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A volumish theorem for alternating virtual links

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ABSTRACT. Dasbach and Lin proved a "volumish theorem" for alternating links. We prove the analogue for alternating link diagrams on surfaces, which provides bounds on the hyperbolic volume of a link in a thickened surface in terms of coefficients of its reduced Jones-Krushkal polynomial. Along the way, we show that certain coefficients of the 4–variable Krushkal polynomial express the cycle rank of the reduced Tait graph on the surface.

CONTENTS

1. Introduction

In [8], Dasbach and Lin proved the following "volumish" theorem for any hyperbolic alternating knot K in S^3: Let

$$V_K(t) = a_n t^n + \cdots + a_m t^m$$

be the Jones polynomial of K, with sub-extremal coefficients a_{n+1} and a_{m-1}. Let $v_{\text{tet}} \approx 1.01494$ and $v_{\text{oct}} \approx 3.66386$ be the hyperbolic volumes of the regular ideal tetrahedron and octahedron, respectively. Then

$$v_{\text{oct}}(\max(|a_{n+1}|, |a_{m-1}|) - 1) \leq \text{vol}(S^3 - K) \leq 10 v_{\text{tet}}(|a_{n+1}| + |a_{m-1}| - 1).$$

Their proof relied on volume bounds proved in [13, 1], which showed that the hyperbolic volume of $S^3 - K$ is linearly bounded above and below by the twist number $t(K)$. Dasbach and Lin proved that for any reduced alternating diagram of K, the twist number $t(K) = |a_{n+1}| + |a_{m-1}|$.

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Recently in [9, 10], similar linear volume bounds in terms of twist number were proved for certain alternating links in thickened surfaces, but the twist number was not proved to be a link invariant. For alternating links in S^3, the invariance of $t(K)$ follows from the proof of the Tait flyping conjecture in [14], but the Tait flyping conjecture remains open for alternating virtual links (see [4]).

Let F be a closed orientable surface of positive genus. Let K be a link in the thickened surface $F \times I$, which admits a reduced alternating surface link diagram D on F. In Section 3 below, we define a *homological twist number* $\tau_F(D)$. In Section 4, we give a sufficient condition for $\tau_F(D)$ to be a link invariant of K in $F \times I$ by expressing $\tau_F(D)$ in terms of specific coefficients of the reduced Jones-Krushkal polynomial of K. Using the new volume bounds in terms of twist number, we prove a "volumish" theorem for alternating links on surfaces, which extends to virtual links.

There is an underlying similarity between the proofs of the two volumish theorems. For alternating links in S^3, to prove that the twist number is expressed by the sub-extremal coefficients of the Jones polynomial, Dasbach and Lin relied on two key facts: (1) the Jones polynomial of an alternating link is a specialization of the two-variable Tutte polynomial of its Tait graph, and (2) certain coefficients of the Tutte polynomial express the cycle rank of the reduced Tait graph. For alternating links in thickened surfaces, we rely on two similar facts: (1) the reduced Jones-Krushkal polynomial is a specialization of the Krushkal polynomial, which extends the Tutte polynomial to a 4–variable polynomial invariant of graphs on surfaces, and (2) certain coefficients of the Krushkal polynomial express the cycle rank of the reduced Tait graph on the surface (see Definition 3.1). The latter claim for the Krushkal polynomial is Theorem 2.3, which is of independent interest, and is proved in Section 2 below.

Let $J_K(t,z)$ denote the reduced Jones-Krushkal polynomial, defined in Section 4 below. Boden and Karimi [4] proved that $J_K(t,z)$ is an invariant of oriented links under isotopy and diffeomorphism of the thickened surface. In Theorem 4.3, we express the homological twist number in terms of specific coefficients of $J_K(t,z)$. This provides linear bounds on the hyperbolic volume of the link K in the thickened surface in terms of the sub-extremal terms of $J_K(t,0)$ using the following geometric results.

A surface link diagram D on F is *weakly prime* if any embedded disc on F whose boundary intersects D exactly twice contains no crossings of D. The diagram D is *cellularly embedded* if the regions $F - D$ are disks. A crossing c is called *nugatory* if there exists a separating simple closed curve on F that intersects D only at c. A surface link diagram D is called *reduced* if it is cellularly embedded and has no nugatory crossings. Note that every reduced alternating diagram D on F is checkerboard-colorable. Additionally, D is *strongly reduced* if there do not exist any simple closed curves on F that intersect D at only one crossing; i.e., neither Tait graph of D on F has loops. The diagram D on F

is called *weakly generalized alternating* (WGA) if D is both weakly prime and reduced alternating; note that D must be cellularly embedded but it may not be strongly reduced. See [10, Section 2] for a more general definition of WGA diagrams.

For a link K in $F \times I$ with a WGA diagram D on F, Howie and Purcell [9] defined the twist number $t_F(D)$ on the projection surface $F \times \{0\}$, and showed there is a lower bound on volume in terms of the twist number. Kalfagianni and Purcell [10] proved there is also an upper bound on volume. Note that if F is a torus, then $F \times I - K$ has a unique hyperbolic structure; for $g \geq 2$, we consider the unique hyperbolic structure for which the boundary surfaces $F \times \{\pm 1\}$ are totally geodesic.

We now combine the hyperbolicity and lower bound from [9], the upper bound from [10] modified for the homological twist number, and our Theorem 4.3 below to state the volumish theorem for alternating virtual links:

Theorem 1.1. *For a closed orientable surface F of genus $g \geq 1$, let K be a non-split oriented link in $F \times I$ that admits a strongly reduced WGA diagram D on $F \times \{0\}$. Let $\tau_F(K)$ be the homological twist number of D. Let $J_K(t, 0) = a_n t^n + \cdots + a_m t^m$, with sub-extremal coefficients a_{n+1} and a_{m-1}. Then*

$$\tau_F(K) = |a_{n+1}| + |a_{m-1}| - 2g,$$

$\tau_F(K)$ is an invariant of K in $F \times I$, and $F \times I - K$ is hyperbolic with

$$\frac{v_{\mathrm{oct}}}{2}\tau_F(K) \leq \mathrm{vol}(F \times I - K) < 10 v_{\mathrm{tet}}\, \tau_F(K) \quad \text{if } g = 1,$$
$$\frac{v_{\mathrm{oct}}}{2}(\tau_F(K) - 3\chi(F)) \leq \mathrm{vol}(F \times I - K) < 12 v_{\mathrm{oct}}\, \tau_F(K) \quad \text{if } g \geq 2.$$

We prove Theorem 1.1 in Section 4 below. The strongly reduced condition on D can be weakened to allow certain loops in the Tait graph if we use the expression for $\tau_F(D)$ in Theorem 4.3. See Corollary 4.4 for cases with loops such that $\tau_F(D)$ is a link invariant.

Virtual links. Virtual links and links in thickened surfaces are compared in detail in [4]. Kuperberg [12] proved that virtual links are in one-to-one correspondence with stable equivalence classes of links in thickened surfaces, and each such class has a unique irreducible representative. For any virtual link diagram, there is an explicit construction to associate a cellularly embedded link diagram on a minimal genus surface. Moreover, a virtual link is alternating if and only if it can be represented by an alternating surface link diagram. Any reduced alternating surface link diagram is checkerboard colorable, but alternating virtual links also admit alternating surface diagrams which are not checkerboard colorable. The main result of [4] is the following diagrammatic characterization of alternating links in thickened surfaces: If K is a non-split alternating link in $F \times I$, then any connected reduced alternating diagram D on F has minimal crossing number $c(K)$, and any two reduced alternating diagrams of K have the same writhe $w(K)$.

The uniqueness statement in Kuperberg's theorem implies that any two minimal-genus diagrams of a virtual link are isotopic on F. Since the reduced Jones-Krushal polynomial is invariant under isotopy of surface link diagrams, we obtain an invariant of virtual links by computing $J_K(t,z)$ on a minimal genus link diagram. By [5, Corollary 8], if a non-split link K in $F\times I$ is represented by a reduced alternating diagram D on F, then it is the minimal genus representative of K. The genus of F is encoded as the highest power of z in $J_K(t,z)$. For an appropriate alternating surface link diagram, Corollary 4.4 below implies that the homological twist number of D on F is also an invariant of the virtual link. Thus, Theorem 1.1 extends to any alternating virtual link that admits an appropriate alternating surface link diagram.

