New York Journal of Mathematics

New York J. Math. **27** (2021) 1134–1172.

The Dyer-Lashof algebra and the hit problems

Hadi Zare

Abstract. We use work of Curtis and Wellington on $\mathcal{A}$-annihilated classes in H_*QS^0 together with Priddy's computations on the action of the Dyer-Lashof algebra on $H_*(\mathbb{Z}\times BO)$ to provide new examples of $\mathcal{A}$-annihilated classes in H_*BO as well $H_*BO(2^r-1)$.We then consider the Becker-Gottlieb transfer associated to $O(1)^{\times n}\to O(n)$ and speculate on the possible applications of our computations to obtain new examples of $\mathcal{A}$-annihilated classes in $H_*BO(1)^n$. Our results, all at the prime $p=2$, provide new numerical conditions which seem to show up for the first time in this context.

Contents

1. Introduction and statement of results

The hit problem of Peterson is about finding a basis for the vector space $\mathbb{Z}/2\otimes_{\mathcal{A}} H^*X$ when $X=\mathbb{R}P^{\times k}$; $\mathbb{Z}/2$ is an $\mathcal{A}$-module concentrated in degree 0 [40, Section 7] (see also [36],[37] for a very recent account on the problem). We work at the prime 2 writing $\mathbb{Z}/2$ for the field of two elements, $\mathcal{A}$ and $\mathcal{R}$ for the mod 2 Steenrod and Dyer-Lashof algebras respectively, H^* (resp. H_*) for $H^*(-;\mathbb{Z}/2)$ (resp. $H_*(-;\mathbb{Z}/2)$). We write $\mathbb{R}P$ for the infinite dimensional real projective space $\mathbb{R}P^\infty$. For a pointed space X, $X^{\times k}$ and $X^{\wedge k}$ denote the k-fold Cartesian and smash products of X with itself, respectively.

Received May 17, 2018.

2010 *Mathematics Subject Classification.* 55S10, 55S12, 55P47.

Key words and phrases. Dyer-Lashof algebra, Becker-Gottlieb transfer, infinite loop space.

This work is partially supported by a grant from the Iranian National Science Foundation (INSF) Grant No. 98023105.

ISSN 1076-9803/2021

An equivalent problem, in a homological setting, is to study the submodule of $\mathcal{A}$-annihilated elements of H_*X where $x \in H_*X$ is called $\mathcal{A}$-annihilated if $Sq^i_* x = 0$ for all $i > 0$. Here, duality of vector spaces over $\mathbb{Z}/2$, provided by the Universal Coefficient Theorem over $\mathbb{Z}/2$, allows to consider $Sq^i_* : H_*X \to H_{*-i}X$ as the dual operation to $Sq^i : H^*X \to H^{*+i}X$. Our aim in this paper is to study this problem by means of looking at the dual problem for H_*BO which is the stable case of the symmetric hit problem of Janfada and Wood [16], [17].

Conventions and notations. For most of the paper we work with sequences of nonnegative/positive integers. For this purpose, we fix some notations to be used through the paper. We write $\phi = (\,)$ for the empty sequence. Given $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$, we abbreviate $(I, J) = (i_1, \dots, i_r, j_1, \dots, j_s)$ allowing I or J or both to be the empty sequence. If $I = (i)$ we write (i, J) for $((i), J)$; we use similar notation if $J = (j)$. For $I = (i_1, \dots, i_r)$ a sequence of nonnegative integers, we define and denote excess of I by $\mathrm{ex}(I) = i_1 - (i_2 + \cdots + i_r)$, dimension of I by $|I| = i_1 + \cdots + i_r$, and length of I by $l(I) = r$; if necessary for the empty sequence $\phi = ()$ we use the conventions that $\mathrm{ex}(\phi) = +\infty$, $|\phi| = l(\phi) = 0$. We shall say I is admissible if either $l(I) > 1$ and $i_j \leqslant 2i_{j+1}$ for all $1 \leqslant j \leqslant r-1$ or $l(I) = 1$ or $I = \phi$. For a nonempty sequence $I = (i_1, \dots, i_r)$ we write $I = I_0$ and $I_j = (i_{j+1}, \dots, i_r)$ for $j < r$, and define $\mathrm{ex}_j(I) = \mathrm{ex}(I_j)$; note that if I is admissible so is I_j for all $j < r$. We allow ourselves to use the abbreviation ex_j instead of $\mathrm{ex}_j(I)$ if there is no confusion about the sequence I.

The following observation is a consequence of definitions and conventions above which we wish to highlight due to its usefulness. We defer its proof to the following sections.

Lemma 1.1. *(i) $I = (i_1, \dots, i_r)$ is admissible if and only if $\mathrm{ex}_j \leqslant \mathrm{ex}_{j+1}$ for $j = 0, \dots, r-1$.*
(ii) Suppose $I = (i_1, \dots, i_r)$ is an admissible sequence such that all of its entries are odd. Then the sequence $(\mathrm{ex}_0, \mathrm{ex}_1, \dots, \mathrm{ex}_{r-1})$ is strictly increasing.
(iii) If $I = (i_1, \dots, i_r)$ is admissible with $\mathrm{ex}(I) > 0$ then $i_j > i_{j+1}$ for all $j = 1, \dots, r-1$.

Next, we describe $H_*(\mathbb{Z}\times BO)$ which is required to state our first result. Write $\iota : \mathbb{R}P \to \{1\} \times BO \to \mathbb{Z} \times BO$ for the inclusion and $a_i \in H_i\mathbb{R}P$ for a generator with $i > 0$, and let $e_i = \iota_* a_i$ for $i > 0$. There is a certain infinite loop map $\chi : QS^0 \to \mathbb{Z} \times BO$ (see Section 3 for more details) and certain elements $[1], [-1] \in H_0 QS^0$ with $[1]^{-1} = [-1]$ (see comments after Remark 2.1 for the definition of these classes). Let $e_0 = \chi_*[1]$ and $e_0^{-1} = \chi_*[-1]$. The space $\mathbb{Z}\times BO$ is an infinite loop space by Bott periodicity which furnishes $H_*(\mathbb{Z} \times BO)$ with a ring structure. It is known that [29] $H_*(\mathbb{Z} \times BO) \simeq \mathbb{Z}/2[e_0, e_0^{-1}, e_i : i > 0, \deg e_i = i]$. Sometimes, in order to stress the role of multiplication by some specific element ξ, we choose to write $*$ for the product operation in homology and $*\,\xi$ means multiplying by ξ. We also assume that while writing an arbitrary monomial $e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ the variables are ordered so that $i_1<i_2<\cdots<i_r$. Although, sometimes it makes it easy to allow cases with $i_1\leqslant\cdots\leqslant i_r$ which will be specified whenever it happens. Our first result reads as follows.

Theorem 1.2. *(i) Suppose $I=(i_1,\ldots,i_r)$ is a sequence of positive integers satisfying*

$$(1)\ 0<\mathrm{ex}(I)<2^{\phi(i_1)},\ (2)\ 0\leqslant 2i_{j+1}-i_j<2^{\phi(i_{j+1})}$$

if $r>1$, and (1) if $r=1$. Then there exists an $\mathcal{A}$-annihilated class, say $\xi_I^{\mathbb{Z}\times BO}\in H_{|I|}(\mathbb{Z}\times BO)$ so that

$$\xi_I^{\mathbb{Z}\times BO}=e_0e_{\mathrm{ex}_0}e_{\mathrm{ex}_1}^2\cdots e_{\mathrm{ex}_{r-1}}^{2^{r-1}}+\ \textit{other terms}\neq 0$$

with

$$0<\mathrm{ex}_0<\mathrm{ex}_1<\cdots<\mathrm{ex}_{r-1}.$$

(ii) If $l(I)=r$ then the class $\xi_I^{BO}:=e_0^{-2^r}\xi_I^{\mathbb{Z}\times BO}$ pulls back to $H_*BO(2^r-1)$.
(iii) The classes $\xi_I^{\mathbb{Z}\times BO}$ where I ranges over all admissible sequences are linearly independent. A similar statement holds for the classes ξ_I^{BO} in $H_*BO(2^r-1)$
Here, for $n\in\mathbb{N}$ with binary expansion $n=\sum_{i=0}^{+\infty}n_i2^i$ with $n_i\in\{0,1\}$ we define $\phi(n)=\min\{i:n_i=0\}$.

We prove the above Theorem in Section 3.

Note 1.3. An important question is about the existence of the sequences mentioned in Theorem 1.2 of a given dimension and length. The tables provided by [39] give a list of such sequences up to dimension 200 where it seems that in these dimensions, omitting sequences with $\mathrm{ex}(I)=0$, there is only at most one sequence in each dimension satisfying conditions of Theorem 1.2. However, this does not appear to be true in all dimensions as in a work in progress [11] and with computer assisted computations, we have determined all such sequences, with $\mathrm{ex}(I)>0$, up to dimension 1.1×10^7 (the highest dimension which we did manage to get with our laptop running for 30 hours). It is observed that in this range there exist less than 8×10^6 number of such sequences. As an outcome of our computations, we have obtained sequences of the same dimension, some of the same length and some with different lengths. For instance, the sequences

$$(1091,547,287,159,95),(1091,547,275,139,127)$$

are both of dimension 2179 and length 5 and satisfy our conditions. As another example, in a low dimension such as 4353 there exist only three such sequences, namely

$$(2177,1089,545,287,255),$$
$$(2177,1089,545,273,143,79,47),\ (2177,1089,545,273,137,69,63).$$

The methodological outcome is that our computations in this paper could produce more precise bounds on the dimension of modules related to the symmetric and nonsymmetric hit problems.

As another application of Priddy's formula, we obtain yet another family of $\mathcal{A}$-annihilated classes. For a nonempty sequence of nonnegative integers $K = (k_1, \dots, k_n)$ we write e^K to denote $e_1^{k_1} \cdots e_n^{k_n}$ in $H_*(\mathbb{Z} \times BO)$. We use the convention that $e_i^0 = 1$; for instance if $K = (1,0,2)$ then $e^K = e_1^1 e_2^0 e_3^2 = e_1 e_3^2$. We define and denote the dimension of e^K by $|e^K| = \sum_{j=1}^n jk_j$. We also define length of e^K by $l(e^K) = \sum_{i=1}^n k_i$ which corresponds to the length filtration function on $H_*(\mathbb{Z}\times BO)$ (see comments after Remark 2.3 for more discussions). We refer to a positive integer n as a spike if $n = 2^t - 1$ for some $t > 0$. Given a natural number, we may consider its *partition into spikes* as $n = \sum_{i=1}^d (2^{\lambda_i} - 1)$ where $d > 0$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_d > 0$ where unlike [17, Section 2] we consider only positive λ_i in order to avoid trivial cases. Note that any such partition corresponds to a decreasing sequence $(\lambda_1, \dots, \lambda_d)$; we define d to be length of our partition. We define the spike partition number of n, denoted $sp(n)$, to be the number of distinct partitions of n into spikes where two partitions are assumed to be distinct if either they are of different lengths or they have the same length but differ at least in one term. More precisely, define

$$\mathrm{Spike}^d(n) = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{Z}_+^{\times d} : \sum_{i=1}^d (2^{\lambda_i} - 1) = n\}$$

and note that the symmetric group on d elements Σ_d acts on this set. It follows that

$$sp(n) = \sum_{d=1}^{+\infty} |\mathrm{Spike}^d(n)/\Sigma_d|.$$

We shall use this function in the statement of our next result. We urge the reader to note that our use of spikes in studying the dual hit problem is different from its applications in the hit problem (compare to [17]). For nonempty sequences $K = (k_1, \dots, k_n)$ and $M = (m_1, \dots, m_n)$, we shall write $K = 2M$ if $k_j = 2m_j$ for all j. For positive integers k, m if k divides m then we write $k|m$. For $i > 0$, we shall write $\overline{e_i} = e_0^{-1} e_i$ noting that $H_*BO \simeq \mathbb{Z}/2[\overline{e_i} : i > 0]$ (see Section 2.4 for more details). Our next result reads as follows.

Theorem 1.4. *(i) For $n > 0$ let $\mathcal{K}_n$ be the set of all sequences of nonnegative integers $K = (k_1, \dots, k_n)$ such that*

$$\binom{l(e^K)}{k_1, \dots, k_n} := \frac{l(e^K)!}{k_1! \cdots k_n!} \equiv 1 \bmod 2, \ \sum_{i=1}^n ik_i = n.$$

For $t > 0$, there exists a nonzero $\mathcal{A}$-annihilated class $\xi_t^{\mathbb{Z}\times BO} \in H_{2^t-1}(\mathbb{Z} \times BO)$ such that

$$\xi_t^{\mathbb{Z}\times BO} = \sum_{K \in \mathcal{K}_{2^t-1}} e_0^{-l(e^K)-2} e^K.$$

In particular, the sum involves the terms $e_0^{-(2^t-1)-2}e_1^{2^t-1}$ *and* $e_0^{-3}e_{2^t-1}$*. Consequently, we have a nonzero* $\mathcal{A}$*-annihilated class* $\xi_t^{BO}\in H_{2^t-1}BO$ *given by*

$$\xi_t^{BO}=e_0^2\xi_t^{\mathbb{Z}\times BO}.$$

In particular, the sum involves terms such as $\overline{e_1}^{2^t-1}$ *and* $\overline{e_{2^t-1}}$*. Moreover, this class is in the image of* $H_*BO(2^t-1)\to H_*BO$ *and it does not pull back to* $H_*BO(s)$ *with* $s<2^t-1$*.*
(ii) Suppose $n>0$ *is given. Then, for each partition of* n *into spikes* $n=\sum_{i=1}^d(2^{\lambda_i}-1)$ *with* $\lambda_1\geqslant\cdots\geqslant\lambda_d>0$ *there exists a nonzero* $\mathcal{A}$*-annihilated class*

$$\xi_{\lambda_1,\dots,\lambda_d}^{\mathbb{Z}\times BO}=\prod_{i=1}^d\xi_{\lambda_i}^{\mathbb{Z}\times BO}.$$

In particular, up to multiplication by a power of e_0*,* $\xi_{\lambda_1,\dots,\lambda_d}^{\mathbb{Z}\times BO}$ *involves terms of the form* e_1^n *and* $\prod_{i=1}^d e_{2^{\lambda_i}-1}$*. Moreover, there are* $sp(n)$ *number of linearly independent nonzero* $\mathcal{A}$*-annihilated classes, living in* $H_n(\mathbb{Z}\times BO)$*, that are obtained in this way. Furthermore, by a translation map, we obtain* $sp(n)$ *number of nonzero* $\mathcal{A}$*-annihilated classes of the form*

$$\xi_{\lambda_1,\dots,\lambda_d}^{BO}=\prod_{i=1}^d\xi_{\lambda_i}^{BO}$$

so that each $\xi_{\lambda_1,\dots,\lambda_d}^{BO}$ *involves terms of the form* $\overline{e_1}^n$ *and* $\prod_{i=1}^d\overline{e_{2^{\lambda_i}-1}}$*.*

The above Theorem allows to identify $\mathcal{A}$-annihilated classes in $H_*(\mathbb{Z}\times BO)$, and consequently H_*BO, which are not products of classes of the form e_{2^t-1}. For instance, consider the class $\xi_4^{\mathbb{Z}\times BO}\in H_{15}BO$. We can see that up to a factor of e_0 we have

$$\xi_4^{BO}=e_1^{15}+e_{15}+e_1e_7^2+e_3^5+(e_5^3+e_3e_6^2)+\text{other terms}$$

where the first four terms are products of classes of the form e_{2^t-1}. Now, the nontrivial interesting class in this case is $e_5^3+e_3e_6^2$ where $Sq_*^2(e_5^3)=Sq_*^2(e_3e_6^2)=e_3e_5^2$ so $Sq_*^2(e_5^3+e_3e_6^2)=0$. One can show that in fact $e_5^3+e_3e_6^2$ is $\mathcal{A}$-annihilated which is a nontrivial example. In fact, whenever 2^t-1 is not prime, we have nontrivial such terms in $\xi_t^{\mathbb{Z}\times BO}$.

Next, we wish to investigate the possible applications of the above computations to study $\mathcal{A}$-annihilated classes in $H_*BO(1)^{\times k}$. By the Künneth formula over $\mathbb{Z}/2$ we have $H_*X^{\times k}\simeq(H_*X)^{\otimes k}$ where the latter denotes the k-fold tensor product of H_*X with itself. Recall that the Becker-Gottlieb transfer map associated to the fibre bundle $i:BO(1)^{\times n}\to BO(n)$, induced by the diagonal embedding $O(1)^{\times n}\to O(n)$, is a stable map $t:\Sigma^\infty BO(n)_+\to\Sigma^\infty BO(1)_+^{\times n}$ where X_+ is the union of X with a disjoint base-point [4] (see also [30] and [1, Chapter 4]). In particular, it satisfies $H_*\Sigma^\infty X_+\simeq\widetilde{H}_*X_+\simeq H_*X$. The representation $\Sigma_n\to\mathrm{Gl}_n(\mathbb{Z}/2)$, sending each permutation to its corresponding permutation matrix, allows to define $\overline{\Sigma_n} = \sum_{\sigma\in\Sigma_n}\mathrm{sgn}(\sigma)\sigma \in \mathbb{Z}/2[\mathrm{Gl}_n(\mathbb{Z}/2)]$ where sgn is the sign function. It is known that $i^*t^* = \overline{\Sigma_n}$ [25, Lemma 6.2]. In particular, this shows that $t^* \neq 0$ and consequently $t_* \neq 0$. For the transfer t_* we have identified $\ker(t_*)$ and $\mathrm{coker}(t_*)$ as well as a submodule $H_*^{\bullet}BO(k)$ of $H_*BO(k)$ on which the restriction of t_* is a monomorphism (see Proposition 4.2 and Corollary 4.3). Assuming that Q^Ie_0 has some terms which belong to this submodule, we automatically obtain some nonzero $\mathcal{A}$-annihilated classes in $H_*\mathbb{R}P^{\times n}$. Let us write $a_{i_1}\otimes\cdots\otimes a_{i_k}$ for the elements of $H_*X^{\times k}$ which we denote by $a_i^{\otimes k}$ if $i_1=\cdots=i_k=i$. Using this notation, we have the following.

