 12), 4], [(0, 1), (7, 12), 12])$	$[(2, 2, 1); (12, 0; (7, 12), (11, 12), (1, 2))]$

TABLE 1. The weak conjugacy classes of finite non-abelian split metacyclic subgroups of  $\text{Mod}(S_3)$ . Note that each data set of type  $\dagger$  is quaternionic, and therefore corresponds to the weak conjugacy action of a  $Q_8$ -action on  $S_2$ . (\*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)



Group	Weak conjugacy classes in $\text{Mod}(S_5)$	Cyclic factors $[D_G; D_F]$
$Z_3 \rtimes_{-1} Z_2$	$((2 \cdot 3, -1), 1; [(0, 1), (1, 3), 3]_2)^*$ $((2 \cdot 3, -1), 0; [(1, 2), (0, 1), 2]_4, [(0, 1), (1, 3), 3], [(0, 1), (2, 3), 3])$ $((2 \cdot 4, -1), 1; [(1, 2), (0, 1), 2]_2)$	$[(2, 3, 1); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 2; ((1, 2), 4)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 2; ((1, 2), 4)); (4, 2, 1);$ $[(2, 3, 1); (4, 2, 1);$
$Z_4 \rtimes_{-1} Z_2$	$((2 \cdot 4, -1), 1; [(1, 2), (1, 4), 2]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 4), 4]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (1, 4), 2]_2, [(0, 1), (1, 2), 2], [(0, 1), (1, 4), 4]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_4, [(1, 2), (1, 4), 2]_2)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 4), 2]_4)$ $((2 \cdot 4, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 4), 2]_2, [(0, 1), (1, 2), 2]_2)$ $((2 \cdot 5, -1), 1; [(0, 1), (1, 5), 5])$ $((2 \cdot 5, -1), 1; [(0, 1), (2, 5), 5])$	$[(2, 3, 1); (4, 2, 1);$ $[(2, 2; ((1, 2), 4)); (4, 0; ((1, 2), 2), ((1, 4), 2), ((3, 4), 2))]$ $[(2, 3, 1); (4, 0; ((1, 2), 2), ((1, 4), 2), ((3, 4), 2))]$ $[(2, 1; ((1, 2), 8)); (4, 2, 1);$ $[(2, 2; ((1, 2), 4)); (4, 2, 1);$ $[(2, 2; ((1, 2), 4)); (4, 1; ((1, 2), 4))]$ $[(2, 3, 1); (5, 1; (1, 5), (4, 5))]$ $[(2, 3, 1); (5, 1; (2, 5), (3, 5))]$
$Z_5 \rtimes_{-1} Z_2$	$((2 \cdot 5, -1), 0; [(1, 2), (0, 1), 2]_3, [(1, 2), (4, 5), 2], [(0, 1), (1, 5), 5])$ $((2 \cdot 5, -1), 0; [(1, 2), (0, 1), 2]_3, [(1, 2), (3, 5), 2], [(0, 1), (2, 5), 5])$ $((4 \cdot 3, -1), 1; [(0, 1), (1, 3), 3])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(1, 4), (0, 1), 4], [(1, 4), (2, 3), 4])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(3, 4), (0, 1), 4], [(3, 4), (2, 3), 4])$ $((2 \cdot 6, -1), 1; [(0, 1), (1, 3), 3])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (1, 6), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 3), 2], [(1, 6), 2]_2, [(0, 1), (1, 3), 3])$ $((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$ $((2 \cdot 8, 5), 1; [(0, 1), (1, 2), 2])$	$[(2, 2; ((1, 2), 4)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 3, 1); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 2; ((1, 2), 4)); (6, 1; (1, 3), (2, 3))]$ $[(4, 2, 1); (6, 1; (1, 3), (2, 3))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$