Related results. Recently, several preprints have appeared with related results.

In [6], Boden, Karimi and Sikora prove the analogues of the Tait conjectures for adequate links in thickened surfaces. Any alternating link diagram in a thickened surface is adequate, so a natural question is how to extend Theorem 4.3 to adequate links in thickened surfaces.

In [2], a general equivalence is established between ribbon graphs and virtual links. As our main results rely on the Krushkal polynomial, which is an invariant of ribbon graphs, this philosophy underlies our results as well.

In [3], Bavier and Kalfagianni prove results similar to Theorem 1.1 without using polynomial invariants of ribbon graphs. Note that in [3], *reduced* is the same as *strongly reduced* here. Their proof relies on the *guts* of a 3–manifold cut along an essential surface, which is the union of all hyperbolic pieces in its JSJ-decomposition, and the Euler characteristic of the guts is related to the twist number using results in [6]. Significantly, to prove that the twist number is invariant, Bavier and Kalfagianni used another part of the Kauffman bracket skein module $\mathcal{S}(F\times I)$, which has a basis of all multi-loops on F, including $\varnothing$. Let $J_0(K)=b_nt^n+\cdots+b_mt^m$ be the normalized invariant of K in $F\times I$ coming from the coefficient in $\mathbb{Z}[A^{\pm 1}]$ of $\varnothing$, so just the contractible states on F. They proved $t_F(K)=|b_{n+1}|+|b_{m-1}|-2g$. In contrast, the Jones-Krushkal polynomial $J_K(t,0)$ uses states on F that are null-homologous, including non-contractible states on F. Thus, $J_K(t,0)\neq J_0(K)$ if $g\geq 2$, and in Proposition 3.3 below, we show that $\tau_F(K)\neq t_F(K)$ if $g\geq 2$. For links in thickened surfaces, we prove invariance of the homological twist number in Corollary 4.4 for more general alternating link diagrams than just strongly reduced ones because loops in Tait graphs are allowed, as long as there are no genus-generating loops.

In [16], Will proves results similar to Theorem 1.1 using inequalities for the twist number obtained from a new 3–variable polynomial that extends Krushkal's homological Kauffman bracket polynomial, which is discussed in Section 4 below.

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2. The Krushkal polynomial

Krushkal [11] introduced a 4–variable polynomial invariant of a graph G embedded in a closed orientable surface F. We will denote this polyomial by $p_G(x, y, u, v)$ and refer to it as the *Krushkal polynomial*. The variables x and y play the same role as in the Tutte polynomial, while u and v reflect how G is embedded on F. If G is cellularly embedded (i.e., the faces of G on F are disks), and G^* denotes the dual graph on F, then the Krushkal polynomial generalizes the Tutte polynomial, satisfying both of its key properties: contraction-deletion and a duality relation, $p_G(x, y, u, v) = p_{G^*}(y, x, v, u)$.

The Krushkal polynomial is defined as the following sum over spanning subgraphs, such that every subgraph contributes a monomial weight $x^a y^b u^c v^d$, where the exponents are topological quantities related to the embedding of this subgraph.

Definition 2.1 ([11]). Let G be a graph cellularly embedded in a closed orientable surface F. The genus of a subsurface $S \subset F$ is the genus of the closed surface obtained from S by capping off all the boundary components of S by disks. For a spanning subgraph H of G, let $\mathcal{H}$ denote the regular neighborhood of H on F. Let $i: \ G \to F$ denote the embedding, and let $i: \ H \to F$ denote its restriction to H. Define:

$$\begin{aligned} c(H) &= \text{number of components of } H, \\ s(H) &= \text{twice the genus of } \mathcal{H}, \\ s^{\perp}(H) &= \text{twice the genus of the subsurface } F - \mathcal{H}, \\ k(H) &= \dim(\ker(i_* : \ H_1(H; \mathbb{R}) \to H_1(F; \mathbb{R}))). \end{aligned}$$

The Krushkal polynomial is defined as the following sum over all spanning subgraphs $H \subset G$:

$$p_G(x, y, u, v) = \sum_{H \subset G} x^{c(H)-c(G)} y^{k(H)} u^{s(H)/2} v^{s^{\perp}(H)/2}. \tag{1}$$

We will refer to the monomial terms in (1) as *weights* on corresponding subgraphs of G.

The Tutte polynomial $T_G(X, Y)$ is related to the Whitney rank generating function $R_G(x, y)$ by $T_G(X, Y) = R_G(X - 1, Y - 1)$ (see [15, § 15.4]), which are extensively studied polynomial invariants of graphs and matroids. If g denotes the genus of F, by [11, Lemma 2.3],

$$R_G(x, y) = y^g p_G(x, y, y, y^{-1}), \text{ and } T_G(X, Y) = R_G(X - 1, Y - 1) \tag{2}$$

The substitution $x = X - 1$ and $y = Y - 1$ will play a key role in the proof of Theorem 2.3. So we define

$$P_G(X, Y, U, V) = p_G(X - 1, Y - 1, U, V).$$

Another specialization to obtain the Jones-Krushkal polynomial is discussed in Section 4.

Definition 2.2. Two edges in G are *parallel* if they are homologous on F. Note that parallel non-loop edges connect the same vertices, but parallel loops may be disjoint. Let G' denote the *reduced graph* of G obtained by deleting all but one edge in each set of parallel edges in G, and deleting all homologically trivial loops, such that the vertex set $V(G') = V(G)$. Note that in the definition of G', edges include loops, so that the interiors of all but one loop in each set of parallel loops are deleted while preserving all vertices of G. Let $G = (V, E)$ and $G' = (V, E')$. Let $\ell = \ell(G')$ denote the subgraph of loops in G'. Let $G' - \ell = (V, E' - \ell)$ denote the graph obtained by removing the interior of each loop in ℓ. Let

$$\mu = b_1(G' - \ell) = |E' - \ell| - |V| + c(G') \quad \text{and} \quad \lambda = b_1(\ell) = |\ell|.$$

Note that although G' is not uniquely determined, μ and λ are invariants of G.

Theorem 2.3. *Let G be a graph embedded in a surface F of genus $g \geq 1$. Let ℓ_0 be the set of homologically trivial loops in G. Let $k = |\ell_0|$ and $n = |V(G)| - c(G)$. Then $P_G(X, Y, U, V)$ includes terms with exactly the following coefficients:*

$$\mu V^g X^{n-1} Y^k + \lambda V^{g-1} X^n Y^k.$$

Proof. By [11, Lemma 2.2], $p_G(x, y, u, v)$ has the property that if e is a loop in G which is trivial in $H_1(F)$, then $p_G = (1 + y)p_{G-e}$, so that $P_G = YP_{G-e}$. Thus, we only need to prove the case $|\ell_0| = 0$, so we will consider only loops in G that are non-trivial in $H_1(F)$.

The unique spanning subgraph H_0 of G which consists of only vertices and no edges has weight $v^g x^n$. Since any other subgraph has a non-empty edge set, its weight has a lower exponent of x (if it has non-loop edges), or a lower exponent of v (if it has homologically non-trivial loops). Thus, the term $v^g x^n$ occurs in $p_G(x, y, u, v)$ with coefficient 1.

Let e' be a non-loop edge of G', and let $\{e_1, \ldots, e_m\}$ be the set of all edges of G parallel to e', which we call the edge class of e'. For $1 \leq j \leq m$, let H_j denote one of the spanning subgraphs of G which consists of j edges from the edge class of e', and no other edges. The weight of each H_j is $v^g x^{n-1} y^{j-1}$. Summing over the weights of all such spanning subgraphs $\{H_j \subset G\}$, we get the following contribution to $p_G(x, y, u, v)$:

$$\sum_{j=1}^{m} \binom{m}{j} v^g x^{n-1} y^{j-1} = \frac{v^g x^{n-1}}{y} \left(\sum_{j=1}^{m} \binom{m}{j} y^j \right) = \frac{v^g x^{n-1}}{y} ((1 + y)^m - 1). \tag{3}$$

Thus, for every non-loop edge e' in G', its edge class in G contributes the expression (3) to $p_G(x, y, u, v)$.