Theorem 1.5. *(i) Suppose $I=(i_1,\dots,i_r)$ is a sequence of positive integers satisfying*

$$(1)\ 0<\mathrm{ex}(I)<2^{\phi(i_1)},\ (2)\ 0\leqslant 2i_{j+1}-i_j<2^{\phi(i_{j+1})}$$

if $r>1$, and (1) if $r=1$. If the pullback of ξ_I^{BO} to $H_*BO(2^r-1)$ projects nontrivially into $H_*^{\diamond}BO(2^r-1)=\mathrm{coker}(t_*)$ then there exists a nonzero $\mathcal{A}$-annihilated class $\xi_I\in H_{|I|}\mathbb{R}P^{\times(2^r-1)}$ so that for any permutation $\tau\in\Sigma_{2^r-1}$ we have $\tau\xi_I=\xi_I$.
(ii) Suppose $k=\sum_{j=1}^l(2^{r_j}-1)$ so that for each j there exists a sequence I^j of positive integers with $l(I^j)=r_j$ and $\mathrm{ex}(I^j)>0$ satisfying the conditions in part (i). Then, for $\sigma\in\Sigma_l$ there exists a nonzero $\mathcal{A}$-annihilated class $\xi_\sigma\in H_d\mathbb{R}P^{\times k}$ with $d=\sum|I^j|$ which is invariant under the action of $\Sigma_{2^{r_{\sigma(1)}}-1}\times\cdots\times\Sigma_{2^{r_{\sigma(l)}}-1}$. Moreover, depending on the partition chosen for k, subject to the existence of sequences I^j, this would lead to at least $l!$ distinct $\mathcal{A}$-annihilated classes in $H_*\mathbb{R}P^{\times k}$.

The $\mathcal{A}$-annihilated classes that we have obtained in the above theorems seem to provide new families of $\mathcal{A}$-annihilated classes in the homology of the relevant spaces of which we don't know of any published account. Note that earlier versions of this paper included a claim that the classes ξ_I are linearly independent when dimension and length of the sequences is fixed, which we believe to be true. However, the provided proof was based on the wrong assumption that the transfer $t_*: H_*BO(n)\to H_*BO(1)^{\times n}$ is a monomorphism. Besides this, the numerical assumptions of Theorem 1.2 allowed us to do machine based computations [11] which suggest that sequences I leading to these classes are often very rare; so far we have found 4 sequences living in the same dimension with our search going up to dimension 1.1×10^7. Hence, computationally this claim seems rather trivial and we hope to come back to it in another work.

Previous similar works. Pengelley and Williams have considered using the infinite loop space structure on BO as well as BU to study the problem [26] (see also [28] as well as [27]), however, it seems that they have not used Wellington's computations [39], [38] in their work, neither have they considered work of Priddy [29]. So, our methodology and the results we have obtained, in terms of the numerical conditions, is different from theirs. Singer [33] considers an extension of the Steenrod algebra, denoted by $\mathcal{H}$, together with its action on $H_*BO(k)$; the algebra $\mathcal{H}$ is a bigraded algebra with generators $\mathbb{S}q^i$ for $i > -1$, subject to the Adem relations formally the same as $\mathcal{A}$, with $\mathbb{S}q^0$ treated in the same level as $\mathbb{S}q^i$ (not necessarily as the identity operator). The bigrading comes from degree and length so that $\mathbb{S}q^i$ lives in bidegree $(i,1)$ of $\mathcal{H}$. According to Singer (see comments after [33, Theorem 1.4]) there is an action of $\mathcal{H}$ on $\bigoplus H_*BO(s)$ which in terms of grading shifts is given by

$$\mathbb{S}q^i : H_nBO(s) \to H_{2n+s-i}BO(s+i).$$

Clearly, this action, if likely to give an action of $\mathcal{H}$ on H_*BO or $H_*(\mathbb{Z}\times BO)$, will increase the homological degree if $i < s+n$. More precisely, the homological dimension of $\mathbb{S}q^i(x)$ is always greater than n for any $x \in H_nBO(s)$ if $i < s+n$. However, in our setting given by the left-action of $\mathcal{A}^{\mathrm{op}}$ or the right-action of $\mathcal{A}$ on $H_*BO(s)$ which is expressed in terms of the Sq^i_* operations, for $x \in H_nBO(s)$ the class Sq^i_*x is of dimension $n-i$ and for the action to be non-trivial we always need $i < n$ which is definitely included in the cases $i < n+s$. Therefore, we argue that in Singer's work [33], the action of $\mathcal{H}$ most of the time increases the homological degree whereas in our case we have a decrease in the homological degree. Another related work is by Ault and Singer [3]. From our point of view, their approach is mainly algebraic where the authors consider the collection $\{H_*\mathbb{R}P^{\times k}\}_{k\geqslant 0}$ as well as $\{H_*\mathbb{R}P^{\wedge k}\}_{k\geqslant 0}$ as a graded algebra with the multiplicative structure coming from natural pairing $\mathbb{R}P^{\times k}\times\mathbb{R}P^{\times l} \to \mathbb{R}P^{\times(k+l)}$. They then consider the actions of $Sq^{2^i}_*$ operations between various gradings of this graded algebra and prove a kind of freeness result for it, generalising a result of Anick. The geometry hidden in this work comes from the James splitting of $\Omega\Sigma X$ which is stably weak homotopy equivalent to $\bigvee_{k=1}^{+\infty} X^{\wedge k}$. Hence, this work is different from ours in the sense that in our approach we mainly consider the homology that is arising from geometry of the spaces. We note that there also exists extensive literature on the hit problem for the Dickson algebra such as [10],[14],[13] where the objects under study are rings such as $(H_*\mathbb{R}P^{\times k})^{\mathrm{Gl}_k(\mathbb{Z}/2)} \simeq D_k^*$ and the interest is in their $\mathcal{A}$-module structure, hence relating Dickson algebras D_k to the dual of the Dyer-Lashof algebra by work of Madsen [21, Corollary 3.3]. The difference of such works with ours is that we work prior to the quotient maps under the $\mathrm{Gl}_k(\mathbb{Z}/2)$-action and we work in the dual setting and our starting point is the action of the Dyer-Lashof algebra rather than the algebra itself. Finally, we have to recall the very much related work of Repka and Selick [32] which is in the same line as our work trying to identify $\mathcal{A}$-annihilated subalgebras of $H_*\mathbb{R}P^{\times k}$; our results differ from theirs in numerical conditions as well as the methods we have employed to prove our result.

Final comments. We note that a general approach to the hit problems, other than providing a $\mathbb{Z}/2$-basis for the vector spaces $Q(k) := \mathbb{Z}/2\otimes_{\mathcal{A}} H^*B\mathbb{Z}/2^{\times k}$ and $\mathbb{Z}/2\otimes_{\mathcal{A}} H^*BO(k)$, is to find upper/lower bounds on the dimension of these spaces. Most notably, we have conjectures of Peterson and Kameko on bounds on $\dim_{\mathbb{Z}/2} Q(k)$ [41] (see [35, Theorem 1.2] for a negative answer to Kameko's conjecture for $k > 3$). It is possible that our work allows one to improve the known lower bounds. However, in this work, we have not studied this side of the problem.

Acknowledgements. The author is grateful to Grant Walker and Reg Wood, whom he has learnt from about the problem, for various comments and correspondences on the hit problem, as well as their interest in the result of this project. He is also grateful to Takuji Kashiwabara for valuable and helpful comments and notes on early versions of this note. We are very grateful to an anonymous referee for his/her useful comments and critical reading which resulted in a much improved version of the paper.

2. Preliminaries

2.1. The action of the Steenrod algebra. For an arbitrary space X, the Steenrod square $Sq^i : H^nX \to H^{n+i}X$ is a linear map of $\mathbb{Z}/2$-vector spaces. Since we work over the field $\mathbb{Z}/2$, then by duality we have $Sq^i_* : H_nX \to H_{n-i}X$. The left action of the Steenrod algebra $\mathcal{A}$ on H^*X, through the operations Sq^i_*, furnishes the homology H_*X with a right $\mathcal{A}$-action, or equivalently a left action of the opposite algebra $\mathcal{A}^{\mathrm{op}}$ on H_*X. We abuse the language to say that $\mathcal{A}$ acts on H_*X referring to the action given by the Sq^i_* operations. Below, we list some of the properties of these operations, or the formulae that these operations satisfy, which will be used throughout the paper implicity or explicitly.

Unstable $\mathcal{A}^{\mathrm{op}}$-modules. The instability condition for the action of $\mathcal{A}$ on H^*X, $Sq^i\xi = 0$ for $i > \dim \xi$, in the homological setting reads as $Sq^t_*x = 0$ if $2t > n$ with $x \in H_nX$. This can be simply verified using Kronecker pairing with $\langle x, Sq^i\xi\rangle = \langle Sq^i_*x, \xi\rangle$ (see also [39]). This could be used to define unstable $\mathcal{A}^{\mathrm{op}}$-modules in purely algebraic terms where an $\mathcal{A}^{\mathrm{op}}$-module M is unstable if $Sq^t_*x = 0$ for any $x \in M$ with $2t > \dim x$. In particular, H_*X is an unstable $\mathcal{A}^{\mathrm{op}}$-module.

Cartan formulae. For space X and Y, the external Cartan formula is give by

$$Sq^t_*(x \otimes y) = \sum_{t_1+t_2=t} (Sq^{t_1}_* x) \otimes (Sq^{t_2}_* y). \tag{1}$$

If we are equipped with a pairing $m : X \times Y \to Z$ then the naturality of the Steenrod operations together with the above formula implies that

$$Sq^t_*(xy) = \sum_{t_1+t_2=t} (Sq^{t_1}_* x)(Sq^{t_2}_* y)$$

where $xy := m_*(x \otimes y)$. In particular, for $X = Y = Z$ being an H-space with m the multiplication of the H-space, the above formula proves to be useful (see also [39, Chapter 5]). We may refer to this latter relation as the internal Cartan formula in order to distinguish it from the former.

2.2. The Dyer-Lashof algebra. We give a brief account on the Kudo-Araki operations over $\mathbb{Z}/2$ and the Dyer-Lashof algebra generated by these operations. The Kudo-Araki operations, first noted by Kudo and Araki [19] (see also [20]), are additive operations which act on $\mathbb{Z}/2$-homology of iterated loop spaces amongst which we are interested in infinite loop spaces such as $\mathbb{Z}\times BO$, BO, and $QS^0 = \operatorname{colim} \Omega^i S^i$ [23]; the infinite loop space structure on $\mathbb{Z}\times BO$ as well as BO arise from the Whitney sum of vector bundles. These operations fit into an algebra over $\mathbb{Z}/2$, known as the Dyer-Lashof algebra denoted by $\mathcal{R}$; $\mathcal{R}$ can be constructed from the associative algebra generated by the operations Q^i (see [39] for a detailed description and construction of this algebra, see also [9, Section 3.2]). The action of the Kudo-Araki operations on homology of an infinite loop space X induces an action of $\mathcal{R}$ on H_*X which turns H_*X into a left $\mathcal{R}$-module. This action has many useful properties of which we list some below (see [6, Part I, Theorem 1.1] for a full list of these properties).
Suppose X is an infinite loop space. A Kudo-Araki operation $Q^n : H_*X \to H_{*+n}X$ is an additive homomorphism such that
(i) Q^n is natural with maps of infinite loop spaces;
(ii) if $n < \dim x$ then $Q^n x = 0$;
(ii) if $n = \dim x$ then $Q^n x = x^2$ where the squaring is with respect to the Pontrjagin product on H_*X induced by the addition of loops.
(iv) For $x, y \in H_*X$, $Q^n(xy)$ is computed by the Cartan formula given by

$$Q^n(xy) = \sum_{i+j=n} (Q^i x)(Q^j y).$$

We note that there are various other properties which we do not use in this paper and refer the reader to [6] for more details.

Remark 2.1. (i) For $x_1, \dots, x_r \in H_*X$ the Cartan formula implies that

$$Q^n(x_1 \cdots x_r) = \sum_{i_1+\cdots+i_r=n} (Q^{i_1}x_1)\cdots(Q^{i_r}x_r).$$

(ii) As we work over $\mathbb{Z}/2$, for an arbitrary class $\xi \in H_*X$ we have

$$Q^{2n+1}\xi^2 = 0,\ Q^{2n}\xi^2 = (Q^n\xi)^2.$$

(iii) By the above properties if I is a sequence so that $\mathrm{ex}(I) < \dim x$ then $Q^I x = 0$ in H_*X.

Note that $\pi_i QS^0 \simeq \pi_i^s$ and in particular $\pi_0 QS^0 \simeq \mathbb{Z}$. Write $(QS^0)_n$ for the path component of QS^0 corresponding to $n \in \pi_0 QS^0$. If $n : S^0 \to QS^0$ corresponds to $n \in \pi_0 QS^0 \simeq \mathbb{Z}$, as QS^0 is an infinite loop space, then we may extend n to an infinite loop map $n : QS^0 \to QS^0$ (in fact a homotopy equivalence whose inverse is $-n$) so that if $\alpha \in (QS^0)_m$ then $\alpha * n \in (QS^0)_{m+n}$. Here, in this subsection, by abuse of notation $*$ denotes the loop sum in QS^0 and the Pontrjagin product induced by it in H_*QS^0. We write for the translation

map $*\,[n] : H_*QS^0 \to H_*QS^0$ induced by $n : QS^0 \to QS^0$ which when restricted to the component $(QS^0)_m$ looks like $*\,[n] : H_*(QS^0)_m \to H_*(QS^0)_{m+n}$. Let's note that we may consider a generator $[n] \in H_0((QS^0)_n;\mathbb{Z}/2) \simeq \mathbb{Z}/2$ and think of $*\,[n]$ as multiplication by $[n]$. The classes $[n]$ (or the homomorphisms $*\,[n]$) have the property that $[n]*[m] = [n+m]$ in H_*QS^0 with $[0]$ playing the role of neutral element [21]. According to Dyer and Lashof [8, Section 5, Corollary 2] (see also [34, Page 33]) H_*QS^0 is a polynomial algebra (under the Pontrjagin product coming from the loop sum) generated by elements of the form $Q^I[1] \in H_{|I|}QS^0$ where I is any nonempty admissible sequence and $[1],[-1] \in H_0QS^0$ subject to $[1]^{-1} = [-1]$ (see also [6, Part I] for more details). Together with the Cartan formula and the Adem relations for the Kudo-Araki operations, this also describes H_*QS^0 as a module over $\mathcal{R}$.
In this paper, for the purpose of applications in $H_*(\mathbb{Z}\times BO)$, we are interested in the submodule of $\mathcal{A}$-annihilated classes in H_*QS^0 whose complete description is unknown. But, there are some sufficient conditions that allow one to identify some of these classes. The following is due to Curtis [7, Lemma 6.2, Theorem 6.3] (see also Wellington [38, Theorem 5.6] as well as [39]).

Theorem 2.2. *Define $\phi : \mathbb{N} \longrightarrow \mathbb{N}\cup\{0\}$ by $\phi(n) = \min\{i : n_i = 0\}$ for $n = \sum_{i=0}^{\infty} n_i 2^i$ with $n_i \in \{0,1\}$. For a generator $Q^I[1]$ of H_*QS^0, suppose $I = (i_1,\dots,i_s)$ with $s > 1$ is a sequence so that $\mathrm{ex}(I) < 2^{\phi(i_1)}$ and $0 \leqslant 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for $1 \leqslant j \leqslant s-1$. Then $Q^I[1]$ is $\mathcal{A}$-annihilated. If $I = (i)$ with $i < 2^{\phi(i)}$, i.e. $i = 2^t - 1$ for some $t > 0$, then $Q^i[1]$ is $\mathcal{A}$-annihilated. Here, $\mathrm{ex}(Q^I x) = i_1 - (i_2 + \cdots + i_s)$.*

The above theorem can be generalised to describe $\mathcal{A}$-annihilated monomials $Q^I x$ in homology of QX. We refer the reader to [42, Theorem 2] for more details (see also [43]).

2.3. $H_*(\mathbb{Z}\times BO)$. We follow [29] to complete our description of $H_*(\mathbb{Z}\times BO)$. Write $\iota : \mathbb{R}P \to \{1\}\times BO \to \mathbb{Z}\times BO$ for the inclusion and $a_i \in H_i\mathbb{R}P$ for a generator with $i > 0$, and let $e_i = \iota_* a_i$ for $i > 0$. For $S^0 = \{0,1\}$, let $\chi : S^0 \to \mathbb{Z}\times BO$ send 0 into $\{0\}\times BO$ and 1 into $\{1\}\times BO$. By the infinite loop space structure of $\mathbb{Z}\times BO$, provided by Bott periodicity, there exists an infinite loop map $\overline{\chi} : QS^0 \to \mathbb{Z}\times BO$ [23]. Setting $e_0 = \overline{\chi}_*[1]$, $\overline{\chi}_*[-1] = e_0^{-1}$, we have an isomorphism of algebras

$$H_*(\mathbb{Z}\times BO) \cong \mathbb{Z}/2[e_0, e_0^{-1}, e_i : \deg e_i = i].$$

The product in this case is coming from the loop structure induced by the Whitney sum. The elements e_0^n, $n \in \mathbb{Z}$, provide translation maps between different path components given by $*\,e_0^n : H_*(\{m\}\times BO) \to H_*(\{m+n\}\times BO)$ which is simply multiplying by e_0^n whose inverse is $*\,e_0^{-n}$; this is analogous to the role of $[n]$ in H_*QS^0. We may also write $e_0^n\,*$ for $*\,e_0^n$ when it is more convenient with $*$ referring to the product on $H_*(\mathbb{Z}\times BO)$ under the additive structure.

2.4. H_*BO inside $H_*(\mathbb{Z}\times BO)$. The aim of this section is to clear up some subtlety which arises from geometric considerations. The subtlety of presenting H_*BO as a subalgebra of $H_*(\mathbb{Z}\times BO)$ is that the space $\mathbb{Z}\times BO$ is an E_∞ ring space [24] with two different products coming from the Whitney sum of virtual vector bundles (the additive structure) and tensor product of virtual vector bundles (the multiplicative structure); the delooping of $\mathbb{Z}\times BO$ provided by the Bott periodicity corresponds to the additive structure. At the prime $p=2$ these provide two different loop structures on $\mathbb{Z}\times BO$, consequently two different ring structures on $H_*(\mathbb{Z}\times BO)$ [22]. Now, the base point component of $\mathbb{Z}\times BO$ when equipped with the additive structure is $\{0\}\times BO$ (denoted by $BO_\oplus$ in the literature) on which the product comes from Whitney sum of virtual vector bundles of dimension 0, whereas the basepoint component of $\mathbb{Z}\times BO$ under the multiplicative structure is $\{1\}\times BO$ (denoted by $BO_\otimes$ in the literature) on which the product is the tensor product of virtual vector bundles of dimension 1. Moreover, the translation map $*\, e_0 : H_*BO_\oplus \to H_*BO_\otimes$ is not a ring map and $H_*(\{0\}\times BO)$ and $H_*(\{1\}\times BO)$ are not isomorphic as rings (although they are additively isomorphic under $*\, e_0$). Hence, since we equip $H_*(\mathbb{Z}\times BO)\simeq\mathbb{Z}/2[e_0,e_0^{-1},e_i : i>0]$ with the product induced by the Whitney sum then $H_*(\{1\}\times BO)$ is not an honest subalgebra of $H_*(\mathbb{Z}\times BO)$ under the additive structure. So, from now on we fix the additive structure on $\mathbb{Z}\times BO$. Our favourite copy of BO in $\mathbb{Z}\times BO$ is $\{0\}\times BO$. Following [29, Section 2], for $i>0$, we set $\overline{e_i}=e_0^{-1}e_i$ which allows us to identify

$$H_*BO\simeq H_*(\{0\}\times BO)\simeq\mathbb{Z}/2[\overline{e_i} : i>0]$$

as a subalgebra of $H_*(\mathbb{Z}\times BO)$.