$Z_3 \rtimes_{-1} Z_4$	$((4 \cdot 3, -1), 1; [(0, 1), (1, 3), 3])$ $((4 \cdot 3, -1), 0; [(1, 2), (0, 1), 2], [(0, 1), (1, 3), 3], [(3, 4), (0, 1), 4], [(3, 4), (2, 3), 4])$ $((2 \cdot 6, -1), 1; [(0, 1), (1, 3), 3])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (1, 6), 2]_2, [(0, 1), (1, 6), 6], [(0, 1), (5, 6), 6])$ $((2 \cdot 6, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 3), 2], [(1, 6), 2]_2, [(0, 1), (1, 3), 3])$ $((4 \cdot 4, -1), 1; [(0, 1), (1, 2), 2])$ $((2 \cdot 8, 5), 1; [(0, 1), (1, 2), 2])$	$[(2, 3, 1); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(4, 0; ((1, 4), 2), ((1, 2), 5)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(4, 0; ((3, 4), 2), ((1, 2), 5)); (3, 1; ((1, 3), 2), ((2, 3), 2))]$ $[(2, 3, 1); (6, 1; (1, 3), (2, 3))]$ $[(2, 2; ((1, 2), 4)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 3, 1); (6, 0; ((1, 6), 2), ((5, 6), 2))]$ $[(2, 2; ((1, 2), 4)); (6, 1; (1, 3), (2, 3))]$ $[(4, 2, 1); (4, 1; ((1, 2), 4))]$ $[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$
$Z_8 \rtimes_5 Z_2$	$((2 \cdot 8, 5), 0; [(1, 2), (1, 4), 4], [(0, 1), (1, 8), 8], [(1, 2), (1, 8), 8])$ $((2 \cdot 8, 5), 0; [(1, 2), (1, 4), 4], [(1, 2), (3, 8), 8], [(0, 1), (3, 8), 8])$ $((2 \cdot 8, 3), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 4), 2], [(1, 2), (1, 8), 4], [(1, 2), (3, 8), 4])$ $((2 \cdot 8, -1), 0; [(1, 2), (0, 1), 2]_2, [(1, 2), (1, 8), 2], [(1, 2), (5, 8), 2], [(0, 1), (1, 2), 2])$ $((4 \cdot 5, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 5), 4], [(1, 2), (4, 5), 10])$ $((4 \cdot 5, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (2, 5), 4], [(1, 2), (3, 5), 10])$ $((4 \cdot 5, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (1, 5), 4], [(1, 2), (4, 5), 10])$ $((4 \cdot 5, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (2, 5), 4], [(1, 2), (3, 5), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (1, 5), 2], [(0, 1), (1, 2), 2], [(0, 1), (3, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (0, 1), 2], [(1, 2), (2, 5), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (3, 10), 2], [(0, 1), (1, 2), 2], [(0, 1), (3, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 2), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$ $((2 \cdot 10, -1), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 2), 2], [(0, 1), (1, 2), 2], [(0, 1), (1, 10), 10])$	$[(2, 3, 1); (8, 1; ((1, 2), 2))]$ $[(2, 3, 1); (8, 0; (1, 2), (3, 4), (1, 8), (5, 8))]$ $[(2, 3, 1); (8, 0; (1, 2), (1, 4), (3, 8), (7, 8))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(2, 2; ((1, 2), 4)); (8, 1; ((1, 2), 2))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (1, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (2, 5), (3, 5))]$ $[(4, 0; (3, 4), 2), (1, 2), 5); (5, 1; (1, 5), (4, 5))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 2; ((1, 2), 4)); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (3, 10), (7, 10))]$ $[(2, 3, 1); (10, 0; ((1, 2), 2), (1, 10), (9, 10))]$