If H is a spanning subgraph of G with the factor x^{n-1} in its weight, then $c(H) = |V| - 1$. Hence, H has the form of some H_j, possibly with loops added.

If H has any loops, then since the loops are homologically non-trivial by assumption, the weight of H has an exponent of v which is strictly less than g. Thus, any term in $p_G(x,y,u,v)$ with a $v^g x^{n-1}$ factor is contributed only by the subgraphs H_j, so the term must be $v^g x^{n-1} y^{j-1}$ for $j \geq 1$.

Let's see how these terms transform in $P_G(X,Y,U,V)$. With the substitution $x = X-1$ and $y = Y-1$, the expression (3) simplifies to

$$\frac{V^g(X-1)^{n-1}}{Y-1}(Y^m - 1) = V^g X^{n-1}(1 + Y^2 + \dots + Y^{m-1}) + O(X^{n-2}).$$

Every non-loop edge in G' contributes such an expression to $P_G(X,Y,U,V)$. Moreover, as discussed above, the weight for H_0 is $v^g x^n$, which then becomes $V^g(X-1)^n$. Since $v^g x^n$ always has coefficient 1 in p_G, H_0 contributes an additional coefficient $-n$ to the term $V^g X^{n-1}$ in P_G. Finally,

$$n = |V(G)| - c(G) = |V(G')| - c(G').$$

Therefore, if $|\ell_0| = 0$, the coefficient on $V^g X^{n-1}$ in $P_G(X,Y,U,V)$ is

$$|E' - \ell| - n = |E' - \ell| - |V(G')| + c(G') = b_1(G' - \ell) = \mu.$$

This proves the claim for μ.

We now proceed similarly for loops in G'. Let f' be a loop of G', and let $\{f_1, \dots, f_m\}$ be the set of all loops of G parallel to f', which we call the edge class of f'. For $1 \leq j \leq m$, let L_j denote one of the spanning subgraphs of G which consists of j loops from the edge class of f', and no other edges. Since we assumed that all loops in G are homologically non-trivial, the weight of L_j is $v^{g-1} x^n y^{j-1}$. By summing over the weights of all such spanning subgraphs $\{L_j \subset G\}$, we get the following contribution to $p_G(x,y,u,v)$:

$$\sum_{j=1}^{m} \binom{m}{j} v^{g-1} x^n y^{j-1} = \frac{v^{g-1}x^n}{y}\left(\sum_{j=1}^{m}\binom{m}{j} y^j\right) = \frac{v^{g-1}x^n}{y}((1+y)^m - 1). \quad (4)$$

Thus, for every loop f' in G', its edge class in G contributes the expression (4) to $p_G(x,y,u,v)$.

If H is a spanning subgraph of G with the factor x^n in its weight, then $c(H) = |V|$. Hence, H consists of only homologically non-trivial loops. We have three cases:

(a) All loops in H are in one edge class of G',
(b) H has loops in distinct edge classes of G', and $g(\mathcal{H}) = 0$,
(c) H has loops in distinct edge classes of G', and $g(\mathcal{H}) > 0$.

In case (a), H is one of the subgraphs L_j. In case (b), H has at least one pair of homologically non-trivial and non-homologous loops, so $g(\mathcal{H}) = 0$ implies that $F - \mathcal{H}$ has genus strictly less than $g-1$. Hence, the weight of H has an exponent of v which is strictly less than $g-1$. In case (c), the weight of H has a factor u^i with $i > 0$. Therefore, any term in $p_G(x,y,u,v)$ with a $v^{g-1}x^n$ factor and without a u factor is contributed only by the subgraphs L_j, so the term must be $v^{g-1}x^n y^{j-1}$ for $j \geq 1$.

With the substitution $x = X - 1$ and $y = Y - 1$, the expression (4) simplifies to

$$\frac{V^{g-1}(X-1)^n}{Y-1}(Y^m - 1) = V^{g-1}X^n(1 + Y^2 + ... + Y^{m-1}) + O(X^{n-1}). \quad (5)$$

Every loop in G' contributes such an expression to $P_G(X, Y, U, V)$, so if $|\ell_0| = 0$, the coefficient on $V^{g-1}X^n$ in $P_G(X, Y, U, V)$ is λ. This completes the proof of the theorem. □

Below, we will need another coefficient of $P_G(X, Y, U, V)$, using the following definition.

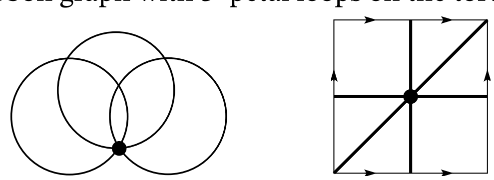
Definition 2.4. For a graph G on the surface F, let $\ell(G)$ be the subgraph of loops in G. We call $\{e_1, e_2\} \subset \ell(G)$ *genus-generating loops* if $g(\mathcal{H}(e_1 \cup e_2)) > 0$. Let G' be the reduced graph of G. Define

$$\gamma(G) = \#\{\{e_1, e_2\} \subset \ell(G') \mid g(\mathcal{H}(e_1 \cup e_2)) > 0\}.$$

We will say that $\{e_1, e_2, e_3\} \subset \ell(G)$ are 3–*petal loops* if no pair of loops is parallel and

$$g(\mathcal{H}(e_1 \cup e_2 \cup e_3)) > 0 \quad \text{and} \quad k(e_1 \cup e_2 \cup e_3) > 0.$$

Note that if $\gamma(G) = 0$, then G has no 3–petal loops. The following figure shows an example of a ribbon graph with 3–petal loops on the torus:



Lemma 2.5. *Let G be a graph embedded in a surface F of genus g, such that G has no* 3*–petal loops. Let $k = |\ell_0|$, $n = |V(G)| - c(G)$, and $\gamma = \gamma(G)$. Then $P_G(X, Y, U, V)$ includes a term with exactly the following coefficient:*

$$\gamma\, UV^{g-1}X^nY^k.$$

Proof. As in the proof above, it suffices to prove the case $k = 0$, so we can assume that all loops in G are homologically non-trivial. We now determine all possible $H \subset G$ that can contribute to the term $UV^{g-1}X^n$ in $P_G(X, Y, U, V)$. Due to the substitution $x = X - 1$ and $y = Y - 1$, we need to consider $H \subset G$ with weight $uv^{g-1}x^iy^j$. Since $i \leq n$, the factor X^n implies that H can contribute to the term $UV^{g-1}X^n$ only if $i = n$. Hence, $c(H) = |V(G)|$ so that $H \subset \ell(G)$ with weight $uv^{g-1}x^ny^j$.

Let $H' \subset G'$ be the reduced graph of H, as in Definition 2.2. Let $\mathcal{H}'$ be the regular neighborhood of H' in F. The condition that G has no 3–petal loops implies that G' and hence H' have no 3–petal loops. By [11, Equation (4.7)],

$$k(H') + g(F) + g(\mathcal{H}') - g(F - \mathcal{H}') = b_1(H').$$

The factor UV^{g-1} implies that $g(\mathcal{H}') = 1$ and $g(F - \mathcal{H}') = g(F) - 1$. Thus, $k(H') = b_1(H') - 2 = |E(H')| - 2$. Since $g(\mathcal{H}') = 1$, the condition that H'

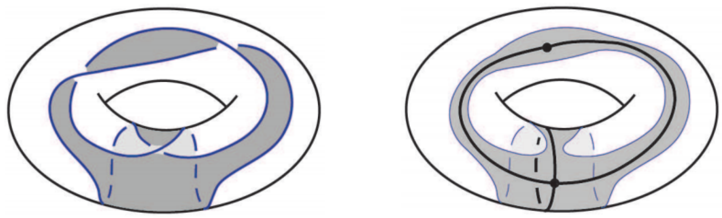


FIGURE 1. An alternating link diagram (left) and its Tait graph G_A (right) on the torus [11, Figure 5].

has no 3–petal loops now implies $k(H') = 0$, so that $|E(H')| = 2$. So the only possible $H' \subset G'$ are the subgraphs $\{e_1 \cup e_2\} \subset \ell(G')$ such that $g(\mathcal{H}') = 1$. Therefore, if $H \subset G$ contributes to the term $UV^{g-1}X^n$ in $P_G(X, Y, U, V)$, then H' is a pair of genus-generating loops.