Remark 2.3. For the sake of presentation, it could be more natural to take $H_*(\{0\}\times BO)\simeq\mathbb{Z}/2[\overline{e_i} : i>0]$ with $\deg\overline{e_i}=i$ and then write $H_*(\mathbb{Z}\times BO)\simeq H_*(\{0\}\times BO)[e_0,e_0^{-1}]$ as a Laurent polynomial algebra which is similar to [34, Page 33]. However, we have followed the tradition in our presentation.

Note that additively $H_*(\mathbb{Z}\times BO)\simeq\bigoplus_{n\in\mathbb{Z}}H_*(\{n\}\times BO)$. Since we work with the additive infinite loop structure on $\mathbb{Z}\times BO$, the ring structure provides various compatible pairings

$$H_*(\{n\}\times BO)\otimes H_*(\{m\}\times BO)\to H_*(\{n+m\}\times BO).$$

In particular, for $e_{i_1}^{k_1}\dots e_{i_r}^{k_r}\in H_*(\mathbb{Z}\times BO)$ with $0\leqslant i_1<\cdots<i_r$ we have $e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}\in H_*(\{k\}\times BO)$ where $k=\sum k_j$. Note that here only k_1 is allowed to be negative if $i_1=0$ and $k_j>0$ for all $j>1$. This induces a filtration on $H_*(\mathbb{Z}\times BO)$, say the length filtration, which assigns length $k=\sum k_j$ to a monomial $e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ which we denote by $l(e_{i_1}^{k_1}\cdots e_{i_r}^{k_r})=k$. We may also introduce a reduced length filtration on $H_*(\mathbb{Z}\times BO)$ which is defined as follows

$$\bar{l}(e_{i_1}^{k_1}\cdots e_{i_r}^{k_r})=\begin{cases} k & \text{if } i_1>0,\\ \sum_{j>1}k_j & \text{if } i_1=0.\end{cases}$$

Let $\xi = \sum_{\text{finite}} \xi_k$ be given where $\xi_k \in H_*(\mathbb{Z}\times BO)$ is nonzero for all k. We write $l(\xi) = l$ if and only if $l(\xi_k) = l$ for all k. We use a similar convention for the reduced length filtration. The following is about the relation between the action of $\mathcal{A}^{\mathrm{op}}$ on $H_*(\mathbb{Z}\times BO)$ and these filtrations.

Lemma 2.4. *(i) The action of $\mathcal{A}^{\mathrm{op}}$ on $H_*(\mathbb{Z}\times BO)$ respects the length filtration in the sense that if $Sq^t_*\xi \neq 0$ and $l(\xi) = k$ then $l(Sq^t_*\xi) = k$ where ξ is any monomial in $H_*(\mathbb{Z}\times BO)$.*
(ii) The action of $\mathcal{A}^{\mathrm{op}}$ on $H_*(\mathbb{Z}\times BO)$ preserves the reduced length filtration in the sense that if $Sq^t_*\xi \neq 0$ and $\bar{l}(\xi) = r$ then $\bar{l}(Sq^t_*\xi) = r$ where ξ is any monomial in $H_*(\mathbb{Z}\times BO)$.

Proof. (i) This is evident from the Cartan formula.
(ii) By definition, for $\xi = e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ we have

$$\bar{l}(\xi) = \begin{cases} l(\xi) & \text{if } i_1 > 0,\\ l(e_{i_2}^{k_2}\cdots e_{i_r}^{k_r}) & \text{if } i_1 = 0.\end{cases}$$

So we only have to verify the claim in the case $i_1 = 0$. Note that $Sq^t_* e_{i_1}^{k_1} = 0$ for all $t > 0$. Hence, by the Cartan formula, $Sq^t_*\xi = e_{i_1}^{k_1} Sq^t_*(e_{i_2}^{k_2}\cdots e_{i_r}^{k_r})$. Our claim, now follows from part (i) noting that by instability conditions it is impossible to have $e_{i_j}^{k_j}$ with $i_j > 0$ such that $Sq^t_* e_{i_j}^{k_j} = e_0^{\alpha_j}$ for some nonnegative integer α_j. □

Remark 2.5. Since $H_*(\mathbb{Z}\times BO)$ is a polynomial algebra then any $\xi \in H_*(\mathbb{Z}\times BO)$ can be written as a sum of finite number of monomials of the form $e_I^K := e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ with $r > 0$ where $I = (i_1,\dots,i_r)$ and $K = (k_1,\dots,k_r)$. So, for any $\xi \in H_*(\mathbb{Z}\times BO)$ we may write

$$\xi = \sum \epsilon_I^K(\xi) e_I^K$$

where the sum runs over all sequences $I = (i_1,\dots,i_r)$ with $0 \leqslant i_1 < \cdots < i_r$ as well as all sequences of integers $K = (k_1,\dots,k_r)$ so that k_1 is allowed to be negative if $i_1 = 0$ and $k_j \geqslant 0$ for all $j > 1$. Here, $e_I^K = e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ and $\epsilon_I^K(\xi) \in \mathbb{Z}/2$ so that except for finitely many I and K we have $\epsilon_I^K(\xi) = 0$. We may rearrange terms and use the length filtration to write

$$\xi = \sum_{k\in\mathbb{Z}} \sum_{k_1+\cdots+k_r=k} \epsilon_I^K(\xi) e_I^K.$$

We write $\xi(k) = \sum_{k_1+\cdots+k_r=k} \epsilon_I^K(\xi) e_I^K$ to be the sum of all terms of ξ of length k so $\xi = \sum \xi(k)$. Moreover, among terms of the same length k, we may impose the reduced length filtration, and set

$$\xi(k,l) = \sum \epsilon_I^K(\xi) e_I^K$$

where the sum runs over all terms of ξ with length equal to k and reduced length equal to l. In these terms, we have $\xi(k) = \sum_{l\in\mathbb{Z}} \xi(k,l)$.

The splitting of terms in the above remark has an immediate consequence which we record as follows.

Lemma 2.6. *(i) Suppose $\xi = \sum \xi(k)$ as in Remark 2.5. Then ξ is $\mathcal{A}$-annihilated if and only if $\xi(k)$ is for all k.*
(ii) Suppose $\xi(k) = \sum_{l\in\mathbb{Z}} \xi(k,l)$ as in Remark 2.5. Then, $\xi(k)$ is $\mathcal{A}$-annihilated if and only if $\xi(k,l)$ is for all l.

Proof. (i) By the Cartan formula, the action of Sq^t_* operations respects length. Therefore if ξ is $\mathcal{A}$-annihilated so is ξ_r. The converse follows from the additivity of Sq^t_* operations.
(ii) The action of the Steenrod algebra on H^*X respects reduced length. This implies our claim in one direction now follows. The converse follows from the additivity of Sq^t_* operations. □

Remark 2.7. Note that $H_*(\mathbb{Z}\times BO)$ is equipped with three filtrations provided by dimension, length and reduced length. This is compatible with the filtration of [33] and the new ingredient here is the reduced length filtration. The reduced length of a monomial e_I^K tells us its "length" when translated back into $H_*BO_\oplus$ and written in terms of $\overline{e_i}$ classes. In fact, if $l(e_I^K) = m$ then $\overline{e_I}^K = e_0^{-m}e_I^K \in H_*BO_\oplus$. Moreover, if $\bar{l}(e_I^K) = n$ then $\overline{e_I}^K$ is in the image of $H_*BO(n) \to H_*BO$ (see Section 2.5 below for more discussion on this).

Finally, we wish to record the following logarithmic property of length and reduced length functions.

Lemma 2.8. *Given two monomials $\xi, \eta \in H_*(\mathbb{Z}\times BO)$ we have*

$$l(\xi\eta) = l(\xi) + l(\eta),\ \bar{l}(\xi\eta) = \bar{l}(\xi) + \bar{l}(\eta).$$

The proof is immediate from definitions.

2.5. Presenting $H_*BO(k)$ inside H_*BO. In order to avoid any confusion with the existing literature, we need to fix our notations and choose our presentations of homologies very carefully. As mentioned at the introduction, while writing monomials $e_{i_1}^{k_1}\cdots e_{i_r}^{k_r}$ in $H_*(\mathbb{Z}\times BO)$, we fix the order $0 \leqslant i_1 < \cdots < i_r$ on variables where the role of e_0 is to change the path component. Since we have chosen $\{0\}\times BO$ as our favourite copy of BO in $\mathbb{Z}\times BO$, we need to describe $H_*BO(k)$ in accordance with this choice.

Recall that $\overline{e_i} := e_0^{-1}e_i$ for $i > 0$. We write $\overline{e_0} \in H_0BO(k) \simeq \mathbb{Z}/2$ for a generator. Note that we may define $\overline{e_0} = e_0^{-1}e_0 = 1$ where 1 is the unit element of the ring $H_*(\mathbb{Z}\times BO)$, and by abuse of notation write $\overline{e_0} \in H_0BO(k)$ for a generator which maps to $1 \in H_0(\mathbb{Z}\times BO)$ under the inclusion $BO(k) \to \{0\}\times BO \to \mathbb{Z}\times BO$. This latter would be equivalent to defining $\overline{e_0} = \chi_*[0]$ which would imply $Q^i\overline{e_0} = 0$ for all $i > 0$. Any of these definitions for $\overline{e_0}$ would be fine.

For any $k > 0$ the inclusion $\iota_k : BO(k) \to BO$ induces a monomorphism of $\mathcal{A}^{op}$-modules in homology. In this case, $H_*BO(k)$ is identified with the $\mathcal{A}^{\mathrm{op}}$-submodule of H_*BO generated by all monomials $\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r}$ with $0 \leqslant i_1 < \cdots < i_r$ and length $\sum_{i=1}^{r} k_i \leqslant k$ (compare to [2, Page 154] noting that their e_i coincides with our $\overline{e_i}$).

The point of choosing $\overline{e_0}$ is that since it is the unit element in $H_*(\mathbb{Z} \times BO)$ then $\overline{e_0}^{i_0} = \overline{e_0}$. Hence, given an element $\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r}$ which we know falls into the image of $(\iota_n)_* : H_*BO(n) \to H_*BO$ for any n with $k := \sum k_j \leqslant n$, we may regard this monomial as of full length, i.e. of length n, just by adding the required number of e_0 in front of it as

$$\overline{e_0}^{n-k}\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r} = \overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r}$$

in $H_*(\mathbb{Z} \times BO)$ as well as H_*BO. We use this convention which allows to think of elements of $H_*BO(n)$ to always have length n.

3. The stable symmetric hit problem

3.1. Preparatory observations. We prove Theorem 1.2 in a few steps. We begin by proving Lemma 1.1.

Proof of Lemma 1.1. (i) By definition of ex_j it follows that $\mathrm{ex}_j - \mathrm{ex}_{j-1} = 2i_{j+1} - i_j$. The immediately implies the I is admissible if and only if $\mathrm{ex}_j \leqslant \mathrm{ex}_{j+1}$ for all $j = 0, \dots, r-1$.
(ii) Since all entries of I are odd then $i_j = 2i_{j+1}$ cannot happen and the admissibility condition $i_j \leqslant 2i_{j+1}$ implies that $2i_{j+1} - i_j \geqslant 1$. Noting that $\mathrm{ex}_j = \mathrm{ex}_{j-1} + (2i_{j+1} - i_j)$ we deduce that $\mathrm{ex}_j > \mathrm{ex}_{j-1}$. This completes the proof.
(iii) By part (i) $\mathrm{ex}_{j-1} \geqslant \mathrm{ex}(I) > 0$ for all $j = 1, \dots, r-1$. It follows that $i_j > (i_{j+1} + \cdots + i_r) > i_{j+1}$ as we work with sequence only having positive entries. □

Next, we record some formulae on the action of Kudo-Araki operations. The following is a corollary of Theorem 2.2 and the above description of $H_*(\mathbb{Z}\times BO)$.

Lemma 3.1. *Suppose I is a sequence satisfying the conditions of Theorem 2.2. Then, $Q^I e_0, Q^I e_0^{-1}$ are $\mathcal{A}$-annihilated classes in $H_*(\mathbb{Z} \times BO)$.*

Proof. Since $\overline{\chi}$ is an infinite loop map, then $\overline{\chi}_* Q^I[1] = Q^I e_0$ and $\overline{\chi}_* Q^I[-1] = Q^I e_0^{-1}$. The mapping $\overline{\chi}_*$ is an $\mathcal{A}^{\mathrm{op}}$-module homomorphism by naturality of the Steenrod operations. The classes $Q^I[1]$ and $Q^I[-1]$ satisfying the conditions of Theorem 2.2 are $\mathcal{A}$-annihilated in H_*QS^0. Therefore, $Q^I e_0 = \overline{\chi}_* Q^I[1]$ and $Q^I e_0^{-1} = \overline{\chi}_* Q^I[-1]$ are $\mathcal{A}$-annihilated in $H_*(\mathbb{Z} \times BO)$. □

The proof of Theorem 1.2(i) follows from evaluating $Q^I e_0$ in $H_*(\mathbb{Z} \times BO)$ for I admissible for which we appeal to Priddy's computations (see Theorem 3.2) together with an application of Lemma 2.6. The proof of Theorem 1.2(ii) is immediate upon applying the length filtration mentioned in Note 2.5 and having part (i). The proof of Theorem 1.2(iii) regarding the linear independence needs a bit of work which will follow later on.

The evaluation of the Q^i operations on the generators of $H_*(\mathbb{Z}\times BO)$ has been carried out by Priddy [29, Theorem 1.1 and Corollary 2.3] (see also work of Kochman [18] and compare to [34, Chapter 1, Proposition 5.11]).

Theorem 3.2. *(i) For $n > k \geqslant 0$ we have*

$$Q^n e_k = \sum_{u=0}^{k} \binom{n-k+u-1}{u} e_{n+u} e_{k-u}.$$

(ii) For $n > 0$ we have

$$Q^n e_0^{-1} = \sum \binom{k}{k_1,\dots,k_n} e_1^{k_1}\cdots e_n^{k_n} e_0^{-k-2}$$

where the sum is over all sequences $k_1,\dots,k_n$ of nonnegative integers with $\sum ik_i = n$, $k=\sum k_i$ and

$$\binom{k}{k_1,\dots,k_n} = \frac{k!}{k_1!\cdots k_n!}.$$

By properties recalled in Section 2.2 it follows immediately that

$$Q^k e_k = e_k^2,\ Q^0 e_0^{-1} = e_0^{-2}$$

which are the obvious cases that are not considered by part (i) and part (ii) of the above theorem, respectively.

Remark 3.3. By Theorem 3.2(i), by choosing the term with $u=0$ in Priddy's formula, for any $n > k \geqslant 0$

$$Q^n e_k = e_k e_n$$

modulo terms $e_{k-u}e_{n+u}$, hence modulo terms of the form e_ie_j with $j > n$ and $i < k$ corresponding to the cases with $u > 0$. Consequently, none of the other terms cancels out with e_ke_n showing that $Q^n e_k \neq 0$.

Next, we show that for I admissible $Q^I e_0 \neq 0$. We freely use the observation and notations of Lemma 1.1 throughout the statement of the lemma and its proof.

Lemma 3.4. *Suppose $I=(i_1,\dots,i_r)$ is an admissible sequence with $\mathrm{ex}(I) > 0$. Then, in $H_*(\mathbb{Z}\times BO)$ we have*

$$Q^I e_0 = e_0 e_{\mathrm{ex}_0} e_{\mathrm{ex}_1}^2 \cdots e_{\mathrm{ex}_{r-1}}^{2^{r-1}} + O_I \neq 0.$$

For $r=1$, $O_I=0$. For $r>1$,

$$O_I = \sum \epsilon_L e_L$$

where $\epsilon_L \in \mathbb{Z}/2$ and the sum runs over all nondecreasing sequences of nonnegative integers $L=(l_1,\dots,l_{2^r})$ so that $l_{2^r} > \mathrm{ex}_{r-1} = i_r$ and $e_L = e_{l_1}\cdots e_{l_{2^r}}$.

Proof. For $r=1$, $I=(i_r)$ and by Theorem 3.2(i) we have $Q^{i_r}e_0 = e_0e_{i_r} \neq 0$. In particular, there are no other terms as $u \in \{0,k\}$ with $k=0$ in Priddy's formula. So, the statement holds for $r=1$.