TABLE 2. The weak conjugacy classes of finite non-abelian split metacyclic subgroups of  $\text{Mod}(S_5)$ . (\*The suffix refers to the multiplicity of the tuple in the split metacyclic data set.)

Continuation of Table 2.

Group	Weak conjugacy classes in $\text{Mod}(S_5)$	Cyclic factors $[D_G; D_F]$
$\mathbb{Z}_6 \rtimes_{-1} \mathbb{Z}_4$	$((4 \cdot 6, -1), 0; [(1, 4), (0, 1), 4], [(3, 4), (1, 6), 4], [(0, 1), (5, 6), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 5)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$
	$((4 \cdot 6, -1), 0; [(3, 4), (0, 1), 4], [(1, 4), (1, 6), 4], [(0, 1), (5, 6), 6])$	$[(4, 0; ((3, 4), 2), ((1, 2), 5)); (6, 0; ((1, 6), 2), ((5, 6), 2))]$
	$((4 \cdot 6, -1), 0; [(1, 4), (0, 1), 4], [(1, 4), (1, 6), 4], [(1, 2), (5, 6), 6])$	$[(4, 0; ((1, 4), 2), ((1, 2), 5)); (6, 1; (1, 3), (2, 3))]$
	$((4 \cdot 6, -1), 0; [(3, 4), (0, 1), 4], [(3, 4), (1, 6), 4], [(1, 2), (5, 6), 6])$	$[(4, 0; ((3, 4), 2), ((1, 2), 5)); (6, 1; (1, 3), (2, 3))]$
$\mathbb{Z}_{15} \rtimes_4 \mathbb{Z}_2$	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (14, 15), 6], [(0, 1), (1, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (1, 15), (4, 15), (2, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (13, 15), 6], [(0, 1), (2, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (2, 15), (8, 15), (1, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (8, 15), 6], [(0, 1), (7, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (7, 15), (13, 15), (2, 3))]$
	$((2 \cdot 15, 4), 0; [(1, 2), (0, 1), 2], [(1, 2), (4, 15), 6], [(0, 1), (11, 15), 15])$	$[(2, 2; ((1, 2), 4)); (15, 0; (11, 15), (14, 15), (1, 3))]$
$\mathbb{Z}_{20} \rtimes_9 \mathbb{Z}_2$	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (19, 20), 4], [(0, 1), (1, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (1, 20), (9, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (1, 20), 4], [(0, 1), (1, 20), 20])$	$[(2, 3, 1; (20, 0; (1, 20), (9, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (17, 20), 4], [(0, 1), (3, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (3, 20), (7, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (19, 20), 4], [(0, 1), (3, 20), 20])$	$[(2, 3, 1; (20, 0; (3, 20), (7, 20), (1, 2))]$
$\mathbb{Z}_{20} \rtimes_9 \mathbb{Z}_2$	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (9, 20), 4], [(0, 1), (11, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (11, 20), (19, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (11, 20), 4], [(0, 1), (11, 20), 20])$	$[(2, 3, 1; (20, 0; (11, 20), (19, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (0, 1), 2], [(1, 2), (7, 20), 4], [(0, 1), (13, 20), 20])$	$[(2, 2; ((1, 2), 4)); (20, 0; (13, 20), (17, 20), (1, 2))]$
	$((2 \cdot 20, 9), 0; [(1, 2), (1, 10), 2], [(1, 2), (9, 20), 4], [(0, 1), (13, 20), 20])$	$[(2, 3, 1; (20, 0; (13, 20), (17, 20), (1, 2))]$

## Acknowledgements

The first and third authors were supported by the UGC-JRF fellowship. The authors would also like to thank Dr. Siddhartha Sarkar for helpful discussions.