Let $\{e'_1 \cup e'_2\} \subset \ell(G')$ be a pair of genus-generating loops, and suppose for $I = 1, 2$, G has m_I parallel loops in the edge class e'_I. Let $H_{i,j} \subset G$ denote the subgraph with i loops (resp. j loops) in the edge class e'_1 (resp. e'_2), which has weight $uv^{g-1}x^n y^{(i-1)+(j-1)}$. As in (4), summing over the weights of all $H_{i,j} \subset G$, we get the following contribution to $p_G(x, y, u, v)$:

$$\sum_{\substack{1 \le i \le m_1 \\ 1 \le j \le m_2}} \binom{m_1}{i}\binom{m_2}{j} uv^{g-1}x^n y^{(i-1)+(j-1)} = \frac{uv^{g-1}x^n}{y^2}((1+y)^{m_1} - 1)((1+y)^{m_2} - 1). \quad (6)$$

Thus, for every pair of genus-generating loops in G', its edge class in G contributes the expression (6) to $p_G(x, y, u, v)$. As in (5), with the substitution $x = X - 1$ and $y = Y - 1$, the expression (6) simplifies to

$$UV^{g-1}X^n(1 + Y^2 + ... + Y^{m_1-1})(1 + Y^2 + ... + Y^{m_2-1}) + O(X^{n-1}).$$

Every pair of genus-generating loops in G' contributes such an expression to $P_G(X, Y, U, V)$, so if $k = |\ell_0| = 0$, the coefficient on $UV^{g-1}X^n$ in $P_G(X, Y, U, V)$ is $\gamma(G)$. □

3. The homological twist number

In this section, we introduce the homological twist number $\tau_F(D)$, which counts sets of homologically twist-equivalent crossings. In contrast, the usual twist number $t_F(D)$, defined in [10, Definition 2.4], counts *twist regions* (maximal strings of bigons) of D on F. Every twist region contributes one homological twist to $\tau_F(D)$, but some crossings of D which are in distinct twist regions can be homologically twist-equivalent. An important advantage of Definition 3.2 below is that $\tau_F(D)$ is invariant for any reduced alternating surface link diagram D, without the need for D to be twist-reduced.

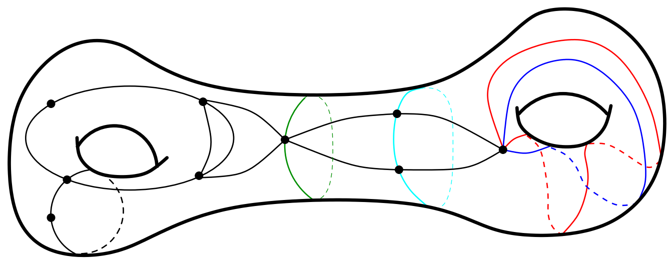


FIGURE 2. Different kinds of cycles in the Tait graph are shown in different colors. From left to right: nugatory crossing (green), null-homologous 2-cycle (cyan), genus-generating loops (blue and red).

Definition 3.1. Let D be a reduced alternating surface link diagram on F. Fix a checkerboard coloring on D. Let G_A (resp. G_B) be the Tait graph (i.e., checkerboard graph) of D on F, whose edges correspond to crossings of D, and whose vertices correspond to shaded (resp. unshaded) regions of $F - D$, such that G_A and G_B are dual graphs on F. See Figure 1. Note that the Tait graph of a reduced alternating surface link diagram may contain loops, but only homologically non-trivial ones. Let G'_A and G'_B be the *reduced Tait graphs* obtained by deleting all but one edge in each set of parallel edges in G_A and G_B, as in Definition 2.2.

See Figure 2 for several examples of different kinds of cycles in the Tait graph on the surface F.

Definition 3.2. Recall, two edges in G are parallel if they are homologous on F. Two crossings of D are *homologically twist-equivalent* if their corresponding edges are parallel in either G_A or G_B. The *homological twist number* $\tau_F(D)$ is defined as the number of homological twist-equivalence classes of crossings of D. Thus, each homological twist corresponds to one set of parallel edges in G_A or G_B, which is one edge in G'_A or G'_B.

See Figure 3 for two examples of homologically twist-equivalent crossings of D on F, which do not form a twist region on F. Figure 3 (a) also provides an example of a reduced alternating link diagram which is not strongly reduced.

Proposition 3.3. *If $t_F(D)$ denotes the twist number, as in* [10, Definition 2.4]*, of a strongly reduced, twist-reduced WGA diagram, then*

$$\tau_F(D) \leq t_F(D) \leq 2\tau_F(D).$$

Moreover, if F is a torus, then $\tau_F(D) = t_F(D)$.

Proof. Let G_A and G_B be the Tait graphs of D on F, which do not contain loops since D is strongly reduced. A pair of edges in G_A or G_B is parallel if and only if they form a null-homologous 2–cycle. If it bounds a disk Δ on F, then the hypothesis that D is twist-reduced, as in [10, Definition 2.5], implies that Δ or

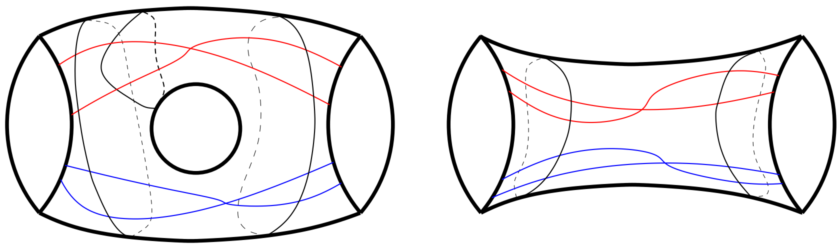


FIGURE 3. In both (a) and (b), an alternating link diagram is projected on the surface F, shown on a part of F which disconnects the surface. In both cases, the red crossing and the blue crossing are homologically twist-equivalent. One of the Tait graphs in (a) has homologous loops, and in (b) has a null-homologous 2–cycle. Neither pair of crossings forms a twist region on F.

a disk in $F - \Delta$ contains a twist region of D, which is the same as a homological twist-equivalence class of crossings of D. Thus, the two definitions of twist number agree in this case.

On the other hand, suppose the null-homologous 2–cycle bounds a subsurface $F' \subset F$ which is not a disk, so it forms an essential separating curve on F. Hyperbolicity precludes both vertices from being 2–valent, but if one vertex is 2–valent, then D has a bigon on F and the two crossings are homologically twist-equivalent. So the two definitions of twist number agree in this case as well.

However, if neither vertex is 2–valent, then the two crossings are homologically twist-equivalent, but are not part of a twist region because D is twist-reduced. Moreover, this discrepancy occurs for every essential null-homologous 2–cycle without 2–valent vertices in G_A or G_B. This proves the inequality.

Finally, if F is a torus, neither G_A nor G_B admits an essential null-homologous 2–cycle, so $\tau_F(D) = t_F(D)$. □

4. The Jones-Krushkal polynomial

In [11], Krushkal defined a homological Kauffman bracket derived from his 4-variable polynomial $p_G(x, y, u, v)$, and proved the invariance of a two-variable generalization of the Jones polynomial for links in thickened surfaces. We will use a later variant $J_K(t, z)$, called the reduced Jones-Krushkal polynomial, which was introduced by Boden and Karimi [4]. Following [11], it is proved in [4] that $J_K(t, z)$ is an invariant of oriented links under isotopy and diffeomorphism of the thickened surface.