We now prove the case for $r>1$. We proceed by induction, and in order to make an illustration, we do the cases $r=2,3$. For $r=2$ with $I=(i_{r-1},i_r)$, using the Cartan formula and the computation for the case $r=1$, we compute that

$$\begin{aligned} Q^{i_{r-1}}Q^{i_r}e_0 &= Q^{i_{r-1}}(e_0e_{i_r}) \\ &= \textstyle\sum_{\beta_{r-1}=0}^{i_{r-1}}(Q^{i_{r-1}-\beta_{r-1}}e_0)(Q^{\beta_{r-1}}e_{i_r}). \end{aligned}$$

We compute each term $(Q^{i_{r-1}-\beta_{r-1}}e_0)(Q^{\beta_{r-1}}e_{i_r})$ of the above sum as follows. By Priddy's formula, Theorem 3.2(i), $Q^{i_{r-1}-\beta_{r-1}}e_0 = e_0e_{i_{r-1}-\beta_{r-1}}$. For the term $Q^{\beta_{r-1}}e_{i_r}$ we proceed as follows. By Lemma 1.1(iii), as I is admissible of positive excess, $i_{r-1} > i_r$. In the above sum, for β_{r-1} running from 0 to i_{r-1}, if $\beta_{r-1} < i_r$ then $Q^{\beta_{r-1}}e_{i_r} = 0$ by basic properties of Kudo-Araki operation recalled in Section 2.2. Hence, in order to get nontrivial terms in the above sum, we have to restrict to $\beta_{r-1} \geqslant i_r$. Again by the properties recalled in Section 2.2, for $\beta_{r-1} = i_r$ we have $Q^{\beta_{r-1}}e_{i_r} = e_{i_r}^2$. Consequently, after separating the term with $\beta_{r-1} = \mathrm{ex}_{r-1} = i_r$, hence $i_{r-1} - \beta_{r-1} = i_{r-1} - i_r = \mathrm{ex}_{r-2}$, we obtain

$$Q^{i_{r-1}}Q^{i_r}e_0 = e_0e_{\mathrm{ex}_{r-2}}e_{i_r}^2 + \sum_{\beta_{r-1}=i_r+1}^{i_{r-1}} e_0e_{i_{r-1}-\beta_{r-1}}(Q^{\beta_{r-1}}e_{i_r}) \tag{2}$$

where by Remark 3.3 we see that $Q^{\beta_{r-1}}e_{i_r} = e_{i_r}e_{\beta_{r-1}}$ modulo terms $e_{l_{r-1}}e_{l_r}$ with $l_r > \beta_{r-1}$ and $l_{r-1} < i_r$. Note that by our choice of β_{r-1} these inequalities combine and yield

$$l_r > \beta_{r-1} \geqslant i_r + 1 > i_r.$$

Our claim now follows.

For $r=3$ with $I=(i_{r-2},i_{r-1},i_r)$, using our computations for the case of $r=2$, we have

$$\begin{aligned} Q^{i_{r-2}}Q^{i_{r-1}}Q^{i_r}e_0 &= Q^{i_{r-2}}(e_0e_{\mathrm{ex}_1(I)}e_{\mathrm{ex}_2(I)}^2 + \textstyle\sum \epsilon_Le_L) \\ &= Q^{i_{r-2}}(e_0e_{\mathrm{ex}_1(I)}e_{\mathrm{ex}_2(I)}^2) + Q^{i_{r-2}}(\textstyle\sum \epsilon_Le_L) \end{aligned} \tag{3}$$

where $L=(l_1,l_2,l_3,l_4)$, $\epsilon_L \in \mathbb{Z}/2$, and $l_4 > i_r = \mathrm{ex}_2(I)$. Applying the Cartan formula (see Remark 2.1), we see that the first term of (3) can be written as

$$Q^{i_{r-2}}(e_0e_{\mathrm{ex}_1(I)}e_{\mathrm{ex}_2(I)}^2) = \sum_{j_1+j_2+j_3=i_{r-2}} (Q^{j_1}e_0)(Q^{j_2}e_{\mathrm{ex}_1(I)})(Q^{j_3}e_{\mathrm{ex}_2(I)}^2). \tag{4}$$

First we extract the leading term of our expression. Choose

$$\begin{aligned} j_3 &= \dim(e_{\mathrm{ex}_2(I)}^2) = 2\mathrm{ex}_2(I) = 2i_r, \\ j_2 &= \dim(e_{\mathrm{ex}_1(I)}) = i_{r-1} - i_r. \end{aligned}$$

Note that by our conventions, for $I=(i_{r-2},i_{r-1},i_r)$ we have $I_1=(i_{r-1},i_r)$ and $I_2=(i_r)$ which yield $\mathrm{ex}_1(I) = \mathrm{ex}(I_1) = i_{r-1}-i_r$ and $\mathrm{ex}_2(I) = i_r$. Moreover, by the hypothesis and Lemma 1.1(i) we have $0 < \mathrm{ex}(I) \leqslant \mathrm{ex}_1(I) \leqslant \mathrm{ex}_2(I)$. These choices together with $\sum j_k = i_{r-2}$ give $j_1 = i_{r-2} - (i_{r-1} + i_r) = \mathrm{ex}_{r-3}(I)$. As recalled in Remark 2.1 we have $Q^{2t+1}\xi^2 = 0$ and $Q^{2t}\xi^2 = (Q^t\xi)^2$. This implies that for the above choices of j_1, j_2, j_3 we have

$$(Q^{j_1}e_0)(Q^{j_2}e_{\mathrm{ex}_1(I)})(Q^{j_3}e^2_{\mathrm{ex}_2(I)}) = e_0e_{\mathrm{ex}_0(I)}e^2_{\mathrm{ex}_1(I)}e^{2^2}_{\mathrm{ex}_2(I)}$$

which is our leading term. For the other terms of (4), note that in order to have a nontrivial term we need $j_3 > \dim e^2_{\mathrm{ex}_2(I)} = 2i_r$. By the Cartan formulae, we see that

$$Q^{j_3}(e^2_{\mathrm{ex}_2(I)}) = \begin{cases} (Q^{j_3'}e_{\mathrm{ex}_2(I)})^2 & \text{if } j_3 = 2j_3', \\ 0 & \text{otherwise} \end{cases}$$

where $j_3 > 2i_r$ implies that $j_3' > i_r$. Hence, all of the terms in the other terms of (4) are of the claimed forms. Note that by Priddy's formula, applying one single operation Q^n doubles the length, that is Q^ne_k is a sum of terms of length 2. Hence, taking care of the powers of variables, all terms in the equation (4) are of length $(2^2)^2$. In order to complete our verification, consider the second term of (3). By the Cartan formula, for any single L, we have

$$Q^{i_{r-2}}e_L = \sum_{\sum j_k = i_{r-2}} (Q^{j_1}e_{l_1})\cdots(Q^{j_4}e_{l_4}).$$

It is immediate that this leads to an expression with terms of length $(2^2)^2$. Moreover, if a term in the above expression is nontrivial then $j_4 \geqslant l_4 > i_r$. Consequently, all terms in this expression have the claimed form.

For the general case, suppose $I' = (i', i_1, \dots, i_r)$ is an admissible sequence with $\mathrm{ex}(I') > 0$. This means $I = (i_1, \dots, i_r)$ is also admissible with $\mathrm{ex}(I) > 0$ and by the induction hypothesis we have

$$Q^Ie_0 = e_0e_{\mathrm{ex}(I_0)}e^2_{\mathrm{ex}(I_1)}\cdots e^{2^{r-1}}_{\mathrm{ex}(I_{r-1})} + \sum \epsilon_Le_L$$

with $l(L) = 2^r$ and $l_{2^r} > i_r$ which allows us to write

$$\begin{aligned} Q^{I'}e_0 &= Q^{i'}Q^Ie_0 \\ &= Q^{i'}(e_0e_{\mathrm{ex}_0(I)}e^2_{\mathrm{ex}_1(I)}\cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I)} + \sum \epsilon_Le_L) \\ &= Q^{i'}(e_0e_{\mathrm{ex}_0(I)}e^2_{\mathrm{ex}_1(I)}\cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I)}) + \sum \epsilon_LQ^{i'}e_L. \end{aligned}$$

By the Cartan formula for the first term we have

$$Q^{i'}(e_0e_{\mathrm{ex}_0(I)}e^2_{\mathrm{ex}_1(I)}\cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I)}) = \sum_{t_0+\cdots+t_{r-1}=i'} (Q^{t_0}e_0)\cdots(Q^{t_{r-1}}e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I)}).$$

We may write i' as

$$\begin{aligned} &2^{r-1}i_r + 2^{r-2}(i_{r-1} - i_r) + \cdots + (i_1 - i_2 - \cdots - i_r) + (i' - i_1 - i_2 - \cdots - i_r) \\ &= 2^{r-1}\mathrm{ex}_{r-1}(I) + \cdots + \mathrm{ex}_0(I) + \mathrm{ex}(I') \end{aligned}$$

which allows us to separate one term of the sum and the other terms and write

$$Q^{i'}(e_0e_{\mathrm{ex}_0(I)}e^2_{\mathrm{ex}_1(I)}\cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I)}) = e_0e_{\mathrm{ex}(I')}e^2_{\mathrm{ex}_0(I)}\cdots e^{2^r}_{\mathrm{ex}_{r-1}(I)} + \sum(Q^{t_0}e_0)\cdots(Q^{t_{r-1}}e^{2^{r-1}}_{\mathrm{ex}_{r-1}}) \tag{5}$$

where the sum is running over all sequences $(t_0,\dots,t_{r-1})$ of nonnegative integers with $\sum t_j = i'$ so that

$$(t_0,t_1,\dots,t_{r-1}) \neq (\mathrm{ex}(I'),\mathrm{ex}_0(I),2\mathrm{ex}_1(I),\dots,2^{r-1}\mathrm{ex}_{r-1}(I)).$$

The term $e_0e_{\mathrm{ex}(I')}e^2_{\mathrm{ex}_0}\cdots e^{2^r}_{\mathrm{ex}_{r-1}}$ is the 'leading term' as claimed. For the second sum, for any single term to be nontrivial we need $Q^{t_{r-1}}e^{2^{r-1}}_{\mathrm{ex}_{r-1}} \neq 0$ which having excluded the cases with $t_{r-1} = 2^{r-1}\mathrm{ex}_{r-1}(I) = 2^{r-1}i_r$ leaves us with the cases $t_{r-1} > 2^{r-1}i_r$. An iterated application of the Cartan formula $Q^{2t+1}\xi^2 = 0$ shows that if there exists $j\in\{0,\dots,r-1\}$ such that 2^j does not divide t_j then $Q^{t_j}e^{2^j}_{\mathrm{ex}_j(I)} = 0$ and consequently

$$(Q^{t_0}e_0)\cdots(Q^{t_{r-1}}e^{2^{r-1}}_{\mathrm{ex}_{r-1}}) = 0.$$

Therefore, the only nontrivial terms in the above sum correspond to r-tuples $(t_0,\dots,t_{r-1})$ such that 2^j divides t_j for all $0\leqslant j\leqslant r-1$. Moreover, the Cartan formula $Q^{2t}\xi^2 = (Q^t\xi)^2$ shows that if $t_j = 2^jp_j$ for some positive integer p_j then $Q^{t_j}e^{2^j}_{\mathrm{ex}_j(I)} = (Q^{p_j}e_{\mathrm{ex}_j(I)})^{2^j}$. Notice that $t_{r-1} > 2^{r-1}i_r$ which implies that $p_{r-1} > i_r$. Hence, the only possible nontrivial terms in the above sum must include a product factor as

$$Q^{p_{r-1}}e_{i_r} = e_{i_r}e_{p_{r-1}} \text{ modulo terms } e_\alpha e_\beta \text{ with } \beta > p_{r-1} > i_r.$$

Consequently, any term in $Q^{i'}e_L$ would include a factor e_{l_j} with $l_j > i_r$. Hence, so far, $Q^{(i',I)}e_0$ satisfies the claimed expression. Next, we deal with the terms in $\sum \epsilon_L Q^{i'}e_L$ where by the inductive assumption there exists j so that $l_j > i_r$; for each L we denote such a choice of j by j_0. By the Cartan formula, for each term $Q^{i'}e_L$ we have

$$Q^{i'}e_L = \sum_{t_1+\cdots+t_{2^r}=i'}(Q^{t_1}e_{l_1})\cdots(Q^{t_{2^r}}e_{l_{2^r}}).$$

If a term in the above sum is to be nontrivial we need $t_{j_0} > l_{j_0}$. As above, the factor $Q^{t_{j_0}}e_{l_{j_0}}$ gives the desired factors $e_\alpha e_\beta$ with $\beta > i_r$. The last claim that $l_{2^r} > \mathrm{ex}_{r-1}(I) = i_r$ can be shown by induction on r and a similar analysis as we did above which we leave to the reader. This completes the proof. □

Note 3.5. By iterated application of Cartan formula together with Theorem 3.2(i), also recorded in Lemma 3.4, all terms of Q^Ie_0 are of the same length filtration, namely filtration 2^r. But, different terms may have different reduced length filtration. For example consider $I = (5,3)$. By iterated application of Priddy's formula, we compute that

$$Q^5Q^3e_0 = e_0e_2e_3^2 + e_0e_1e_3e_4 + e_0e_1e_2e_5 + e_0e_1^2e_6 + e_0^2e_3e_5.$$

The last term has reduced length equal to 2 whereas all of other terms are of reduced length 3.

The implicit property which seems to be useful in some of our computations is that the reduced length of each term in $Q^5Q^3e_0$ is bounded below. We wish to have this property in full generality. We don't have a proof in the general case, however, and we provide a proof when $l(I)$ is small.

Lemma 3.6. *Suppose $I=(i_1,\dots,i_r)$ is an admissible sequence of positive dimension, having only odd entries,* $\mathrm{ex}(I)>0$ *and $r\in\{1,2\}$. If $Q^Ie_0=\sum\epsilon_Ke_K$ where K runs over all increasing sequences with $l(K)=2^r$ and $\epsilon_K\in\mathbb{Z}/2$ then $\bar{l}(e_K)\geqslant 2^{r-1}$ whenever $\epsilon_K=1$.*

We remark that a partial proof we had for $l(I)=3$ depends on too much detailed analysis, hence does not seem too hopeful to be generalised in an inductive manner.

Proof. By Lemma 3.4 we have $Q^Ie_0\neq 0$. Hence, there exists a K such that $\epsilon_K=1$.
Case of $l(I)=1$**.** For any $I=(i)$ with $i>0$, we have $Q^ie_0=e_0e_i$ which satisfies $\bar{l}(e_0e_i)=1=2^0=2^{l(I)-1}$. Hence, the above inequality holds in this case.
Case of $l(I)=2$**.** Suppose $I=(i_1,i_2)$ is an admissible as required by the lemma. Then, by Priddy's formula and the Cartan formula we have

$$\begin{aligned} Q^{i_1}Q^{i_2}e_0 &= Q^{i_1}(e_0e_{i_2})\\ &= \textstyle\sum_{j_1=i_2}^{i_1}(Q^{i_1-j_1}e_0)(Q^{j_1}i_2)\\ &= \textstyle\sum_{j_1=i_2}^{i_1}\sum_{u=0}^{i_2}\binom{j_1-i_2+u-1}{u}e_0e_{i_1-j_1}e_{i_2-u}e_{j_1+u}. \end{aligned}$$

First, suppose $j_1<i_1$. In this case $i_1-j_1>0$. Moreover, for any u we have $j_1+u\geqslant i_2>0$. Hence, any term with $j_1\neq i_1$ in the above sum has at least two e_α's with $\alpha>0$. Hence, $\bar{l}(e_0e_{i_1-j_1}e_{i_2-u}e_{j_1+u})\geqslant 2=2^{l(I)-1}$. Next, suppose $j_1=i_1$ where the corresponding terms are $\binom{i_1-i_2+u-1}{u}e_0^2e_{i_2-u}e_{j_1+u}$ with $0\leqslant u\leqslant i_2$. Obviously, for $u<i_2$ any term, if with nonzero coefficient, has reduced length equal to 2. For $u=i_2$ the term is $\binom{i_1-1}{i_2}e_0^3e_{i_1+i_2}=0$ and the binomial coefficient is trivial since i_1 and i_2 are both odd by hypothesis. Hence, any term in the expression for $Q^{i_1}Q^{i_2}e_0$ is either of reduced length 3 or 2. In particular, this proves our claim on the lower bound for the reduced length. □

3.2. Proof of Theorem 1.2. This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) Let $\xi_I^{\mathbb{Z}\times BO}:=Q^Ie_0\in H_{|I|}Q(\mathbb{Z}\times BO)$ where I is chosen as in Theorem 1.2. By Lemma 3.1 this class is $\mathcal{A}$-annihilated. By Lemma 3.4

$$\xi_I^{\mathbb{Z}\times BO}=e_0e_{\mathrm{ex}_0}e_{\mathrm{ex}_1}^2\cdots e_{\mathrm{ex}_{r-1}}^{2^{r-1}}+O_I\neq 0$$

where $O_I = 0$ if $r = 1$ and for $r > 1$,

$$O_I = \sum \epsilon_L e_L$$

where $\epsilon_L \in \mathbb{Z}/2$ and the sum runs over all sequences of nonnegative integers $L = (l_1, \dots, l_{2^r})$ satisfying conditions of Lemma 3.4. Since I satisfies conditions of Theorem 1.2 then by Lemma A.2 all entries of I are odd. It follows from Lemma 1.1(ii) that $\mathrm{ex}_{j-1} < \mathrm{ex}_j$ for $0 < j < r$.

(ii) We note that $e_{\mathrm{ex}_j} = e_0\overline{e_{\mathrm{ex}_j}}$ for $0 \leqslant j < r$. Note that by hypothesis $\mathrm{ex}_0 = \mathrm{ex}(I) > 0$. Hence,

$$\xi_I^{\mathbb{Z}\times BO} = e_0 e_0^{2^r-1}\overline{e_{\mathrm{ex}_0}}\,\overline{e_{\mathrm{ex}_1}}^2 \cdots \overline{e_{\mathrm{ex}_{r-1}}}^{2^{r-1}} + O_I.$$

Let $\xi_I^{BO} = e_0^{-2^r}\xi_I^{\mathbb{Z}\times BO}$ which is given by

$$\xi_I^{BO} = \overline{e_{\mathrm{ex}_0}}\,\overline{e_{\mathrm{ex}_1}}^2 \cdots \overline{e_{\mathrm{ex}_{r-1}}}^{2^{r-1}} + e_0^{-2^r}O_I.$$

The class e_0 is also $\mathcal{A}$-annihilated, so is $e_0^{-2^r}$. Hence, by the Cartan formula ξ_I^{BO} is also $\mathcal{A}$-annihilated, which obviously lives in $H_*(\{0\}\times BO)$. Now, it is obvious that being a sum of monomials of length $2^r - 1$ the class ξ_I^{BO} is in the image of $H_*BO(2^r-1) \to H_*BO$, induced by the inclusion $BO(2^r-1) \to BO$, which is a monomorphism of $\mathcal{A}$-modules, i.e. ξ_I^{BO} pulls back to an $\mathcal{A}$-annihilated class in $H_*BO(2^r-1)$.