## References

- [1] BHATTACHARYA, ATREYEE; PARSAD, SHIV; RAJEEVSARATHY, KASHYAP. Geometric realizations of cyclic actions on surfaces–II. Preprint, 2018. [arXiv:1803.00328](https://arxiv.org/abs/1803.00328). 623
- [2] BIRMAN, JOAN S. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.* **22** (1969), 213–238. [MR243519](#), [Zbl 0167.21503](#), doi: [10.1002/cpa.3160220206](https://doi.org/10.1002/cpa.3160220206). 619, 634
- [3] BIRMAN, JOAN S.; HILDEN, HUGH M. On the mapping class groups of closed surfaces as covering spaces. *Advances in the theory of Riemann surfaces* (Proc. Conf., Stony Brook, N.Y., 1969), 81–115. *Ann. of Math. Studies*, 66. *Princeton Univ. Press, Princeton, N.J.*, 1971. [MR0292082](#), [Zbl 0217.48602](#). 619
- [4] BIRMAN, JOAN S.; HILDEN, HUGH M. Isotopies of homeomorphisms of Riemann surfaces and a theorem about Artin’s braid group. *Bull. Amer. Math. Soc.* **78** (1972), 1002–1004. [MR307217](#), [Zbl 0255.57002](#), doi: [10.1090/S0002-9904-1972-13082-9](https://doi.org/10.1090/S0002-9904-1972-13082-9). 619
- [5] BIRMAN, JOAN S.; HILDEN, HUGH M. On isotopies of homeomorphisms of Riemann surfaces. *Ann. of Math. (2)* **97** (1973), 424–439. [MR325959](#), [Zbl 0237.57001](#), doi: [10.2307/1970830](https://doi.org/10.2307/1970830). 619
- [6] BOGOPOLSKI, OLEG V. Classifying the actions of finite groups on orientable surfaces of genus 4. *Siberian Adv. Math.* **7** (1997), no. 4, 9–38. [MR1604157](#), [Zbl 0926.57011](#). 620
- [7] BREUER, THOMAS. Characters and automorphism groups of compact Riemann surfaces. *London Mathematical Society Lecture Note Series*, 280. *Cambridge University Press, Cambridge*, 2000. xii+199 pp. ISBN: 0-521-79809-4. [MR1796706](#), [Zbl 0952.30001](#). 622, 624
- [8] BROUGHTON, S. ALLEN. Classifying finite group actions on surfaces of low genus. *J. Pure Appl. Algebra.* **69** (1991), no. 3, 233–270. [MR1090743](#), [Zbl 0722.57005](#), doi: [10.1016/0022-4049\(91\)90021-S](https://doi.org/10.1016/0022-4049(91)90021-S). 620
- [9] BROUGHTON, S. ALLEN; WOOTTON, AARON. Finite abelian subgroups of the mapping class group. *Algebr. Geom. Topol.* **7** (2007), 1651–1697. [MR2366175](#), [Zbl 1126.14038](#), doi: [10.2140/agt.2007.7.1651](https://doi.org/10.2140/agt.2007.7.1651). 618
- [10] BUJALANCE, EMILIO; CIRRE, FRANCISCO-JAVIER; GAMBOA, JOSÉ M.; GROMADZKI, GRZEGORZ. On compact Riemann surfaces with dihedral groups of automorphisms. *Math. Proc. Cambridge Philos. Soc.* **134** (2003), no. 3, 465–477. [MR1981212](#), [Zbl 1059.30030](#), doi: [10.1017/S030500410200662X](https://doi.org/10.1017/S030500410200662X). 619
- [11] DHANWANI, NEERAJ K.; RAJEEVSARATHY, KASHYAP. Commuting conjugates of finite-order mapping classes. *Geom. Dedicata.* **209** (2020), 69–93. [MR4163393](#), [Zbl 1451.57012](#), doi: [10.1007/s10711-020-00523-9](https://doi.org/10.1007/s10711-020-00523-9). 618, 624, 625
- [12] FARB, BENSON; MARGALIT, DAN. A primer on mapping class groups. *Princeton Mathematical Series*, 49. *Princeton University Press, Princeton, NJ*, 2012. xiv+472 pp. ISBN: 978-0-691-14794-9. [MR2850125](#), [Zbl 1245.57002](#). 639
- [13] GILMAN, JANE. Structures of elliptic irreducible subgroups of the modular group. *Proc. London Math. Soc. (3)* **47** (1983), no. 1, 27–42. [MR698926](#), [Zbl 0522.30035](#), doi: [10.1112/plms/s3-47.1.27](https://doi.org/10.1112/plms/s3-47.1.27). 619, 623
- [14] HARVEY, WILLIAM J. Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math. Oxford Ser. (2)* **17** (1966), 86–97. [MR0201629](#), [Zbl 0156.08901](#), doi: [10.1093/qmath/17.1.86](https://doi.org/10.1093/qmath/17.1.86). 618, 621, 622
- [15] HEMPEL, CHARLES E. Metacyclic groups. *Comm. Algebra.* **28** (2000), no. 8, 3865–3897. [MR1767595](#), [Zbl 0993.20013](#), doi: [10.1080/00927870008827063](https://doi.org/10.1080/00927870008827063). 620, 634