We briefly recall the homological Kauffman bracket due to Krushkal [11]. Let F be a closed orientable surface of genus g. Let K be a link in $F \times I$, with a link diagram D on F. Suppose that D has c crossings, each of which can be

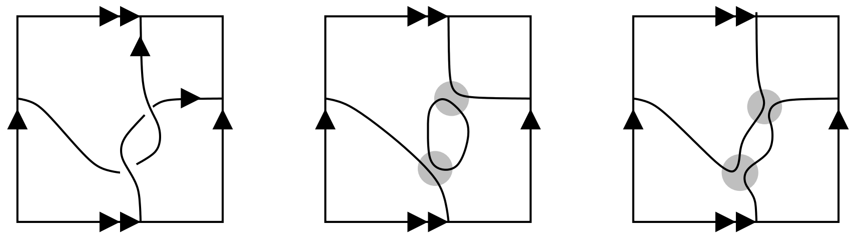


FIGURE 4. For D on the torus (left), states s_A (middle) and s_B (right) are shown. Here, $|s_A| = 2$, $|s_B| = 1$, $r(s_A) = r(s_B) = 1$, $k(s_A) = 1$, $k(s_B) = 0$.

resolved by an A–smoothing or B–smoothing. A *state* s of D is a collection of simple closed curves on F that results from smoothing each crossing of D. See Figure 4. Let $a(s)$ and $b(s)$ be the number of A and B–smoothings, and let $|s|$ be the number of closed curves in s. Let s_A and s_B denote the all–A and all–B states of D, so that for the Tait graphs G_A and G_B, we have $|V(G_A)| = |s_A|$ and $|V(G_B)| = |s_B|$. Let $n = |V(G_A)| - 1$ and $N = |V(G_B)| - 1$. Define

$$k(s) = \dim(\mathrm{kernel}(i_* : \ H_1(s) \to H_1(F))),$$
$$r(s) = \dim(\mathrm{image}(i_* : \ H_1(s) \to H_1(F))),$$

where $i : \ s \to F$ is the inclusion map. We call $r(s)$ the *homological rank* of s, so that $k(s) + r(s) = |s|$. The homological Kauffman bracket is defined as follows:

$$\langle D \rangle_F = \sum_s A^{(a(s)-b(s))}(-A^{-2} - A^2)^{k(s)} z^{r(s)}.$$

To recover the usual Kauffman bracket for a virtual link diagram D, we set $z = -A^{-2} - A^2$ and divide by one factor of $-A^{-2} - A^2$ (see Example 2 below). To obtain the Jones-Krushkal polynomial, which was the original link invariant defined in [11], we normalize by the writhe as usual, $(-A)^{-3w(D)}\langle D \rangle_F$, and set $A = t^{-1/4}$.

If D is checkerboard colorable, then $[K] = 0$ in $H_1(F \times I)$ by [4], so it follows that $k(s) \geq 1$ for every state s of D. So we can instead use the following version of the Jones-Krushkal polynomial due to Boden and Karimi:

Definition 4.1 ([4])**.** Suppose K is an oriented link in $F \times I$, represented by a checkerboard-colorable link diagram D on F. The *reduced Jones-Krushkal polynomial* is defined by

$$J_K(t, z) = (-1)^{w(D)} t^{3w(D)/4} \sum_s t^{(b(s)-a(s))/4} (-t^{-1/2} - t^{1/2})^{(k(s)-1)} z^{r(s)}.$$

The reduced Jones-Krushkal polynomial specializes to the usual Jones polynomial $V_K(t)$ by setting $z = -t^{-1/2} - t^{1/2}$. Any classical diagram will have $r(s) = 0$ for all states, so that $J_K(t, z) = V_K(t)$ for every classical link K. However, there exist alternating virtual knots with $V_K(t) = 1$ but non-trivial $J_K(t, z)$.

By [11, Theorem 6.1] for non-split D, we obtain $\langle D\rangle_F$ from $P_{G_A}(X,Y,U,V)$ as follows:

$$\langle D\rangle_F(A,z) = A^{(2g+2n-c)} d\, z^g P_{G_A}\left(-A^{-4}, -A^4, \frac{A^2}{z}, \frac{1}{A^2 z}\right).$$

With the additional normalization as in Definition 4.1, we obtain $J_K(t,z)$ by

$$J_K(t,z) = (-1)^w t^{(3w-2g-2n+c)/4} z^g P_{G_A}\left(-t, -t^{-1}, \frac{1}{z\sqrt{t}}, \frac{\sqrt{t}}{z}\right). \tag{7}$$

Recall the definition of *genus-generating loops* and 3–*petal loops* from Definition 2.4.

Definition 4.2. For a reduced alternating diagram D on F, let $\ell(G'_A)$ and $\ell(G'_B)$ be the subgraphs of loops in the reduced Tait graphs G'_A and G'_B. Define

$$\gamma(D) = \#\{\{e_1,e_2\} \subset \ell(G'_A) \mid g(\mathcal{H}(e_1 \cup e_2)) > 0\},$$
$$\bar{\gamma}(D) = \#\{\{e_1,e_2\} \subset \ell(G'_B) \mid g(\mathcal{H}(e_1 \cup e_2)) > 0\}.$$

Theorem 4.3. *For a closed orientable surface F of genus $g \geq 0$, let K be a non-split oriented link in $F \times I$ that admits a reduced alternating diagram D on F, such that neither of its Tait graphs has 3–petal loops. Let*

$$\lambda = |\ell(G'_A)|,\ \bar{\lambda} = |\ell(G'_B)|,\ \mu = b_1(G'_A - \ell(G'_A)),\ \bar{\mu} = b_1(G'_B - \ell(G'_B)),$$

$\gamma = \gamma(D)$, $\bar{\gamma} = \bar{\gamma}(D)$. *Then*

$$\tau_F(D) = b_1(G'_A) + b_1(G'_B) - 2g = \lambda + \mu + \bar{\lambda} + \bar{\mu} - 2g \tag{8}$$

and the reduced Jones-Krushkal polynomial $J_K(t,z)$ has the following coefficients:

$$(-1)^{(w+n)} t^{\frac{3w+2n+c}{4}} \left((-1)^c t^{(g-c)} \left(\bar{\lambda} z t^{\frac{1}{2}} - (\bar{\mu} - \bar{\gamma}) t\right) - (\mu - \gamma) t^{-1} + \lambda z t^{-\frac{1}{2}}\right), \tag{9}$$

where c and w are the crossing number and writhe of D, and $n = |V(G_A)| - 1$.

We prove Theorem 4.3 after the following corollary, which is important for Theorem 1.1. Recall that D is strongly reduced when neither G_A nor G_B has loops, so in particular, $\gamma(D) = \bar{\gamma}(D) = 0$. In addition, $\gamma(D) = \bar{\gamma}(D) = 0$ implies that neither Tait graph of D has 3–petal loops.

Corollary 4.4. *If D is a reduced alternating diagram on F, such that $\gamma(D) = \bar{\gamma}(D) = 0$, then $\tau_F(D)$ is a link invariant of K in $F \times I$.*

Proof. For $g(F) = 0$, the twist number is a link invariant by the proof of the Tait flyping conjecture in [14], so we may assume $g(F) \geq 1$. By [4], $J_K(t,z)$ is an invariant of K in $F \times I$. Thus, by Theorem 4.3, $\tau_F(D)$ is a link invariant when $\gamma(D) = \bar{\gamma}(D) = 0$, and the terms in (9) are distinct terms in $J_K(t,z)$.

The terms in (9) coincide when $(-1)^c t^{(g-c)} = \pm t^{-1}$ or $\pm t^{-2}$; i.e., when $c = g+1$ or $c = g+2$. As D is cellularly embedded, $c = |V_A| + |V_B| + 2g - 2$ with $|V_A|, |V_B| \geq 1$, which allows only the cases: $(g,c) \in \{(1,2),(1,3),(2,4)\}$. Moreover, both $c = g+1$ and $c = g+2$ imply that either $|V_A| = 1$ or $|V_B| = 1$.