(iii) We wish to show that the set $\{\xi_I^{\mathbb{Z}\times BO} : I \text{ admissible}\}$ is a linearly independent set. By filtration considerations as well as dimensional reasons, it is enough to show that

$$\mathcal{B}_{r,d} = \{\xi_I^{\mathbb{Z}\times BO} : I \text{ admissible where } l(I) = r, |I| = d\}$$

is a linearly independent set where r and d are arbitrary positive integers. This is equivalent to showing that every finite subset of $\mathcal{B}_{r,d}$ is a linearly independent set. As we work over $\mathbb{Z}/2$, it is enough to show that given a finite set $\{\xi_{I_k}^{\mathbb{Z}\times BO}\}_{k=1}^n \subseteq \mathcal{B}_{r,d}$ no nontrivial linear combination of its elements adds up to 0. More precisely, given a finite set $\{\xi_{I_k}^{\mathbb{Z}\times BO}\}_{k=1}^n \subseteq \mathcal{B}_{r,d}$ we have to show if $\{c_k : c_k \in \mathbb{Z}/2\}_{k=1}^n$ is given so that at least one c_k is nonzero then

$$\sum_{k=1}^n c_k \xi_{I_k}^{\mathbb{Z}\times BO} \neq 0.$$

Obviously, we can drop the terms with $c_k = 0$. Since we choose $c_k \in \mathbb{Z}/2$ then this boils down to showing that for every finite subset $\{\xi_{I_k}^{\mathbb{Z}\times BO}\}_{k=1}^n \subseteq \mathcal{B}_{r,d}$ we have

$$\sum_{k=1}^n \xi_{I_k}^{\mathbb{Z}\times BO} \neq 0.$$

By definition of the classes $\xi_I^{\mathbb{Z}\times BO}$ as well as by Lemma 3.4 we have

$$\xi_{I_k}^{\mathbb{Z}\times BO} = Q^{I_k}e_0 = e_0 e_{\mathrm{ex}_0(I_k)} e_{\mathrm{ex}_1(I_k)}^2 \cdots e_{\mathrm{ex}_{r-1}(I_k)}^{2^{r-1}} + \sum \epsilon_L e_L$$

so that $l_{2^r} > \mathrm{ex}_{r-1}(I_k)$. Here, $l(I) = r$ and $|I| = d$ is fixed. We have two cases.
Case 1. Suppose sequences $I_1, \dots, I_n$ are given with $l(I_k) = r$ for all $1 \leqslant k \leqslant n$ so that there is a unique sequence say I_{k_m} among them with $\mathrm{ex}_{r-1}(I_{k_m}) < \mathrm{ex}_{r-1}(I_k)$ for all $k \neq k_m$. Then the leading term of $Q^{I_{k_m}} e_0$, namely $e_0 e_{\mathrm{ex}_0(I_{k_m})} e^2_{\mathrm{ex}_1(I_{k_m})} \cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I_{k_m})}$, is not cancelled out by the leading term of $Q^{I_k} e_0$, namely $e_0 e_{\mathrm{ex}_0(I_k)} e^2_{\mathrm{ex}_1(I_k)} \cdots e^{2^{r-1}}_{\mathrm{ex}_{r-1}(I_k)}$, as $\mathrm{ex}_{r-1}(I_{k_m}) < \mathrm{ex}_{r-1}(I_k)$. Moreover, other terms in $Q^{I_k} e_0$ are of the form e_L with $l_{2^r} > \mathrm{ex}_{r-1}(I_k) > \mathrm{ex}_{r-1}(I_{k_m})$. Hence, none of these terms also cancels out with the leading term of $Q^{I_{k_m}} e_0$. This completes the proof in this case.
Case 2. Suppose sequences $I_1, \dots, I_n$ are given so that sequences $I_{k_{m_1}}, \dots, I_{k_{m_s}}$ exist with $s > 1$, $\mathrm{ex}_{r-1}(I_{k_{m_1}}) = \cdots = \mathrm{ex}_{r-1}(I_{k_{m_s}})$ and $\mathrm{ex}_{r-1}(I_{k_{m_\alpha}}) < \mathrm{ex}_{r-1}(I_k)$ for all $k \notin \{k_{m_1}, \dots, k_{m_s}\}$. It is enough to show that the sum of leading terms of $Q^{I_{k_{m_\alpha}}} e_0$ is nonzero. However, this is obvious since if $I \neq J$ and $l(I) = l(J) = r$ then $(\mathrm{ex}_0(I), \dots, \mathrm{ex}_{r-1}(I)) \neq (\mathrm{ex}_0(J), \dots, \mathrm{ex}_{r-1}(J))$. This implies that in the leading terms there will always be different indices, hence the sum of them is nonzero. This completes the proof in this case.

Finally, to see that the classes ξ_I^{BO} are linearly independent, notice that $* \ e_0^{-2^r}$ is an isomorphism of $\mathbb{Z}/2$-modules whose inverse is $* \ e_0^{2^r}$. Consequently, it sends linearly independent sets to linearly independent sets, and in particular sends $\{\xi_I^{\mathbb{Z}\times BO}\}$ to $\{\xi_I^{BO}\}$, and the linear independence of the former implies the linear independence of the latter. Finally, note that the inclusion $\iota_k : BO(k) \to BO$ induces a monomorphism of $\mathcal{A}^{\mathrm{op}}$-modules. Hence, the pull back of the classes ξ_I^{BO} to $H_*BO(2^r - 1)$ are linearly independent. This completes the proof of part (iii). □

The proof has an interesting side result which we wish to record. A careful analysis of the above proof provides the following.

Corollary 3.7. *For any I chosen as in Theorem* 1.2 *with $l(I) = r$, the class ξ_I^{BO} is a sum of monomials $\overline{e_K}$ each of which is not divisible by e_0 neither by $\overline{e_0}$. More precisely*

$$Q^I e_0 = \sum \epsilon_K \overline{e_K}$$

where $\epsilon_K \in \mathbb{Z}/2$, $\overline{e_K} = \overline{e_{k_1}} \cdots \overline{e_{k_{2^r}}}$, where K runs over all admissible nondecreasing sequences of length 2^r with $k_1 > 0$.

The proof is easy and we leave it to the reader. After Lemma 3.4 one may ask about the cases when $Q^I e_0 = 0$ in $H_*(\mathbb{Z} \times BO)$ or in general $Q^I e_k = 0$. By Lemma 3.4 if $Q^I e_0 = 0$ then I is not admissible. It seems to the author that we can answer this question in the same line as [31, Lemma 3.5]. These computations require more topological tools to be recalled or introduced. So, we leave further investigation on this to a future work.

3.3. Proof of Theorem 1.4. The proof we present is an application of Theorem 3.2(ii).

Proof of Theorem 1.4. (i) An application of Lemma 3.1 together with Theorem A.1(i) shows that the class $Q^n e_0^{-1}$ is $\mathcal{A}$-annihilated if and only if $n = 2^t - 1$ for some $t > 0$. Let $\xi_t^{\mathbb{Z}\times BO} = Q^{2^t-1}e_0^{-1}$. The equation

$$\xi_t^{\mathbb{Z}\times BO} = \sum_{K\in\mathcal{K}_{2^t-1}} e_0^{-l(e^K)-2}e^K$$

is merely a restatement of Theorem 3.2(ii). For $i|2^t-1$ let K be a sequence of length $l(e^K) = 2^t-1$ so that $k_j = 0$ for all $j \neq i$ and k_i be defined by $2^t - 1 = ik_i$. Obviously the coefficient $\binom{l(e^K)}{k_1,\ldots,k_n} = \binom{k_i}{k_i} = 1$. The corresponding term would be $e_0^{-k_i-2}e_i^{k_i}$. In particular, our sum would involve the terms $e_0^{-(2^t-1)-2}e_1^{2^t-1}$ and $e_0^{-3}e_{2^t-1}$. Hence, $\xi_t^{\mathbb{Z}\times BO}$ involves some nontrivial terms. Also, note that $\xi_t^{\mathbb{Z}\times BO}$ is a sum of monomials e^K when K varies in $\mathcal{K}_{2^t-1}$, hence a sum of different monomials. Therefore, $\xi_t^{\mathbb{Z}\times BO} \neq 0$. Since $e_i = e_0\overline{e_i}$ we see that

$$\xi_t^{\mathbb{Z}\times BO} = \sum_{K\in\mathcal{K}_{2^t-1}} e_0^{-l(e^K)-2}e_0^{l(e^K)}\overline{e^K} = \sum_{K\in\mathcal{K}_{2^t-1}} e_0^{-2}\overline{e^K}$$

where $\overline{e^K} = \overline{e_1}^{k_1}\cdots\overline{e_{2^t-1}}^{k_{2^t-1}}$. Consequently, $*\, e_0^2$ being an isomorphism of $\mathbb{Z}/2$-vector spaces,

$$\xi_t^{BO} = e_0^2\xi_t^{\mathbb{Z}\times BO}$$

is a nontrivial $\mathcal{A}$-annihilated class. The terms $e_0^{-(2^t-1)-2}e_1^{2^t-1}$ and $e_0^{-3}e_{2^t-1}$ of $\xi_{2^t-1}^{\mathbb{Z}\times BO}$ translate to $\overline{e_1}^{2^t-1}$ and $\overline{e_{2^t-1}}$ in H_*BO as claimed. By filtration considerations, this class is in the image of $H_*BO(2^t-1) \to H_*BO$. Moreover, the presence of $\overline{e_1}^{2^t-1}$ does not allow this class to pull back to $H_*BO(s)$ for $s < 2^t-1$.
(ii) It immediately follows from the Cartan formula that given the classes $\xi_{\lambda_i}^{\mathbb{Z}\times BO} \in H_{2^{\lambda_i}-1}(\mathbb{Z}\times BO)$ which are $\mathcal{A}$-annihilated then the class

$$\xi_{\lambda_1,\ldots,\lambda_d}^{\mathbb{Z}\times BO} = \prod_{i=1}^{d}\xi_{\lambda_i}^{\mathbb{Z}\times BO}$$

is $\mathcal{A}$-annihilated. We only note that for two different partitions of n into spikes such as

$$(\lambda_1,\ldots,\lambda_d), (\mu_1,\ldots,\mu_s)$$

then the fact that the classes $\xi_{\lambda_1,\ldots,\lambda_d}^{\mathbb{Z}\times BO}$ and $\xi_{\mu_1,\ldots,\mu_s}^{\mathbb{Z}\times BO}$ involve the leading terms $\prod_{i=1}^{d} e_{2^{\lambda_i}-1}$ and $\prod_{j=1}^{s} e_{2^{\mu_j}-1}$, respectively, shows that $\xi_{\lambda_1,\ldots,\lambda_d}^{\mathbb{Z}\times BO} \neq \xi_{\mu_1,\ldots,\mu_s}^{\mathbb{Z}\times BO}$. We therefore obtain $sp(n)$ different $\mathcal{A}$-annihilated classes in $H_n(\mathbb{Z}\times BO)$. The case of H_*BO is similar and we leave it to the reader. $\square$

Remark 3.8. (i) The class $\xi := e_0^2Q^{2^t-1}e_0^{-1}$ lives in H_*BO and being expressed in terms of monomials e^K with $l(K) = 2^t - 1$ pulls back to $H_*BO(2^t-1)$. However, by Proposition 4.2, all of its terms, apart from $\overline{e_{2^t-1}}$, maps trivially under the transfer $t_*: H_*BO(2^t-1) \to H_*\mathbb{R}P^{\times(2^t-1)}$. This implies that

$$t_*\xi = \sum_{\tau\in\Sigma_{2^t-1}} \tau(a_{2^t-1}\otimes 1^{\otimes(2^t-2)}).$$

(ii) In theory, we may apply Theorem 1.4 to produce more $\mathcal{A}$-annihilated classes in H_*BO. Suppose d is given and assume $d = \sum_{j=1}^n (2^{t_j}-1)$. Then, we may look for sequences K^j with $|e^{K_j}| = 2^{t_j}-1$. Then the product of these elements is nontrivial. Moreover, a partition of d into spikes, i.e. writing d as $d = \sum_{j=1}^n (2^{t_j}-1)$, is not unique and we may look for various ways that can give us new $\mathcal{A}$-annihilated classes. This seems to be related to the μ function of Wood [41] but we have not exploited the idea in this paper.

4. The (nonsymmetric) hit problem

This section demonstrates how one can use the action of the Dyer-Lashof algebra to the study of the hit problem as promised at the beginning. The proof of Theorem 1.5 is a standard argument based on transfer and the idea is to use the transfer map $BO(k)_+ \to \mathbb{R}P_+^{\times k}$ and its effect in reduced homology. The classes that we wish to obtain are the images of classes ξ_I^{BO} where we simply define $\xi_I = t_*\xi_I^{BO}$. By stability of the Sq_*^t operations ξ_I is $\mathcal{A}$-annihilated. It remains to show that these classes are nontrivial.

4.1. The transfer $BO(k)_+ \to BO(1)^{\times k}{}_+$. Consider the fibre bundle $i: BO(1)^{\times k} \to BO(k)$ and the Becker-Gottlieb transfer associated to it which is a stable map $t: \Sigma^\infty BO(k)_+ \to \Sigma^\infty BO(1)^{\times k}{}_+$ [4] (see also [30] and [1, Chapter 4]). Here, for a space X, X_+ denotes X with an added disjoint base point which satisfies $H_*\Sigma^\infty X_+ \simeq \widetilde{H}_*X_+ \simeq H_*X$. We begin with the following which has to be well known. For instance, it immediately follows from [5, Theorem 3.11].

Lemma 4.1. *Suppose $0 \leqslant i_1 < \cdots < i_r$. The map $t_*: H_*BO(k) \to H_*BO(1)^{\times k} \simeq (H_*BO(1))^{\otimes k}$ induced by the transfer satisfies*

$$t_*(\overline{e_{i_1}}^{k_1}\cdots\overline{e_{i_r}}^{k_r}) = \sum_{\tau\in\Sigma_k}\tau(a_{i_1}^{\otimes k_1}\otimes\cdots\otimes a_{i_r}^{\otimes k_r})$$

where $k_1+\cdots+k_n = k$ and Σ_k is the permutation group on k elements which acts by permutation of the factors.

This allows the following description of $\ker t_*$, as well as an implicit description of coker t_*.

Proposition 4.2. (i) *Consider a monomial $\overline{e_{i_1}}^{k_1}\cdots\overline{e_{i_r}}^{k_r} \in H_*BO(k)$ where $k = \sum_{j=1}^r k_j$ and $k_j > 0$ for all j. Suppose $k_j > 1$ for some j. Then for the transfer $t_*: H_*BO(k) \to H_*\mathbb{R}P^{\times k}$ we have $t_*(\overline{e_{i_1}}^{k_1}\cdots\overline{e_{i_r}}^{k_r}) = 0$.*
(ii) *Let $k > 0$. Let $H_*^\bullet BO$ be the submodule of H_*BO generated by all monomials $\overline{e_I} = \overline{e_{i_1}}\cdots\overline{e_{i_r}}$ with I being strictly increasing and $l(I) = k$; in the previous notation this corresponds to the cases with $k_j = 1$ for all j and $r = k$. Let*

$H_*^\bullet BO(k) = H_*BO(k) \cap H_*^\bullet BO$. Then the restriction of $t_* : H_*^\bullet BO(k) \to H_*\mathbb{R}P^{\times k}$ is a monomorphism.

Proof. (i) Suppose $\tau(a_{i_1}^{\otimes k_1} \otimes \cdots \otimes a_{i_r}^{\otimes k_r})$ is a term of the above expression for $t_*(\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r})$ where $\tau \in \Sigma_k$. For j_0 with $k_{j_0} > 1$ and let $\sigma \in \Sigma_{k_1} \times \cdots \times \Sigma_{k_r}$ be a permutation which on the blocks coming from Σ_{k_j} with $j \neq j_0$ acts as identity and on the j_0-th block a nonidentity permutation; in matrix representation it is a block diagonal matrix $\sigma = \mathrm{diag}(1_{k_1 \times k_1}, \dots, \sigma_{k_{j_0}}, \dots, 1_{k_r \times k_r})$ where $\sigma_{k_{j_0}} \in \Sigma_{k_{j_0}}$ is a nonidentity element. It is evident that

$$\tau(a_{i_1}^{\otimes k_1} \otimes \cdots \otimes a_{i_r}^{\otimes k_r}) = \tau\sigma(a_{i_1}^{\otimes k_1} \otimes \cdots \otimes a_{i_r}^{\otimes k_r}).$$

Hence, the terms corresponding to τ and $\tau\sigma$ cancel out in the expression for $t_*(\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r})$. Moreover, we can do this for any $\tau \in \Sigma_k$. These together with the fact that the above expression has even number of terms, proves our claim. (ii) This is immediate from the formula. Since S is strictly increasing, then for any $\sigma, \tau \in \Sigma_k$ we have $\tau(a_{i_1} \otimes \cdots \otimes a_{i_k}) \neq \sigma(a_{i_1} \otimes \cdots \otimes a_{i_k})$. In particular, $\tau(a_{i_1} \otimes \cdots \otimes a_{i_k}) \neq a_{i_1} \otimes \cdots \otimes a_{i_k}$ for any $\tau \in \Sigma_k$. This together with the above formula completes the proof. □

Corollary 4.3. *Consider the transfer homomorphism $t_* : H_*BO(k) \to H_*\mathbb{R}P^{\times k}$. Then $\ker(t_*)$ is precisely the submodule of $H_*BO(k)$ that is generated by all monomials $\overline{e_{i_1}}^{k_1} \cdots \overline{e_{i_r}}^{k_r} \in H_*BO(k)$ where $k = \sum_{j=1}^r k_j$ and $k_j > 0$ for all j such that $k_j > 1$ for some j. Let $H_*^\diamond BO(k) = \mathrm{coker}(t_*)$ which is defined by the following short exact sequence*

$$0 \to \ker t_* \to H_*BO(k) \xrightarrow{p} H_*^\diamond BO(k) \to 0.$$

Then t_* extends to a nontrivial monomorphism $\overline{t_*} : H_*^\diamond BO(k) \to H_*\mathbb{R}P^{\times k}$.

Proof. This immediately follows from Proposition 4.2. □

It would be ideal to determine all admissible sequences $I = (i_1, \dots, i_r)$ satisfying conditions of Theorem 1.5 so that ξ_I^{BO} satisfies $p(\xi_I^{BO}) \neq 0$ in $H_*^\diamond BO(k)$, hence also mapping nontrivially under $t_* : H_*BO(k) \to H_*\mathbb{R}P^{\times k}$. Notice that for $l(I) = r$ the class $\xi_I = e_0^{-2^r} Q^I e_0$ is a sum of (formal) monomials in $\overline{e_i}$'s which has a square-free term if and only if $Q^I e_0$ has a square-free term up a multiplication by $e_0^{-2^r}$. Therefore, it is enough to have some criteria for $Q^I e_0$ having a square-free term, up to multiplication by $e_0^{-2^r}$. Any class $Q^I e_0$ can be expressed as a sum of classes e_K with K a non-decreasing sequence such that $l(K) = 2^r$ if $l(I) = r$. The map $* \, e_0^{-2^r}$ then rewrites these classes in $H_*(\{0\} \times BO)$ as a sum of monomials in $\overline{e_i}$ variables. It is easy to show that the length and reduced length are related by $l(e_0^{-2^r} e_K) = \overline{l}(e_K)$. Note that $* \, e_0^{-2^r}$ is a ring map. The Cartan formulae $Q^{2n+1}\xi^2 = 0$ and $Q^{2n}\xi^2 = (Q^n\xi)^2$ imply that if the part of K consisting of nonzero entries is not strictly increasing, i.e. e_K has a square, then depending on the parity of n, $Q^n e_K$ is either 0 or has a square in it. This reduces our search for square-free terms to the following cases.