- [16] KATOK, SVETLANA. Fuchsian groups. Chicago Lectures in Mathematics. *University of Chicago Press, Chicago, IL*, 1992. x+175 pp. ISBN: 0-226-42582-7; 0-226-42583-5. [MR1177168](#), [Zbl 0753.30001](#). 621
- [17] KERCKHOFF, STEVEN P. The Nielsen realization problem. *Ann. of Math. (2)* **117** (1983), no. 2, 235–265. [MR690845](#), [Zbl 0528.57008](#), doi: [10.2307/2007076](#). 618, 619, 621, 623
- [18] KIMURA, HIDEYUKI. Classification of automorphism groups, up to topological equivalence, of compact Riemann surfaces of genus 4. *J. Algebra*. **264** (2003), no. 1, 26–54. [MR1980684](#), [Zbl 1027.30063](#), doi: [10.1016/S0021-8693\(03\)00138-8](#). 620
- [19] MACBEATH, ALEXANDER M.; WILKIE, H. C. Discontinuous groups and birational transformations: Summer School. Queen's College Dundee, University of St. Andrews. *Department of Math., Queen's College*, 1961. 619, 621
- [20] MACLACHLAN, COLIN. Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London Math. Soc. (3)* **15** (1965), 699–712. [MR0179348](#), [Zbl 0156.08902](#), doi: [10.1112/plms/s3-15.1.699](#). 618
- [21] MARGALIT, DAN; WINARSKI, REBECCA R. Braids groups and mapping class groups: the Birman-Hilden theory. *Bull. Lond. Math. Soc.* **53** (2021), no. 3, 643–659. [MR4275077](#), [Zbl 1470.57045](#), doi: [10.1112/blms.12456](#). 619
- [22] NIELSEN, JAKOB. Abbildungsklassen endlicher Ordnung. *Acta Math.* **75** (1943), 23–115. [MR0013306](#), [Zbl 0027.26601](#), doi: [10.1007/BF02404101](#). 618, 621, 623
- [23] PARSAD, SHIV; RAJEEVSARATHY, KASHYAP; SANKI, BIDYUT. Geometric realizations of cyclic actions on surfaces. *J. Topol. Anal.* **11** (2019), no. 4, 929–964. [MR4040017](#), [Zbl 1432.57047](#), doi: [10.1142/s1793525319500365](#). 618, 620, 623, 624
- [24] RAJEEVSARATHY, KASHYAP. Fractional powers of Dehn twists about nonseparating curves. *Glasg. Math. J.* **56** (2014), no. 1, 197–210. [MR3137860](#), [Zbl 1286.57017](#), doi: [10.1017/S0017089513000177](#). 620, 642
- [25] RAJEEVSARATHY, KASHYAP; VAIDYANATHAN, PRAHLAD. Roots of Dehn twists about multicurves. *Glasg. Math. J.* **60** (2018), no. 3, 555–583. [MR3830536](#), [Zbl 1409.57022](#), doi: [10.1017/S0017089517000283](#). 620, 622, 640, 641, 642
- [26] THURSTON, WILLIAM P. Geometry and topology of three-manifolds. Notes available at: <http://library.msri.org/books/gt3m/>. 619, 621
- [27] TUCKER, THOMAS W. Finite groups acting on surfaces and the genus of a group. *J. Combin. Theory Ser. B.* **34** (1983), no. 1, 82–98. [MR701174](#), [Zbl 0521.05027](#), doi: [10.1016/0095-8956\(83\)90009-6](#). 624

(N. K. Dhanwani) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI, KNOWLEDGE CITY, SECTOR 81, MANAULI, PO, SAHIBZADA AJIT SINGH NAGAR, MOHALI 140306, PUNJAB, INDIA  
[neerajk.dhanwani@gmail.com](mailto:neerajk.dhanwani@gmail.com)

(K. Rajeevsarathy) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL, BHOPAL BYPASS ROAD, BHAURI, BHOPAL 462 066, MADHYA PRADESH, INDIA  
[kashyap@iiserb.ac.in](mailto:kashyap@iiserb.ac.in)

(A. Sanghi) DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL, BHOPAL BYPASS ROAD, BHAURI, BHOPAL 462 066, MADHYA PRADESH, INDIA  
[apeksha16@iiserb.ac.in](mailto:apeksha16@iiserb.ac.in)

This paper is available via <http://nyjm.albany.edu/j/2022/28-25.html>.