So one Tait graph G consists of only loops, and as D is reduced alternating, these loops are homologically non-trivial.

Let $H \subset G$. For $[\partial\mathcal{H}]$ in $H_1(F)$, let $\Lambda(H) = \dim([\partial\mathcal{H}])$. By [11, Equation (5.5)],

$$g(\mathcal{H}) + g(F - \mathcal{H}) + \Lambda(H) = g(F).$$

Since D is cellularly embedded, then so is G. Thus, for $H = G$ consisting of homologically non-trivial loops, we have $g(F-\mathcal{H}) = \Lambda(H) = 0$. Hence, $g(\mathcal{H}) > 0$, which implies that at least one pair of loops in G must be genus-generating loops, which are excluded by the condition $\gamma(D) = \bar{\gamma}(D) = 0$.

Therefore, when $\gamma(D) = \bar{\gamma}(D) = 0$, the terms in (9) are distinct terms in $J_K(t,z)$. □

The proof of Corollary 4.4 relies on the condition $\gamma(D) = \bar{\gamma}(D) = 0$, but it may not be necessary.

Question 4.5. If D is a reduced alternating diagram on F, is $\tau_F(D)$ a link invariant of K in $F \times I$?

Proof of Theorem 4.3. If $g = 0$, then D is a classical link diagram. In this case, $\lambda = \bar{\lambda} = 0$ since loops in its Tait graph can only come from nugatory crossings, so $\gamma = \bar{\gamma} = 0$. For classical links, $J_K(t,z) = V_K(t)$, so now both (8) and (9) follow from [8].

To prove (8) for $g > 0$, we extend the argument in [8] to links in thickened surfaces. Let $G_A = (V_A, E_A)$, $G'_A = (V_A, E'_A)$, $G_B = (V_B, E_B)$, $G'_B = (V_B, E'_B)$. Since G_A and G_B are dual graphs on F, $|E_A| = |E_B|$ and $|V_A| + |V_B| = |E_A| + 2-2g$. The homological twist number $\tau_F(D)$ counts sets of homologically twist-equivalent crossings, which we can count using sets of parallel edges in G_A and G_B, as follows:

$$\begin{aligned}
\tau_F(D) &= |E_A| - (|E_A| - |E'_A|) - (|E_B| - |E'_B|) \\
&= |E'_A| + |E'_B| - |E_A| \\
&= |E'_A| + |E'_B| - (|V_A| + |V_B| - 2 + 2g) \\
&= (|E'_A| - |V_A| + 1) + (|E'_B| - |V_B| + 1) - 2g \\
&= b_1(G'_A) + b_1(G'_B) - 2g \\
&= \lambda + \mu + \bar{\lambda} + \bar{\mu} - 2g.
\end{aligned}$$

We now prove (9) for $g > 0$. Let $P_{G_A}(X,Y,U,V)$ be as in Theorem 2.3, with $G = G_A$. By [11, Theorem 3.1], $P_{G_B}(X,Y,U,V) = P_{G_A}(Y,X,V,U)$. Therefore, by Theorem 2.3, $\lambda, \bar{\lambda}, \mu, \bar{\mu}$ are exactly the coefficients of the following terms of $P_{G_A}(X,Y,U,V)$:

$$\mu\, V^g X^{n-1} + \lambda\, V^{g-1} X^n + \bar{\mu}\, U^g Y^{N-1} + \bar{\lambda}\, U^{g-1} Y^N, \tag{10}$$

where $n = |V_A| - 1$ and $N = |V_B| - 1$. Using $\chi(F) = |V_A| + |V_B| - c$, we have $n + N = c - 2g$.

Let $\pi(X^\alpha Y^\beta U^i V^j) \in \mathbb{Z}[t^{\pm 1/2}, z]$ denote the term in $J_K(t,z)$ obtained from $X^\alpha Y^\beta U^i V^j$ by the substitutions in (7). We evaluate each term in (10):

$$\begin{aligned}
\pi(V^g X^{n-1}) &= (-1)^w t^{\frac{3w-2g-2n+c}{4}} z^g \left(\frac{\sqrt{t}}{z}\right)^g (-t)^{n-1} = (-1)^{(w+n)}\, t^{\frac{3w+2n+c}{4}} (-t^{-1}),\\
\pi(V^{g-1} X^{n}) &= (-1)^w t^{\frac{3w-2g-2n+c}{4}} z^g \left(\frac{\sqrt{t}}{z}\right)^{g-1} (-t)^{n} = (-1)^{(w+n)}\, t^{\frac{3w+2n+c}{4}} (z t^{-\frac{1}{2}}),\\
\pi(U^g Y^{N-1}) &= (-1)^w t^{\frac{3w-2g-2n+c}{4}} z^g \left(\frac{1}{z\sqrt{t}}\right)^g (-t)^{2g+n-c+1}\\
&= (-1)^{(w+n)}\, t^{\frac{3w+2n+c}{4}} (-1)^c t^{(g-c)} (-t),\\
\pi(U^{g-1} Y^{N}) &= (-1)^w t^{\frac{3w-2g-2n+c}{4}} z^g \left(\frac{1}{z\sqrt{t}}\right)^{g-1} (-t)^{2g+n-c}\\
&= (-1)^{(w+n)}\, t^{\frac{3w+2n+c}{4}} (-1)^c t^{(g-c)} (z t^{\frac{1}{2}}).
\end{aligned}$$

This verifies that the terms in (9) come from the corresponding terms in (10). We now find the other terms in $P_{G_A}(X,Y,U,V)$ that overlap with these terms in $J_K(t,z)$.

For the μ–term, suppose $\pi(X^\alpha Y^\beta U^i V^j) = \pm\pi(V^g X^{n-1})$. Since the RHS has no z factor, it follows that $i + j = g$. From exponents on t, we have

$$\alpha - \beta - i/2 + j/2 = g/2 + n - 1 \implies \alpha + j + 1 = \beta + g + n.$$

If $\alpha = n - \kappa$ for some integer $\kappa \geq 0$, then

$$n - \kappa + j + 1 = \beta + (i + j) + n \geq 0 \implies \kappa = 0 \text{ or } \kappa = 1.$$

If $\alpha = n$ then $\beta + i = 1$, so $\beta, i \in \{0,1\}$. If $\alpha = n-1$ then $\beta + i = 0$, so $\beta = i = 0$. We are left with only three possibilities:

$$\begin{aligned}
\alpha = n-1,\ \beta = 0,\ i = 0,\ j = g &\implies V^g X^{n-1}\\
\alpha = n,\ \beta = 0,\ i = 1,\ j = g-1 &\implies U V^{g-1} X^n\\
\alpha = n,\ \beta = 1,\ i = 0,\ j = g &\implies V^g X^n Y
\end{aligned}$$

We already know $\mu V^g X^{n-1}$ is in $P_{G_A}(X,Y,U,V)$. Since G_A does not have 3–petal loops, we can apply Lemma 2.5 to see that $UV^{g-1}X^n$ has coefficient γ in $P_{G_A}(X,Y,U,V)$. As a term in $J_K(t,z)$, $\pi(V^g X^{n-1}) = -\pi(UV^{g-1}X^n)$ because X^n and X^{n-1} contribute opposite signs, so we call it the $(\mu-\gamma)$–term in $J_K(t,z)$. For the final case above, we claim that $V^g X^n Y$ cannot be a term in $P_{G_A}(X,Y,U,V)$. Suppose there exists $H \subset G_A$ whose weight contributes to $V^g X^n Y$. As in the proof of Lemma 2.5, the factor X^n implies $H \subset \ell(G_A)$. Because D is reduced alternating on F, all loops in G_A are homologically non-trivial. The factor Y implies that H has weight with a factor y^k for $k > 0$, so H must contain 3-petal loops, which are excluded by hypothesis. Thus, $V^g X^n Y$ cannot be a term in $P_{G_A}(X,Y,U,V)$. With the cases exhausted, we see that no other terms in $P_{G_A}(X,Y,U,V)$ besides $V^g X^{n-1}$ and $UV^{g-1}X^n$ contribute to the $(\mu-\gamma)$–term in $J_K(t,z)$.