Lemma 4.4. *Suppose* $0 < i_1 < \cdots < i_k$ *and* $n > i_1 + \cdots + i_k$. *If* $Q^n(e_{i_1} \cdots e_{i_k})$ *has a square-free term then* $n > (i_1 + \cdots + i_k) + k - 1$.

Proof. A proof by contradiction argument, applying the Cartan formula 2.1(i), proves the Lemma. We leave the details to the reader. □

Note that, read in words, the above lemma says that for $\xi = e_{i_1} \cdots e_{i_k}$ with $0 < i_1 < \cdots < i_k$ if $Q^n\xi$ has a square-free term then

$$n \geqslant \dim \xi + \text{ number of variables in } \xi = \dim \xi + l(\xi).$$

In fact, what k counts is the number of variables e_i with $i > 0$. This immediately proves the following.

Corollary 4.5. *Suppose I is an increasing sequence of nonnegative integers. If* $Q^n(e_I)$ *has a square-free term then* $n > \dim(e_I) + \bar{l}(e_I) - 1$. *In particular, for* $I = (i_1, \dots, i_r)$ *if* $s \geqslant 1$ *is the largest integer such that* $i_s = 0$ *then* $(i_{s+1}, \dots, i_r)$ *is strictly increasing.*

We wish to use this result to decide about the square-free terms of $Q^I e_0$, albeit up to a power of e_0. We note that the case of $l(I) = 1$ is a trivial case as $Q^i e_0 = e_0 e_i$ which after translation by e_0 pulls back to e_i in $H_*BO(1)$. It is known that the transfer in the case of $n = 1$ is the same as the identity. Hence, we focus on the cases with $l(I) \geqslant 2$. We have the following.

Lemma 4.6. *Suppose* $I = (i_1, \dots, i_r)$ *is an admissible sequence with* $r \geqslant 2$. *For* $I_1 = (i_2, \dots, i_r)$, *suppose* $Q^{I_1} e_0 = \sum \epsilon_K e_K$ *where* $\epsilon_K \in \mathbb{Z}/2$ *and K runs over all increasing sequences of non-negative integers of length* 2^{r-1}. *If* $Q^I e_0$ *has a square-free term, up to a multiplication by a power of* e_0, *then there exists a K with* $\epsilon_K = 1$ *and*

$$\mathrm{ex}(I) > \bar{l}(e_K) - 1.$$

Proof. For any e_K in the expression for $Q^{I_2} e_0$ we have $\dim(e_K) = \dim(I_1) = i_2 + \cdots + i_r$. Noting that $\mathrm{ex}(I) = i_1 - (i_2 + \cdots + i_r)$ the claimed inequality follows the inequality of Corollary 4.5. This completes the proof. □

4.2. Proof of Theorem 1.5. The above observations on the transfer map $t_* : H_*BO(k) \to H_*\mathbb{R}P^{\times k}$ allow us to prove Theorem 1.5(i). We have the following.

Lemma 4.7. *Suppose I is a nonempty sequence, with* $l(I) > 1$, *as in Theorem 1.5(i). Then* $\xi_I \neq 0$. *Moreover, the class* ξ_I *is invariant under the action of* Σ_{2^r-1}.

Proof. The point that $\xi_I \neq 0$ is obvious from the definitions. The class $\xi_I^{BO} = e_0^{-2^r} \xi_I^{\mathbb{Z}\times BO}$ pulls back to $H_{|I|}BO(2^r - 1)$ and by definition $\xi_I = t_* \xi_I^{BO}$. Obviously, $p(e_0^{-2^r} \xi_I^{\mathbb{Z}\times BO}) \neq 0$ is the same as $\xi_I \neq 0$. We proceed to show that invariance under the action of Σ_{2^r-1}. We may always force the expression of Lemma 3.4 to split and write

$$\xi_I^{\mathbb{Z}\times BO} = Q^I e_0 = e_0 e_{\mathrm{ex}_0} e_{\mathrm{ex}_1}^2 \cdots e_{\mathrm{ex}_{r-1}}^{2^{r-1}} + \sum_S \epsilon_S e_S + O$$

where $\epsilon_S \in \mathbb{Z}/2$ and $S = (s_1, \dots, s_{2^r})$ is running over all strictly increasing sequences with $l(S) = 2^r$ such that $s_1 = 0$, and O consists of the remaining terms. Note that for $r = 1$ the middle sum and O are empty. For $r > 1$, the assumption that $p(\xi) \neq 0$ in $H_*^{\diamond} BO(2^r - 1)$ guarantees ξ has some terms which belong to $H_*^{\bullet} BO(2^r - 1)$ so that the middle sum is not empty, i.e. if $r > 1$ then at least for one S we have $\epsilon_S = 1$. For $r > 1$, $O = \sum \epsilon_L e_L$ where $\epsilon_L \in \mathbb{Z}/2$ and the sum runs over all increasing (but not strictly) sequences of nonnegative integers $L = (l_1, \dots, l_{2^r})$ so that $l_{2^r} > \mathrm{ex}_{r-1}(I) = i_r$ and $e_L = e_{l_1} \cdots e_{l_{2^r}}$. Using $\overline{e_i} = e_0^{-1} e_i$, this implies that

$$\xi_I^{BO} = \overline{e_{\mathrm{ex}_0}}\, \overline{e_{\mathrm{ex}_1}}^2 \cdots \overline{e_{\mathrm{ex}_{r-1}}}^{2^{r-1}} + \sum_S \epsilon_S \overline{e_{S_1}} + e_0^{-2^r} O$$

where $S_1 = (s_2, \dots, s_{2^r})$ by the notation introduced at the beginning and $\overline{e_{S_1}} = \overline{e_{s_2}}\, \overline{e_{s_3}} \cdots \overline{e_{s_{2^r}}}$. It is now easy to see that

$$\xi_I^{BO} = \sum_S \epsilon_S \overline{e_{S_1}} + R$$

where R is a sum of terms of the form $\overline{e_L}$ with L increasing but not strictly. By Proposition 4.2 all terms that have an index with at least two repeated entries map trivially under t_*, i.e. $t_* R = 0$, and we have

$$\xi_I = t_* \sum_S \epsilon_S \overline{e_{S_1}}.$$

Moreover, the sum over S's includes at least one nontrivial term. With all sequences being strictly increasing, it is evident that if S, S' are two distinct strictly increasing sequences then $S \neq \tau S'$ for all $\tau \in \Sigma_{2^r-1}$ where the permutation group acts by permuting the factors. Moreover, $S \neq \tau S$ for all nonidentity permutations $\tau \in \Sigma_{2^r-1}$. This then proves that $\xi_I \neq 0$. Finally, note that by the formula for t_* as in Lemma 4.1 the element $\xi_I = t_* \xi_I^{BO}$ with ξ_I^{BO} being a sum of monomials satisfies the relation $\tau \xi_I = \xi_I$, so ξ_I is invariant under the action of Σ_{2^r-1}. □

Next, we turn to using these classes to construct more $\mathcal{A}$-annihilated classes. Let's note that there exist obvious pairings $\mathbb{R}P^{\times n} \times \mathbb{R}P^{\times m} \to \mathbb{R}P^{\times (n+m)}$ which induce $H_* \mathbb{R}P^{\times n} \otimes H_*\ RP^{\times m} \to H_* \mathbb{R}P^{n+m}$. These also induce pairings $\mathbb{R}P^{\wedge n} \wedge \mathbb{R}P^{\wedge m} \to \mathbb{R}P^{\wedge (n+m)}$ inducing $\widetilde{H}_* \mathbb{R}P^{\wedge n} \otimes \widetilde{H}_* \mathbb{R}P^{\wedge m} \to \widetilde{H}_* \mathbb{R}P^{\wedge (n+m)}$. Both pairings on the level of homology are given by juxtaposition of elements. The geometry behind this is provided by James' splitting that for a path connected space X we have $\Sigma\Omega\Sigma X$ is weak homotopy equivalent to $\bigvee_{r=1}^{+\infty} \Sigma X^{\wedge r}$ [15]. Moreover, we have $H_* \Omega\Sigma X \simeq T(\widetilde{H}_* X)$, the tensor algebra generated by $\widetilde{H}_* X$, where the latter isomorphism holds in $\mathbb{Z}$-homology. Consequently, working with such pairings is quite natural and corresponds to working in the homology ring of $\Omega\Sigma X$. We have the following.

Theorem 4.8. *Suppose $k = \sum_{j=1}^{l}(2^{r_j} - 1)$ so that for each j there exists a sequence I^j of positive integers with $l(I^j) = r_j$ and $\mathrm{ex}(I^j) > 0$ satisfying the conditions of Theorem 1.5. Then, for any $\sigma \in \Sigma_l$ there exists an $\mathcal{A}$-annihilated class $\xi_\sigma \in H_d\mathbb{R}P^{\times k}$ with $d = \sum |I^j|$ which is invariant under the action of $\Sigma_{2^{r_{\sigma(1)}}-1}\times\cdots\times\Sigma_{2^{r_{\sigma(l)}}-1}$. Moreover, depending on the partition chosen for k, subject to the existence of sequences I^j, this would lead to at least $l!$ distinct $\mathcal{A}$-annihilated classes in $H_*\mathbb{R}P^{\times k}$.*

Proof. Since I^j satisfies the conditions of Theorem 1.5, and consequently satisfies the conclusions of Lemma 4.7, then we have nontrivial $\mathcal{A}$-annihilated classes, namely $\xi_{I^j} \in H_*\mathbb{R}P^{\times(2^{r_j}-1)}$. By the external Cartan formula, the class $\xi_{I^{\sigma(1)}} \otimes \cdots \otimes \xi_{I^{\sigma(i)}}$ is $\mathcal{A}$-annihilated. We set $\xi_\sigma = \xi_{I^{\sigma(1)}} \cdots \xi_{I^{\sigma(i)}}$ to be the image of $\xi_{I^{\sigma(1)}} \otimes \cdots \otimes \xi_{I^{\sigma(i)}}$ under the pairing

$$H_*\mathbb{R}P^{\times(2^{r_{\sigma(1)}}-1)} \otimes \cdots \otimes H_*\mathbb{R}P^{\times(2^{r_{\sigma(l)}}-1)} \to H_*\mathbb{R}P^k.$$

As the pairing is induced by a mapping of spaces, hence by naturality of the Steenrod operations the class ξ_σ is an $\mathcal{A}$-annihilated class. The invariance under the action of $\Sigma_{2^{r_{\sigma(1)}}-1}\times\cdots\times\Sigma_{2^{r_{\sigma(l)}}-1}$ is clear since each factor in ξ_σ is invariant under the action of the relevant permutation group. Since $H_*\mathbb{R}P^{\times k}$ is a tensor algebra, it is not commutative. So, while taking the product of various elements the order matters. Because of the invariance under $\Sigma_{2^{r_{\sigma(1)}}-1} \times \cdots \times \Sigma_{2^{r_{\sigma(l)}}-1}$ the only order changing permutations can arise from Σ_l. By suitable choices for the sequences I^j, say choosing them to be from different lengths and dimensions, and having no entries in common, one can see that the number $l!$ distinct classes can be attained. We leave the details to the reader. □

Finally, we turn to some computations in low lengths. The results of the appendix (see Theorem A.1) determine all sequences I of positive excess with $l(I) \leqslant 4$ with $Q^I e_0$ being $\mathcal{A}$-annihilated, hence all nontrivial classes ξ_I^{BO} classes with $l(I) \leqslant 4$. We may ask if in these cases $\xi_I = t_*\xi_I^{BO} \neq 0$? We provide an answer to this when $l(I) \leqslant 3$. First, note that it is a general fact that the transfer associated to the identity is the identity. Hence, in the case of $n = 1$ we have the $\mathcal{A}$-annihilated classes $\overline{e_{2^t-1}} \in H_{2^t-1}BO$ which pull back to $a_{2^t-1} \in H_{2^t-1}BO(1)$ with $t_*a_{2^t-1} = a_{2^t-1}$. According to Theorem A.1 there is no sequence with $l(I) = 2$ so that $Q^I e_0$ is $\mathcal{A}$-annihilated. Hence, we do not expect any ξ_I class with $l(I) = 2$. The following lemma eliminates existence of ξ_I classes with $l(I) = 3$.

Lemma 4.9. *Suppose I is an admissible sequence of positive integers with $l(I) = 3$ such that $Q^I e_0$ is $\mathcal{A}$-annihilated, that is I satisfies conditions of Theorem 1.2 which are the same as conditions of Theorem 2.2. Then the pull back of $e_0^{-2^3}Q^I e_0$*

to $H_*BO(2^{l(I)}-1)$ maps trivially under the transfer $t_* : H_*BO(2^{l(I)}-1) \to H_*\mathbb{R}P^{\times(2^{l(I)}-1)}$.

Proof. The proof is by contradiction. Suppose there exists an admissible sequence $I=(i_1,i_2,i_3)$ so that the pull back of $e_0^{-2^{l(I)}}Q^Ie_0$ maps nontrivially under the transfer t_*. By Proposition 4.2 and Corollary 4.3 and the explanations afterwards any elements non-square-free monomial in $H_*BO(k)$ maps trivially under the transfer $t_* : BO(k) \to H_*\mathbb{R}P^{\times k}$. Hence, $\xi_I \neq 0$ implies that $Q^{I_1}e_0$ must have at least one square-free term. We show that the existence of any square-free term would contradict Lemma 4.6. By Lemma 4.6 if $Q^{i_2}Q^{i_3}e_0$ is a sum of monomials e_K then there exists K with $l(K)=2^2$ such that $\bar{l}(e_K)-1<\mathrm{ex}(I)$, i.e. $\bar{l}(e_K)\leqslant \mathrm{ex}(I)$. On the other hand, by Theorem A.1, there exist positive integers m and n with $m<n-1$ so that

$$I=(2^{n+1}+2^m-1, 2^n+2^m-1, 2^n-1).$$

Consequently $\mathrm{ex}(I)=1$. However, since I satisfies conditions of Theorem 1.2 then by Lemma A.2 all of its entries are odd allowing us to use Lemma 3.6 to $I_1=(i_2,i_3)$ with $l(I_1)=2$ which shows that $\bar{l}(e_K)\geqslant 2^{l(I_1)-1}=2$. This contradicts the inequality $\bar{l}(e_K)\leqslant \mathrm{ex}(I)$ and completes the proof. □

Appendix A. Closed forms for low lengths

By Seyyed Mohammad Ali HasanZadeh
School of Mathematics
Sharif University of Technology
Tehran, Iran
m.ali.hasanzadeh1@gmail.com.

The contents of this appendix are identical to some parts of [12] which we have put online only to share and publicise our result; in particular [12] is not intended to be published. We refer the interested reader to [12] for more details as well as tables which computes sequences satisfying conditions of Theorem 2.2 up to dimension 2^{17}.

It is possible to use the conditions of Theorem 2.2 to put more restrictions on the sequences I. From computational point of view putting more restrictions on the set of all such sequences is the same as making the space in which we have to search for such sequences smaller. Our main result in this appendix determines all sequences of small length.

Theorem A.1. *Suppose I is an admissible sequence of positive excess satisfying conditions of Theorem 2.2.*
(i) If $l(I)=1$ then $I=(2^t-1)$ for some $t>0$. Moreover, any $I=(2^t-1)$ satisfies

conditions of Theorem 2.2.
(ii) For $l(I) = 2$ *there is no sequence which satisfies conditions of Theorem* 2.2.
(iii) If $l(I) = 3$ *then there exist positive integers* m *and* n *with* $m \leqslant n-1$ *so that*

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$

Moreover, any sequence I *of the above form satisfies conditions of Theorem* 2.2.
(iv) For $l(I) = 4$ *if* I *satisfies conditions of Theorem* 2.2 *then* I *has one of the following forms*

$$\begin{aligned} I &= (2^{n+3} + 2^{n-1} - 1, 2^{n+2} + 2^{n-1} - 1, 2^{n+1} + 2^{n-1} - 1, 2^n + 2^{n-1} - 1) \\ &\quad \text{with } n > 2, \\ I &= (2^{n+2} + 2^{m+1} + 2^{m'} - 1, 2^{n+1} + 2^m + 2^{m'} - 1, 2^n + 2^m - 1, 2^n - 1) \\ &\quad \text{with } n > m > m' > 1. \end{aligned}$$

Moreover, any sequence of the above forms satisfies conditions of Theorem 2.2.

Let's note that part (i) and (ii) of the above Theorem are almost trivial and were known to Wellington. We provide a proof for the case of $l(I) = 3$ as an illustration. We leave the case of $l(I) = 4$ to the reader as it is lengthy and tedious. We only prove the assertions in the direction that takes us from knowing that I satisfies conditions of Theorem 2.2 with $l(I) = r$ to getting closed form for I. The proof in the other direction is immediate as from given closed forms one can readily verify that the satisfy conditions of Theorem 2.2; we leave the details to the reader. Next, we give a list of elementary properties that these sequence enjoy and will be use in the coming proofs.

Lemma A.2. *(i) For* $r > 1$, *if* $I = (i_1, \dots, i_r)$ *is an admissible sequence with* $\mathrm{ex}(I) > 0$ *then* I *is strictly decreasing, i.e.* $i_j > i_{j+1}$ *for all* $1 \leqslant j < r$.
(ii) Suppose $I = (i_1, \dots, i_s)$ *is a sequence with* $\mathrm{ex}(I) > 0$ *which satisfies conditions of Theorem* 2.2. *Then, all entries of* I *are odd.*

Proof. (i) This easily follows by induction from the admissibility and positivity of excess.
(ii)It is straightforward to see that if $s = 1$ then $I = (2^t - 1)$ for some $t > 0$.
Suppose $s > 0$ and $I = (i_1, \dots, i_r)$ is an admissible sequence with $\mathrm{ex}(I) > 0$ which satisfies condition of Theorem 2.2. First, note that if i_1 is even then $\phi(i_1) = 0$ which together with condition $0 < \mathrm{ex}(I) < 2^{\phi(i_1)}$ shows that $0 < \mathrm{ex}(I) < 1$ which is a contradiction. Hence, i_1 must be odd. For $j > 1$ suppose i_j is even, hence $\phi(i_j) = 0$. The condition $0 \leqslant 2i_j - i_{j-1} < 2^{\phi(i_j)} = 1$ implies that $i_{j-1} = 2i_j$. By iterating this process we see that i_1 must be even which is a contradiction. Hence, I consists of only odd entries. □

The following is now evident.