For the $\bar{\mu}$–term, we can use duality [11, Theorem 3.1]: $P_{G_A}(X,Y,U,V) = P_{G_B}(Y,X,V,U)$. If $\pi(X^\alpha Y^\beta U^i V^j) = \pm\pi(U^g Y^{N-1})$, the argument above for the dual graph G_B again implies only three possibilities:

$$\alpha = 0,\ \beta = N-1,\ i = g,\ j = 0 \implies U^g Y^{N-1}$$
$$\alpha = 0,\ \beta = N,\ i = g-1,\ j = 1 \implies U^{g-1} V Y^N$$
$$\alpha = 1,\ \beta = N,\ i = g,\ j = 0 \implies U^g X Y^N$$

By the same arguments on the dual graph, for D reduced alternating, only $U^g Y^{N-1}$ and $U^{g-1}VY^N$ are terms in $P_{G_A}(X,Y,U,V)$. Therefore, no other terms in $P_{G_A}(X,Y,U,V)$ besides these terms contribute to the $(\bar{\mu}-\bar{\gamma})$–term in $J_K(t,z)$.

For the λ–term, suppose $\pi(X^\alpha Y^\beta U^i V^j) = \pm\pi(V^{g-1}X^n)$. Since the RHS has a z factor, it follows that $i + j = g-1$. From exponents on t, we have

$$\alpha - \beta - i/2 + j/2 = (g-1)/2 + n \implies \alpha = \beta + i + n.$$

If $\alpha = n - \kappa$ for some integer $\kappa \geq 0$, then

$$n - \kappa = \beta + i + n \geq 0 \implies \beta = i = \kappa = 0.$$

This leaves only one possibility:

$$\alpha = n,\ \beta = 0,\ i = 0,\ j = g-1 \implies V^{g-1}X^n.$$

Therefore, no other terms in $P_{G_A}(X,Y,U,V)$ besides $V^{g-1}X^n$ contribute to the λ–term in $J_K(t,z)$. For the $\bar{\lambda}$–term, we can use a similar argument or use duality again.

This completes the proof of (9). □

Lemma 4.6. *For K in $F\times I$ as in Theorem 4.3, only the terms V^gX^n and U^gY^N of $P_{G_A}(X,Y,U,V)$ contribute the extremal terms of $J_K(t,1)$, which has span $(c-g)$.*

Proof. By [4, Theorem 2.9], and dividing by one factor of $-A^{-2} - A^2$ for the reduced polynomial, the span of $J_K(t,1)$ is exactly $(c-g)$. We now identify the subgraphs of G_A that contribute the two extremal terms of $J_K(t,1)$. By (7), the term in $P_{G_A}(X,Y,U,V)$ which contributes the highest t–degree term of $J_K(t,1)$ has the highest X–degree and highest V–degree. Namely, the unique spanning subgraph H_0 in G_A with an empty edge set has weight $v^g x^n$. Similarly, $H = G_A$ has weight $u^g y^N$, which contributes the the lowest t–degree term of $J_K(t,1)$. Thus, $P_{G_A}(X,Y,U,V)$ has the terms V^gX^n and U^gY^N, which contribute the extremal terms of $J_K(t,1)$.

We claim that no other terms of $P_{G_A}(X,Y,U,V)$ contribute the extremal terms of $J_K(t,1)$. Suppose there exists $H \subset G_A$ whose weight also contributes to V^gX^n. As in the proof of Lemma 2.5, the factor X^n implies H has weight with factor x^n and $H \subset \ell(G_A)$. Thus, H has weight $v^g x^n y^k$ for $k \geq 0$. Because D is reduced alternating on F, all loops in G_A are homologically non-trivial. If $k > 0$ then H must contain 3–petal loops, which are excluded by hypothesis. Thus, only H_0 contributes the term V^gX^n in $P_{G_A}(X,Y,U,V)$. The argument for $H = G_A$ follows by duality [11, Theorem 3.1], $P_{G_B}(X,Y,U,V) = P_{G_A}(Y,X,V,U)$. □

Proof of Theorem 1.1. Since D is strongly reduced, $\lambda = \bar{\lambda} = 0$. Thus, by (8),

$$\tau_F(D) = \mu + \bar{\mu} - 2g,$$

which is a link invariant of K in $F \times I$ by Corollary 4.4.

We claim that the μ and $\bar{\mu}$ terms in (9) with $\lambda = \bar{\lambda} = 0$ and $\gamma = \bar{\gamma} = 0$ are exactly the sub-extremal terms of $J_K(t, 0)$. By Lemma 4.6, only the terms $V^g X^n$ and $U^g Y^N$ of $P_{G_A}(X, Y, U, V)$ contribute the extremal terms of $J_K(t, 1)$, which has span $(c - g)$. The μ and $\bar{\mu}$ terms in (10) differ from $V^g X^n$ and $U^g Y^N$, and in $J_K(t, 1)$ they have span $(c - g - 2)$ by (9), so they are the sub-extremal terms of $J_K(t, 1)$. Moreover, by (7) neither the extremal terms nor the μ and $\bar{\mu}$ terms have a z factor in $J_K(t, z)$. Thus, the μ and $\bar{\mu}$ terms in (9) are exactly the sub-extremal terms of $J_K(t, 0)$. This proves the first part of Theorem 1.1.

By Proposition 3.3,

$$\tau_F(D) \leq t_F(D) \leq 2\tau_F(D).$$

The volume bounds in Theorem 1.1 now follow from [10, Theorem 1.4]. Since essential null-homologous cycles occur only for $g \geq 2$, the bounds for $g = 1$ are the same as in [10, Theorem 1.4] with $\tau_F(D) = t_F(D)$. Since $\tau_F(D) \leq t_F(D)$, the lower bound for $g \geq 2$ is the same as in [10, Theorem 1.4]. Since $t_F(D) \leq 2\tau_F(D)$, the upper bound for $g \geq 2$ must be doubled. □

5. Examples

Below we confirm Theorem 2.3 and Theorem 4.3 for three virtual links.

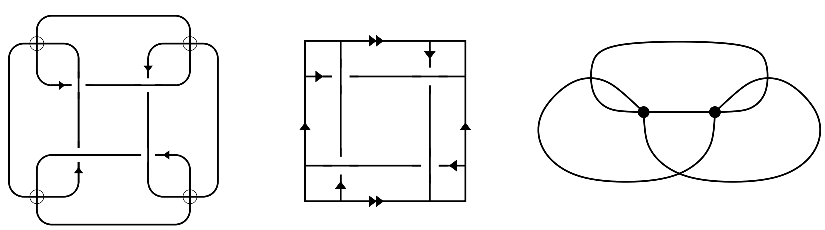


FIGURE 5. The 2×2 square weave on the torus as a virtual link, its diagram D with $\tau_F(D) = 4$, and self-dual Tait graphs $G_A = G_B$ shown as a ribbon graph.

Example 1. The 4–component virtual link K_1 shown in Figure 5 is also discussed in [4, Example 3.10]. For its 2×2 square weave diagram D on the torus, $\tau_F(D) = 4$. We have the following data from this diagram: $g = 1$,

$$\mu = 3,\ \lambda = 0,\ \gamma = 0,\ \bar{\mu} = 3,\ \bar{\lambda} = 0,\ \bar{\gamma} = 0,\ c = 4,\ w = -4,\ n = 1,\ N = 1.$$

Eqn (8) : $\tau_F(D) = \lambda + \mu + \bar{\lambda} + \bar{\mu} - 2g = 4$

Eqn (10) : $\mu V^g X^{n-1} + \lambda V^{g-1} X^n + \bar{\mu} U^g Y^{N-1} + \bar{\lambda} U^{g-1} Y^N = 3V + 3U$

$P_{G_A}(X, Y, U, V) = VX + 6 + UY + \boxed{3V + 3U}$

$$\text{Eqn (9)}: \ -\bar{\lambda}(zt^{-4}) + (\bar{\mu} - \bar{\gamma})t^{-7/2} + (\mu - \gamma)t^{-5/2} - \lambda(zt^{-2})$$
$$= 0 + 3t^{-7/2} + 3t^{-5/2} + 0$$
$$J_K(t,z) = -t^{-9/2} + \boxed{3t^{-7/2} + 3t^{-5/2}} - t^{-3/2} + \left(6zt^{-3}\right)$$

These results agree with Theorem 2.3 and Theorem 4.3. Note that $\lambda = \bar{\lambda} = 0$, so we can compute $\tau_F(K_1)$ directly from the sub-extremal coefficients of $J_K(t,0)$. As discussed in [7], the hyperbolic volume of $T^2 \times I - K_1$ is $4v_{\text{oct}}$, which is within the bounds of Theorem 1.1 for $\tau_F(K_1) = 4$.