Corollary A.3. *If* I *satisfies conditions of Theorem* 2.2 *then* $l(I)$ *and* $|I|$ *have the same parity.*

Next, note that for a sequence $I = (i_1, \dots, i_r)$ we have its dimension $|I| = i_1 + \cdots + i_r$. We have the following.

Lemma A.4. *Suppose $I = (i_1, \dots, i_r)$ is an admissible sequence of positive excess. Then I satisfies conditions of Theorem 2.2 if and only if for $(i_0, I) := (i_0, i_1, \dots, i_r)$ we have $0 < 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for all $j \in \{0, \dots, r-1\}$ where $i_0 = |I|$.*

Proof. The inequality $0 < \mathrm{ex}(I) < 2^{\phi(i_1)}$ is the same as $0 < 2i_1 - i_0 < 2^{\phi(i_1)}$ if we replace $\mathrm{ex}(I)$ by $2i_1 - i_0$. The result now follows. □

The above observations are important as they replace two conditions of Theorem 2.2 only with one condition, so that search for sequences satisfying condition of Lemma A.4 would also provide us with all sequences which satisfy conditions of Theorem 2.2 and the refinement would be $i_0 = |I| = i_1 + \cdots + i_r$.

The main ingredient of Theorem 2.2 is the function ϕ which allows us to express the result in a combinatorial manner. We are therefore interested in studying properties of this function. To begin with, let's recall the following which is implicit in Curtis's work and follows by looking at the binary expansion of numbers.

Lemma A.5. *Suppose $I = (i_1, \dots, i_r)$ is an admissible sequence with $\mathrm{ex}(I) > 0$ satisfying conditions of Theorem 2.2. Then,*

$$\phi(i_1) \leqslant \cdots \leqslant \phi(i_r).$$

We also define $\psi : \mathbb{N} \to \mathbb{N}$ by

$$\psi(n) = \max\{i : n_i = 1\} + 1 = \min\{i : \forall j \geqslant i,\ n_j = 0\}.$$

In fact, writing $n \in \mathbb{N}$ in binary expansion $\cdots n_i \cdots n_1 n_0$, that is $n = \sum_{i=0}^{+\infty} n_i 2^i$ with $n_i \in \{0, 1\}$, the function ψ assigns to n the 'length' of its binary expansion.

We call n a spike if $n = 2^t - 1$ for some $t > 0$. The following is immediate.

Lemma A.6. *(i) For $n \in \mathbb{N}$, $\phi(n) \leqslant \psi(n)$. Moreover, $\phi(n) = \psi(n)$ if and only if $n = 2^t - 1$ with $t = \psi(n)$.*
(ii) If n is a non-spike then $\phi(n) < \psi(n) - 1$.

Proof. Both parts immediately follow from definitions. We only note that, for (ii), by definition $\psi(n) - 1$ is the last place in the binary expansion of n which a 1 appears whereas $\phi(n)$ is the first place in the binary expansion a 0 shows up, hence $\phi(n) \neq \psi(n) - 1$. □

We also have the following characterisation of non-spike integers.

Lemma A.7. *(i) Let n be a non-spike positive integer, that is $n \neq 2^t - 1$ for all t. Then, there exists a natural number $N_n > 1$ so that $n = 2^{\phi(n)} N_n + 2^{\phi(n)} - 1$. In particular, N_n is an even number.*
(ii) If n is a non-spike then there exists a nonnegative integer $B(n)$ with either $B(n) = 0$ or $B(n) > 2^{\phi(n)}$ such that

$$n = 2^{\psi(n)-1} + B(n) + 2^{\phi(n)} - 1.$$

Proof. (i) Note that for any positive integer k we have $2^k - 1 = \sum_{j=0}^{k-1} 2^j$. This together with definition of $\phi(n)$ shows that for $n = \sum_{i=0}^{+\infty} n_i 2^i$ we have

$$N_n = (n - 2^{\phi(n)} - 1)/2^{\phi(n)} = \sum_{i=\phi(n)}^{+\infty} n_i 2^{i-\phi(n)}.$$

If N_n is an odd number then $n_{\phi(n)} = 1$ which contradicts definition of $\phi(n)$. In particular, N_n cannot be 1.
(ii) by definition of ψ we have $n_{\psi(n)-1} = 1$ and $n_i = 0$ for all $i \geqslant \psi(n)$. For $n = \sum_{i=0}^{+\infty} n_i 2^i$ written in binary form, $B(n) = \sum_{i=\phi(n)+1}^{\psi(n)-2} n_i 2^i$ is the required value. □

Sometimes we refer to $B(n)$ as the (undetermined-) block of n.

The functions ϕ and ψ provide some useful upper and lower bounds for non-spike numbers. Before proceeding further, let's recall that for all $i > 0$ we have

$$2^i - 1 = \sum_{j=0}^{i-1} 2^j.$$

Most of what we say below are consequences of this equality. For instance, we have the following.

Lemma A.8. *(i) For $n \in \mathbb{N}$, $n_{\psi(n)} = 0$ and $n_{\psi(n)-1} = 1$.*
(ii) For any non-spike positive integer $n \in \mathbb{N}$ we have

$$2^{\psi(n)-1} + 2^{\phi(n)} - 1 \leqslant n \leqslant 2^{\psi(n)} - 1 - 2^{\phi(n)} < 2^{\psi(n)} - 1.$$

(iii) For any non-spike positive integer n we have $\phi(n) < \psi(n) - 1$.

Proof. (i) This follows from the definition of ψ.
(ii) Using the equality $2^i - 1 = \sum_{j=0}^{i-1} 2^j$ the inequality is immediate.
(iii) This is also immediate from the definitions. □

The following provides an application of the inequalities of Lemma A.8.

Lemma A.9. *Suppose $I = (i_1, \dots, i_r)$ is an admissible sequence with $\mathrm{ex}(I) > 0$ which satisfies conditions of Theorem 2.2. If $i_j = 2^n - 1$ for some $n > 0$ then $j = r$.*

Proof. By Lemma A.2(i) the sequence I is strictly decreasing. Suppose $j < r$ and $i_j = 2^n - 1$ for some $n > 0$. Then $i_{j+1} < 2^n - 1$. We compare $\psi(i_{j+1})$ to n and proceed to show that any possible choice for $\psi(i_{j+1})$ leads to a contradiction. We consider the following cases.
Case of $\psi(i_{j+1}) \geqslant n+1$. In this case, $\psi(i_{j+1}) - 1 \geqslant n$. If i_{j+1} is non-spike then by Lemma A.8(ii) we have

$$i_{j+1} \geqslant 2^{\psi(i_{j+1})-1} + 2^{\phi(i_{j+1})} - 1 > 2^{\psi(i_{j+1})-1} - 1 \geqslant 2^n - 1 = i_j$$

which is a contradiction. If i_{j+1} is a spike then by Lemma A.6 $\phi(i_{j+1}) = \psi(i_{j+1})$ and

$$i_{j+1} = 2^{\psi(i_{j+1})} - 1 > 2^n - 1 = i_j$$

which is a contradiction.

Case of $\psi(i_{j+1}) \leqslant n-1$**.** Using the inequalities of Lemma A.8(ii) we see that

$$i_{j+1} \leqslant 2^{\psi(i_{j+1})} - 1 \leqslant 2^{n-1} - 1$$

which together with $i_j = 2^n - 1$ and the admissibility condition implies that

$$2i_{j+1} \leqslant 2^n - 2 = i_j - 1 < i_j \leqslant 2i_{j+1}$$

which is an obvious contradiction.

Case of $\psi(i_{j+1}) = n$**.** First, suppose $\phi(i_{j+1}) < \psi(i_{j+1})$, i.e. i_{j+1} is not a spike. By Lemma A.8(ii) we have

$$i_{j+1} \geqslant 2^{\psi(i_{j+1})-1} + 2^{\phi(i_{j+1})} - 1 = 2^{n-1} + 2^{\phi(i_{j+1})} - 1.$$

Noting that $i_j = 2^n - 1$, and multiplying both sides of this inequality by 2, we have

$$2i_{j+1} \geqslant 2^n + 2^{\phi(i_{j+1})+1} - 2$$

which implies that $2i_{j+1} - i_j \geqslant 2^{\phi(i_{j+1})+1} - 1$. For I satisfying conditions of Theorem 2.2 we have $2i_{j+1} - i_j \leqslant 2^{\phi(i_{j+1})}$. Moreover, by Lemma A.2(ii) all entries of I are odd and this latter inequality reads as $2i_{j+1} - i_j < 2^{\phi(i_j+1)}$ which is the same as $2i_{j+1} - i_j \leqslant 2^{\phi(i_j+1)} - 1$. It is impossible to have both inequalities

$$2i_{j+1} - i_j \geqslant 2^{\phi(i_{j+1})+1} - 1,\ 2i_{j+1} - i_j \leqslant 2^{\phi(i_j+1)} - 1$$

holding together. Hence, we have a contradiction.

The only possible remaining case is the case of $\phi(i_{j+1}) = \psi(i_{j+1}) = n$. In this case, by Lemma A.6, $i_{j+1} = 2^n - 1 = i_j$. But this contradicts Lemma 1.1(iii), unless $j = r$. This completes the proof. □

A very immediate corollary of the proof is the following.

Corollary A.10. *If $I = (i_1, \dots, i_r)$ is an admissible sequence of positive excess, satisfying conditions of Theorem 2.2, with $i_r = 2^{\phi(i_r)} - 1$ then $\phi(i_{r-1}) < \phi(i_r)$.*

Proof. By Lemma A.5 $\phi(i_{r-1}) \leqslant \phi(i_r)$. We have to eliminate the possibility of $\phi(i_{r-1}) = \phi(i_r)$. By Lemma A.8(ii), $i_{r-1} \geqslant 2^{\psi(i_{r-1})-1} + 2^{\phi(i_{r-1})} - 1$. Also, using the upper bound of Lemma A.8(ii) for i_r, noting that by Lemma A.6(i) $\phi(i_r) = \psi(i_r)$, we see that

$$2i_r - i_{r-1} \leqslant 2^{\psi(i_r)+1} - 2 - (2^{\psi(i_{r-1})-1} + 2^{\phi(i_{r-1})} - 1) = 2^{\phi(i_r)} - 2^{\psi(i_{r-1})-1} - 1.$$

By Lemma A.9 i_{r-1} is not a spike. Hence, by Lemma A.6(ii), $\phi(i_{r-1}) < \psi(i_{r-1}) - 1$. If $\phi(i_r) = \phi(i_{r-1})$ then the above inequality reads as

$$2i_r - i_{r-1} \leqslant 2^{\phi(i_r)} - 2^{\psi(i_{r-1})-1} - 1 = 2^{\phi(i_{r-1})} - 2^{\psi(i_{r-1})-1} - 1 < -1$$

which contradicts the admissibility of I. Hence, it is not possible to have $\phi(i_r) = \phi(i_{r-1})$. This completes the proof. □

We can prove a more practical criterion of the sequence of ψ's. We have the following.

Theorem A.11. *Suppose $I = (i_1, \dots, i_r)$ is an admissible sequence of positive excess, satisfying conditions of Lemma 2.2. Then for all $j \in \{2, \dots, r\}$ we have $\psi(i_j) = \psi(i_{j-1}) - 1$.*

Proof. We eliminate the cases with $\psi(i_j) \leqslant \psi(i_{j-1}) - 2$ and $\psi(i_j) \geqslant \psi(i_{j-1})$ as follows.
Case of $\psi(i_j) \leqslant \psi(i_{j-1}) - 2$. We have

$$i_j \leqslant 2^{\psi(i_j)} - 1 \leqslant 2^{\psi(i_{j-1})-2} - 1$$

which implies that $2i_j \leqslant 2^{\psi(i_{j-1})-1} - 2$. Now, for any $j \in \{2, \dots, r\}$ by Lemma A.9 i_{j-1} is not a spike. Hence, $i_{j-1} \geqslant 2^{\psi(i_{j-1})} + 2^{\phi(i_{j-1})} - 1$. These together imply that

$$2i_j - i_{j-1} < -2^{\phi(i_{j-1})} < 0$$

which is a contradiction.
Case of $\psi(i_j) = \psi(i_{j-1})$. First, suppose i_j is not a spike. Then,

$$i_j \geqslant 2^{\psi(i_j)-1} + 2^{\phi(i_j)} - 1 = 2^{\psi(i_{j-1})-1} + 2^{\phi(i_j)} - 1.$$

Moreover, since by Lemma A.9 i_{j-1} is not a spike, then $i_{j-1} < 2^{\psi(i_{j-1})} - 1$. These together imply that $2i_j - i_{j-1} > 2^{\phi(i_j)}$ which contradicts the hypothesis of I satisfying conditions of Theorem 2.2. Hence, i_j has to be a spike and $j = r$ by Lemma A.9, that is $i_r = 2^{\phi(i_r)} - 1$. In this case, as i_{j-1} is not a spike, we have

$$i_{j-1} \leq 2^{\psi(i_{j-1})} - 1 - 2^{\phi(i_{j-1})}.$$

These together imply that

$$2i_r - i_{r-1} \geqslant 2^{\phi(i_r)} + 2^{\phi(i_{r-1})} - 1 > 2^{\phi(i_r)}$$

which contradicts conditions of Theorem 2.2. Note that the possibility of $2^{\phi(i_{r-1})} - 1 = 0$, hence $\phi(i_{r-1}) = 0$ or equivalently i_{r-1} being even, is eliminated by Lemma A.2(ii) as I satisfying conditions of Theorem 2.2 consists only of odd entries.
Finally, note that it is impossible to have $\psi(i_j) > \psi(i_{j-1})$ as this would imply that $i_j > i_{j-1}$ which contradicts Lemma A.2(i). Consequently, we are left with $\psi(i_j) = \psi(i_{j-1}) - 1$ as the only choice. This completes the proof. □

A.1. Case of $l(I) = 1, 2$.

Lemma A.12. *Suppose $I = (i_1, \dots, i_s)$ is an admissible sequence.*
(i) For $l(I) = 1$, I satisfies conditions of Theorem 2.2 if and only if $I = (2^t - 1)$ for some $t > 0$.
(ii) If $l(I) = 2$ then I does not satisfy conditions of Theorem 2.2.

Proof. (i) It is clear from binary expansion.
(ii) By Lemma A.7, for any non-spike positive integer i, we have $i = 2^{\phi(i)} - 1 + N_i 2^{\phi(i)}$ for some positive integer $N_i \neq 1$ (if i is a spike then the expression still is valid with $N_i = 0$). If $I = (i_1, i_2)$ is a sequence satisfying conditions of Theorem 2.2 then by Lemma A.5 $\phi(i_1) \leqslant \phi(i_2)$. The conditions

$$0 \leqslant 2i_2 - i_1 < 2^{\phi(i_2)},\ i_1 - i_2 < 2^{\phi(i_1)}$$

imply that

$$i_2 = 2^{\phi(i_2)} - 1 + N_{i_2} 2^{\phi(i_2)} < 2^{\phi(i_2)+1} \Rightarrow N_{i_2} \leqslant 1.$$

If i_2 is not a spike then this is a contradiction by Lemma A.7. The only remaining case is that i_2 is a spike which corresponds to the case $N_{i_2} = 0$ and $i_2 = 2^{\phi(i_2)} - 1$. Note that $i_1 = 2^{\phi(i_1)} - 1 + N_{i_1} 2^{\phi(i_1)}$. Using the conditions of the theorem simultaneously, yields

$$\begin{array}{rclcrcl} i_1 - i_2 & < & 2^{\phi(i_1)} & \Rightarrow & N_{i_1} & < & 2^{\phi(i_2)-\phi(i_1)}, \\ 2i_2 - i_1 & < & 2^{\phi(i_2)} & \Rightarrow & N_{i_1} & > & 2^{\phi(i_2)-\phi(i_1)} - 1. \end{array}$$

This gives the desired contradiction. □

A.2. Case of $l(I) = 3$.

Lemma A.13. *If $I = (i_1, i_2, i_3)$ is an admissible sequence which satisfies conditions of Theorem* 2.2 *then $i_3 = 2^n - 1$ for some positive integer n.*

Proof. Suppose $I = (i_1, i_2, i_3)$ is a sequence which satisfies conditions of Theorem 2.2. Note that by Lemma A.2(ii) all entries of I are odd, consequently $\phi_j > 0$ for all j. Since all entries of I are odd, we may sharpen the inequalities of Theorem 2.2 and write

$$(1)\ i_1 - (i_2 + i_3) < 2^{\phi_1},\ (2)\ 2i_2 - i_1 < 2^{\phi_2},\ (3)\ 2i_3 - i_2 < 2^{\phi_3}$$

as the left side of either inequality is meant to be odd. Here, we write $\phi_j = \phi(i_j)$ for brevity. By adding the three inequalities above, we have

$$i_3 < 2^{\phi_1} + 2^{\phi_2} + 2^{\phi_3}.$$

By Lemma A.8(ii) we have

$$2^{\psi_3 - 1} + 2^{\phi_3} - 1 \leqslant i_3.$$

These together imply that

$$\begin{array}{rcl} 2^{\psi_3-1} + 2^{\phi_3} - 1 \leqslant i_3 < 2^{\phi_1} + 2^{\phi_2} + 2^{\phi_3} & \Rightarrow & 2^{\psi_3-1} - 1 < 2^{\phi_1} + 2^{\phi_2} \\ & \Rightarrow & 2^{\psi_3-1} \leqslant 2^{\phi_1} + 2^{\phi_2}. \end{array}$$

Suppose i_3 is not a spike. By Lemma A.6(ii) $\phi_3 < \psi_3 - 1$ or equivalently $\phi_3 + 1 \leqslant \psi_3 - 1$. Consequently,

$$2^{\phi_3+1} \leqslant 2^{\phi_1} + 2^{\phi_2}.$$

Note that by Lemma A.5 $\phi_1 \leqslant \phi_2 \leqslant \phi_3$ which implies that $2^{\phi_3+1} \geqslant 2^{\phi_1} + 2^{\phi_2}$. Consequently,

$$2^{\phi_3+1} = 2^{\phi_1} + 2^{\phi_2}$$

which is possible if and only if $\phi_1 = \phi_2 = \phi_3$. Since i_1, i_2, i_3 are not spike then applying Lemma A.7 we have

$$\mathrm{ex}(I) = i_1 - (i_2 + i_3) = (N_1 - N_2 - N_3)2^{\phi_1} - 2^{\phi_1} + 1.$$

Since N_1, N_2, N_3 are all even then either $N_1 - N_2 - N_3 \leqslant 0$ or $N_1 - N_2 - N_3 \geqslant 2$. If $N_1 - N_2 - N_3 \geqslant 2$ then

$$\mathrm{ex}(I) \geqslant 2^{\phi_1+1} - 2^{\phi_1} - 1 = 2^{\phi_1} + 1 > 2^{\phi_1}$$

which contradicts (1), hence contradicting I satisfying conditions of Theorem 2.2. If $N_1 - N_2 - N_3 \leqslant 0$ then bearing in mind that $\phi_1 > 0$ we see that $\mathrm{ex}(I) < 0$ which contradicts the hypothesis. We then conclude that i_3 is a spike. □

Next, we compute i_1 and i_2.

Lemma A.14. *Suppose $I = (i_1, i_2, i_3)$ is an admissible sequence satisfying conditions of Theorem 2.2. Then, for some $m \leqslant n - 1$*

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$

Proof. We shall write $\psi_j = \psi(i_j)$ and $\phi_j = \phi(i_j)$. By Lemma A.13 $i_3 = 2^n - 1$ for some $n > 0$. It follows that $\phi_3 = \psi_3 = n$. By Lemma A.11 it follows that $\psi_2 = n + 1$ and $\psi_1 = n + 2$. Moreover, since i_3 is a spike then by Lemmata A.10 and A.5 we have $\phi_1 \leqslant \phi_2 < \phi_3 = n$. By Lemma A.9 both i_1 and i_2 are non-spike. Applying Lemma A.6(ii) we see that

$$i_1 = 2^{n+1} + B_1 + 2^{\phi_1} - 1,\ i_2 = 2^n + B_2 + 2^{\phi_2} - 1$$

where $B_1 = \sum_{k_1=\phi_1+1}^{n} \alpha_{k_1} 2^{k_1}$ and $B_2 = \sum_{k_2=\phi_2+1}^{n-1} \alpha_{k_2} 2^{k_2}$. We claim that $B_1 = B_2 = 0$. We proceed as follows.
First, suppose $B_2 > 0$. The conditions $0 < 2i_2 - i_1 < 2^{\phi_2}$ and $0 < \mathrm{ex}(I) < 2^{\phi_1}$ imply that

$$\begin{array}{ll}
2^{n+1} + 2B_2 + 2^{\phi_2+1} - 2 - (2^{n+1} + B_1 + 2^{\phi_1} - 1) < 2^{\phi_2} & \Rightarrow \\
2B_2 - B_1 + 2^{\phi_2} - 1 < 2^{\phi_1}, & \\
0 < 2^{n+1} + B_1 + 2^{\phi_1} - 1 - (2^n + B_2 + 2^{\phi_2} - 1 + 2^n - 1) < 2^{\phi_1} & \Rightarrow \\
B_1 - B_2 - 1 < 2^{\phi_2}. &
\end{array}$$

By adding the resulting inequalities we have $B_2 - 2 < 2^{\phi_1} \leqslant 2^{\phi_2}$, so from B_2 being even we deduce that $B_2 \leqslant 2^{\phi_2}$. But, this is a contradiction as if $B_2 > 0$ then from its expression $B_2 > 2^{\phi_2}$. Hence, $B_2 = 0$ and consequently $i_2 = 2^n + 2^{\phi_2} - 1$. Therefore, for some $m \leqslant n - 1$ we have $\phi_2 = m$ and

$$I = (2^{n+1} + B_1 + 2^{\phi_1} - 1, 2^n + 2^m - 1, 2^n - 1)$$

where $\phi_1 \leqslant m$. Next, we turn to B_1.
First, suppose $B_1 = 0$. In this case if $\phi_1 < m$ then $\mathrm{ex}(I) < 0$ which is a contradiction. Hence, in the case of $B_1 = 0$ we have $\phi_1 = m$ and

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$

We show that the assumption that $B_1 > 0$ leads to a contradiction. The condition $0 < 2i_2 - i_1 < 2^{\phi_2} = 2^m$ implies that

$$2^m \leqslant B_1 + 2^{\phi_1} < 2^{m+1}.$$

Consider the binary expansion of B_1 as $B_1 = \sum_{k_1=\phi_1+1}^{\psi_1-2} \alpha_{k_1} 2^{k_1}$ bearing in mind that $\psi_1 - 2 = n$. The above inequalities show that

$$\alpha_{k_1} = 0 \text{ for all } k_1 \geqslant m+1.$$

Hence, $B_1 = \sum_{k_1=\phi_1+1}^{m} \alpha_{k_1} 2^{k_1}$. If $\phi_1 < m$ then the condition $B_1 + 2^{\phi_1} \geqslant 2^m$ reads as $\sum_{k_1=\phi_1+1}^{m} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} \geqslant 2^m$ which is impossible if $\alpha_m = 0$. Hence, $\alpha_m = 1$. Therefore,

$$i_1 = 2^{n+1} + 2^m + \sum_{k_1=\phi_1+1}^{m-1} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} - 1.$$

The condition $\mathrm{ex}(I) < 2^{\phi_1}$ reads as

$$\sum_{k_1=\phi_1+1}^{m-1} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} + 1 < 2^{\phi_1}$$

which is an obvious false inequality. Hence, $B_1 = 0$. This completes the proof. $\square$

As an example where the boundary value $m = n-1$ could be attained, consider $I = (19, 11, 7)$ where $11 = 2^3 + 2^{3-1} - 1$. The process of eliminating B_1 in the above proof is more of a intuitive nature and would be immediate if one has the experience and passion for working with binary expansions.

References

[1] Adams, John Frank. Infinite loop spaces. Annals of Mathematics Studies, 90. *Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo*, 1978. x+214 pp. ISBN: 0-691-08207-3; 0-691-08206-5. MR0505692, Zbl 0398.55008, doi: 10.1515/9781400821259. 1138, 1156

[2] Asadi-Golmankhaneh, Mohammad A.; Eccles, Peter J. Double point self-intersection surfaces of immersions. *Geom. Topol.* **4** (2000), 149–170. MR1742556, Zbl 0941.57025, arXiv:math/0003236, doi: 10.2140/gt.2000.4.149. 1147

[3] Ault, Shaun V.; Singer, William. On the homology of elementary Abelian groups as modules over the Steenrod algebra. *J. Pure Appl. Algebra* **215** (2011), no. 12, 2847–2852. MR2811567, Zbl 1238.55008, arXiv:1105.1139, doi: 10.1016/j.jpaa.2011.04.004. 1140

[4] Becker, James C.; Gottlieb, Daniel H. The transfer map and fiber bundles. *Topology* **14** (1975), 1–12. MR0377873, Zbl 0306.55017, doi: 10.1016/0040-9383(75)90029-4. 1138, 1156

[5] Brumfiel, Gregory W.; Madsen, Ib Henning. Evaluation of the transfer and the universal surgery classes. *Invent. Math.* **32** (1976), no. 2, 133–169. MR0413099, Zbl 0319.55027, doi: 10.1007/BF01389959. 1156

[6] Cohen, Frederick R.; Lada, Thomas J.; May, J. Peter. The homology of iterated loop spaces. Lecture Notes in Mathematics, 533. *Springer-Verlag, Berlin-New York*, 1976. vii+490 pp. MR0436146, Zbl 0334.55009. 1142, 1143

[7] CURTIS, EDWARD B. The Dyer–Lashof algebra and the Λ-algebra. *Illinois J. Math.* **19** (1975), 231–246. MR0377885, Zbl 0311.55007, doi: 10.1215/ijm/1256050812. 1143

[8] DYER, ELDON; LASHOF, RICHARD K. Homology of iterated loop spaces. *Amer. J. Math.* **84** (1962), 35–88. MR0141112, Zbl 0119.18206, doi: 10.2307/2372804. 1143

[9] GALATIUS, SØREN. Mod p homology of the stable mapping class group. *Topology* **43** (2004), no. 5, 1105–1132. MR2079997, Zbl 1074.57013, doi: 10.1016/j.top.2004.01.011. 1142

[10] GIAMBALVO, VINCENT; PETERSON, FRANKLIN P. A-generators for ideals in the Dickson algebra. *J. Pure Appl. Algebra* **158** (2001), no. 2–3, 161–182. MR1822839, Zbl 0984.55012, doi: 10.1016/S0022-4049(00)00051-7. 1140

[11] HASANZADEH, SEYYED MOHAMMAD ALI; ZARE, HADI. An algorithmic search for A-annihilated classes in $H_*Q_0S^0$. Work in progress. 1136, 1139

[12] HASANZADEH, SEYYED MOHAMMAD ALI; ZARE, HADI. An algorithmic search for A-annihilated classes in the Dyer–Lashof algebra and H_*QS^0 I. Closed form for low lengths and tables in low dimensions. Preprint, 2019. arXiv:1905.00611. 1161

[13] HU'NG, NGUYEN H.V.; NAM, TRAN NGOC. The hit problem for the Dickson algebra. *Trans. Am. Math. Soc.* **353** (2001), no. 12, 5029–5040. MR1852092, Zbl 0979.55011, doi: 10.1090/S0002-9947-01-02705-2. 1140

[14] HU'NG, NGUYEN H.V.; PETERSON, FRANKLIN P. A-generators for the Dickson algebra. *Trans. Amer. Math. Soc.* **347** (1995), no. 12, 4687–4728. MR1316852, Zbl 0864.55016, doi: 10.2307/2155059. 1140

[15] JAMES, IOAN. M. Reduced product spaces. *Ann. of Math.* (2) **62** (1955), 170–197. MR0073181, Zbl 0064.41505, doi: 10.2307/2007107. 1159

[16] JANFADA, ALI S.; WOOD, REGINALD M. W. The hit problem for symmetric polynomials over the Steenrod algebra. *Math. Proc. Cambridge Philos. Soc.* **133** (2002), no. 2, 295–303. MR1912402, Zbl 1012.55023, doi: 10.1017/S0305004102006059. 1135

[17] JANFADA, ALI S.; WOOD, REGINALD M. W. Generating $H^*(BO(3);\mathbb{F}_2)$ as a module over the Steenrod algebra. *Math. Proc. Cambridge Philos. Soc.* **134** (2003), no. 2, 239–258. MR1972137, Zbl 1032.55017, doi: 10.1017/S0305004102006394. 1135, 1137

[18] KOCHMAN, STANLEY O. Homology of the classical groups over the Dyer–Lashof algebra. *Trans. Amer. Math. Soc.* **185** (1973), 83–136. MR0331386, Zbl 0271.57013, doi: 10.2307/1996429. 1148

[19] KUDO, TATSUJI; ARAKI, SHÔRÔ. On $H_*(\Omega^N S^n; Z_2)$. *Proc. Japan Acad.* **32** (1956), 333–335. MR0079768, Zbl 0072.17904, doi: 10.3792/pja/1195525378. 1142

[20] KUDO, TATSUJI; ARAKI, SHÔRÔ. Topology of H_n-spaces and H-squaring operations. *Mem. Fac. Sci. Kyūsyū Univ. A.* **10** (1956), 85–120. MR0087948, Zbl 0074.38502, doi: 10.2206/kyushumfs.10.85. 1142

[21] MADSEN, IB. On the action of the Dyer–Lashof algebra in $H_*(G)$. *Pacific J. Math.* **60** (1975), no. 1, 235–275. MR0388392, Zbl 0313.55018, doi: 10.2140/pjm.1975.60.235. 1140, 1143

[22] MADSEN, IB H.; SNAITH, VICTOR P.; TORNEHAVE, JØRGEN. Infinite loop maps in geometric topology. *Math. Proc. Cambridge Philos. Soc.* **81** (1977), no. 3, 399–430. MR0494076, Zbl 0357.55015, doi: 10.1017/S0305004100053482. 1144

[23] MAY, J. PETER. The geometry of iterated loop spaces. Lecture Notes in Mathematics, 271. *Springer-Verlag, Berlin-New York*, 1972. viii+175 pp. MR0420610, Zbl 0244.55009. 1142, 1143

[24] MAY, J. PETER. E_∞ ring spaces and E_∞ ring spectra. Lecture Notes in Mathematics, 577. *Springer-Verlag, Berlin-New York*, 1977. 268 pp. MR0494077, Zbl 0345.55007, doi: 10.1007/BFb0097608. 1144

[25] MITCHELL, STEPHEN A.; PRIDDY, STEWARD B. Stable splittings derived from the Steinberg module. *Topology* **22** (1983), no. 3, 285–298. MR0710102, Zbl 0526.55010, doi: 10.1016/0040-9383(83)90014-9. 1139

[26] PENGELLEY, DAVID J.; WILLIAMS, FRANK. Global structure of the mod two symmetric algebra, $H_*(BO;\mathbb{F}_2)$, over the Steenrod algebra. *Algebr. Geom. Topol.* **3** (2003), 1119–1138. MR2012968, Zbl 1057.55004, arXiv:math/0312220, doi: 10.2140/agt.2003.3.1119. 1139

[27] PENGELLEY, DAVID; WILLIAMS, FRANK. A new action of the Kudo–Araki–May algebra on the dual of the symmetric algebras, with applications to the hit problem. *Algebr. Geom. Topol.* **11** (2011), no. 3, 1767–1780. MR2821440, Zbl 1220.55009, doi: 10.2140/agt.2011.11.1767. 1139

[28] PENGELLEY, DAVID; WILLIAMS, FRANK. Sparseness for the symmetric hit problem at all primes. *Math. Proc. Cambridge Philos. Soc.* **158** (2015), no. 2, 269–274. MR3310245, Zbl 1371.55013, doi: 10.1017/S0305004114000668. 1139

[29] PRIDDY, STEWART. Dyer–Lashof operations for the classifying spaces of certain matrix groups. *Quart. J. Math. Oxford Ser.* (2) **26** (1975), no. 102, 179–193. MR0375309, Zbl 0311.55013, doi: 10.1093/qmath/26.1.179. 1135, 1139, 1143, 1144, 1148

[30] PRIDDY, STEWART. Lectures on the stable homotopy of BG. *Proceedings of the School and Conference in Algebraic Topology*, 289–308, Geom. Topol. Monogr., 11. *Geom. Topol. Publ., Coventry*, 2007. MR2402810, Zbl 1139.55301, arXiv:0710.5942, doi: 10.2140/gtm.2007.11.289. 1138, 1156

[31] RANDAL-WILLIAMS, OSCAR. The homology of the stable nonorientable mapping class group. *Algebr. Geom. Topol.* **8** (2008), no. 3, 1811–1832. MR2448874, Zbl 1211.55007, arXiv:0803.3825, doi: 10.2140/agt.2008.8.1811. 1154

[32] REPKA, JOE; SELICK, PAUL S. On the subalgebra of $H_*((RP^\infty)^n;F_2)$ annihilated by Steenrod operations. *J. Pure Appl. Algebra* **127** (1998), no. 3, 273–288. MR1617199, Zbl 0926.55013, doi: 10.1016/S0022-4049(96)00177-6. 1140

[33] SINGER, WILLIAM M. Rings of symmetric functions as modules over the Steenrod algebra. *Algebr. Geom. Topol.* **8** (2008), no. 1, 541–562. MR2443237, Zbl 1137.13005, doi: 10.2140/agt.2008.8.541. 1139, 1140, 1146

[34] SNAITH, VICTOR P. Stable homotopy around the Arf–Kervaire invariant. Progress in Mathematics, 273. *Birkhäuser Verlag, Basel*, 2009. xiv+239 pp. ISBN: 978-3-7643-9903-0. MR2498881, Zbl 1169.55001, doi: 10.1007/978-3-7643-9904-7. 1143, 1144, 1148

[35] SUM, NGUYEN. The negative answer to Kameko's conjecture on the hit problem. *Adv. Math.* **225** (2010), no. 5, 2365–2390. MR2680169, Zbl 1203.55009, doi: 10.1016/j.aim.2010.04.026. 1141

[36] WALKER, GRANT; WOOD, REGINALD M. W. Polynomials and the mod 2 Steenrod algebra. Vol. 1. The Peterson hit problem. London Mathematical Society Lecture Note Series, 441. *Cambridge University Press, Cambridge,* 2018. xxiv+346 pp. ISBN: 978-1-108-41448-7; 978-1-108-41406-7. MR3729477, Zbl 1387.55001, doi: 10.1017/9781108333368. 1134

[37] WALKER, GRANT; WOOD, REGINALD M. W. Polynomials and the mod 2 Steenrod algebra. Vol. 2. Representations of $GL(n;\mathbb{F}_2)$. London Mathematical Society Lecture Note Series, 442. *Cambridge University Press, Cambridge*, 2018. xxiv+355 pp. ISBN: 978-1-108-41445-6; 978-1-108-41406-7. MR3729478, Zbl 1390.55001, doi: 10.1017/9781108304092. 1134

[38] WELLINGTON, ROBERT J. The A-algebra $H_*\Omega^{n+1}\Sigma^{n+1}X$, the Dyer-Lashof algebra, and the Λ-algebra. Ph.D. Thesis, *The University of Chicago*, 1977. MR2611762. 1139, 1143

[39] WELLINGTON, ROBERT J. The unstable Adams spectral sequence for free iterated loop spaces. *Mem. Am. Math. Soc.* **36** (1982), no. 258, viii+225 pp. MR0646741, Zbl 0488.55006, doi: 10.1090/memo/0258. 1136, 1139, 1141, 1142, 1143

[40] WOOD, REGINALD M. W. Problems in the Steenrod algebra. *Bull. Lond. Math. Soc.* **30** (1998), no. 5, 449–517. MR1643834, Zbl 0924.55015, doi: 10.1112/S002460939800486X. 1134

[41] WOOD. REGINALD M. W. Hit problems and the Steenrod algebra. Lecture notes, University of Ioannina, Greece, June 2000. 1141, 1156

[42] ZARE, HADI. On spherical classes in H_*QS^n. PhD thesis, *University of Manchester*, 2009. 1143

[43] ZARE, HADI. A note on A-annihilated generators of H_*QX. *Glasg. Math. J.* **62** (2020), no. 2, 281–295. MR4085043, Zbl 1442.55013, doi: 10.1017/s0017089519000090. 1143

(Hadi Zare) SCHOOL OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, COLLEGE OF SCIENCE, UNIVERSITY OF TEHRAN, TEHRAN 14174, IRAN
hadi.zare@ut.ac.ir

This paper is available via http://nyjm.albany.edu/j/2021/27-45.html.