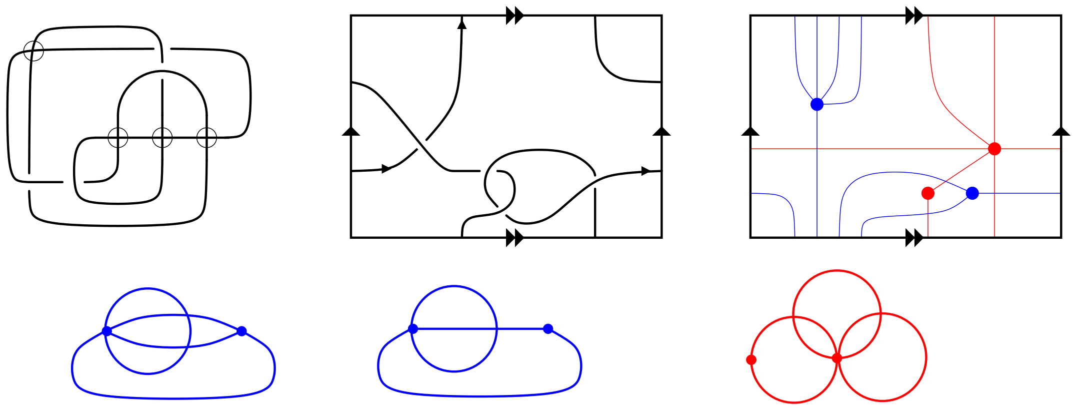


FIGURE 6. First row, left to right: Virtual knot 4.106, its diagram D on the torus with $\tau_F(D) = 3$, and its Tait graphs G_A (red) and G_B (blue) on the torus. Second row, left to right, shown as ribbon graphs: Tait graph G_B and its reduction G'_B (blue), and G_A (red) which is already reduced. Note the pair of genus-generating loops in G_A.

Example 2. The virtual knot $K_2 = 4.106$ is shown in Figure 6, with a diagram D shown on the torus. From the diagram we have: $g = 1$,

$$\mu = 1,\ \lambda = 2,\ \gamma = 1,\ \bar{\mu} = 1,\ \bar{\lambda} = 1,\ \bar{\gamma} = 0,\ c = 4,\ w = -2,\ n = 1,\ N = 1.$$
$$\text{Eqn (8)}: \ \tau_F(D) = \lambda + \mu + \bar{\lambda} + \bar{\mu} - 2g = 3$$
$$\text{Eqn (10)}: \ \mu V^g X^{n-1} + \lambda V^{g-1} X^n + \bar{\mu} U^g Y^{N-1} + \bar{\lambda} U^{g-1} Y^N$$
$$= V + 2X + U + Y$$
$$P_{G_A}(X,Y,U,V) = UX + UY + VX + \boxed{V + 2X + U + Y} + 2$$
$$\text{Eqn (9)}: \ \bar{\lambda}(-zt^{-5/2}) + (\bar{\mu} - \bar{\gamma})t^{-2} + (\mu - \gamma)t^{-1} + \lambda(-zt^{-1/2})$$
$$= -zt^{-5/2} + t^{-2} + 0 - 2zt^{-1/2}$$
$$J_K(t,z) = -t^{-3} + \boxed{t^{-2}} - 1 + \left(\boxed{-zt^{-5/2}} + 2zt^{-3/2}\ \boxed{-2zt^{-1/2}}\right)$$

These results agree with Theorem 2.3 and Theorem 4.3. Note that one of the coefficients in (9) is zero because $\mu = \gamma = 1$. In this case, $\tau_F(K_2)$ cannot be computed directly from the coefficients of $J_K(t,z)$. Also, note that if we set $z = -t^{-1/2} - t^{1/2}$, then $J_K(t,z) = 1$, so the virtual knot 4.106 has trivial Jones polynomial.

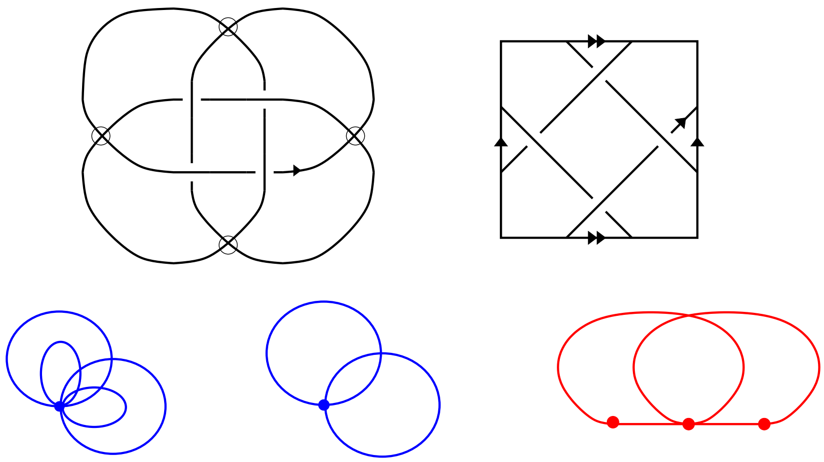


FIGURE 7. First row, left to right: Virtual knot 4.105, and its diagram D on the torus with $\tau_F(D) = 2$. Second row, left to right, shown as ribbon graphs: Tait graph G_B and its reduction G'_B (blue), and G_A (red) which is already reduced. Note the pair of genus-generating loops in G'_B.

Example 3. The virtual knot $K_3 = 4.105$ is shown in Figure 7, with a diagram D shown on the torus. From the diagram on the torus, we can see $\tau_F(D) = 2$, but it is less apparent from the virtual link diagram which evokes the knot 8_{18}. We have the following data from this diagram: $g = 1$,

$$\mu = 2,\ \lambda = 0,\ \gamma = 0,\ \bar{\mu} = 0,\ \bar{\lambda} = 2,\ \bar{\gamma} = 1,\ c = 4,\ w = -4,\ n = 2,\ N = 0.$$

$$\text{Eqn (8)}:\ \tau_F(D) = \lambda + \mu + \bar{\lambda} + \bar{\mu} - 2g = 2$$

$$\text{Eqn (10)}:\ \mu V^g X^{n-1} + \lambda V^{g-1} X^n + \bar{\mu} U^g Y^{N-1} + \bar{\lambda} U^{g-1} Y^N = 2VX + 2$$

$$P_{G_A}(X,Y,U,V) = VX^2 + U + V + 2X + \boxed{2VX + 2}$$

$$\text{Eqn (9)}:\ \bar{\lambda}(zt^{-7/2}) - (\bar{\mu} - \bar{\gamma})t^{-3} - (\mu - \gamma)t^{-2} + \lambda(zt^{-3/2}) = 2zt^{-7/2} + t^{-3} - 2t^{-2} + 0$$

$$J_K(t,z) = t^{-4} + \boxed{t^{-3} - 2t^{-2}} + t^{-1} + \left(\boxed{2zt^{-7/2}} - 2zt^{-5/2}\right)$$

These results agree with Theorem 2.3 and Theorem 4.3. Note that because $\bar{\gamma} = 1$, $\tau_F(K_3)$ cannot be computed directly from the coefficients of $J_K(t,z)$.

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